A new series of axial algebras of Monster type $(2\eta, \eta)$ related to the extended symmetric groups

by

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Abstract

The main aim of this work is to construct several new series of axial algebras of Monster type $(2\eta, \eta)$. These arise as subalgebras of the Matsuo algebras generated by singles and doubles with respect to a flip, an automorphism of order 2. First, we construct a new series of algebras of dimension n^2 in an *ad hoc* way, as subalgebras of the Matsuo algebra $M_\eta(O^{\epsilon}_{n+1}(3))$ and investigate the properties of these new algebras. Towards the end of the thesis, we determine all classes of flips σ for the extended symmetic group $2^{n-1} : S_n$. Then, for each class of flips, we construct the fixed subalgebras M_σ and determine their basic combinatorial properties. Furthermore, we identify the values of the parameter η, for which these new algebras are not simple and determine the dimension of the radical in each case.

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CHAPTER 1

INTRODUCTION

1.1 Background

The field of axial algebras has sprung, in a sense, from a single example. In 1973, Fischer and Griess predicted the existence of the largest sporadic finite simple groups, the *Monster* M of order approximately 8×10^{53}. The character table of M was found, predicated on the existence of the smallest non-trivial irreducible representation of dimension 196883. It was Norton, who observed that this irreducible representation carries the structure of a commutative non-associative algebra and it was exactly this algebra that Griess [16] used in 1982 to show the existence of M.

Shortly after that, Conway [7] improved the construction and produced the unital *Griess algebra* V of dimension $1 + 196883 = 196884$. Discovery of M led to a host of conjectures, the main one of them being the *Monstrous Moonshine* conjecture of Conway and Norton [8], generalising earlier observations of McKay and Thompson, namely, that M acts on a graded module, whose graded dimension is the modular invariant

$$J(q) = q^{-1} + 196884q + 21493760q^2 + \dots$$

and the entire graded character consists of modular functions known as *Hauptmoduln*.

Initially, this conjecture was verified by Atkin, Fong and Smith using some recurrence relations satisfied by the coefficients of modular functions. The first explicit construction was proposed by Frenkel, Lepowsky and Meurman [12] using vertex operators from quantum field theory. Later this was generalised by Borcherds [4] who developed the theory of vertex algebras and observed that the graded module acted on by M is in fact a vertex operator algebra (VOA) $V^\natural$. As the coefficients above suggest, the Griess algebra is simply the weight 2 component of the VOA $V^\natural$.

Nearly ten years later, Miyamoto [34] realised that the special elements of V, the $2A$-axes of Norton, are conformal elements of $V^\natural$, rescaled and renamed as *Ising vectors*, generating Virasoro subalgebras of $V^\natural$ in a specific action. Moreover, he noticed that the Ising vectors satisfy specific C_2-graded *fusion laws* (Definition 2.1.1) and this forces $V^\natural$ (and its component V) to have automorphisms, called *Miyamoto involutions* (Section 2.4), one for each Ising vector. In the Griess algebra V, the adjoint of a $2A$-axis a satisfies the following *Monster fusion law*:

$*$	1	0	$\frac{1}{4}$	$\frac{1}{32}$
1	1		$\frac{1}{4}$	$\frac{1}{32}$
0		0	$\frac{1}{4}$	$\frac{1}{32}$
$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$1, 0$	$\frac{1}{32}$
$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{32}$	$1, 0, \frac{1}{4}$

Table 1.1: Monster fusion law

Miyamoto asked which VOA are generated by a pair of Isiang vectors and he managed to do some cases. In 2007, Sakuma [36] completed this project, having classified weight 2 subalgebras in the VOA generated by two Ising vectors. Namely he proved that every such weight 2 component is isomorphic to one of eight specific algebras, all of them arising as subalgebras of the Griess algebra V.

Ivanov [23] extracted the properties used by Sakuma in his proof. These properties became the axioms of a new class of commutative non-associative algebras, called the *Majorana algebras*. Naturally, the Griess algebra V is a Majorana algebra. Majorana algebras are generated by special idempotents whose adjoint action is semi-simple with all eigenvalues in $\{1, 0, \frac{1}{4}, \frac{1}{32}\}$ and multiplication of eigenvectors is controlled by the Monster fusion law from Table 1.1. Thus, Majorana algebras became the first class of non-associative algebras, whose axioms involve a fusion law. There were also further axioms, which we omit for now. The graded nature of the Monster fusion law means that, for every axis a in a Majorana algebra, it admits an automorphism τ_a of order two, called the *Majorana involution* of a. Thus, every Majorana algebra admits a significant (definitely, non-trivial) automorphism group.

In 2011, Ivanov, Pasechnik, Seress, and Shpectorov [22] proved a version of the Sakuma theorem for Majorana algebras, i.e., they classified Majorana algebras generated by two Majorana axes. Equivalently, since two involutions always generate a dihedral group, these are the algebras having dihedral automorphism groups. In the same paper [22] the authors also determined Majorana algebras for the first non-dihedral group, the symmetric group S_4. Further assumptions were utilised in this additional proof and they were added to the axioms of Majorana algebras.

At about the same time, competing programs in the computer algebra system GAP were created by Seress [38] and by Shpectorov, that were able to find Majorana algebras for various further small groups. This experience of explicitly computing Majorana convinced Shpectorov that the fusion law itself was strong enough in most cases and the additional axioms were not necessary. Thus, the axial algebras were born. They were defined in 2015 in broad generality by Hall, Rehren, and Shpectorov in [17, 18]. These allowed an arbitrary fusion law, arbitrary field, and no further axioms, apart from commutativity and primitivity of axes. According to this new paradigm, different fusion laws define different subclasses of axial algebras.

It turned out the class of Jordan algebras also exhibits axial behaviour. They were introduced in 1933 by Pascual Jordan [26, 27] to represent algebras of states of quantum systems. They are commutative algebras A that satisfy the Jordan identity:

$$x^2(yx) = (x^2y)x,$$

for all $x, y \in A$

Idempotents in a Jordan algebra satisfy the Peirce decomposition amounting to the following fusion law:

$*$	1	0	$\frac{1}{2}$
1	1		$\frac{1}{2}$
0		0	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$1, 0$

Table 1.2: Fusion law $\mathcal{J}(\frac{1}{2})$

A *group of* 3*-transpositions* is a pair (G, C), where G is a group, C a normal set (union of conjugacy classes) of involutions, i.e., elements of order 2, such that $G = \langle C \rangle$ and for all $a, b \in C$ we have that $|ab| \leq 3$. Given a group of 3-transpositions, a field $\mathbb{F}$ of characteristic not two, and $\eta \in \mathbb{F}$ with $\eta \neq 1, 0$, we can define the Matsuo algebra $M_\eta(G, C)$ (Definition 3.3.1). This is an *algebra of Jordan type* η, the class of axial algebras introduced in [18], which is defined by the fusion law as in Table 1.2, but with η in place of $\frac{1}{2}$. (In particular, Jordan algebras generated by primitive axes are algebras of Jordan type $\frac{1}{2}$.) In [28, 29] Joshi introduced the concept of a double axis, and in [14], Galt, Joshi, Mamontov, Shpectorov, and Staroletov introduced the flip construction that produces, for an involution (flip) σ of (G, C) a subalgebra generated by single and double axes (see Section 5.1 for the definitions) fixed by σ. This flip subalgebra satisfies the fusion law of Monster type $(2\beta, \beta)$ (as in Table 1.1, but with 2β and β in place of $\frac{1}{4}$ and $\frac{1}{32}$). In this way, we obtain a rich family of examples of algebras of Monster type. It is a very interesting task to study such flip subalgebras and determine their properties.

The case of $G = S_n$ was completed in [29, 14]. In particular, they constructed a new infinite series of algebras $Q_k(\eta)$ of dimension k^2 generated by k single axes and $k^2 - k$ double axes. Other cases of almost simple 3-transposition groups and their flips have also been considered by various authors, with papers currently being prepared for publication. In [1], I constructed an infinite series of algebras $Q^k(\eta)$ within the Matsuo algebra for the group $O^+_{k+1}(3)$, which is, in a sense, dual to $Q_k(\eta)$. Namely, $Q^k(\eta)$ is generated by $k^2 - k$ single axes and k double axes. It was later understood that the origin of this new series $Q^k(\eta)$ is not so much the Matsuo algebra of $O^+_{k+1}(3)$, but rather its proper Matsuo subalgebra $M_\eta(2^k : S_{k+1})$.

In this thesis, we completely classify all flips of $G = 2^{n-1} : S_n$ (where we take $n = k + 1$), determine the corresponding flip subalgebras and discuss their properties, including their critical values of η and the dimension of the corresponding radical.

1.2 Discussion

As discussed above, in this thesis, we utilise the flip construction to build new series of axial algebras of Monster type $(2\eta, \eta)$ generated by single and double axes. First, we deal with the case of Matsuo algebras $M_\eta({}^-O^+_{n+1}(3))$ corresponding to a class of reflections of the isometry group of a non-degenerate orthogonal space over $\mathbb{F}_3$. In these Matsuo algebras, we create a new series of subalgebras of dimension n^2.

Namely, let $G = GO^+_{n+1}(3)$ be a group of orthogonal transformations of a vector space V of dimension $n + 1$ over $\mathbb{F}_3$, and let $\{e_0, e_1, \ldots, e_n\}$ be an orthonormal basis (this means that e_i are pairwise orthogonal and

$$(e_0, e_0) = (e_1, e_1) = \cdots = (e_n, e_n) = 1.$$

Suppose that C is the set of reflections and let $G = \langle C \rangle \leq GO^+_{n+1}(3)$. Then (G, C) is a 3-transposition group.

Again suppose that $\mathbb{F}$ is a field of characteristic not equal to 2 and let $\eta \in \mathbb{F}$, $\eta \neq 0, 1$. Recall from ([18], Section 6) that the Matsuo algebra $M_\eta(G, C)$ over $\mathbb{F}$ corresponding to the 3-transposition group (G, C) has a basis C and the multiplication of two basis elements $a, b \in C$ is given by

$$a \cdot b = \begin{cases} a, & \text{if } a = b; \\ 0, & \text{if } a \neq b \text{ and } ab = ba; \\ \frac{\eta}{2}(a + b - c), & \text{if } a \neq b \text{ and } a^b = b^a =: c. \end{cases}.$$

The elements $a \in C$ are the axes of the axial algebra $M_\eta(G, C)$, and we call them *single axes*. The sums of two orthogonal single axes $a + b$ are called *double axes* (Definition 5.1.1). In the following theorem, we determine elements of C with the corresponding 1-spaces in the orthogonal space V and construct primitive axial subalgebras of dimension n^2 that are generated by single and double axes.

Theorem 1.2.1. Let A be the subspace of the Matsuo algebra $M_\eta({}^-O^+_{n+1}(3))$ spanned by the set of single axes $S = \{\langle e_i + \epsilon e_j \rangle : 1 \leq i < j \leq n, \epsilon = \pm 1\}$ and the set of double axes $D = \{\langle e_0 + e_i \rangle + \langle e_0 - e_i \rangle \mid 1 \leq i \leq n\}$. Then A is a primitive axial algebra of Monster type $(2\eta, \eta)$ of dimension $|S| + |D| = n(n-1) + n = n(n-1+1) = n^2$.

Moreover, we study the simplicity of the constructed subalgebras A. In Proposition 6.4.2 we show that no proper ideal of A can contain one of the generating axes in $S \cup D$. By [30], this means that A is simple exactly when the Frobenius form on A has zero radical (equivalently the Gram matrix of the Frobenius form on A has non-zero determinant). We compute in GAP [15], the determinant as a polynomial in η and we find its roots for $n \leq 14$. Based on this, we have several precise conjectures describing the roots and their multiplicities for arbitrary n.

Conjecture 1.2.2. The determinant of the Gram matrix G is a polynomial of degree $\frac{n(n+1)}{2}$, unless $n = 3$.

Conjecture 1.2.3. The multiplicity of the eigenvalue $\frac{1}{2}$ is

$$\frac{n(n-1)}{2} - 1 = \frac{n^2 - n - 2}{2} = \frac{(n+1)(n-2)}{2}.$$

Conjecture 1.2.4. The Gram matrix G has eigenvalue $-\frac{1}{n-3}$ with multiplicity n.

Conjecture 1.2.5. There is just one further simple eigenvalue $-\frac{1}{2(n-1)}$ with multiplicity 1.

Next, we consider the extended symmetric group $2^{n-1} : S_n$. Let $v_i = (1, \dots, 1, -1, 1, \dots, 1)$, where -1 appears in the ith position. (This notation is re-introduced and utilised in Section 7.1.) Suppose that $G = \langle C \rangle$ and C is the set consisting of all elements $e_{ij} = (i, j)$ and $f_{ij} = v_{ij}(i, j)$, where $1 \leq i < j \leq n$ and $v_{ij} = v_i v_j = (1, \dots, 1, -1, 1, \dots, 1, -1, 1, \dots, 1)$, where -1 are in the ith and jth positions. Then (G, C) is a 3-transposition group. Furthermore, in this case we have four types of flips σ:

1. $\sigma = g_{k,0} = \sigma_k = (1,2)(3,4)\cdots(2k-1, 2k)$, $n = 2k$.
2. $\sigma = g_{0,r} = w_r = (1, 1, \dots, 1, -1, -1, \dots, -1)$, where r is the number of -1s and $r \leq \frac{n}{2}$.
3. $\sigma = g_{k,r} = (1, 1, \dots, 1, -1, \dots, -1) : (1,2)(3,4)\cdots(2k-1, 2k)$, $n \geq 2k + r$.
4. $\sigma = h_k = (-1, 1, -1, 1, \dots, -1, 1) : (1,2)(3,4)\cdots(2k-1, 2k)$, $n = 2k$.

For each class of flips σ we have the corresponding flip subalgebaras as follows:

Theorem 1.2.6. The fixed subalgebra M_σ of the Matsuo algebra $M = M_\eta(2^{2k-1} : S_{2k})$ contains precisely $2k$ single axes, $2k(k-1)$ double axes, and no extras. Therefore, the dimension of the flip subalgebra $A_\sigma = M_\sigma$ is $2k + 2k(k-1) = 2k^2$.

Theorem 1.2.7. The fixed subalgebra M_σ of the Matsuo algebra $M = M_\eta(2^{n-1} : S_n)$ contains $r(r-1) + (n-r)(n-r-1)$ single axes, $r(n-r)$ double axes, and no extras. Therefore, the dimension of $A_\sigma = M_\sigma$ is $r^2 + n(n-r-1)$.

Theorem 1.2.8. The flip subalgebra A_σ is the direct sum of two subalgebras isomorphic to $2Q_k(\eta)$ and $R_{n-2k,r}$.

Theorem 1.2.9. The fixed subalgebra M_σ of the Matsuo algebra $M = M_\eta(2^{n-1} : S_n)$ contains no single axes, $\frac{n(n-1)}{2}$ double axes, and no extras. Therefore, the dimension of the flip subalgebra $A_\sigma = M_\sigma$ is $\frac{n(n-1)}{2}$.

In Chapter 2, we give the basic background of axial algebras and it is divided into four sections. In the first section, we provide definitions and some examples of axial algebras. After that, we introduce Seress property and its application. Then, we discuss the primitive axial algebra of Jordan type η and finding the eigenspaces for the adjoint map ad_a. We conclude this chapter by defining the grading and automorphisms.

The goal of Chapter 3 is to study Matsuo algebras. First, we discuss the groups of 3-transpositions and their correspondence with Fischer spaces. Then we define Matsuo algebras.

In Chapter 4, we study the structure of axial algebras. In particular, we define closed set of axes, the radical, Frobenius form, and Gram matrix. Later, in Chapter 6, we use the computer algebra system GAP [15] to determine the Gram matrix for subalgebras we construct, for small n.

In Chapter 5, we review some of Joshi's work from [29]. In particular, we discuss double axes, fixed subalgebra $Q_k(\eta)$ of dimension k^2. We conclude this chapter by describing the ideals and the radical of $M_\eta(S_{2k})$.

The aim of Chapter 6 is to construct a new series of axial algebras of Monster type $(2\eta, \eta)$ related to the orthogonal groups $O^{\epsilon}_{n+1}(3)$. Firstly, we define the involutions (classes of reflections). Then we construct the fixed subalgebra generated by $n(n_1)$ single axes and n double axes. Furthermore, we investigate when these algebras are simple. We formulate three conjectures based on the calculation, using GAP, of the roots of the determinant of the Gram matrix of the Frobenius form for several small values of n.

In Chapter 7, we construct new series of axial algebras of Monster type $(2\eta, \eta)$ related to the extended symmetric groups $2^{n-1} : S_n$. We begin this chapter by determining the class C and we prove the key lemmas. In the next section, we discuss the automorphism of (G, C). Then we determine the classes of flips. In particular, we have two classes of flips $\sigma = g_{k.r}$ and $\sigma = h_k$. Moreover, we construct the fixed subalgebras M_σ corresponding to σ. Finally, we calculate the critical values of η and then discuss when these new fixed algebras are simple.

In Chapter 8, we provide a conclusion of this work and also may include some conjectures. Finally, we provide recommendations for the future study.

1.3 Notation

In the following table we list the most common used notations.

Notation	**Description**
$A, B, C, \ldots$	Algebraic structure.
$a, b, c, \ldots$	Elements in algebraic structure.
$\alpha, \beta, \gamma, \ldots$	Scalars.
$\mathcal{G}, \mathcal{L}, \ldots$	Graphs.
$\mathbb{F}$	A field.
$\mathcal{F}$	Fusion rules $* : \mathcal{F} \times \mathcal{F} \to 2^{\mathcal{F}}$.
ad_a	The adjoint map in $\mathrm{End}(A)$.
$\lambda, \mu, \ldots$	Eigenvalues in $\mathcal{F}$.
$A_\lambda(a)$	λ-eigenspace of ad_a.
τ_a	Miyamoto involution.
X	The set of generating axes.
$G(X)$	Miyamoto group.
T_a	Axis subgroups.
C	A set of 3-transposition.
(G, C)	A 3-transposition group.
$M_\eta(\mathcal{G}, \mathbb{F})$	Matsuo algebra of the Fischer space $\mathcal{G}$ over a field $\mathbb{F}$.
$(.,.)$	The bilinear form.
$A^\perp$	The radical of the Frobenius form.
$Ann(A)$	The annihilator of A.
Gr	The Gram matrix.
$Q_k(\eta)$	Fixed subalgebra of dimension k^2 in the Matsuo algebra $M_\eta(S_{2k})$.
$Q^k(\eta)$	The subalgebra A of $M_\eta(O^+_{k+1}(3))$ of dimension k^2.
$2Q_k(\eta)$	Flip subalgebra of dimension $2k^2$ in the Matsuo algebra $M_\eta(2^{2k-1} : S_{2k})$.
$M_\eta({}^-O^+_{n+1}(3))$	Matsuo algebra of the orthogonal group over $\mathbb{F}_3$.
$R_{n,r}$	Flip subalgebra of dimension $r^2 + n(n-r-1)$ in the Matsuo algebra $M_\eta(2^{n-1} : S_n)$.
H_k	Flip subalgebra of dimension $\frac{n(n-1)}{2}$ in the Matsuo algebra $M_\eta(2^{n-1} : S_n)$.
r_u	Reflection in a nonsingular vector u.
$\det(G)$	The determinant of G.
$G = 2^{n-1} : S_n$	The extended symmetric group.
(v, σ)	Elements in G, where $v = (\delta_1, \delta_2, \ldots, \delta_n) \in V$ and $\sigma \in S_n$.
$\hat{A}$	Ambient subalgebra.
η_i	Critical values.
n_i	Multiplicity of η_i.

CHAPTER 2

AXIAL ALGEBRAS

In this chapter we provide basic definitions related to axial algebras, introduce key examples and develop their properties.

2.1 Definitions and examples of axial algebras

In this thesis an algebra is a vector space with a bilinear product. In particular, algebras are non-associative, i.e., associativity is not assumed. Suppose that A is a commutative algebra over a field $\mathbb{F}$. For an arbitrary $a \in A$, we write ad_a for the *adjoint map* in $\mathrm{End}(A)$ that is given by $\mathrm{ad}_a : b \mapsto ab$. The eigenvalues, eigenvectors and eigenspaces of a are the eigenvalues, eigenvectors and eigenspaces of ad_a, respectively. The element a is said to be diagonalisable if ad_a is diagonalisable as a matrix, that is, there exists a basis of A consisting of eigenvectors of ad_a. For $\lambda \in \mathbb{F}$, we write

$$A_\lambda(a) = \{b \in A | ab = \lambda b\}$$

for the λ-eigenspace of ad_a. This is trivial when λ is not an eigenvalue of ad_a. For $\mathcal{F} \subset \mathbb{F}$, we set

$$A_\mathcal{F}(a) = \oplus_{\lambda \in \mathcal{F}} A_\lambda(a).$$

Note that $A_\emptyset(a) = 0$.

Definition 2.1.1 ([18])**.** A *fusion law* is a set $\mathcal{F}$ together with a map $* : \mathcal{F} \times \mathcal{F} \to \mathcal{P}(\mathcal{F})$, where $\mathcal{P}(\mathcal{F})$ is the power set of $\mathcal{F}$.

We will represent fusion laws by tables similar to group multiplication tables. In the cell corresponding to $\lambda, \mu \in \mathcal{F}$, we simply list the elements of the subset $\lambda * \mu$. In particular, if $\lambda * \mu = \emptyset$ then we leave the cell empty.

Definition 2.1.2. The fusion law $\mathcal{F}$ is *symmetric* (or commutative) if $\lambda * \mu = \mu * \lambda$ for all $\lambda, \mu \in \mathcal{F}$,

Example 2.1.3. For $\mathcal{A} = \{0, 1\} \subseteq \mathbb{F}$, consider the fusion law in Table 2.1. Here $1*1 = \{1\}$,

$*$	1	0
1	1	
0		0

Table 2.1: Fusion law $\mathcal{A}$

$0 * 0 = \{0\}$ and $1 * 0 = 0 * 1 = \emptyset$. Manifestly, this fusion law is symmetric.

Let again A be a commutative algebra over $\mathbb{F}$ and let $\mathcal{F} \subseteq \mathbb{F}$ be a symmetric fusion law.

Definition 2.1.4. An non-zero idempotent $a \in A$ is an *$\mathcal{F}$-axis* if

(1) $A = A_{\mathcal{F}}(a)$; that is, a is diagonalisable and all of its eigenvalues lie in $\mathcal{F}$; and

(2) for all $\lambda, \mu \in \mathcal{F}$, $A_\lambda(a)A_\mu(a) \subseteq A_{\lambda*\mu}(a)$; that is, every product uv of a λ-eigenvector u and μ-eigenvector v is a sum of some ν-eigenvectors, for $\nu \in \lambda * \mu$.

Notice that if a is an idempotent then 1 is an eigenvalue of ad_a. Hence we always assume that $1 \in \mathcal{F}$.

Definition 2.1.5. An axis $a \in A$ is *primitive* if $A_1(a) = \mathbb{F}a$; i.e., it is 1-dimensional.

Definition 2.1.6. The algebra A is an *$\mathcal{F}$-axial algebra* if it is generated by a set of $\mathcal{F}$-axes. The algebra A is said to be *primitive* if the generating axes are primitive.

We will speak of simply axes and axial algebras when the fusion law $\mathcal{F}$ is clear from the context.

Example 2.1.7. The *Griess algebra* V over $\mathbb{R}$ is of dimension $196,884$. This algebra is primitive axial for the fusion law given in Table 1.1.

Example 2.1.8. Jordan algebras are also examples of axial algebras. Recall that a Jordan algebra is a commutative algebra A satisfying the Jordan identity:

$$x(yx^2) = (xy)x^2,$$

for all $x, y \in A$.

If A is a Jordan algebra and $a \in A$ is an idempotent then we have the *Peirce decomposition*:

$$A = A_1(a) \oplus A_0(a) \oplus A_{\frac{1}{2}}(a),$$

where the summands satisfy the multiplication properties:

$$A_1(a)A_1(a) \subseteq A_1(a), \quad A_1(a)A_0(a) = 0, \quad A_1(a)A_{\frac{1}{2}}(a) \subseteq A_{\frac{1}{2}}(a),$$

$$A_0(a)A_{\frac{1}{2}}(a) \subseteq A_{\frac{1}{2}}(a), \quad A_{\frac{1}{2}}(a)A_{\frac{1}{2}}(a) \subseteq A_1(a) \oplus A_0(a).$$

One can easily see that this amounts to the condition that every non-zero idempotent in a Jordan algebra is a $\mathcal{J}(\frac{1}{2})$-axis, where the fusion law $\mathcal{J}(\frac{1}{2})$ is given in Table 1.2.

Note that not every Jordan algebra contains non-zero idempotents. For example, in a nilpotent Jordan algebra, every idempotent must be zero. However, when a Jordan algebra is generated by a set of idempotents, it is an example of an axial algebra for the above fusion law.

Based on this example, Hall, Rehren, and Shpectorov in [18] introduced the class of axial algebras of Jordan type η (for $\eta \in \mathbb{F}$, $\eta \neq 0, 1$) as primitive axial algebras for the fusion law $\mathcal{J}(\eta)$, shown in Table 2.2. Clearly, a Jordan algebra generated by primitive idempotents is

$*$	1	0	η
1	1		η
0		0	η
η	η	η	$1+0$

Table 2.2: Fusion law $J(\eta)$

an example of algebras of Jordan type $\frac{1}{2}$. We will introduce further examples of algebras of Jordan type in the next chapter.

The following is an example of an algebra of Jordan type $\frac{1}{2}$.

Example 2.1.9. Consider a field $\mathbb{F}$ of characteristic not equal to two and let V be a vector space with a quadratic form $q : V \to \mathbb{F}$. The associated bilinear form $F(u, v)$ is given by:

$$q \to F(u, v) := \frac{1}{2}(q(u + v) - q(u) - q(v)).$$

Then the *Clifford algebra* $C = Cl(V, q)$ is defined as;

$$C := T(V)/\langle v^2 - q(v)\rangle,$$

where $T(V)$ is the tensor algebra,

$$T(V) = \mathbb{F} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \dots,$$

and $q(v) = F(v, v)$. This is an associative non-commutative algebra of dimension equal to $2^{\dim(V)}$. Define a Jordan algebra $C^J = (C, \frac{u\otimes v + v\otimes u}{2})$, let B be a subspace $\mathbb{F} \oplus V$ of C^J. It is easy to check that B is a subalgebra. Indeed, for every $u, v \in V$ we have that $u \circ v = \frac{uv+vu}{2} = F(u, v) \in \mathbb{F}$. Then take two elements $(\alpha, u), (\beta, v) \in B$, so we have

$$\begin{aligned}
(\alpha, u) \circ (\beta, v) &= \frac{1}{2}((\alpha, u)(\beta, v) + (\beta, v)(\alpha, u)),\\
&= \frac{1}{2}(\alpha\beta + \alpha v + u\beta + uv + \alpha\beta + \beta u + v\alpha + vu),\\
&= \alpha\beta + \frac{1}{2}(uv + vu) + \alpha v + \beta u.
\end{aligned}$$

Since $\alpha\beta + \frac{1}{2}(uv + vu) \in \mathbb{F}$ and $\alpha v + \beta u \in V$, we have that $(\alpha, u) \circ (\beta, v) \in B$. So B is a subalgebra of C^J. Since C^J is a Jordan algebra, we also have that B is a Jordan algebra.

To find the idempotents in B, let $a = \alpha + u$, then

$$\begin{aligned}
a^2 &= (\alpha + u)(\alpha + u),\\
&= \alpha^2 + 2\alpha u + u^2,\\
&= \alpha^2 + q(u) + u^2.
\end{aligned}$$

If $a^2 = a$, so we have that $2\alpha u = u$ and this implies to $\alpha = \frac{1}{2}$. Also, $\alpha^2 + q(u) = \alpha$, hence $q(u) = \frac{1}{4}$. Define $\bar{a} = \alpha - u$. If a is an idempotent, then $\bar{a}$ is also an idempotent.

Now, we determine the eigenspaces for ad_a. Let $a = \frac{1}{2} + u \in B$ be an idempotent and suppose that $w \in u^\perp$ (this means that u and w are orthogonal). Then

$$aw = (\frac{1}{2} + u)w = \frac{w}{2} + uw = \frac{w}{2}.$$

Hence, we deduce that $u^\perp \subseteq B_{\frac{1}{2}}(a)$ and similarly we have $u^\perp \subseteq B_{\frac{1}{2}}(\bar{a})$.

Moreover, $a\bar{a} = (\frac{1}{2} + u)(\frac{1}{2} + u) = \frac{1}{4} + q(u) = 0$, since $q(u) = \frac{1}{4}$. This means that $\bar{a} \in B_0(a)$.

Since $\dim(B) = \dim(V) + 1$ and $\dim(u^\perp) = \dim(V) - 1$, we have that

$$B_1(a) = \langle a \rangle = B_0(\bar{a}),$$
$$B_0(a) = \langle \bar{a} \rangle = B_1(\bar{a}),$$
$$B_{\frac{1}{2}}(a) = \langle u^\perp \rangle = B_0(\bar{a}).$$

Then, by the Peirce decomposition, we have that $B = B_1(a) \oplus B_0(a) \oplus B_{\frac{1}{2}}(a)$.

2.2 Seress property

Suppose that A is a commutative algebra and $a \in A$ is a primitive axis for a fusion law $\mathcal{F}$.

Proposition 2.2.1. Under the above assumptions, for $\lambda \in \mathcal{F}$, we have

$$A_1(a)A_\lambda(a) = \begin{cases} A_\lambda(a) & \text{if } \lambda \neq 0, \\ 0 & \text{if } \lambda = 0. \end{cases}$$

Proof. Take $u \in A_1(a)$ and $v \in A_\lambda(a)$. By primitivity, $A_1(a) = \mathbb{F}a$. That is, $u = \alpha a$ for some $\alpha \in \mathbb{F}$. Therefore, $uv = \alpha av = \alpha \mathrm{ad}_a(v) = \alpha\lambda v$, since $v \in A_\lambda(a)$. Hence, $uv = \alpha\lambda v$. When $\lambda = 0$, this clearly means that $A_1(a)A_0(a) = 0$. If $\lambda \neq 0$ then, first of all, we have $\alpha\lambda v \in A_\lambda(a)$, i.e., $A_1(a)A_\lambda(a) \subseteq A_\lambda(a)$. On the other hand, for any $w \in A_\lambda(a)$, we can select $u = a$ (i.e., $\alpha = 1$) and $v = \frac{1}{\lambda}w$. Then we have that $uv = \alpha\lambda v = 1\lambda\frac{1}{\lambda}w = w$, which means that $A_1(a)A_\lambda(a) = A_\lambda(a)$. □

In view of this, for primitive axial algebras, we will assume that in the corresponding fusion law we have that

$$1 * \lambda = \begin{cases} \{\lambda\} & \text{if } \lambda \neq 0, \\ \emptyset & \text{if } \lambda = 0. \end{cases}$$

We can see this to hold in the examples of fusion laws above. We also note that, in those fusion laws, the element 0 behaves in a similar way to the element 1. Fusion laws having this additional property have a special name.

Definition 2.2.2. A fusion law $\mathcal{F}$ is *Seress* if $0 \in \mathcal{F}$ and

$$0 * \lambda = \begin{cases} \{\lambda\} & \text{if } \lambda \neq 1, \\ \emptyset & \text{if } \lambda = 1. \end{cases}$$

So we have the following.

Example 2.2.3. The fusion laws $\mathcal{A}$, $\mathcal{J}(\eta)$, and the fusion law of the Griess algebra in Table 1.1 are Seress.

Note that an axial algebra for a Seress fusion law will be also be called Seress.

Definition 2.2.4. Suppose that A is an algebra and $u, v \in A$. We say that u and v *associate* if $(uw)v = u(wv)$ for all $w \in A$.

The following is an important property of axial algebras with a Seress fusion law.

Proposition 2.2.5 (Seress)**.** Suppose that A is a Seress axial algebra and $a \in A$ is one of the generating axes. Then a associates with $A_1(a) \oplus A_0(a)$, that is,

$$a(uv) = (au)v,$$

for all $u \in A$ and $v \in A_1(a) \oplus A_0(a)$.

Proof. Since the equality $a(xu) = (ax)u$ is linear in x, it suffices to take x from any spanning set of A. Since the eigenvectors of ad_a form a spanning set of A, we can assume that x is an eigenvector, that is, $x \in A_\lambda(a)$ for some $\lambda \in \mathcal{F}$. In view of the Seress property, $A_\lambda(a)A_1(a) \subseteq A_\lambda(a)$ and $A_\lambda(a)A_0(a) \subseteq A_\lambda(a)$. This means that $xu \in A_\lambda(a)$. Therefore, $a(xu) = \lambda xu$. Also, $(ax)u = \lambda xu$. Hence the claim holds. □

Now, we provide an application of Seress property. Note that the fusion law $\mathcal{A}$ in Table 2.1 is Seress.

Proposition 2.2.6. Every primitive axial algebra with fusion law $\mathcal{A}$ is a direct sum of a number of copies of $\mathbb{F}$.

Proof. We need to show that for any algebra A with a primitive $\mathcal{A}$-axis a, we can write $A = \langle a \rangle \oplus A_0(a)$. It is easy to check that $A_0(a)$ is a subalgebra, and so A is a direct sum of decomposition subalgebras. Furthermore, every non-zero primitive idempotent b lies in $A_0(a)$, where $b \neq a$. If A is a primitive $\mathcal{A}$-axial algebra then $A_0(a)$ is also a primitive $\mathcal{A}$-axial algebra. To prove this, we write $b = \lambda a + u$, where $u \in A_0(a)$ and $\lambda \in \mathbb{F}$.

$$b = b^2 = \lambda^2 a^2 + 2au + u^2,$$

since $u \in A_0(a)$ we have $au = 0$. So $\lambda a = \lambda^2 a^2$ and $u = u^2$. Since $\lambda = 1, 0$ and $\langle a \rangle \cong \mathbb{F}a$ we only have two idempotents 0 and a. If $\lambda a = 0$, then $b \in A_0(a)$ as claimed. Assume that

$\lambda a = a$ and since $b \neq a$ we should have $u \neq 0$. Note that $ba = a$ and $bu = u$ and hence $a, u \in A_1(b)$. Thus b is not primitive, this is a contradiction. □

2.3 Algebras of Jordan type η

In this section we classify all 2-generated axial algebras A of Jordan type η. This means that there are two axes that together generate the whole algebra. Consider the fusion law of Jordan type $J(\eta)$ as shown in the example 2.2. Thus we write $A = \langle a, b \rangle$, for a and b are primitive axes. Set

$$\sigma := ab - \eta a - \eta b.$$

We need to prove that $\sigma \in A_1(a) + A_0(a)$, and also by symmetry $\sigma \in A_1(b) + A_0(b)$. Indeed, by using the fact that $A = A_1(a) \oplus A_0(a) \oplus A_\eta(a)$, we can write

$$b = \phi_a(b) \cdot a + b_0 + b_\eta,$$

where $\phi_a(b) \in \mathbb{F}, b_0 \in A_0(a)$ and $b_\eta \in A_\eta(a)$. We write ϕ to denote $\phi_a(b)$. Then we have

$$\begin{aligned}
\sigma = ab - \eta a - \eta b &= a(\phi a + b_0 + b_\eta) - \eta a - \eta(\phi a + b_0 + b_\eta), \\
&= (\phi - \eta - \eta\phi)a + (-\eta)b_0 + (\eta - \eta)b_\eta, \\
&\in A_1(a) \oplus A_0(a).
\end{aligned}$$

Since the fusion law $J(\eta)$ is Seress, and by Corollary 2.2.5, a associates with σ. By symmetry we deduce that $\sigma \in A_1(a) \oplus A_0(a)$.

Set $\pi = \pi_a(b) = \phi - \eta - \eta\phi$. Since a and b are interchanged, so we can write $\phi_b(a) = \phi'$ and then $\pi' = \pi_b(a) = \phi' - \eta - \eta\phi'$. Hence $\sigma = \pi a - \eta b_0$.

Now, we claim that A is spanned by a, b and σ. Thus $A = \langle a, b, \sigma \rangle$, this means A is at most 3-dimensional.

Firstly, we prove that $\pi = \pi'$ and $\phi = \phi'$. Notice that

$$\sigma a = (\pi a - \eta b_0)a = \pi a,$$

since $a^2 = a, ab_0 = 0$. Also, by symmetry $\sigma b = \pi' b$. Then we can write $\sigma(ab) = (\sigma a)b = \pi ab$, since σ and b associate. Thus,

$$\sigma^2 = \sigma(ab - \eta a - \eta b) = \pi ab - \eta\pi a - \eta\pi' b.$$

Then we have

$$\pi ab - \eta\pi a - \eta\pi' b = \pi' ab - \eta\pi' b - \eta\pi a,$$

so either $ab = 0$ or $\pi = \pi'$. If $ab = 0$, then $b = b_0$ and $\phi = \phi' = 0$. So $\pi = \pi' = -\eta$. Therefore, in both cases we get $\pi = \pi'$. Furthermore, if $\pi = \pi'$ then

$$\phi - \eta - \eta\phi = \pi = \phi' - \eta - \eta\phi',$$

so we have $(\phi - \phi')(1 - \eta) = 0$, but $\eta \neq 1$. Then $\phi = \phi'$. Therefore,

$$\sigma^2 = \pi ab - \eta\pi a - \eta\pi' b = \pi ab - \eta\pi a - \eta\pi b = \pi(\pi ab - \eta a - \eta b) = \pi\sigma.$$

Then there are two cases:

(i) If $ab = 0$, then A is spanned by a and b as a vector space, and $A_\eta(a) = A_\eta(b) = \{0\}$. The fusion law in this case as in Table 2.1..

(ii) If $ab \neq 0$, then A has a spanning set $\{a, b, \sigma\}$ and the multiplication between the elements is given in the folowing table:

$*$	a	b	σ
a	a	$\sigma + \eta a + \eta b$	πa
b	$\sigma + \eta a + \eta b$	b	πb
σ	πa	πb	$\pi\sigma$

Table 2.3: Multiplication table of the primitive axial algebras A

Consider a 2-generated primitive axial algebra $A = \langle\langle a, b\rangle\rangle = \langle a, b, \sigma\rangle$, and suppose that $\dim(A) = 3$. Then we determine the eigenspaces for ad_a. Since A is primitive, we have $A_1(a) = \langle a\rangle$.

Recall that $\sigma \in A_1(a) \oplus A_0(a)$, and let $u = \sigma + \alpha a \in A_0(a)$, for some $\alpha \in \mathbb{F}$. So

$$0 = ad_a u = au = a(\sigma + \alpha a) = \pi a + \alpha a,$$

then $\alpha = -\pi$ and so $u = \sigma - \pi a$. Hence $A_0(a) = \langle \sigma - \pi a \rangle$.

Now, we calculate the spanning set in $A_\eta(a)$. Let $v = \alpha a + \beta b + \gamma\sigma \in A_\eta(a)$, for some $\alpha, \beta, \gamma \in \mathbb{F}$. Then

$$\begin{aligned} \eta v &= av, \\ \eta\alpha a + \eta\beta b + \eta\gamma\sigma &= \alpha a + \beta ab + \gamma a\sigma, \\ &= \alpha a + \beta(\sigma + \eta a + \eta b) + \gamma\pi a, \\ &= (\alpha + \beta\eta + \gamma\pi)a + \beta\eta b + \beta\sigma. \end{aligned}$$

So, $\alpha\eta = \alpha + \beta\eta + \gamma\pi$. Take $\gamma = 1$ and $\beta = \eta$. Then we get $\alpha = \frac{\eta^2+\pi}{\eta-1}$. Since $\pi = \phi - \eta - \eta\phi$, we have that $\alpha = \frac{\eta^2+\phi-\eta-\eta\phi}{\eta-1} = \frac{(\eta-1)(\eta-\phi)}{\eta-1} = \eta - \phi$. Hence $v = (\eta - \phi)a + \eta b + \sigma$. Therefore, $A_\eta(a) = \langle (\eta - \phi)a + \eta b + \sigma \rangle$.

2.4 Grading and automorphisms

In this section we define a grading and related automorphisms. First, we provide the definition of a morphism of the fusion laws and then we define a group fusion law. The definitions and results in this part come from [10] and [30].

Definition 2.4.1. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two fusion laws. A *morphism* from $\mathcal{F}_1$ to $\mathcal{F}_2$ is a map $\alpha : \mathcal{F}_1 \to \mathcal{F}_2$ such that

$$\alpha(f_1 * f_2) \subseteq \alpha(f_1) * \alpha(f_2),$$

for all $f_1, f_2 \in \mathcal{F}_1$.

We also consider the natural extension of α to a map $2^{\mathcal{F}_1} \to 2^{\mathcal{F}_2}$, which will also be denoted by α. This makes the collection of all fusion laws into a category **Fus**.

Definition 2.4.2. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two fusion laws. Then we define the *product* of $(\mathcal{F}_1, *)$ and $(\mathcal{F}_2, *)$ to be the fusion law $(\mathcal{F}_1 \times \mathcal{F}_2, *)$ given by

$$(f_1', f_2') * (f_1'', f_2'') := \{(f_1, f_2) | f_1 \in f_1' * f_1'', f_2 \in f_2' * f_2''\}.$$

Furthermore, we define the *union* of $(\mathcal{F}_1, *)$ and $(\mathcal{F}_2, *)$ to be the fusion law $(\mathcal{F}_1 \cup \mathcal{F}_2, *)$, such that $*$ extends the fusion laws on $\mathcal{F}_1, \mathcal{F}_2$ and is given by

$$f_1 * f_2 := \emptyset$$

for all $f_1 \in \mathcal{F}_1, f_2 \in \mathcal{F}_2$,

Definition 2.4.3. Let T be a group. Then T together with the map

$$* : T \times T \to 2^T : (t_1, t_2) \to \{t_1 t_2\}$$

is a *group fusion law*. The identity element of a group T is the unique of the fusion law $(T, *)$.

Notice that the category of groups is a full subcategory of **Fus**. Namely, if T_1 and T_2 are groups, then the fusion law morphisms from $(T_1, *)$ to $(T_2, *)$ are precisely the homomorphisms arising from homomorphisms from T_1 to T_2.

Example 2.4.4. Suppose that G is a group and let X be the set of conjugacy classes of G. Then we define a fusion law on X as:

$$A \in B * C \Leftrightarrow A \cap BC \neq \emptyset$$

where BC is the setwise product of B and C in the group G. The conjugacy class $\{1\} \in G$ is a unit of this fusion law. If G is a finite abelian group, then this fusion law coincides with the group fusion law which is defined in the previous definition.

Definition 2.4.5. Let $\mathcal{F}$ be a fusion law and let $(T, *)$ be a group fusion law. A T*-grading* of $\mathcal{F}$ is a morphism $\alpha : \mathcal{F} \to (T, *)$. We call the grading *abelian* if T is an abelian group and we call it *adequate* if $\alpha(\mathcal{F})$ generates T.

Every fusion law admits a T-grading where T is the trivial group; we call this the *trivial* grading.

Let $\mathcal{F}$ be a fusion law. We say that a T-grading α of $\mathcal{F}$ is a *finest* grading of $\mathcal{F}$ if every grading of $\mathcal{F}$ factors uniquely through $(T, *)$, in other words, if for each T'-grading β of $\mathcal{F}$, there is a unique group homomorphism $\rho : T \to T'$ such that $\beta = \rho \circ \alpha$. (In categorical terms, this can be rephrased as the fact that α is an initial object in the category of gradings of $\mathcal{F}$.)

Similarly, we say that an abelian T-grading α of $\mathcal{F}$ is a finest abelian grading of $\mathcal{F}$ if every abelian grading of $\mathcal{F}$ factors uniquely through $(T, *)$

The proof of the next lemma is given in [10].

Lemma 2.4.6. Every fusion law $\mathcal{F}$ admits a unique finest grading given by the group with presentation

$$T_{\mathcal{F}} := \langle \gamma_f, f \in \mathcal{F} | \gamma_{f_1}\gamma_{f_2} = \gamma_{f_3} : f_3 \in f_1 * f_2 \rangle,$$

with grading map $\alpha : \mathcal{F} \to (T_{\mathcal{F}}, *) : f \to \gamma_f$. Similarly, there is a unique finest abelian grading, given by the abelianization $T_{\mathcal{F}}/[T_{\mathcal{F}}, T_{\mathcal{F}}]$ of $T_{\mathcal{F}}$. Both gradings are adequate.

Example 2.4.7. In the fusion law as in Table 2.2, let $T = \{1, -1\} \cong C_2$. Then the fusion law of V admits a C_2-grading. Indeed, the map $\alpha : \mathcal{F} \to C_2$ mapping $1, 0, \frac{1}{4}$ to 1 and $\frac{1}{32}$ to -1 is a morphism. Notice that this is the finest grading of the fusion law.

Similarly, the fusion law $\mathcal{J}(\eta)$ is C_2-graded: the map $\alpha : \mathcal{F} \to C_2$ mapping 1 and 0 to 1 and η to -1 is a morphism. Again, this is the finest grading of the Jordan fusion law.

Lemma 2.4.8. Suppose that $\mathcal{F}$ is a fusion law graded by T. If $\mathcal{F}$ is Seress, then the map $\alpha : \mathcal{F} \to C_2$ mapping 1 and 0 to 1.

Proof. Assume that the map α mapping 1 to t, where $t \in T$. Since $1 * 1 = \{1\}$, $t^2 = t$ and so $t = 1$. If $\mathcal{F}$ is Seress then $0 \in \mathcal{F}$ and $0 * 0 = \{0\}$, and so again if the map α mapping 0 to t then again $t^2 = t$. Then $t = 0$. Therefore, α mapping 1 and 0 to 1. $\square$

Consider again an arbitrary fusion law $\mathcal{F}$ and assume that A is an $\mathcal{F}$-axial algebra, and let $\alpha : \mathcal{F} \to T$ be a morphism.

Suppose that $\mathcal{F}$ is T-graded for some group T. Let T^* denotes the group of linear characters of T over $\mathbb{F}$ (i. e., homomorphisms $T \to (\mathbb{F}^*, \cdot)$).

For an axis a in A, we define a map $\gamma_a : T^* \to Aut(A)$, namely each $\chi \in T^*$ is mapped to the automorphism of A defined by

$$v^{\gamma_a(\chi)} := \chi(t)v,$$

where $v \in A_\mu(a)$. So $\gamma_a(\chi)$ acts on $A_\mu(a)$ as $\chi(t) \cdot Id$.

Since A is T-graded, the map $\gamma_a(\chi)$ is an automorphism. Indeed, since $\gamma_a(\chi)$ is $\mathbb{F}$-linear map, then we just need to check the product of two eigenvectors. Again let $\alpha : \mathcal{F} \to C_2 \cong \{t, t'\}$ be a fusion law morphism. Take $u \in A_\lambda(a)$ with $\lambda \mapsto t$, and $v \in A_\mu(a)$ with $\mu \mapsto t'$. Then

$uv \in \oplus A_v(a), v \in \lambda * \mu$. Hence by the grading rule $\lambda * \mu \mapsto tt'$. So

$$\begin{aligned} u^{\gamma_a(\chi)} \cdot v^{\gamma_a(\chi)} &= (\chi(t)a)(\chi(t')a), \\ &= \chi(t)\chi(t')uv, \\ &= \chi(tt')uv, \\ &= (uv)^{\gamma_a(\chi)}. \end{aligned}$$

Therefore, $\gamma_a(\chi)$ is an automorphism.

Proposition 2.4.9. Let a be an $\mathcal{F}$-axis and $s \in Aut(A)$. Then a^s is also an $\mathcal{F}$-axis. Moreover,

$$\alpha_{a^s}(\chi) = \alpha_a(\chi)^s.$$

Proof. Since s is an automorphism, then $A_\lambda(a)^s = A_\lambda(a^s)$.This means that s permutes the axes of A. Furthermore, for $u \in A$ we can write u as $u = \sum_{t \in T} u_t$, where $u_t \in Aut_\lambda(a), \lambda \mapsto t$. Also, we have $u^s = \sum_{t \in T} u_t^s$, where $u_t^s \in Aut_\lambda(a^s), \lambda \mapsto t$. Hence

$$\begin{aligned} \alpha_{a^s}(\chi) &= \sum_{t \in T} \chi(t) \cdot u_t^s \\ &= (\sum_{t \in T} \chi(t) \cdot u)^s \\ &= u^{\alpha_a(\chi)^s} \end{aligned}$$

Therefore, $\alpha_{a^s}(\chi) = \alpha_a(\chi)^s$. ☐

Definition 2.4.10. We call the image T_a of the map γ_a the *axis subgroup* of $Aut(A)$ corresponding to a.

Assume that $T \cong C_2$. If char $(\mathbb{F})$ is equal to 2, then $T^* = 1$ so in this case there is no automorphisms. Then suppose that char $(\mathbb{F}) \neq 2$, then $T^* = \{\chi_1, \chi_{-1}\}$, with $\gamma(\chi_1)$ is the identity automorphism, and

$$\gamma_a(\chi_{-1}) = \begin{cases} Id & \text{on} \oplus A_\lambda(a), \lambda \mapsto \chi_1 \\ -Id & \text{on} \oplus A_\lambda(a), \lambda \mapsto \chi_{-1} \end{cases}.$$

Moreover, $\gamma_a(\chi_{-1}) = Id$ if all odd eigenspaces are trivial.

Since we mostly deal with a C_2-graded fusion law, we will use the notation

$$\tau_a = \gamma_a(\chi_{-1}) \in Aut(A).$$

This is called the τ-involution corresponding to the axis a. Also it is known as the *Miyamoto involution*.

Definition 2.4.11. We define the *Miyamoto group* $G(X)$ of A with respect to the set of generating axes X as the subgroup of $Aut(A)$ generated by the axis subgroups T_a, where $a \in X$.

CHAPTER 3

MATSUO ALGEBRAS

The main purpose of this chapter is to introduce Matsuo algebras. First, we define groups of 3-transpositions and Fischer spaces.

3.1 Groups of 3-transpositions

Let G be a group generated by a normal set $C \subseteq G$ of involutions. Then C is called a set of 3-transpositions in G if $|ab| \in \{1, 2, 3\}$, for all $a, b \in C$. A pair (G, C) is called a 3-*transposition group* (cf. [3]).

Example 3.1.1. Any symmetric group S_n with the class C of transpositions is a group of 3-transpositions. Indeed, let $a, b \in C$, if $a = b$, then ab has order 1. Now take any two distinct transpositions $a, b \in S_n$, then ab has order 2, if a and b have disjoint support, and ab has order 3, if the supports of a and b meet in a single point.

H. Cuypers and J.I. Hall in [6] introduced two complementely graphs having the conjugacy class C of 3-transpositions as the set of vertices. One is the *commuting graph* on C, where two distinct involutions from C are adjacent whenever they commute. The other graph is the *complement* of the commuting graph and it is known as the *diagram* of C. In the diagram two involutions are adjacent whenever they commute. So, for each $a \in C$ we denote by C_a the set of neighbors of a in the commuting graph on C and by A_a the neighbours of a in complement graph, two elements $a, b \in C$ are equivalent if and only if they have the same set of neighbours.

For $c \in C$ define $A(c) = \{d \in C : |cd| = 2\}$ and $B(c) = \{d \in C : |cd| = 3\}$. We define two

G-invariant equivalent relations τ and θ on C as follows:

$$a\tau b \text{ if } b \in A(a),$$

and

$$a\theta b \text{ if } b \in B(a),$$

for $a, b \in C$. Correspondingly, we define two (normal) subgroups of G:

$$\tau(G) = \langle ab : a, b \in C, a\tau b\rangle,$$

and

$$\theta(G) = \langle ab : a, b \in C, a\theta b\rangle.$$

We also define $\beta(G) = \tau(G)\theta(G)$. The 3-transposition group is *irreducible* if $\beta(G) = \tau(G) = \theta(G) = 1$.

Fischer in 1971, proved (for finite G) that $\tau(G) = [O_2(G), G]$ and $\theta(G) = [O_3(G), G]$. Clearly, this means that all finite 3-transposition groups with no non-trivial solvable normal subgroups are irreducible. See [6]. The following theorem provides the statement of the classification of the finite irreducible 3-transposition groups and it is also known as Fischer's theorem:

Theorem 3.1.2 ([6])**.** Let (G, C) be a finite 3-transposition group, and suppose that G is irreducible. Then up to a center [1], the class C is identified as one of:

1. The transposition class of a symmetric group;
2. The transvection class of the isometry group of a non-degenerate orthogonal space over $\mathbb{F}_2$;
3. The transvection class of the isometry group of a non-degenerate symplectic space over $\mathbb{F}_2$;
4. One of the two reflection classes of the isometry group of a non-degenerate orthogonal space over $\mathbb{F}_3$;
5. The transvection class of the isometry group of a non-degenerate unitary space over $\mathbb{F}_4$;
6. Triality case, $O_8^+(2) : S_3$ or $O_8^+(3) : S_3$;

[1] Two groups G and H are isomorphic up to a center if $G/Z(G) \cong H/Z(H)$.

7. A unique class of involutions in one of the three Fischer's *sporadic groups*: F_{22}, F_{23}, F_{24}.

Note that, the first six cases of this list are usually called the *classical Fischer groups*. However, F_{22}, F_{23}, F_{24} are called the *sporadic simple Fischer groups*.

3.2 Fischer spaces

We define a *partial linear space* as a pair $(\mathcal{G}, \mathcal{L})$ consisting of a set of points, $\mathcal{G}$, and a set of lines $\mathcal{L} \subseteq 2^{\mathcal{G}}$ such that every $l \in \mathcal{L}$ has size at least 2, and any two distinct lines intersect in at most one point. Also, we define a *partial triple system* as a partial linear space $(\mathcal{G}, \mathcal{L})$ in which every line has exactly three points. For any two *collinear* points in a partial triple system $a, b \in \mathcal{G}$ there exists a unique line l consisting of a, b and a unique element $a \wedge b \in \mathcal{G}$; i.e., $l = \{a, b, a \wedge b\}$.

Furthermore, we will write $a \sim b$ to indicate that a and b are collinear, for distinct points $a, b \in \mathcal{G}$. If there is no line containing a and b then we write $a \not\sim b$. So, $\mathcal{G}$ partitions into three subsets corresponding to a:

$$\mathcal{G} = \{a\} \cup a^{\sim} \cup a^{\not\sim},$$

where $a^{\sim} = \{b \in \mathcal{G} | a \sim b\}$ and $a^{\not\sim} = \{b \in \mathcal{G} | a \not\sim b\}$. If $a^{\sim} = \phi$, then a is called an *isolated* point, and the space $\mathcal{G}$ is called *non-degenerate* if $\mathcal{G}$ does not contain isolated points.

Note that usually we write $\mathcal{G}$ instead of $(\mathcal{G}, \mathcal{L})$. A subset $\mathcal{H}$ of $\mathcal{G}$ is called a *subspace* if any two collinear points $a, b \in \mathcal{H}$, the entire line through a and b is contained in $\mathcal{H}$. It is easy to see that the intersection of any collection of subspaces is again a subspace. It follows that, for every set of points $X \subseteq \mathcal{G}$, there is a unique smallest subspace containing X. We denote this smallest subspace by $\langle X \rangle$, and we say that it is *generated* by X.

Definition 3.2.1 ([3])**.** A Fischer space consists of a partial triple system for which, if l_1, l_2 are any two lines with distinct intersection, the subspace $\langle l_1 \cup l_2 \rangle$ is isomorphic to the dual affine plane of order 2, denoted by $\mathcal{P}_2^{\vee}$, or to the affine plane of order 3, denoted by $\mathcal{P}_3$. See Figure 3.1.

Note that we say that a Fischer space is *connected* if its collinearity graph is connected.

Proposition 3.2.2 ([6])**.** There is a $1-1$ correspondence between connected Fischer spaces and 3-transposition groups with trivial center. I.e., for each 3-transposition group

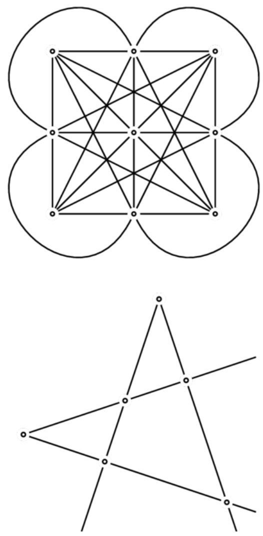

Figure 3.1: The affine plane $\mathcal{P}_3$ and the dual affine plane $\mathcal{P}_2^\vee$

(G, C), the space $\mathcal{G}(G, C)$ is a connected Fischer space and every connected Fischer space is isomorphic to $\mathcal{G}(G, C)$ for some 3-transposition group (G, C).

Example 3.2.3. Suppose that G is a symmetric group S_4 and let C be the set of transpositions in G. Then $\mathcal{G}(G, C)$ is a 3-transposition group, Fischer space contains of 6 points and 4 lines. (I. e., the dual affine plane of order 2).

Furthermore, two 3-transposition groups (G_1, C_1) and (G_2, C_2) have the same Fischer space if and only if they have the same central type, i.e., $(\bar{G}_1, \bar{C}_1) \cong (\bar{G}_2, \bar{C}_2)$, where $\bar{G}_i = G_i/Z(G_i)$.[1]

3.3 Matsuo algebras

Matsuo algebras are a class of non-associative algebras which were defined in 2003 [31].

Definition 3.3.1. Let $\eta \in \mathbb{F}$, where $\eta \neq 0, 1$ and $\mathbb{F}$ be any field with $\operatorname{char} \mathbb{F} \neq 2$, and assume that $\mathcal{G}$ is a partial triple system. Then the Matsuo algebra $M_\eta(\mathcal{G}, \mathbb{F})$ of the Fischer space $\mathcal{G}$ over a field $\mathbb{F}$ is spanned by the points of $\mathcal{G}$ where the bilinear multiplication for

[1]This is the same as saying that the two groups are isomorphic up to the center. For a group of 3-transpositions, we also assume that the isomorphism takes $\bar{C}_1$ to $\bar{C}_2$.

two basis elements $a, b \in \mathcal{G}$ is given by:

$$a \cdot b = \begin{cases} a & \text{if } a = b \\ 0 & \text{if } a \not\sim b \\ \frac{\eta}{2}(a + b - a \wedge b) & \text{if } a \sim b \end{cases}.$$

Note that the dimension of Matsuo algebra $M_\eta(\mathcal{G}, \mathbb{F})$ is equal to $|C|$.

Proposition 3.3.2 ([18])**.** The eigenspaces of a in Matsuo algebra $M_\eta(\mathcal{G}, \mathbb{F})$ are:

- $\langle a \rangle$, 1-eigenspace;
- $\langle b + a \wedge b - \eta a | b \sim a \rangle \oplus \langle b | b \not\sim a \rangle$, 0-eigenspace;
- $\langle b - a \wedge b | b \sim a \rangle$, η-eigenspace.

The algebra $A = M_\eta(\mathcal{G}, \mathbb{F})$ decomposes into a direct sum of the above eigenspaces for any $a \in \mathcal{G}$,

$$A = A_1(a) \oplus A_0(a) \oplus A_\eta(a).$$

Proof. Indeed, the basis for Matsuo algebra A consists of $a, b \in \mathcal{G}$. Then a is a 1-eigenvector, and $a^{\not\sim}$ is a set of 0-eigenvectors. For $b \sim a$, we partition $a^{\sim}$ into $b, a \wedge b$. Hence the subspace $\langle a, b, a \wedge b \rangle$ of A is spanned by $a, b - a \wedge ab$, and $b + a \wedge b - \eta a$, so these are 1, η, and 0 eigenvectors of a, respectively. Since a is idempotent, i.e., $a^2 = aa = a$ and so a is a 1-eigenvector for ad_a. Also, $a(b - a \wedge b) = ab - a(a \wedge b) = \frac{\eta}{2}(a + b - a \wedge b - a - a \wedge b + b) = \frac{\eta}{2}(2b - 2(a \wedge b)) = \eta(b - a \wedge b)$, then $b - a \wedge b$ is η-eigenvector for ad_a. Similarly, $a(b + a \wedge b - \eta a) = ab + a(a \wedge b) - \eta a = \frac{\eta}{2}(a + b - a \wedge b + a + a \wedge b - b) - \eta a = \eta a - \eta a = 0$, and therefore $b + a \wedge b - \eta a$ is a 0-eigenvector for ad_a. □

Recently, T. Yabe [41], has proved the following result:

Proposition 3.3.3. The Matsuo algebra associated with a connected Fischer space is a Jordan algebra over a field $\mathbb{F}$ ($\operatorname{char} \mathbb{F} \neq 3$) if and only if the Fischer space is isomorphic to either the affine space of order 3 or the Fischer space associated with the symmetric group.

Example 3.3.4. Let $A = M_\eta(S_3, (12)^{S_3})$, that is, $A = \langle a, b, c \rangle$ where $a = (12), b = (13)$, and $c = (23)$. Fischer space in this case is a line with three points a, b, and c. Since a, b, and c are idempotents, we get $a^2 = a, b^2 = b$, and $c^2 = c$. Also,

$a \cdot b = \frac{\eta}{2}(a + b - a^b) = \frac{\eta}{2}(a + b - c)$;

$a \cdot c = \frac{\eta}{2}(a + c - a^c) = \frac{\eta}{2}(a + c - b);$

$b \cdot c = \frac{\eta}{2}(b + c - b^c) = \frac{\eta}{2}(b + c - a).$

Hence the multiplication table is given as the following:

$\cdot$	a	b	c
a	a	$\frac{\eta}{2}(a+b-c)$	$\frac{\eta}{2}(a+c-b)$
b	$\frac{\eta}{2}(a+b-c)$	b	$\frac{\eta}{2}(b+c-a)$
c	$\frac{\eta}{2}(a+c-b)$	$\frac{\eta}{2}(b+c-a)$	c

Table 3.1: Multiplication table of the Matsuo algebra for S_3

Example 3.3.5. Consider $A = \langle a, b, c, d, f, g \rangle$ (i.e., $A = M_\eta(S_4)$), where $a = (12), b = (13), c = (23), d = (14), f = (24), g = (34)$. Fischer space consists of 6 points and 4 lines.

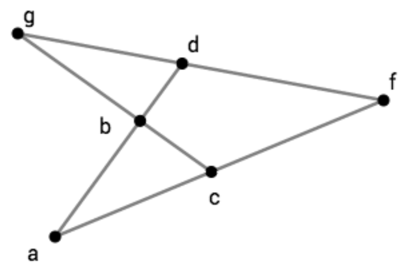

Figure 3.2: Fischer space of Matsuo algebra for S_4

We define the multiplication for any two elements $a_{(ij)}, a_{(kl)} \in A$, where $\{i, j, k, l\} = \{1, 2, 3, 4\}$ as follows:

- $a_{(ij)} \cdot a_{(ij)} = a_{(ij)};$
- $a_{(ij)} \cdot a_{(kl)} = 0;$
- $a_{(ij)} \cdot a_{(ik)} = \frac{\eta}{2}(a_{(ij)} + a_{(ik)} - a_{(jk)}).$

Then the multiplication table below shows the products of two basis elements of A:

$\cdot$	a	b	c	d	f	g
a	a	$\frac{\eta}{2}(a+b-c)$	$\frac{\eta}{2}(a+c-b)$	$\frac{\eta}{2}(a+d-f)$	$\frac{\eta}{2}(a+f-d)$	0
b	$\frac{\eta}{2}(a+b-c)$	b	$\frac{\eta}{2}(b+c-a)$	$\frac{\eta}{2}(b+d-g)$	0	$\frac{\eta}{2}(b+g-d)$
c	$\frac{\eta}{2}(a+c-b)$	$\frac{\eta}{2}(b+c-a)$	c	0	$\frac{\eta}{2}(c+f-g)$	$\frac{\eta}{2}(c+g-f)$
d	$\frac{\eta}{2}(a+d-f)$	$\frac{\eta}{2}(b+d-g)$	0	d	$\frac{\eta}{2}(d+f-a)$	$\frac{\eta}{2}(d+g-b)$
f	$\frac{\eta}{2}(a+f-d)$	0	$\frac{\eta}{2}(c+f-g)$	$\frac{\eta}{2}(d+f-a)$	f	$\frac{\eta}{2}(f+g-c)$
g	0	$\frac{\eta}{2}(b+g-d)$	$\frac{\eta}{2}(c+g-f)$	$\frac{\eta}{2}(d+g-b)$	$\frac{\eta}{2}(f+g-c)$	g

Table 3.2: Multiplication table of Matsuo algebra for S_4

CHAPTER 4

THE STRUCTURE OF AXIAL ALGEBRAS

The definitions and the results of this chapter come from [30].

4.1 Closed set of axes

Let a be an axis and $s \in Aut(A)$, then also a^s is an axis. Indeed, note that $A_\lambda(a^s) = A_\lambda(a)^s$ for all $\lambda \in \mathcal{F}$. Hence we have

$$A_\lambda(a^s)A_\mu(a^s) = A_\lambda(a)^s A_\mu(a)^s = (A_\lambda(a)A_\mu(a))^s \subseteq A_{\lambda*\mu}(a)^s = A_{\lambda*\mu}(a^s),$$

for all $\lambda, \mu \in \mathcal{F}$.

Note that every axial algebra A has a set of generating axes X. We define the Miyamoto group of an arbitrary set of axes (not necessarily generating A).

Definition 4.1.1. Let Y be an arbitrary set of axes, then the Miyamoto group $G(Y)$ of Y is the subgroup of $Aut(A)$ which is generated by the axis subgroups T_a, where $a \in Y$.

Definition 4.1.2. Let Y be an arbitrary set of axes, then Y is *closed* if $Y^\tau = Y$, for all τ in the axis subgroups T_a with $a \in Y$. Equivalently, we can write $Y^{G(Y)} = Y$.

Note that the intersection of two closed set Y and Y' is also closed. Then we define the *closure* $\bar{Y}$ of Y as the unique smallest closed set of axes which contains Y.

Proposition 4.1.3. Let Y be an arbitrary set of axes, then we have $\bar{Y} = Y^{G(Y)}$. Moreover, $G(\bar{Y}) = G(Y)$.

Proof. First, we need to prove that $Y^{G(Y)} \subseteq \bar{Y}$. Then from the definition of the closure of Y, we have $Y \subseteq \bar{Y}$. Hence $G(Y) \leq G(\bar{Y})$ and then $Y^{G(Y)} \subseteq \bar{Y}^{G(\bar{Y})} = \bar{Y}$.

To prove $\bar{Y} \subseteq Y^{G(Y)}$, it suffices to show that $Y^{G(Y)}$ is closed. Let $b \in Y^{G(Y)}$, then $b = a^s$ where $a \in Y$ and $s \in G(Y)$. From Proposition 2.4.9, we have that $\tau_{a^s}(\chi) = \tau_a(\chi)^s$ and so $T_b = T_{a^s} = T_a^s$. Since $T_a \leq G(Y)$ and $s \in G(Y)$, so $T_b = T_a^s \leq G(Y)^s = G(Y)$. Then $G(Y^{G(Y)}) = G(Y)$. This means that $Y^{G(Y)}$ is invariant under $G(Y) = G(Y^{G(Y)})$ and then $Y^{G(Y)}$ is closed. Therefore, $\bar{Y} = Y^{G(Y)}$. Moreover $G(\bar{Y}) = G(Y^{G(Y)}) = G(Y)$. □

Then we have:

$$\bar{Y} = \{a^s : a \in Y, s \in G(Y)\}.$$

Definition 4.1.4. Let Y and Y' be two sets of axes. Then they are *equivalent*, $Y \sim Y'$, if $\bar{Y} = \bar{Y'}$.

We write $A(Y)$ to denote the subalgebra of A which is generated by the set of axes Y.

Proposition 4.1.5. Assume that Y and Y' are two equivalent sets of axes in A. Then the following conditions hold:

(1) $G(Y) = G(Y')$, and

(2) If A is generated by Y then also A is generated by Y'.

Proof. From the definition of the equivalent sets and the Proposition 4.1.3, we have $G(Y) = G(Y')$ if $\bar{Y} = \bar{Y'}$, since $G(Y) = G(\bar{Y}) = G(\bar{Y'}) = G(Y')$. Let $A(Y') = \langle Y' \rangle$ be an axial subalgebra of A generated by the set of axes Y'. It is invariant under T_b for some $b \in Y'$, so it also invariant under $G(Y') = G(Y)$. We claim that $Y \subseteq A(Y')$ and thus $A = A(Y')$. Since $\bar{Y} = \bar{Y'}$ and $Y \subseteq \bar{Y}$, we have that every $a \in Y$ can be written as b^s, where $b \in Y'$ and $s \in G(Y')$. Therefore, $a = b^s \in A(Y')$. □

Note that if a property of an axial algebra is invariant under the equivalence of axes, then it is called *stable*. We have the following result.

Corollary 4.1.6. The Miyamoto group $G(X)$ of an axial algebra is stable.

4.2 Ideals and the radical

Suppose that a is an axis and W is a subspace of A which is invariant under the action of the adjoint of a. Note that if ad_a is semisimple on A, then it is also semisimple on W.

Hence we can write

$$W = \oplus_{\lambda \in \mathcal{F}} W_\lambda(a),$$

where $W_\lambda(a) = W \cap A_\lambda(a) = \{w \in W : aw = \lambda w\}$.

Proposition 4.2.1. Let a be an axis and W be a subspace of A, then if W is invariant under ad_a then it is invariant under $\tau_a(\chi)$ for all $\chi \in T^*$.

Example 4.2.2. For any axis $a \in A$, the ideals of A are invariant under ad_a.

Proposition 4.2.3. Let a be an axis in A, then the ideals I of A are $G(X)$-invariant.

Lemma 4.2.4. Assume that I is an ideal of A and a is a primitive axis. Then either $a \in I$ or $I \subseteq A_{\mathcal{F}/\{1\}}(a)$.

Proof. Suppose that $I \nsubseteq A_{\mathcal{F}/\{1\}}(a)$, so choose $u \in I$ such that $\phi_a(u) \neq 0$. Since I is invariant under the action of ad_a, we have $\phi_a(u)a \in I$. Hence $a \in I$ since I is a subspace of A. □

Corollary 4.2.5. If I_1 and I_2 are two ideals not containing a, then the sum of those ideals is again an ideal not containing any axes from X.

This shows that the radical of A, which we define below, is well defined.

Definition 4.2.6. The *radical* $R(A, X)$ of A with respect to the generating set of primitive axes X is the unique largest ideal of A that does not contain any axes from X.

4.3 Frobenius form

Definition 4.3.1. A *Frobenius form* on an axial algebra A is a non-zero bilinear form $(\cdot, \cdot)$ that associates with the algebra product, that is

$$(uv, w) = (u, vw)$$

for all $u, v, w \in A$.

For some classes of axial algebras (for example, for Majorana algebras), it is additionally assumed that $(a, a) = 1$ for each axis $a \in A$.

Example 4.3.2. Suppose that A is the Matsuo algebra with a 3-transposition group G, then

$$(a,b) = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } ab = ba \\ \frac{\eta}{2} & \text{if } \langle a,b \rangle \cong S_3 \end{cases}.$$

is a Frobenius form.

Lemma 4.3.3. Let $\mathcal{F}$ be the fusion law and let A be an $\mathcal{F}$-axial algebra with a bilinear form $(\cdot,\cdot)$. Then

(i) If $(\cdot,\cdot)$ is a Frobenius form on A then $A_\lambda(a) \perp A_\mu(a)$ for all axis $a \in A$ and $\lambda, \mu \in \mathcal{F}$ with $\lambda \neq \mu$.

(ii) Assume that A has a basis $\mathcal{B}$ of primitive axes, and if $A_\lambda(a) \perp A_\mu(a)$ for all axis $a \in \mathcal{B}$ and $\lambda, \mu \in \mathcal{F}$. Then $(\cdot,\cdot)$ is a Frobenius form.

(iii) Suppose that $(\cdot,\cdot)$ is a Frobenius form on A. If a is a primitive axis and $(a,a) = 1$, then $(a,u) = \phi_a(u)$ for all $u \in A$. Moreover, if $(a,a) = 0$, then $(a,u) = 0$.

Proof. First we prove (i), assume that $u \in A_\lambda(a)$ and $v \in A_\mu(a)$ where $\lambda, \mu \in \mathcal{F}$. Since $(\cdot,\cdot)$ is a Frobenius form and $a \in A$, then

$$\lambda(u,v) = (au,v) = (u,av) = \mu(u,v),$$

and since $\lambda \neq \mu$, we obtain that $(u,v) = 0$, and so $A_\lambda(a) \perp A_\mu(a)$.

From (i), we see that if a is a primitive axis, then $(a,u) = (a, \phi_a(u)a) = \phi_a(u)(a,a)$. Hence $(a,u) = \phi_a(u)$ or 0 if $(a,a) = 1$ or 0, respectively.

It remains to prove (ii), so suppose that a is a primitive axis in the basis $\mathcal{B}$ and let $u \in A_\lambda(a)$, $v \in A_\mu(a)$. If $\lambda \neq \mu$ then $(u,v) = 0$ and so

$$(au,v) = \lambda(u,v) = 0 = \mu(u,v) = (u,av).$$

In the case of $\lambda = \mu$, we see that (au,v) and (u,av) are both equal to $\lambda(u,v)$. Then by bilinearity, $(au,v) = (u,av)$ for all $u,v \in A$. Therefore, $(\cdot,\cdot)$ is a Frobenius form. □

We use the notation $A^\perp$ to denote the radical of the Frobenius form,

$$A^\perp = \{u \in A \mid (u,v) = 0, \text{ for all } v \in A\}.$$

It is easy to check that this radical $A^\perp$ is an ideal of A. Indeed, suppose that $u \in A^\perp$ and $v, w \in A$. Then we have $(uv, w) = (u, vw) = 0$. Hence, $uv \in A^\perp$ and since the Frobenius form is bilinear, we deduce that $A^\perp$ is an ideal.

Proposition 4.3.4 (Theorem 4.8, [30])**.** Suppose that A is a primitive axial algebra generated by a set of axes X. Then the radical $A^\perp$ of the Frobenius form coincides with the radical $R(A)_X$ of A if and only if $(a, a) \neq 0$, for all $a \in X$.

4.4 Projection Frobenius form

Taking Lemma 4.3.3 into account, we define a *projection form* as a Frobenius form that satisfies $(a, a) = 1$, for all $a \in X$. In particular, Hall, Segev and Shpectorov showed that every axial algebra A of Jordan type η admits a projection form, see [19]. So in this case, the radical of A coincides with the radical of its projection form.

Example 4.4.1. Consider the Matsuo algebra $M = M_\eta(G, C)$ with $\eta \neq 0, 1$. The Matsuo algebra M is an example of an axial algebra of Jordan type η (as discussed in the previous chapter). Hence every Matsuo algebra should admit a projection form. And indeed this projection form is defind by:

$$(a, b) = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \not\sim b \\ \frac{\eta}{2} & \text{if } a \sim b, \end{cases}$$

where $a, b \in C$.

Furthermore, the Gram matrix of this form is given by

$$Gr = I + \frac{\eta}{2} T,$$

where T is the adjacency matrix of the collinearity graph of the Fischer space on C, and I is the identity matrix. Clearly, the projection form has a non-trivial radical if and only if the Gram matrix Gr is not of full rank, and this arises if and only if $\eta = -\frac{2}{\lambda}$ for an eigenvalue $\lambda \neq 0$ of T. Moreover, the radical of the projection form coincides with the λ-eigenspace of the matrix T. (See Example 4.11, [30].)

Now, we assume that A is a primitive axial algebra and consider the ideals that contain axes $a \in C$. We have the following definition.

Definition 4.4.2. [1] The *projection graph* $\mathcal{P}$ of A is the graph on C where two vertices a and b are connected by an edge if $(a, b) \neq 0$.

Proposition 4.4.3 (Corollary 4.14, [30])**.** Assume that A is a primitive axial algebra admitting a projection form, where the projection graph is connected. Then every proper ideal of A is contained in the radical.

From Proposition 4.2.3, we show that the ideals of A are invariant under the Miyamoto group $G(X)$. Since $G(X) \leq \mathrm{Aut}(\mathcal{P})$, we have the quotient graph $\bar{\mathcal{P}} := \mathcal{P}/G(X)$. This graph has as vertices the orbits of axes, $a^G(X)$ for $a \in X$. Two orbits are adjacent if they contain adjacent axes. We call the graph $\bar{\mathcal{P}}$ the *orbit projection graph*.

Proposition 4.4.4 (Corollary 4.15, [30])**.** Again let A be a primitive axial algebra with a projection form and a connected orbit projection graph. Then every proper ideal of A is contained in the radical.

4.5 Decompositions of axial algebras

Let A be a commutative non-associative algebra and A_i, $i \in I$, be subalgebras of A.

Definition 4.5.1. An algebra A *decomposes* as $\sum_{i \in I} A_i$ if $A_i A_j = 0$ for all $i \neq j$ and $A = \langle\langle A_i : i \in I \rangle\rangle$.

Notice that the subspace $\sum_{i \in I} A_i$ is a subalgebra. Indeed, take two elements $a = \sum_{i \in I} a_i$ and $b = \sum_{i \in I} b_i$ from the $\sum_{i \in I} A_i$. Then $ab = \sum_{i \in I} a_i \sum_{i \in I} b_i = \sum_{i \in I} a_i b_i$. Since $a, b \in \sum_{i \in I} A_i$, we can see that all pairwise products are zero. Then $\sum_{i \in I} A_i$ is closed under the multiplication. Therefore, $\sum_{i \in I} A_i$ is a subalgebra. Furthermore, this subalgebra contains all A_i. Thus, we deduce that $\sum_{i \in I} A_i = A$. Then every element $a \in A$ can be written as $a = \sum_{i \in I} a_i$, where $a_i \in A_i$. If the decomposition $\sum_{i \in I} a_i$ is unique for each $a \in A$ then A is the *direct sum* of the subalgebras A_i and we write $A = \oplus_{i \in I} A_i$.

Let A be an axial algebra, we define the *annihilator* of A as the set

$$Ann(A) := \{v \in A \mid vA = 0\}.$$

Since A is commutative, we have $Ann(A)$ is an ideal. Also, by the definition of the radical, we prove that $Ann(A)$ is a subset of $R(A)_X$. Indeed, for all $a \in X$, $a \cdot a = a \neq 0$. Then $Ann(A)$ does not contain any axes from X and so $Ann(A) \subseteq R(A, X)$.

However, the annihilator is not always equal to the radical of A. A counterexample is given below:

Example 4.5.2 (Example 4.11 and Example 5.5, [30])**.** Let A be the Matsuo algebra for the symmetric group S_5. The eigenvalues of A are $\lambda = -2, 1, 6$, so $\eta = 1, -2, -\frac{1}{3}$, respectively. For $\eta = -2$, there is a 4-dimensional radical which is spanned by

$$(i, j) + (i, k) + (i, k) - (m, j) - (m, k) - (m, l),$$

where $\{i, j, k, l, m\} = \{1, 2, 3, 4, 5\}$. The authors showed that there exists a vector that does not belong to $Ann(A)$. These vectors do not lie in $Ann(A)$. Hence, $Ann(A) \subsetneq R(A, X)$.

The next proposition shows that we can decompose a primitive axial algebra A as a sum of smaller axial algebras.

Proposition 4.5.3 (Theorem 5.11, [30])**.** Let $A = \sum_{i \in I} A_i$ be a primitive axial algebra with a generating set of axes X. Suppose that X_i is the set of all axes $a \in X$ which are contained in A_i. Then $A = \sum_{i \in I} B_i$, where $B_i = \langle\langle X_i \rangle\rangle$.

Furthermore, in the case where A has two different generating sets X and Y, it is proved in [30] that the decomposition of A into a sum of axial algebras is stable under any change of axes.

CHAPTER 5

DOUBLE AXES AND FLIP SUBALGEBRAS

In this chapter we review some results from [28]. First, we define double axes and the related fusion laws. Then, we define fixed subalgebras. Finally, we discuss the flip subalgebras for the symmetric group S_{2k} determined by Joshi.

5.1 Axes and their fusion laws

The axial algebra of Monster type (α, β) is generated by a set of primitive axes $M(\alpha, \beta)$-axes. Joshi proved that an axis of Monster type $(2\eta, \eta)$ arises as the sum of two orthogonal axes of Jordan type η.

$*$	1	0	α	β
1	1		α	β
0		0	α	β
α	α	α	$1+0$	β
β	β	β	β	$1+0+\alpha$

Table 5.1: The fusion law $M(\alpha, \beta)$

Here we are focussing on the case $\eta \neq \frac{1}{2}$, so that $2\eta \neq 1$. We define double axes as follows.

Definition 5.1.1. Consider a Matsuo algebra $M = M_\eta(G, C)$, where (G, C) is a group of 3-transpositions. Let a, b be any two Matsuo axes such that $a \cdot b = 0$. Then $x = a + b$ will be called a *double axis*.

Note that axes a and b satisfying $a \cdot b = 0$ are called orthogonal.

It is easy to see that a double axis is an idempotent:

$$x^2 = (a+b)^2 = a^2 + ab + ba + b^2 = a + 0 + 0 + b = a + b = x.$$

For a double axis $x = a + b$, we define $M_{\alpha\beta}(a, b)$ as

$$M_{\alpha\beta}(a, b) = M_\alpha(a) \cap M_\beta(b).$$

Note that $x = a + b$ acts on $M_{\alpha\beta}(a, b)$ as the scalar $\alpha + \beta$.

Theorem 5.1.2 ([28])**.** Suppose $a, b \in M = M_\eta(G, C)$ are two orthogonal axes. Then $x = a + b$ satisfies the fusion law $M(2\eta, \eta)$. Furthermore,

$$\begin{aligned} M_0(x) &= M_{00}(a, b); \\ M_1(x) &= M_{10}(a, b) + M_{01}(a, b) = \langle a, b \rangle; \\ M_{2\eta}(x) &= M_{\eta\eta}(a, b); \\ M_\eta(x) &= M_{0\eta}(a, b) + M_{\eta 0}(a, b). \end{aligned}$$

Hence every double axis satisfies the fusion law $\mathcal{M}(2\eta, \eta)$, where $\eta \notin \{0, 1\}$. Hence, the fusion law in this case is given in the following table:

$*$	1	0	2η	η
1	1		2η	η
0		0	2η	η
2η	2η	2η	$1+0$	η
η	η	η	η	$1+0+2\eta$

Table 5.2: The fusion law of Matsuo algebras $\mathcal{M}(2\eta, \eta)$

Remarks:

- Note that the fusion rules $J(\eta)$ satisfied by every single axis $a \in M$ is obtained by dropping a row and a column from $\mathcal{M}(2\eta, \eta)$. This corresponds simply to the 2η-eigenspace being zero.
- Subalgebras of M generated by single axes are Matsuo algebras.
- Double axes are not primitive in M, namely, $M_1(x) = \langle a, b \rangle$ is 2-dimensional. We try to see if M contains a subalgebra in which the double axes generating it are primitive.
- The fusion law $\mathcal{M}(2\eta, \eta)$ is C_2-graded.

Here some examples of the subalgebras generated by single and double axes:

Example 5.1.3. Consider the subalgebra $R = \langle\langle c, a+b \rangle\rangle$ of $M = M_\eta(S_4)$. (See Figure 5.1.) Then R is 4-dimensional with the basis $\{c, x, d, y\}$, where $c = (12), x = a + b = (13) + (24), d = c^{\tau_x} = (34), y = x^{\tau_c} = e + f = (14) + (23)$ and $\tau_c = (12), \tau_x = (13)(24)$.

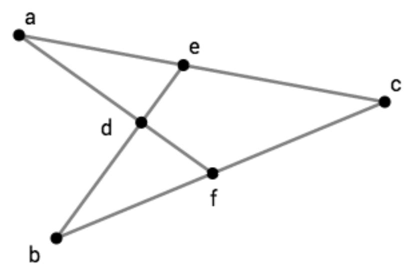

Figure 5.1: Fischer space of the group S_4

Now we calculate the multiplication table of R as the following examples show, where $\cdot$ refers to the Matsuo product.

$$\begin{aligned} c \cdot x &= (12) \cdot ((13) + (24)) = (12) \cdot (13) + (12) \cdot (24) \\ &= \frac{\eta}{2}((12) + (13) - (23)) + \frac{\eta}{2}((12) + (24) - (14)) \\ &= \eta(12) + \frac{\eta}{2}((13) + (24)) - \frac{\eta}{2}((23) + (14)) \\ &= \eta c + \frac{\eta}{2}x - \frac{\eta}{2}y. \end{aligned}$$

For the double axes x and y:

$$\begin{aligned} x \cdot y &= (a+b)\cdot(e+f) = ((13)+(24))\cdot((14)+(23)) \\ &= (13)\cdot(14) + (13)\cdot(23) + (24)\cdot(14) + (24)\cdot(23) \\ &= \frac{\eta}{2}((13)+(14)-(34)) + \frac{\eta}{2}((13)+(23)-(12)) \\ &\quad + \frac{\eta}{2}((24)+(14)-(12)) + \frac{\eta}{2}((24)+(23)-(34)) \\ &= \eta((13)+(24)) + \eta((14)+(23)) - \eta(12) - \eta(34) \\ &= \eta(a+b) + \eta(e+f) - \eta c - \eta d \\ &= \eta x + \eta y - \eta c - \eta d. \end{aligned}$$

Similarly, we can compute the remainder of the products. Hence the complete multiplication table is given in the following table:

$\cdot$	c	x	d	y
c	c	$\eta c + \frac{\eta}{2}(x-y)$	0	$\eta c + \frac{\eta}{2}(y-x)$
x	$\eta c + \frac{\eta}{2}(x-y)$	x	$\eta d + \frac{\eta}{2}(x-y)$	$\eta(x+y-c-d)$
d	0	$\eta d + \frac{\eta}{2}(x-y)$	d	$\eta d + \frac{\eta}{2}(y-x)$
y	$\eta c + \frac{\eta}{2}(y-x)$	$\eta(x+y-c-d)$	$\eta d + \frac{\eta}{2}(y-x)$	y

Table 5.3: Multiplication table for the 4-dimensional subalgebra R

Double axes $x = a + b$ are not primitive in the Matsuo algebra, but they can be primitive in proper subalgebras. Thus, we focus on studying a subalgebra A of a Matsuo algebra M in which the double axes of the generating set are primitive.

5.2 Fixed subalgebras

Suppose that $M = M_\eta(G, C)$ is a Matsuo algebra and let H be a subgroup of $Aut(G, C)$. (The latter denotes the normaliser of C in $Aut(G)$.) Let

$$F = \{v \in M : v^h = v, \text{ for all } h \in H\} \subset M.$$

The subspace F is closed under addition and multiplication. Indeed, for all $v, w \in F$, we see that;

$$(v+w)^h = v^h + w^h = v + w,$$
$$(vw)^h = v^h w^h = vw.$$

Hence, F is a subalgebra of M and it is called the *fixed subalgebra* corresponding to H. We use the notation M_H for this subalgebra.

Proposition 5.2.1 (Proposition 5.2.2, [28])**.** Assume that $F = M_H$ is the fixed subalgebra of the Matsuo algebra M with respect to $H \leq Aut(G, C)$. Then the dimension of F is the same as the number of the orbits of H on C.

Proof. Let $B_1, B_2, \ldots, B_n$ be the orbits of H on C. Since we have $v^h = v$ for all $v \in F$ and $h \in H$, then all the coefficients of elements of B_i where $i \in \{1, 2, \ldots, n\}$ are equal. Let $e_i = \sum_{b \in B_i} b$, for all $i \in \{1, 2, \ldots, n\}$. Hence the set $\mathcal{B} = \{e_1, e_2, \ldots, e_n\}$ spans F and also it is linearly independent. Therefore, $\mathcal{B}$ is a basis for the fixed subalgebra F. □

Notice for any element $v \in F$ we can write:

$$v = \sum_{i=1}^{n} \delta_i \sum_{b \in B_i} b = \sum_{i=1}^{n} \delta_i e_i,$$

where n is the number of the orbits.

Suppose that $\sigma \in \mathrm{Aut}(G, C)$ is an involution and let $H = \langle \sigma \rangle$. Then the action of $\langle \sigma \rangle$ on the class of 3-transposition C has three different types of orbits. Since $|H| = 2$, we have two possible H-orbit lengths: 1 and 2.

1. Singles, orbits of length 1 (in this case, $a^\sigma = a$).
2. Doubles, orbits of length 2 (in this case $aa^\sigma = 0$, a a^σ are orthogonal).
3. Extras, orbits of length 2 (in this case $aa^\sigma \neq 0$, a a^σ are not orthogonal).

Proposition 5.2.2. [28] Let $\sigma \in \mathrm{Aut}(G, C)$ be an involution, and suppose that a is an axis in $\mathcal{B}$ such that a and a^h are orthogonal. Then the double axis $x = a + a^\sigma$ is primitive in $F = M_H$, where $H = \langle \sigma \rangle$.

Proof. Since a and a^σ are orthogonal, the involution σ does not fix a. Let $x = a + a^\sigma$ be a double axis of F. Moreover, $\{a, a^\sigma\}$ is a single orbit (is not a 2-dimensional orbit). So a

and a^σ should belong to F together, not separately. The 1-eigenspace of ad_x of F is

$$F_1(x) = F_1(a + a^\sigma) = M_1(a + a^\sigma) \cap F = \langle a, a^\sigma \rangle \cap F = \langle a + a^\sigma \rangle = \langle x \rangle.$$

Therefore, x is primitive in F. □

We call the subalgebra generated by all single and double axes contained in $F = M_H$ the *flip subalgebra* corresponding to the *flip* σ. By the above, every flip subalgebra is a primitive axial algebra of Monster type $(2\eta, \eta)$.

5.3 The series $Q_k(\eta)$

Here we introduce the subalgebra of dimension k^2 in the Matsuo algebra $M_\eta(S_{2k})$, which was constructed in [28]. Recall that the generating axes of $M_\eta(S_{2k})$ are the transpositions of S_{2k}. We call this algebra $Q_k(\eta)$.

Theorem 5.3.1. (Theorem 6.1.2 [28]) Assume that M is the Matsuo algebra for the symmetric group S_{2k} and let $\sigma = (1,2)(3,4)\dots(2k-1,2k)$ be an involution in S_{2k}. Then the fixed subalgebra relative to $\langle\sigma\rangle$ is spanned by the set containing k single axes and $k(k-1)$ double axes. Moreover, this set of axes forms a basis for the fixed subalgebra M_σ, and hence the dimension is k^2.

The single axes are

$$(2i-1, 2i), \quad i = 1, \dots, n,$$

and the double axes are

$$(2i-1, 2j-1) + (2i, 2j),$$

$$(2i-1, 2j) + (2j-1, 2i),$$

where $1 \le i < j \le n$.

5.4 Ideals and the radical of $M_\eta(S_{2k})$

In this section we will investigate the simplicity of the k^2-dimensional subalgebra $Q_k(\eta)$. First, we find the projection graph $\mathcal{P}$. Let $(G, C) = (S_{2k}, (1,2)^{S_{2k}})$ be the 3-transposition group and let $\sigma = (1,2)(3,4) \cdot (2k-1, 2k)$ be an involution (flip).

Recall that the fixed subalgebra $M_\sigma = Q_k(\eta)$ has k single axes of the form $(2i-1, 2i)$ and $k^2 - k$ double axes of the form $(2i-1, 2j-1) + (2i, 2j)$ and $(2i-1, 2j) + (2j-1, 2i)$, where $1 \leq i < j \leq 2k$.

The projection graph $\mathcal{P}$ has all singles and all doubles as vertices. We say that $\{2i-1, 2i\}$ is the support of the single $(2i-1, 2i)$. Similarly, the support of a double is the union of the two parts. For example, the support of $(1,4) + (2,3)$ is $\{1,2,3,4\}$.

Note that two axes (single or double) are orthogonal if and only if their supports are disjoint. Then in $\mathcal{P}$ two axes are adjacent exactly if their support sets are not disjoint. Hence, we have the following:

1. If all single axes have disjoint support, then they form a coclique in $\mathcal{P}$.
2. Take a single axis $a = (1,2)$, we can see that a is adjacent in $\mathcal{P}$ to all doubles $x = (1, 2i-1) + (2, 2i)$ and $y = (1, 2i) + (2, 2i-1)$ for all $i \geq 2$.
3. The doubles x and y are also adjacent to the single $(2i-1, 2i)$. Since i is arbitrary, we have that a is at distance two from all other singles.
4. Every double axis is adjacent in $\mathcal{P}$ to some single axes. Then it is at distance at most three from the single axis a.

Notice that in this case the fixed subalgebra has no extras. Also, the entire Fischer space of S_{2k} is connected. Then the graph $\mathcal{P}$ is connected. According to [30], this means that the k^2-dimensional subalgebra $Q_k(\eta)$ has no proper ideals containing axes from C.

Definition 5.4.1. A value $\eta = \eta_0$ is called *critical* for A if the algebra A for this value has a non-zero radical.

Proposition 5.4.2. [21] Let $A = M_\sigma$ be a flip subalgebra, then the critical values of η are the same as the special values of Matsuo algebras.

Now, we need to find the special values of η and for each of them find the radical. Recall that the radical of the k^2-dimensional subalgebra $Q_k(\eta)$ coincides with the radical of the Frobenius form. Furthermore, it has a non-trivial radical if and only if the determinant of the Gram matrix is zero, this means that it has zero eigenvalue. [30].

Recently, Hall and Shpectorov in [21] determined the eigenvalues of the diagrams of all 3-transposition groups. In particular, the eigenvalues of the diagram of S_{2k} are $4(k-1)$,

$2(k-2)$, -2. Using the method shown in Example 4.4.1, we now find that the critical values η of the Matsuo algebra $M_\eta(S_{2k})$ are:

$$-\frac{1}{2(k-1)}, -\frac{1}{k-2}.$$

Note that if $\lambda = -2$, then $\eta = 1$, which is discarded as $\eta \neq 0, 1$.

Thus, we have the following result:

Proposition 5.4.3. The algebra $Q_k(\eta)$ of dimension k^2 is simple unless $\eta = -\frac{1}{2(k-1)}$ or $\eta = -\frac{1}{k-2}$.

Here is an example of the Gram matrix and how to find the critical values of η for a flip subalgebra.

Example 5.4.4. Consider the algebra $Q_2(\eta)$ of dimension 4. Then the Gram matrix is given by:

$$G = \begin{pmatrix} 1 & 0 & \eta & \eta \\ 0 & 1 & \eta & \eta \\ \eta & \eta & 2 & 2\eta \\ \eta & \eta & 2\eta & 2 \end{pmatrix},$$

by using GAP, we calculate the determinant and the eigenvalues of the Gram matrix G;

$$\det(G) = 2\eta^3 - 3\eta^2 + 1,$$

then we have that

$$\lambda = [(-1)^2, -\frac{1}{2}].$$

Therefore, the 4-dimensional subalgebra $Q_2(\eta)$ is simple unless $\lambda = -\frac{1}{2}$.

CHAPTER 6

NEW SERIES OF AXIAL ALGEBRAS OF MONSTER TYPE

In this chapter, we construct a similar subalgebra of dimension k^2 in the Matsuo algebra $M_\eta({}^-O^+_{n+1}(3))$. However, our subalgebra contains $k(k-1)$ single axes and k double axes. Some of the definitions, comments and properties are from my recent published paper [1].

Recall that $GO^+_{n+1}(3)$ is the group of all orthogonal transformations of a vector space V of dimension $n+1$ over the finite field $\mathbb{F}_3 = \{-1, 0, 1\}$ with an orthonormal basis $B = \{e_0, e_1, \ldots, e_n\}$. Consider the set C of all reflections with respect to vectors u with $(u,u) = -1$. Let $G = \langle C \rangle \leq GO^+_{n+1}(3)$. We will see below that (G, C) is a 3-transposition group and it is denoted ${}^-O^+_{n+1}(3)$. Let $M = M_\eta(G, C)$ be the corresponding Matsuo algebra. In this section we construct a subalgebra of M of dimension n^2, generated by single and double axes.

6.1 Involutions

Recall that a reflection in a nonsingular vector u (i.e., u satisfies $(u,u) \neq 0$), is given by

$$r_u : v \mapsto v - 2\frac{(v,u)}{(u,u)}u.$$

Remark. For a vector u with $(u,u) = -1$, since $2 = -1$ in $\mathbb{F}_3$, we get the map $r_u : v \mapsto v - (v,u)u$.

Lemma 6.1.1. For every $\alpha \in GO^+_{n+1}(3)$, $r_u^\alpha = r_{u^\alpha}$.

Proof. Let $v \in V$. Then

$$\begin{aligned} v^{r_u^\alpha} &= v^{\alpha^{-1} r_u \alpha} = ((v^{\alpha^{-1}})^{r_u})^\alpha \\ &= (v^{\alpha^{-1}} - 2\frac{(v^{\alpha^{-1}}, u)}{(u,u)} u)^\alpha \\ &= v - 2\frac{(v, u^\alpha)}{(u^\alpha, u^\alpha)} u^\alpha \\ &= v^{r_{u^\alpha}}. \end{aligned}$$

Hence we obtain that $r_u^\alpha = r_{u^\alpha}$. □

Note that $r_u = r_v$ if and only if $v = \pm u$. Indeed, it is easy to see that $r_u = r_{\alpha u}$ for $0 \neq \alpha \in \mathbb{F}_3$. Conversely, if $r_u = r_v$ then $-v = v^{r_v} = v^{r_u} = v - 2\frac{(v,u)}{(u,u)} u$, which immediately implies that u is a multiple of v.

Proposition 6.1.2. Suppose $u, v \in V$ with $(u,u) = -1 = (v,v)$ and suppose that u and v are independent, that is, $u \neq \pm v$. Then $|r_u r_v| = 2$ if $(u,v) = 0$ and $|r_u r_v| = 3$ if $(u,v) = \pm 1$.

Proof. If $(u,v) = 0$ then $u^{r_v} = u$ and so, by Lemma 6.1.1, $r_u^{r_v} = r_u$. This means that $(r_u r_v)^2 = 1$ and so $|r_u r_v| = 2$. Now suppose that $(u,v) \neq 0$. Substituting $-v$ for v if necessary, we may assume that $(u,v) = -1$. Then $u^{r_v} = u + v = v^{r_u}$. Therefore, $r_u^{r_v} = r_{u+v} = r_v^{r_u}$. In particular, $r_u^{r_v} = r_v^{r_u}$, which means that $(r_u r_v)^3 = 1$, and so $|r_u r_v| = 3$. □

This proposition shows that the class C of reflections r_u with $(u,u) = -1$ is a class of 3-transpositions and so ${}^-O^+_{n+1}(3) = (G, C)$, where $G = \langle C \rangle$, is a 3-transposition group, as claimed in the introduction and at the beginning of this section. We recall from the introduction that we identify the element $r_u \in C$ with the 1-dimensional subspace $\langle u \rangle$ as both u and $-u$ define the same element $r_u = r_{-u}$ of C.

If $u, v \in V$ with $(u,u) = -1 = (v,v)$ then

$$\langle u \rangle . \langle v \rangle = \begin{cases} \langle u \rangle & \text{if } u = \pm v, \\ 0 & \text{if } (u,v) = 0, \\ \frac{\eta}{2}(\langle u \rangle + \langle v \rangle - \langle v - (u,v)u \rangle) & \text{if } (u,v) = \pm 1. \end{cases} .$$

Suppose we have a non-degenerate bilinear form on a given space V with a basis $\{e_1, e_2, \dots, e_n\}$. The Gram matrix is given by:

$$Gr = \begin{bmatrix} (e_1, e_1) & (e_1, e_2) & \dots & (e_1, e_n) \\ (e_2, e_1) & (e_2, e_2) & \dots & (e_2, e_n) \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ (e_n, e_1) & \dots & \dots & (e_n, e_n) \end{bmatrix},$$

where $(e_i, e_j) = 0, i \neq j$, and $(e_1, e_1) = (e_2, e_2) = \cdots = (e_{n-1}, e_{n-1}) = 1$, and $(e_n, e_n) = \epsilon$.

Two forms G and G' on V are equivalent if there exists a matrix $A \in GL(V)$ such that $G' = A^t G A$, and the determinant is defined as

$$\det(G') = \det(A)^2 \det(G).$$

Now we discuss the involutions and then determine single and double axes to build new subalgebras.

<u>In the case of $n = 2$</u>:

First, we provide the conjugacy classes of involution in the following table:

O_2^+	O_2^-
$(0, 2; 0, +)$	$(0, 2; 0, -)$
$(1, 1; +, +)$	$(1, 1; +, -)$
$(1, 1; -, -)$	$(1, 1; -, +)$
$(2, 0; +, 0)$	$(2, 0; -, 0)$

where $Id \in 2, 0; (\pm, 0)$ and $-Id \in 0, 2; (0, \pm)$.

For all n, the identity classes are just Matsuo algebra.

Then we discuss how to determine single and double axes to build new subalgebras.

Example 6.1.3. Assume that $\tau \in (1, 1; +, +)$, in this case $(e_1, e_1) = 1 = (e_2, e_2)$ where $e_1 \perp e_2$. So the Gram matrix is given as:

$$Gr = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

To determine the single and double axes, we find the reflection in vectors of length $+1$, and -1:

$$\alpha e_1 + \beta e_2 \implies +1+1=-1,$$

$$\alpha e_1 \implies +1,$$

$$\beta e_2 \implies +1.$$

	Reflections of length 1	Reflections of length -1
Single axes	2 fixed	-
	$\pm e_1, \pm e_2$	-
Double axes	-	1
	-	$a = \pm(e_1+e_2), b = \pm(e_1 - e_2)$
Type of subalgebras	2-dimensional subalgebra	1-dimensional subalgebra

Also, it is easy to verify that the class $(1,1;-,-)$ has two fixed single axes by reflection in vectors of length -1 and one double axis by reflection in vectors of length 1.

Example 6.1.4. The class $(1,1;+,-)$ has no double axes. Indeed, in this case the Gram matrix is given by:

$$Gr = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and

$$\tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then,

$$\alpha e_1 + \beta e_2 \implies +1-1=0,$$

$$\alpha e_1 \implies +1,$$

$$\beta e_2 \implies -1.$$

Therefore, we just have single axes and there is no double axes in this class.

	Reflections of length 1	Reflections of length -1
Single axes	1 fixed $\pm e_1$	1 fixed $a = \pm e_2$
Double axes	- -	- -
Type of subalgebras	1-dimensional subalgebra	1-dimensional subalgebra

Similarly, we can prove that the class $(1, 1; -, +)$ has no double axes.

<u>In the case of $n = 3$</u>:

The conjugacy classes of involution in this case are given in the following table:

O_3^+	O_3^-
$(0, 3; 0, +)$	$(0, 3; 0, -)$
$(1, 2; +, +)$	$(1, 2; +, -)$
$(1, 2; -, -)$	$(1, 2; -, +)$
$(2, 1; +, +)$	$(2, 1; +, -)$
$(2, 1; -, -)$	$(2, 1; -, +)$
$(3, 0; +, 0)$	$(3, 0; -, 0)$

Example 6.1.5. Suppose that $\tau \in (1, 2; -, -)$, so $(e_1, e_1) = -1$, $(e_2, e_2) = 1$, and $(e_3, e_3) = -1$, where $e_1 \perp e_2, e_1 \perp e_3$, and $e_2 \perp e_3$. Then the Gram matrix corresponding to this class is:

$$Gr = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

and

$$\tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Then,

$$\alpha e_1 + \beta e_2 + \gamma e_3 \implies -1 + 1 - 1 = -1,$$

$$0e_1 + \beta e_2 + \gamma e_3 \implies 0 + 1 + (-1) = 0,$$

$$\alpha e_1 + 0e_2 + \gamma e_3 \implies -1 + 0 + (-1) = 1,$$

$$\alpha e_1 + \beta e_2 + 0e_3 \implies -1 + 1 = 0,$$

$$\alpha e_1 \implies -1,$$

$$\beta e_2 \implies 1,$$

$$\gamma e_3 \implies -1.$$

	Reflections of length 1	Reflections of length -1
Single axes	1 fixed	2 fixed
	$\pm e_2$	$\pm e_1, \pm e_3$
Double axes	1	-
	$a = \pm(e_1 + e_3), b = \pm(e_1 - e_3)$	-
Type of subalgebras	2-dimensional subalgebra	-
Fischer space		

Example 6.1.6. In this example we take $\tau \in (2, 1; +, +)$, so $(e_1, e_1) = 1, (e_2, e_2) = 1$, and $(e_3, e_3) = 1$. Then the Gram matrix is given by:

$$Gr = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and

$$\tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Then,

$$\begin{aligned}
\alpha e_1 + \beta e_2 + \gamma e_3 &\implies +1+1+1 = 0, \\
0e_1 + \beta e_2 + \gamma e_3 &\implies 0+1+1 = -1, \\
\alpha e_1 + 0e_2 + \gamma e_3 &\implies +1+0+1 = -1, \\
\alpha e_1 + \beta e_2 + 0e_3 &\implies +1+1+0 = -1, \\
\alpha e_1 &\implies 1, \\
\beta e_2 &\implies 1, \\
\gamma e_3 &\implies 1.
\end{aligned}$$

	Reflections of length 1	Reflections of length -1
Single axes	3 fixed	2
	$\pm e_1, \pm e_2, \pm e_3$	$a = \pm(e_1 + e_2), b = \pm(e_1 - e_2)$
Double axes	-	2
	-	$c = \pm(e_2 + e_3), d = \pm(e_2 - e_3)$
	-	$e = \pm(e_1 + e_3), f = \pm(e_1 - e_3)$
Type of subalgebras	3-dimensional subalgebra	4-dimensional subalgebra

The case of $n = 4$:

The table below represents the conjugacy classes of involution in all $O_4^\epsilon(3)$:

For example, we take $\tau \in (1,3;+,+)$, so $(e_1, e_1) = 1, (e_2, e_2) = 1, (e_3, e_3) = 1$, and $(e_4, e_4) = 1$.

O_4^+	O_4^-
$(0,4;0,+)$	$(0,4;0,-)$
$(1,3;+,+)$	$(1,3;+,-)$
$(1,3;-,-)$	$(1,3;-,+)$
$(2,2;+,+)$	$(2,2;+,-)$
$(2,2;-,-)$	$(2,2;-,+)$
$(3,1;-,-)$	$(3,1;-,+)$
$(3,1;-,-)$	$(3,1;-,+)$
$(4,0;+,0)$	$(4,0;-,0)$

Then the Gram matrix is given as follows:

$$Gr = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and

$$\tau = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Now we discuss the reflections in vectors of length 1 and -1, so suppose that $q(v) = e_1^2 + e_2^2 + e_3^2 + e_4^2$, with $U = \langle e_1, e_2, e_3 \rangle$ and $V = \langle e_4 \rangle$, where $\tau = id$ on U and $\tau = -id$ on V. In total, we have $\frac{3^4-1}{2} = 40$ 1-spaces.

The condition $q(v) = 1$ implies that v has either three or one non-zero coordinates. The four 1-spaces $\langle v \rangle$ with one non-zero coordinate are the $\langle e_i \rangle$ and they are all fixed, so we have four single axes.

Also, we have $\frac{2^4}{2} = 8$ 1-space with 4 non-zero coordinates, they are all in orbits of length 2 and the product is $1 + 1 + 1 - 1 = -1$, so none are orthogonal.

Now we act on $\langle v \rangle$ with $q(v) = -1$. Then we have two non-zero coordinates. If both non-zero's are within $\{e_1, e_2, e_3\}$ then they are fixed, so there are $3 \times 2 = 6$ single axes of such 1-spaces. In addition, we have $v = \alpha e_i + \beta e_4$ with $i \in \{1, 2, 3\}$ then $v^\tau = \alpha e_i - \beta e_4$ where $(v, v^\tau) = 1 - 1 = 0$, so they are orthogonal. Hence, we have three double axes as there are three choices for i.

The following table shows single and double axes in both reflections and then determine new subalgebras:

	Reflections of length 1	Reflections of length -1
Single axes	4 fixed	6
	$\pm e_1, \pm e_2$	$a = \pm(e_1 + e_2), b = \pm(e_1 - e_2)$
	$\pm e_3, \pm e_4$	$c = \pm(e_1 + e_3), d = \pm(e_1 - e_3)$
		$g = \pm(e_2 + e_3), h = \pm(e_2 - e_3)$
Double axes	-	3
	-	$e = \pm(e_1 + e_4), f = \pm(e_1 - e_4)$
	-	$i = \pm(e_2 + e_4), j = \pm(e_2 - e_4)$
	-	$k = \pm(e_3 + e_4), l = \pm(e_3 - e_4)$
Type of subalgebras	4-dimensional subalgebra	9-dimensional subalgebra

We calculate the multiplication table of the 9-dimensional subalgebra. Recall that this subalgebra generated by six single axes:

$a = \pm(e_1 + e_2), b = \pm(e_1 - e_2), c = \pm(e_1 + e_3),$

$d = \pm(e_1 - e_3), g = \pm(e_2 + e_3), h = \pm(e_2 - e_3).$

And three double axes:

$x = e + f$, where $e = \pm(e_1 + e_4)$ and $f = \pm(e_1 - e_4)$,

$y = i + j$, where $i = \pm(e_2 + e_4)$ and $j = \pm(e_2 - e_4)$,

$z = k + l$, where $k = \pm(e_3 + e_4)$ and $l = \pm(e_3 - e_4)$.

6.2 Subalgebras for $O^+_{(k+1)}(3)$

In this section we construct new subalgebras corresponding to the class of involution $(k, 1; ++)$ with $\epsilon = +$ and we consider the reflections in vectors of length -1. By this way we can verify how many single and double axes we have in each case. At the end of the section we will generalize these results and we provide a new subalgebra of dimension k^2 on the orthogonal group $O^+_{(k+1)}(3)$.

- Case of $k = 2$:

  Single axes: $\pm(e_1 + e_2), \pm(e_1 - e_2)$.

  Double axes: $(e_1 + e_3) + (e_1 - e_3), (e_2 + e_3) + (e_2 - e_3)$.

  So, in this case we construct a 4-dimensional subalgebra generated by two single axes and two double axes as shown above.

- Case of $k = 3$:

  Single axes: $\pm(e_1 + e_2), \pm(e_1 - e_2)$,

  $\pm(e_1 + e_3), \pm(e_1 - e_3)$,

  $\pm(e_2 + e_3), \pm(e_2 - e_3)$.

  Double axes: $(e_1 + e_4) + (e_1 - e_4)$,

  $(e_2 + e_4) + (e_2 - e_4)$,

  $(e_3 + e_4) + (e_3 - e_4)$.

  In this case we get a subalgebra of dimension 9 which is generated by those six single axes and three double axes.

- Case of $k = 4$:

  Single axes: $\pm(e_1 + e_2), \pm(e_1 - e_2)$,

  $\pm(e_1 + e_3), \pm(e_1 - e_3)$,

  $\pm(e_1 + e_4), \pm(e_1 - e_4$,

  $\pm(e_2 + e_3), \pm(e_2 - e_3)$,

  $\pm(e_2 + e_4), \pm(e_2 - e_4)$,

  $\pm(e_3 + e_4), \pm(e_3 - e_4)$.

  Double axes: $(e_1 + e_5) + (e_1 - e_5)$,

  $(e_2 + e_5) + (e_2 - e_5)$,

$\cdot$	a	b	c	d	g	h	x	y	z
a	a	0	$\frac{\eta}{2}(a+c-g)$	$\frac{\eta}{2}(a+d-h)$	$\frac{\eta}{2}(a+g-c)$	$\frac{\eta}{2}(a+h-d)$	$\frac{\eta}{2}(2a+x-y)$	$\frac{\eta}{2}(2a+y-x)$	0
b	0	b	$\frac{\eta}{2}(b+c-g)$	$\frac{\eta}{2}(b+d-h)$	$\frac{\eta}{2}(b+g-c)$	$\frac{\eta}{2}(b+h-d)$	$\frac{\eta}{2}(2b+x-y)$	$\frac{\eta}{2}(2b+y-x)$	0
c	$\frac{\eta}{2}(+c-g)$	$\frac{\eta}{2}(b+c-g)$	c	0	$\frac{\eta}{2}(c+g-a)$	$\frac{\eta}{2}(c+h-b)$	$\frac{\eta}{2}(2c+x-z)$	0	$\frac{\eta}{2}(2c+z-x)$
d	$\frac{\eta}{2}(a+d-h)$	$\frac{\eta}{2}(b+d-h)$	0	d	$\frac{\eta}{2}(d+g-b)$	$\frac{\eta}{2}(d+h-a)$	$\frac{\eta}{2}(2d+x-z)$	0	$\frac{\eta}{2}(2d+z-x)$
g	$\frac{\eta}{2}(a+g-c)$	$\frac{\eta}{2}(b+g-c)$	$\frac{\eta}{2}(c+g-a)$	$\frac{\eta}{2}(d+g-b)$	g	0	0	$\frac{\eta}{2}(2g+y-z)$	$\frac{\eta}{2}(2g+z-y)$
h	$\frac{\eta}{2}(a+h-d)$	$\frac{\eta}{2}(b+h-d)$	$\frac{\eta}{2}(c+h-b)$	$\frac{\eta}{2}(d+h-a)$	0	h	0	$\frac{\eta}{2}(2h+y-z)$	$\frac{\eta}{2}(2h+z-y)$
x	$\frac{\eta}{2}(2a+x-y)$	$\frac{\eta}{2}(2b+x-y)$	$\frac{\eta}{2}(2c+x-z)$	$\frac{\eta}{2}(2d+x-z)$	0	0	x	$\eta(-a-b+y+x)$	$\eta(-c-d+x+z)$
y	$\frac{\eta}{2}(2a+y-x)$	$\frac{\eta}{2}(2b+y-x)$	0	0	$\frac{\eta}{2}(2g+y-z)$	$\frac{\eta}{2}(2h+y-z)$	$\eta(-a-b+y+x)$	y	$\eta(-g-h+y+z)$
z	0	0	$\frac{\eta}{2}(2c+z-x)$	$\frac{\eta}{2}(2d+z-x)$	$\frac{\eta}{2}(2g+z-y)$	$\frac{\eta}{2}(2h+z-y)$	$\eta(-c-d+x+z)$	$\eta(-g-h+y+z)$	z

Table 6.1: Multiplication table for the 9-dimensional subalgebra

n	Single axes	Double axes	Subalgebras
2	2	2	4-dimensional subalgebra
3	6	3	9-dimensional subalgebra
4	12	4	16-dimensional subalgebra
5	20	5	25-dimensional subalgebra
6	30	6	36-dimensional subalgebra
7	42	7	49-dimensional subalgebra
$\vdots$	$\vdots$	$\vdots$	$\vdots$
k	$k(k-1)$	k	k^2-dimensional subalgebra

Table 6.2: Subalgebras generated by single and double axes

$(e_3 + e_5) + (e_3 - e_5)$,

$(e_4 + e_5) + (e_4 - e_5)$.

Then we get a 16-dimensional subalgebra which is generated by 12 single axes and four double axes as described above.

In this way we can construct further new subalgebras. The following table lists the number of single and double axes for each case of k and corresponding subalgebras.

Theorem 6.2.1. Let A be the subspace of the Matsuo algebra $M_\eta({}^-O^+_{k+1}(3))$ spanned by the set of single axes $S = \{\langle e_i + \epsilon e_j\rangle : 1 \le i < j \le n, \epsilon = \pm 1\}$ and the set of double axes $D = \{\langle e_0 + e_i\rangle + \langle e_0 - e_i\rangle \mid 1 \le i \le k\}$. Then A is a primitive axial algebra of Monster type $(2\eta, \eta)$ of dimension $|S| + |D| = k(k-1) + k = k(k-1+1) = k^2$.

Proof. To show that A is a subalgebra, we need to check that A is closed under multiplication. We establish this by looking through the possible cases of pairs of axes $a, b \in S \cup D$ and showing in each case that $ab \in A$. Note that every axis, single or double, is an idempotent, so we just need to consider pairs of distinct axes: $a \neq b$.

Let us start with two single axes: $a = \langle e_i + \epsilon e_j\rangle$ and $b = \langle e_{i'} + \epsilon' e_{j'}\rangle$. Then $|\{i, j\} \cap \{i', j'\}|$ is 0, 1 or 2. If $\{i, j\}$ and $\{i', j'\}$ are disjoint then, clearly, $ab = 0$ since $(e_i + \epsilon e_j, e_{i'} + \epsilon' e_{j'}) = 0$. If $|\{i, j\} \cap \{i', j'\}| = 1$, then, without loss of generality, $i = i'$, that is, $b = \langle e_i + \epsilon' e_{j'}\rangle$. In this case, $ab = \frac{\eta}{2}(a+b-c)$, where $c = b^{r_a} = \langle -\epsilon e_j + \epsilon' e_{j'} \rangle$. Manifestly, $c \in S$, and so $ab \in A$.

Finally, suppose that $|\{i,j\} \cap \{i',j'\}| = 2$. Then, without loss of generality, $a = \langle e_i + e_j \rangle$ and $b = \langle e_i - e_j \rangle$. Here, $(e_i + e_j, e_i - e_j) = 0$ and so again, as in the first case, $ab = 0$.

Next, assume that $a = \langle e_i + \epsilon e_j \rangle$ is a single axis and $b = \langle e_0 + e_s \rangle + \langle e_0 - e_s \rangle$ is a double axis. Here we have two options: either $s \notin \{i,j\}$ or $s \in \{i,j\}$ (say, $s = i$). In the the first case, $ab = 0$ since $(e_i \pm e_j, e_0 \pm e_s) = 0$. If $s = i$ then

$$\begin{aligned}
ab &= \langle e_i + \epsilon e_j \rangle (\langle e_0 + e_i \rangle + \langle e_0 - e_i \rangle) \\
&= \frac{\eta}{2}(\langle e_i + \epsilon e_j \rangle + \langle e_0 + e_i \rangle - \langle e_0 + \epsilon e_j \rangle) \\
&+ \frac{\eta}{2}(\langle e_i + \epsilon e_j \rangle + \langle e_0 - e_i \rangle - \langle e_0 - \epsilon e_j \rangle) \\
&= \eta a + \frac{\eta}{2} b - \frac{\eta}{2}(\langle e_0 + e_j \rangle + \langle e_0 - e_j \rangle).
\end{aligned}$$

Clearly, $\langle e_0 + e_j \rangle + \langle e_0 - e_j \rangle \in D$ and so $ab \in A$.

Finally, let $a = \langle e_0 + e_i \rangle + \langle e_0 - e_i \rangle$ and $b = \langle e_0 + e_j \rangle + \langle e_0 - e_j \rangle$ be two double axes, $i \neq j$. Then

$$\begin{aligned}
ab &= (\langle e_0 + e_i \rangle + \langle e_0 - e_i \rangle)(\langle e_0 + e_j \rangle + \langle e_0 - e_j \rangle) \\
&= \frac{\eta}{2}(\langle e_0 + e_i \rangle + \langle e_0 + e_j \rangle - \langle e_i + e_j \rangle) \\
&+ \frac{\eta}{2}(\langle e_0 + e_i \rangle + \langle e_0 - e_j \rangle - \langle e_i - e_j \rangle) \\
&+ \frac{\eta}{2}(\langle e_0 - e_i \rangle + \langle e_0 + e_j \rangle - \langle e_i - e_j \rangle) \\
&+ \frac{\eta}{2}(\langle e_0 - e_i \rangle + \langle e_0 - e_j \rangle - \langle e_i + e_j \rangle) \\
&= \eta a + \eta b - \eta \langle e_i + e_j \rangle - \eta \langle e_i - e_j \rangle.
\end{aligned}$$

Clearly, all summands here are in A, so in this final case $ab \in A$.

We have shown that A is a subalgebra. Manifestly, the vectors in $S \cup D$ are linearly independent, and so they form a basis of A. This yields the claim concerning the dimension of A.

It remains to show that the double axes $x = \langle e_0 + e_i \rangle + \langle e_0 - e_j \rangle$ are primitive in A. Consider $\sigma = r_{e_0}$. This involution fixes all single axes in S and it switches two single axes $a = \langle e_0 + e_i \rangle$ and $b = \langle e_0 - e_i \rangle$ in every $x = a + b \in D$. Hence $S \cup D$ is contained in the fixed subalgebra M_σ, which means that A is contained in M_σ. Recall from Theorem 5.1.2 that $M_1(x) = \langle a, b \rangle$. Within $M_1(x)$, σ fixes $a + b = x$ and inverts $a - b$. Hence $A_1(x) = A \cap M_1(x) = \langle x \rangle$ and so x is indeed primitive in A. $\square$

We use $Q^k(\eta)$ to denote the subalgebra A of $M_\eta(O^+_{k+1}(3))$.

In the next sections, we study some properties of the subalgebras that were discovered in the previous section. Firstly, we compute the Frobenius form for the subalgebra of dimension n^2. Then, using GAP to calculate the determinant and the eigenvalues of the Gram matrix.

Furthermore, we use the theory developed in [30] to investigate the following question: for which values of η is A a simple algebra?

6.3 Frobenius form on A

First of all, note that A inherits from M a Frobenius form (see Definition 4.3.1), a bilinear from associating with the algebra product. In this subsection we compute the values of the Frobenius form on the basis $S \cup D$ of A. For $a \in S \cup D$, let the support $\text{supp}(a)$ be defined as $\{i, j\}$ if $a = \langle e_i + \epsilon e_j \rangle$ is a single axis and as $\{0, i\}$ if $a = \langle e_0 + e_i \rangle + \langle e_0 - e_i \rangle$ is a double axis.

Proposition 6.3.1. Let $a, b \in S \cup D$. Then

- If $a = b$ then $(a, a) = 1$ if $a \in S$ and $(a, a) = 2$ if $a \in D$.
- If $a \neq b$ then $(a, b) = 0$ if $\text{supp}(a) \cap \text{supp}(b) = \emptyset$ or if $\text{supp}(a) = \text{supp}(b)$ (in this case both a and b are single axes).
- If $a \neq b$ and $|\text{supp}(a) \cap \text{supp}(b)| = 1$ then $(a, b) = \frac{\eta}{2}$ if $a, b \in S$; η if $a \in S$ and $b \in D$ (or vice versa); and 2η if $a, b \in D$.

Proof. This follows immediately from the values of the Frobenius form on M, as given in Subsection 4.3. $\square$

6.4 Ideals in A

According to [30], the ideals of A containing axes from $S \cup D$ are controlled by the projection graph on the set $S \cup D$ of axes of A.

Definition 6.4.1. The *projection graph* of A is the graph on $S \cup D$ where two vertices a and b are connected by an edge if $(a, b) \neq 0$.

Proposition 6.4.2. The algebra A has no proper non-zero ideals containing an axis from $S \cup D$.

Proof. According to [30], it suffices to show that the projection graph is connected. By Proposition 6.3.1, we see that the single axis $\langle e_i + \epsilon e_j \rangle$ is connected by edges to both double axes $\langle e_0 + e_i \rangle + \langle e_0 - e_i \rangle$ and $\langle e_0 + e_j \rangle + \langle e_0 - e_j \rangle$. Thus all double axes and all single axes are contained in the same connected component of the projection graph. $\square$

6.5 Radical

We turn now to ideals of A that contain no axes from $S \cup D$. All such ideals are contained in the *radical* of A, which is defined in [30] as the largest ideal not containing any of the generating axes of A. It is also shown in [30] that, in the presence of a Frobenius form having non-zero values (a, a) on all generating axes a, the radical of A coincides with the radical

$$A^{\perp} = \{u \in A : (u, v) = 0 \text{ for all } v \in A\}$$

of the Frobenius form on A. Clearly, this radical is non-zero if and only if the determinant of the Gram matrix of the Frobenius form is non-zero. Clearly, the determinant of the Gram matrix (written with respect to the basis $S \cup D$ of A) is a polynomial in η of degree depending on n. In the next section we compute this polynomial for $n \leq 14$ and based on this we put forward exact conjectures concerning the values of η for which the radical is non-zero (and hence A is not simple).

6.6 Critical values of η

Here we use GAP [15] to compute and factorize the determinant of the Gram matrix of the Frobenius form on A for small values of n. We conclude this section with some conjectures.

Example 6.6.1. If we consider the 4-dimensional subalgebra A, then the Gram matrix is given by:

$$G = \begin{pmatrix} 1 & 0 & \eta & \eta \\ 0 & 1 & \eta & \eta \\ \eta & \eta & 2 & 2\eta \\ \eta & \eta & 2\eta & 2 \end{pmatrix}.$$

The determinant $\det(G)$ and the list of eigenvalues $\bar{\lambda}$ (where the multiplicity of each eigenvalue is shown as the exponent) of the Gram matrix G are;

$$\det(G) = 2\eta^3 - 3\eta^2 + 1,$$

$$\bar{\lambda} = [(-1)^2, -\frac{1}{2}].$$

The radical of A is non-zero and A is not simple if and only if $\eta = -1, \frac{1}{2}$.

Example 6.6.2. In this example, we take the subalgebra A of dimension 9. Then the Gram matrix G as follows;

$$Gr = \begin{pmatrix} 1 & 0 & \frac{\eta}{2} & \frac{\eta}{2} & \frac{\eta}{2} & \frac{\eta}{2} & \eta & \eta & 0 \\ 0 & 1 & \frac{\eta}{2} & \frac{\eta}{2} & \frac{\eta}{2} & \frac{\eta}{2} & \eta & \eta & 0 \\ \frac{\eta}{2} & \frac{\eta}{2} & 1 & 0 & \frac{\eta}{2} & \frac{\eta}{2} & \eta & 0 & \eta \\ \frac{\eta}{2} & \frac{\eta}{2} & 0 & 1 & \frac{\eta}{2} & \frac{\eta}{2} & \eta & 0 & \eta \\ \frac{\eta}{2} & \frac{\eta}{2} & \frac{\eta}{2} & \frac{\eta}{2} & 1 & 0 & 0 & \eta & \eta \\ \frac{\eta}{2} & \frac{\eta}{2} & \frac{\eta}{2} & \frac{\eta}{2} & 0 & 1 & 0 & \eta & \eta \\ \eta & \eta & \eta & \eta & 0 & 0 & 2 & 2\eta & 2\eta \\ \eta & \eta & 0 & 0 & \eta & \eta & 2\eta & 2 & 2\eta \\ 0 & 0 & \eta & \eta & \eta & \eta & 2\eta & 2\eta & 2 \end{pmatrix}$$

Also, using GAP we calculate the determinant of G and the eigenvalues;

$$\det(Gr) = 16\eta^3 - 12\eta^2 + 1,$$

$$\bar{\lambda} = [(\frac{1}{2})^2, -\frac{1}{4}].$$

Then, $A^{\perp} \neq 0$ if and only if $\eta = \frac{1}{2}$ or $\eta = -\frac{1}{4}$. Then for all other values of η the subalgebra A is simple.

Example 6.6.3. Assume that we have the 16-dimensional subalgebra A, then the Gram matrix as follows;

$$Gr = \begin{pmatrix}
1 & 0 & \frac{\eta}{2} & \frac{\eta}{2} & \frac{\eta}{2} & \frac{\eta}{2} & \frac{\eta}{2} & \frac{\eta}{2} & \frac{\eta}{2} & \frac{\eta}{2} & 0 & 0 & \eta & \eta & 0 & 0 \\
0 & 1 & \frac{\eta}{2} & \frac{\eta}{2} & \frac{\eta}{2} & \frac{\eta}{2} & \frac{\eta}{2} & \frac{\eta}{2} & \frac{\eta}{2} & \frac{\eta}{2} & 0 & 0 & \eta & \eta & 0 & 0 \\
\frac{\eta}{2} & \frac{\eta}{2} & 1 & 0 & \frac{\eta}{2} & \frac{\eta}{2} & \frac{\eta}{2} & \frac{\eta}{2} & 0 & 0 & \frac{\eta}{2} & \frac{\eta}{2} & \eta & 0 & \eta & 0 \\
\frac{\eta}{2} & \frac{\eta}{2} & 0 & 1 & \frac{\eta}{2} & \frac{\eta}{2} & \frac{\eta}{2} & \frac{\eta}{2} & 0 & 0 & \frac{\eta}{2} & \frac{\eta}{2} & \eta & 0 & \eta & 0 \\
\frac{\eta}{2} & \frac{\eta}{2} & \frac{\eta}{2} & \frac{\eta}{2} & 1 & 0 & 0 & 0 & \frac{\eta}{2} & \frac{\eta}{2} & \frac{\eta}{2} & \frac{\eta}{2} & \eta & 0 & 0 & \eta \\
\frac{\eta}{2} & \frac{\eta}{2} & \frac{\eta}{2} & \frac{\eta}{2} & 0 & 1 & 0 & 0 & \frac{\eta}{2} & \frac{\eta}{2} & \frac{\eta}{2} & \frac{\eta}{2} & \eta & 0 & 0 & \eta \\
\frac{\eta}{2} & \frac{\eta}{2} & \frac{\eta}{2} & \frac{\eta}{2} & 0 & 0 & 1 & 0 & \frac{\eta}{2} & \frac{\eta}{2} & \frac{\eta}{2} & \frac{\eta}{2} & 0 & \eta & \eta & 0 \\
\frac{\eta}{2} & \frac{\eta}{2} & \frac{\eta}{2} & \frac{\eta}{2} & 0 & 0 & 0 & 1 & \frac{\eta}{2} & \frac{\eta}{2} & \frac{\eta}{2} & \frac{\eta}{2} & 0 & \eta & \eta & 0 \\
\frac{\eta}{2} & \frac{\eta}{2} & 0 & 0 & \frac{\eta}{2} & \frac{\eta}{2} & \frac{\eta}{2} & \frac{\eta}{2} & 1 & 0 & \frac{\eta}{2} & \frac{\eta}{2} & 0 & \eta & 0 & \eta \\
\frac{\eta}{2} & \frac{\eta}{2} & 0 & 0 & \frac{\eta}{2} & \frac{\eta}{2} & \frac{\eta}{2} & \frac{\eta}{2} & 0 & 1 & \frac{\eta}{2} & \frac{\eta}{2} & 0 & \eta & 0 & \eta \\
0 & 0 & \frac{\eta}{2} & \frac{\eta}{2} & \frac{\eta}{2} & \frac{\eta}{2} & \frac{\eta}{2} & \frac{\eta}{2} & \frac{\eta}{2} & \frac{\eta}{2} & 1 & 0 & 0 & 0 & \eta & \eta \\
0 & 0 & \frac{\eta}{2} & \frac{\eta}{2} & \frac{\eta}{2} & \frac{\eta}{2} & \frac{\eta}{2} & \frac{\eta}{2} & \frac{\eta}{2} & \frac{\eta}{2} & 0 & 1 & 0 & 0 & \eta & \eta \\
\eta & \eta & \eta & \eta & \eta & \eta & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2\eta & 2\eta & 2\eta \\
\eta & \eta & 0 & 0 & 0 & 0 & \eta & \eta & \eta & \eta & 0 & 0 & 2\eta & 2 & 2\eta & 2\eta \\
0 & 0 & \eta & \eta & 0 & 0 & \eta & \eta & 0 & 0 & \eta & \eta & 2\eta & 2\eta & 2 & 2\eta \\
0 & 0 & 0 & 0 & \eta & \eta & 0 & 0 & \eta & \eta & \eta & \eta & 2\eta & 2\eta & 2\eta & 2
\end{pmatrix}$$

Again by using GAP, we can calculate the determinant and the eigenvalues of the Gram matrix G:

$$\det(Gr) = \eta^{10} + \frac{5}{3}\eta^9 - \frac{5}{4}\eta^8 - \frac{5}{2}\eta^7 + \frac{15}{16}\eta^6 + \frac{23}{16}\eta^5 - \frac{35}{64}\eta^4 - \frac{5}{16}\eta^3 + \frac{5}{32}\eta^2 - \frac{1}{192},$$

$$\bar{\lambda} = [(\frac{1}{2})^5, -\frac{1}{6}, (-1)^4].$$

Hence, the subalgebra A is simple unless $\eta = \frac{1}{2}, -\frac{1}{6}$ and $\eta = -1$.

Similarly, we compute the eigenvalues of the Gram matrix for the further subalgebras in case of $n = 5, 6, 7, \ldots$. The following table illustrates the special eigenvalues which have been calculated by GAP, (Figure 3.1);

Then, we formulate several conjectures:

Conjecture 6.6.4. The determinant of the Gram matrix Gr is a polynomial of degree $\frac{n(n+1)}{2}$, unless $n = 3$.

n	Degree of the determinant	Special eigenvalues
2	3	$[1^2, -\frac{1}{2}]$
3	3	$[\frac{1}{2}^2, -\frac{1}{4}]$
4	10	$[\frac{1}{2}^5, -\frac{1}{6}, -1^4]$
5	15	$[\frac{1}{2}^9, -\frac{1}{8}, -\frac{1}{2}^5]$
6	21	$[\frac{1}{2}^{14}, -\frac{1}{10}, -\frac{1}{3}^6]$
7	28	$[\frac{1}{2}^{20}, -\frac{1}{12}, -\frac{1}{4}^7]$
8	36	$[\frac{1}{2}^{27}, -\frac{1}{14}, -\frac{1}{5}^8]$
9	45	$[\frac{1}{2}^{35}, -\frac{1}{16}, -\frac{1}{6}^9]$
10	55	$[\frac{1}{2}^{44}, -\frac{1}{18}, -\frac{1}{7}^{10}]$
11	66	$[\frac{1}{2}^{54}, -\frac{1}{20}, -\frac{1}{8}^{11}]$
12	78	$[\frac{1}{2}^{65}, -\frac{1}{22}, -\frac{1}{9}^{12}]$
13	91	$[\frac{1}{2}^{77}, -\frac{1}{24}, -\frac{1}{10}^{13}]$
14	105	$[\frac{1}{2}^{90}, -\frac{1}{26}, -\frac{1}{11}^{14}]$

Table 6.3: Eigenvalues for n^2-dimensional subalgebras

Conjecture 6.6.5. The multiplicity of the eigenvalue $\frac{1}{2}$ is

$$\frac{n(n-1)}{2} - 1 = \frac{n^2 - n - 2}{2} = \frac{(n+1)(n-2)}{2}.$$

Conjecture 6.6.6. The Gram matrix Gr has eigenvalue $-\frac{1}{n-3}$ with multiplicity n.

Conjecture 6.6.7. There is just one further simple eigenvalue $-\frac{1}{2(n-1)}$ with multiplicity 1.

6.7 The dimension of the radicals

In this section we are looking for the conjectures concerning the values of η for which the radical of the algebra is non-zero. Then we calculate the dimension of the radical.

Let A be the subalgebra of the Matsuo algebra $M_\eta(^-O^+_{n+1}(3))$ spanned by the set of single axes $S = \{\langle e_i + \epsilon e_j\rangle : 1 \leq i < j \leq n, \epsilon = \pm 1\}$ and the set of double axes $D = \{\langle e_0 + e_i\rangle \dotplus \langle e_0 - e_i\rangle \mid 1 \leq i \leq n\}$, where $|S| = n(n-1)$ and $|D| = n$, and $\dotplus$ is the addition in the Matsuo algebra.

The set $\mathcal{B} = S \cup D$ is a basis for the subalgebra A. We use $a_i := \langle e_0 + e_i\rangle \dotplus \langle e_0 - e_i\rangle$ to denote the elements of D and, we have single axes;

$$\begin{aligned} b_{ij} &:= \langle e_i + e_j\rangle \\ c_{ij} &:= \langle e_i - e_j\rangle. \end{aligned}$$

Now, we take

$$\begin{aligned} d_{ij} &:= b_{ij} - c_{ij} \\ e_{ij} &:= b_{ij} + c_{ij}. \end{aligned}$$

The next proposition shows how we compute the values of the Frobenius form on the basis $S \cup D$ of A. For $a \in S \cup D$, let the support $\text{supp}(a)$ be defined as $\{i, j\}$ if $a = \langle e_i + \epsilon e_j\rangle$ is a single axis and as $\{0, i\}$ if $a = \langle e_0 + e_i\rangle \dotplus \langle e_0 - e_i\rangle$ is a double axis.

Proposition 6.7.1. Let $a, b \in S \cup D$. Then

- If $a = b$ then $(a, a) = 1$ if $a \in S$ and $(a, a) = 2$ if $a \in D$.
- If $a \neq b$ then $(a, b) = 0$ if $\text{supp}(a) \cap \text{supp}(b) = \emptyset$ or if $\text{supp}(a) = \text{supp}(b)$ (in this case both a and b are single axes).
- If $a \neq b$ and $|\text{supp}(a) \cap \text{supp}(b)| = 1$ then $(a, b) = \frac{\eta}{2}$ if $a, b \in S$; η if $a \in S$ and $b \in D$ (or vice versa); and 2η if $a, b \in D$.

Lemma 6.7.2. The subspace $\mathcal{D}$ spanned by all d_{ij} is orthogonal with respect to the Frobenius form $(,)$ to all other vectors of the basis. Therefore, $\mathcal{D}$ splits off as an orthogonal direct summand.

Proof. For the distinct indices i, j, k in $\{1, \ldots, n\}$ the claim holds as:

$$\begin{aligned}(a_k, d_{ij}) &= (a_k, b_{ij} - c_{ij});\\ &= 0 - 0 = 0.\end{aligned}$$

The other cases are as follows:

$$\begin{aligned}(a_i, d_{ij}) &= (a_i, b_{ij} - c_{ij});\\ &= \eta - \eta = 0.\end{aligned}$$

And similarly, $(a_j, d_{ij}) = 0$.

Now we verify that d_{ij} is orthogonal to all single axes,

$$\begin{aligned}(e_{ij}, d_{ij}) &= (b_{ij} + c_{ij}, b_{ij} - c_{ij})\\ &= 1 - 0 + 0 - 1 = 0.\end{aligned}$$

Also,

$$\begin{aligned}(e_{ik}, d_{ij}) &= (b_{ik} + c_{ik}, b_{ij} - c_{ij})\\ &= \frac{\eta}{2} - \frac{\eta}{2} + \frac{\eta}{2} - \frac{\eta}{2} = 0.\end{aligned}$$

In the same way, we can check that $(e_{jk}, d_{ij}) = 0$.

$$\begin{aligned}(d_{ij}, d_{ij}) &= (b_{ij} - c_{ij}, b_{ij} - c_{ij})\\ &= 1 - 0 - 0 + 1 = 2.\end{aligned}$$

Moreover, if we take $b_{ij} = b_{ji}$ and $c_{ij} = -c_{ji}$ Therefore, $\mathcal{D}$ is orthogonal to all other vectors in the basis of the subalgebra A. □

Let $f_{ij} = e_{ij} + \alpha(a_i + a_j) + \beta(\sum_{s \neq i,j} a_s)$. We want to choose $\alpha, \beta \in \mathbb{F}$ so that all f_{ij} are orthogonal to all a_s.

Hence we have:

$$
\begin{aligned}
0 = (f_{ij}, a_i) &= (b_{ij} + c_{ij} + \alpha(a_i + a_j) + \beta(\sum_{s \neq i,j} a_s), a_i) \\
&= \eta + \eta + \alpha(2 + 2\eta) + \beta((n-2)2\eta) \\
&= 2\eta + 2\alpha(1 + \eta) + 2\beta((n-2)\eta) \\
&= 2(\eta + \alpha(1 + \eta) + \beta((n-2)\eta)).
\end{aligned}
$$

Then,

$$
(1 + \eta)\alpha + (n-2)\eta\beta = -\eta. \tag{6.1}
$$

Also,

$$
\begin{aligned}
0 = (f_{ij}, a_s) &= (b_{ij} + c_{ij} + \alpha(a_i + a_j) + \beta(\sum_{s \neq i,j} a_s), a_s) \\
&= 0 + 0 + \alpha(2\eta + 2\eta) + \beta(2 + (n-3)2\eta) \\
&= 2(2\alpha\eta + \beta(1 + (n-3)\eta).
\end{aligned}
$$

So,

$$
2\alpha\eta + (1 + (n-3)\beta\eta = 0. \tag{6.2}
$$

We solve (6.1) and (6.2) using Cramer's Rule:

$$
\begin{aligned}
w = \det \begin{bmatrix} \eta + 1 & (n-2)\eta \\ 2\eta & 1 + (n-3)\eta \end{bmatrix} &= (\eta + 1)(1 + (n-3)\eta) - 2\eta^2(n-2) \\
&= \eta + (n-3)\eta^2 + 1 + (n-3)\eta - 2(n-2)\eta^2 \\
&= 1 + (n-2)\eta + (-n+1)\eta^2 \\
&= -(\eta - 1)((n-1)\eta + 1).
\end{aligned}
$$

Furthermore,

$$
u = \det \begin{bmatrix} -\eta & (n-2)\eta \\ 0 & 1 + (n-3)\eta \end{bmatrix} = -\eta(1 + (n-3)\eta,
$$

and also,

$$v = \det \begin{bmatrix} \eta+1 & -\eta \\ 2\eta & 0 \end{bmatrix} = 2\eta^2.$$

Then,

$$\alpha = \frac{u}{w} = \frac{\eta(1+(n-3)\eta)}{(\eta-1)(1+(n-1)\eta)}, \tag{6.3}$$

and

$$\beta = \frac{v}{w} = \frac{-2\eta^2}{(\eta-1)(1+(n-1)\eta)}. \tag{6.4}$$

Now,

$$\begin{aligned}(f_{ij}, f_{ij}) &= (b_{ij} + c_{ij} + \alpha(a_i + a_j) + \beta(\sum_{s\neq i,j} a_s), b_{ij} + c_{ij} + \alpha(a_i + a_j) + \beta(\sum_{s\neq i,j} a_s)) \\ &= 2((2\eta+2)\alpha^2 + 4\eta\alpha + 4(n-2)\eta\alpha\beta - 4\eta\beta + 2(1+(n-3)\eta(2+(n-3)\eta))\beta^2 + 1).\end{aligned}$$

By substituting the values of α and β, we get:

$(f_{ij}, f_{ij}) = 2\frac{(n-3)(8n-24)\eta^6+(-2n^2+30n-90)\eta^5+(-2n^2+16n+10)\eta^4+(n^2-13n+10)\eta^3+(-2n^2+3n-2)\eta^2+(n^2-3n)\eta+n}{(\eta-1)^2(1+(n-1)\eta)^2}$.

Similarly, we calculate (f_{ij}, f_{ik}):

$$\begin{aligned}(f_{ij}, f_{ik}) &= (b_{ij} + c_{ij} + \alpha(a_i + a_j) + \beta(\sum_{s\neq i,j} a_s), b_{ik} + c_{ik} + \alpha(a_i + a_k) + \beta(\sum_{s\neq i,k} a_s)) \\ &= 2(\eta + 2\eta\alpha - 2\eta\beta + (2\eta+2)\alpha^2 + (n-2)^2\eta\beta^2 + ((3n-7)\eta+1)\alpha\beta).\end{aligned}$$

Also, from (3) and (4) we get:

$(f_{ij}, f_{ik}) = 2\frac{(3n+5)(n-1)\eta^5+(2n^2+3n-10)\eta^4+(n^2-6n+3)\eta^3+2(n-2)\eta^2+\eta}{(\eta-1)^2(1+(n-1)\eta)^2}$.

Hence, the new basis of the subalgebra A is $\mathcal{B}' = \{d_{ij}\} \cup \{f_{ij}\} \cup a_i$.

We aim to determine the dimension of the radicals of the form, that means where the determinant of the new Gram matrix vanishes with respect to the new basis $\mathcal{B}'$.

The Gram matrix with a new basis is

$$G' = \begin{pmatrix} G_S & & \\ & G_D & \\ & & G_F \end{pmatrix}, \tag{6.5}$$

where $G_S = 2I_{|d_{ij}|}$,

$$G_D = \begin{bmatrix} 2 & \dots & 2\eta \\ \vdots & \ddots & \\ 2\eta & & 2 \end{bmatrix},$$

and G_F is the Gram matrix of the form on the new vector $F = f_{ij}$ and it can be written as $G_F = (f_{ij}, f_{ij})I + (f_{ij}, f_{ik})\Gamma_F$, where I is the identity matrix and Γ_F is the adjacency matrix of a group whose vertex set is $F = \{f_{ij}\}$. There is an edge between the intersection pairs, f_{ij} and f_{ik}.

Take $\zeta = (f_{ij}, f_{ij})$ and $\tau = (f_{ij}, f_{ik})$, as we computed earlier. Then $G_F = \zeta I + \tau X$ where I is the identity matrix and X is the adjacency matrix of the graph on the set of all 2-element subsets $\{i, j\}$ where two such subsets adjacent when they share a single common element. (This graph is known as the Johnson graph $J(n, 2)$.) The eigenvalues of G_F are of the form $\zeta + \tau\lambda$ where λ is an eigenvalue of X. The Johnson graph is a strongly regular graph and it has parameters:

$$(N, K, a, c) = (\frac{n(n-1)}{2}, 2n-4, n-2, 4).$$

It follows that the eigenvalues of X are:

$$\lambda_0 = K = 2n - 4,$$
$$\lambda_1 = \frac{(a-c) + \sqrt{\Delta}}{2},$$
$$\lambda_2 = \frac{(a-c) - \sqrt{\Delta}}{2},$$

where $\Delta = (a-c)^2 + 4(K-c)$, and the multiplicities of the eigenvalues λ_0, λ_1 and λ_2 are as follows:

$$\begin{aligned}
m_0 &= 1,\\
m_1 &= \frac{1}{2}((N-1) - \frac{2k+(N-1)(a-c)}{\sqrt{\Delta}}),\\
m_2 &= \frac{1}{2}((N-1) + \frac{2k+(N-1)(a-c)}{\sqrt{\Delta}}).
\end{aligned}$$

To compute λ_1 and λ_2, first we calculate Δ

$$\begin{aligned}
\Delta &= (a-c)^2 + 4(K-c)\\
&= (n-6)^2 + 4(2n-4-4)\\
&= n^2 - 4n + 4\\
&= (n-2)^2.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\lambda_1 &= \frac{(n-6)+(n-2)}{2} = n-4,\\
\lambda_2 &= \frac{(n-6)-(n-2)}{2} = -2.
\end{aligned}$$

Then the multiplicity of λ_1 is:

$$\begin{aligned}
m_1 &= \frac{1}{2}((N-1) - \frac{2k+(N-1)(a-c)}{\sqrt{\Delta}})\\
&= \frac{1}{2}(\frac{(n+1)(n-2)}{2} - \frac{2(2n-4)+\frac{(n+1)(n-2)(n-6)}{2}}{n-2})\\
&= \frac{1}{2}(\frac{(n+1)(n-2)}{2} - \frac{n^2-5n+2}{2})\\
&= \frac{1}{2}(\frac{n^2-n-2-n^2+5n-2}{2})\\
&= n-1,
\end{aligned}$$

and the multiplicity of λ_2 is:

$$\begin{aligned}
m_2 &= \frac{1}{2}((N-1) + \frac{2k + (N-1)(a-c)}{\sqrt{\Delta}}) \\
&= \frac{1}{2}(\frac{(n+1)(n-2)}{2} + \frac{2(2n-4) + \frac{(n+1)(n-2)(n-6)}{2}}{n-2}) \\
&= \frac{1}{2}(\frac{(n+1)(n-2)}{2} + \frac{n^2-5n+2}{2}) \\
&= \frac{1}{2}(\frac{n^2-n-2+n^2-5n+2}{2}) \\
&= \frac{n(n-3)}{2}.
\end{aligned}$$

From this we can find the eigenvalues $\zeta + \eta\lambda_i$.

CHAPTER 7

FLIP SUBALGEBRAS IN THE EXTENDED SYMMETRIC CASE

In this chapter we construct flip subalgebras of $M_\eta(2^{n-1} : S_n)$. First, we discuss the classes of flips and then, for each representative flip, we describe the corresponding flip subalgebra. Towards the end of the chapter, we find the critical values of η for these new algebras and determine the dimension of the radical when it is non-trivial.

7.1 Setup

Let $S = S_n$ and $V \cong 2^n \cong C_2 \times C_2 \times \dots C_2$ be the multiplicative group of all $\{-1, 1\}$-sequences of length n. Then S acts on V by permuting positions in each sequence, and so we can consider the semi-direct product $\hat{G} = \hat{G}_n := V : S \cong 2^n : S_n$. (Note that this is the wreath product $C_2 \wr S_n$.) Recall that multiplication in $\hat{G}$ is as follows: $(v, \sigma)(v', \sigma') = (v(v')^{\sigma^{-1}}, \sigma\sigma')$. We use right actions and this explains the exponent notation for the action of σ on v'.

It will be convenient to use the following notation: the pair $(v, \sigma) \in \hat{G}$, where $v = (\delta_1, \delta_2, \dots, \delta_n) \in V$ and $\sigma \in S$, will be denoted by $(\delta_1, \delta_2, \dots, \delta_n : \sigma)$. For example, for $n = 3$, $v = (1, -1, 1)$, and $\sigma = (2, 3)$, we write $(1, -1, 1 : (2, 3))$ instead of $((1, -1, 1), (2, 3))$. We will view both V and S as subgroups of $\hat{G}$, that is, we will simplify $(\delta_1, \delta_2, \dots, \delta_n : ())$ to just $(\delta_1, \delta_2, \dots, \delta_n)$ and, similarly, we will simplify $(1, 1, \dots, 1 : \sigma)$ to just σ. So we can also write $(\delta_1, \delta_2, \dots, \delta_n : \sigma) = (\delta_1, \delta_2, \dots, \delta_n)\sigma$.

It will also be convenient to refer to v as the *V-part* of the element $(v, \sigma) \in \hat{G}$, and similarly, σ can be called the *S-part*.

Throughout the chapter, we will be defining various elements of V. In particular, let $v_i = (1, \dots, 1, -1, 1, \dots, 1) \in V$ be the element, where the only -1 appears in the ith position. Clearly, $v_1, v_2, \dots, v_n$ are the generators of V and so every element of V can be expressed as a product of the elements v_i.

We denote by π the natural homomorphism $\hat{G} \to S$ sending every $(\delta_1, \delta_2, \dots, \delta_n : \sigma)$ to its S-part σ. Clearly, π is surjective and V is the kernel of π.

It is well-known that two elements of S are conjugate if and only if they have the same cycle type, describing the cycle lengths of the decomposition of a permutation as a product of independent cycles. For example, the cycle type of $(1,3,7)(2,4)(5,8) \in S_8$ is $2^2 3^1$, since it has two 2-cycles and one 3-cycle. Note that we ignore 1-cycles, i.e., fixed points. For $g \in (v, \sigma) \in \hat{G}$, we define the *cycle type* of g as the cycle type of the permutation σ.

Lemma 7.1.1. If two elements of $\hat{G}$ are conjugate then they have the same cycle type.

Proof. If $g = (v, \sigma)$ and $g' = (v', \sigma')$ are conjugate in $\hat{G}$ then $g' = g^h$ for some $h = (u, \tau) \in \hat{G}$. Applying π, we obtain $\sigma' = \pi(g') = \pi(g^h) = \pi(g)^{\pi(h)} = \sigma^\tau$. So σ and σ' are conjugate in S, yielding that they have the same cycle type. □

We note that the converse is not true and we have elements of $\hat{G}$ with the same cycle type that are not conjugate.

Define C to be the conjugacy class $(1,2)^{\hat{G}}$ of $\hat{G}$. Clearly, C contains all transpositions (2-cycles) from S. Which other elements does it contain? Let

$$v_{i,j} = v_i v_j = (1, \dots, 1, -1, 1, \dots, 1, -1, 1, \dots, 1),$$

where the entries -1 are in the ith and jth positions.

Proposition 7.1.2. The class C consists of all elements $e_{i,j} = (i,j)$ and $f_{i,j} = v_{i,j}(i,j)$ for $1 \leq i < j \leq n$.

Proof. Since $(1,2)$ and (i,j) are conjugate in $S \leq \hat{G}$, it follows that $e_{i,j} = (i,j)$ is an element in C for all $1 \leq i < j \leq n$. Note that $(i,j)^{v_i} = (v_i)^{-1}(i,j)v_i = v_i v_i^{(i,j)}(i,j) = v_i v_j (i,j) = v_{i,j}(i,j)$. So all elements $f_{i,j} = v_{i,j}(i,j)$ are also contained in C.

This gives us $2\binom{n}{2}$ elements of C. Hence, to finish the proof, it suffices to show that $|C| = 2\binom{n}{2}$. Let S_0 be the set-wise stabilizer of $\{1,2\}$ in S (this is equal to the centraliser of $(1,2)$ in S) and $V_0 = \{(\delta_1, \delta_2, \dots, \delta_n) \in V \mid \delta_1 = \delta_2\}$. It is clear that V_0 is a subgroup of V of index 2 and that V_0 centralises $(1,2)$. It follows that $C_{\hat{G}}((1,2)) \geq V_0 S_0$. Therefore,

$|C| = [\hat{G} : C_{\hat{G}}((1,2))] \leq \frac{|V||S|}{|V_0||S_0|} = 2\binom{n}{2}$, since $\frac{|V|}{|V_0|} = 2$ and $\frac{|S|}{|S_0|} = \binom{n}{2}$. Thus, $|C| \leq 2\binom{n}{2}$, and so we have the desired equality. □

Let $G = \langle C \rangle$.

Proposition 7.1.3. (G, C) is a 3-transposition group.

Proof. We need to show that $|cd| \leq 3$ for all $c, d \in C$. Since all elements of C are conjugate in $\hat{G}$, it suffices to take $c = (1,2)$. If $d = c$ then, clearly, $cd = 1$ and so $|cd| = 1$. If $d = v_{1,2}(1,2)$ then $cd = v_{1,2}$ is of order 2. Now we can assume that $(i,j) := \pi(d) \neq (1,2)$. If $\{1,2\}$ and $\{i,j\}$ are disjoint then, clearly, c and d commute, that is, $|cd| = 2$. Finally, suppose that $|\{1,2\} \cap \{i,j\}| = 1$. Without loss of generality we may assume that $(i,j) = (1,j)$ for some $j \geq 3$. Then $d = (1,j)$ or $v_{1,j}(1,j)$. In the first case, $cd = (1,2)(1,j) = (1,2,j)$, and in the second case,

$$\begin{aligned} cd &= (1,2)v_{1,j}(1,j) \\ &= (v_{1,j})^{(1,2)^{-1}}(1,2)(1,j) \\ &= v_{2,j}(1,2,j). \end{aligned}$$

In both cases $|cd| = 3$. □

Let $W = \{(\delta_1, \delta_2, \ldots, \delta_n) \in V \mid \delta_1\delta_2 \cdots \delta_n = 1\}$. Note that W consists of all tuples from V containing an even number of -1s.

Proposition 7.1.4. $G = WS$ has index 2 in $\hat{G}$. In particular, $G \trianglelefteq \hat{G}$.

Proof. First, we need to prove that WS is a subgroup of $\hat{G}$ of index 2. Since $G = \langle C \rangle = \langle (1,2)^{\hat{G}} \rangle$, the class C contains all transpositions (i,j) and they generate S. Hence S is contained in G.

Consider the map $\phi : V \to C_2 = \{1, -1\}$ given by:

$$\phi((\delta_1, \delta_2, \ldots, \delta_n)) = \delta_1\delta_2 \cdots \delta_n.$$

It is easy to check that ϕ is a homomorphism. Indeed, let $v = (\delta_1, \delta_2, \ldots, \delta_n)$ and

$v' = (\delta'_1, \delta'_2, \ldots, \delta'_n)$ be in V. Then

$$\begin{aligned}\phi(vv') &= \phi((\delta_1, \delta_2, \ldots, \delta_n)(\delta'_1, \delta'_2, \ldots, \delta'_n)) \\ &= \phi((\delta_1\delta'_1, \delta_2\delta'_2, \ldots, \delta_n\delta'_n)) \\ &= \delta_1\delta'_1\delta_2\delta'_2 \cdots \delta_n\delta'_n \\ &= (\delta_1\delta_2 \cdots, \delta_n)(\delta'_1\delta'_2 \cdots \delta'_n) = 1.\end{aligned}$$

So ϕ is a homomorphism. Furthermore, ϕ is surjective and $W = \ker \phi$, so W is a subgroup and $[V : W] = \frac{|V|}{|W|} = |\mathrm{im}(\phi)| = |C_2| = 2$ by the First Isomorphism Theorem. We have shown that W is a subgroup of V of index 2.

Also, S normalises W. Indeed, suppose that $(\delta_1, \delta_2, \ldots, \delta_n) \in W$. Then $(\delta_1, \delta_2, \ldots, \delta_n)^\sigma = (\delta_{1^\sigma}, \delta_{2^\sigma}, \ldots, \delta_{n^\sigma})$. In particular, $(\delta_1, \delta_2, \ldots, \delta_n)^\sigma$ has the same number of -1s as $(\delta_1, \delta_2, \ldots, \delta_n)$. Recall that W consists of all tuples from V with an even number of -1s. It follows that $(\delta_1, \delta_2, \ldots, \delta_n) \in W$ if and only if $(\delta_1, \delta_2, \ldots, \delta_n)^\sigma \in W$ for all $\sigma \in S$. Hence $W^\sigma = W$, i.e., S normalises W. Consequently, WS is a subgroup of $\hat{G}$. Moreover, $\frac{|\hat{G}|}{|WS|} = \frac{|VS|}{|WS|} = \frac{|V||S|}{|W||S|} = \frac{|V|}{|W|} = 2$. So, $[\hat{G} : WS] = 2$, which also shows that WS is a normal subgroup of $\hat{G}$.

From Proposition 7.1.2, every element of C is either $x := (i, j)$ or $x' := v_{i,j}(i, j)$ for some $1 \leq i < j \leq n$. Since, in the first of these two, the V-part is the identity, with zero -1's in it, and since $v_{i,j}$ contains two -1s, both x and x' lie in WS. Therefore, $C \subseteq WS$ and this means that $G = \langle C \rangle \leq WS$.

As we previously discussed, $S \subseteq G$. Also, all elements $v_{i,j} = x'x^{-1} \in G$. Next, we prove that $W = \langle v_{i,j} \mid 1 \leq i < j \leq n \rangle$. For $2 \leq k \leq n$, we define $T_k = \langle v_{i,j} \mid 1 \leq i < j \leq k \rangle$ and $W_k = \{(\delta_1, \delta_2, \ldots, \delta_n) \in W \mid \delta_{k+1} = 1 = \delta_{k+2} = \ldots, = \delta_n\}$. We prove by induction that $T_k = W_k$ for all k. If $n = 2$ then the statement is clearly correct since $T_1 = \langle v_{1,2} \rangle = W_2$. Assume now that $k \geq 3$ and $T_s = W_s$ is true for $s = k - 1$. That is, $T_{k-1} = W_{k-1}$. Clearly, $T_k \subseteq W_k$. Now, take an arbitrary $v = (\delta_1, \delta_2, \ldots, \delta_n) \in W_k$. If $\delta_k = 1$, then $v \in W_{k-1} = T_{k-1} \subseteq T_k$. If $\delta_k = -1$, then note that $vv_{k-1,k}$ has the $(k-1)$th entry 1, so $vv_{k-1,k} \in T_k$. Since also $v_{k-1,k} \in T_k$, we conclude that $v \in T_k$. Thus, all elements of W_k are in T_k, and so $W_k \subseteq T_k$, yielding $T_k = W_k$. So indeed $T_k = W_k$ for all k.

Taking $k = n$, we obtain that $T_n = W_n = W$, i.e., W is generated by all elements $v_{i,j}$. Since all $v_{i,j}$ are in G, we conclude that $W \leq G$. We have shown that $WS \leq G$, and so $G = WS$ is of index two in $\hat{G}$. □

We see that G has the structure $2^{n-1} : S_n$. We will call it the *extended symmetric group*.

This is a well-known group that appears, for example, as the Weyl group $W(D_n)$. The 3-transposition group (G, C) appears in [6] in the class PR.2.

7.2 The automorphism group of G

Since G has index 2 in $\hat{G}$, it is normal, and this means that $\hat{G}$ acts on G by conjugation. We note that this action is not faithful. Namely, the element $z = (-1, -1, \ldots, -1) \in V$ is in the centre of $\hat{G}$ and so it acts on G trivially. Hence the first result we need is the following.

Proposition 7.2.1. If $n \geq 3$ then the kernel of $\hat{G}$ acting on G by conjugation coincides with $Z = \langle z \rangle = Z(\hat{G})$.

Proof. Note that the kernel coincides with the centraliser of G in $\hat{G}$ and it contains the centre of $\hat{G}$. Suppose $g \in C_{\hat{G}}(G)$. Then in particular g commutes with all elements of S and this implies that $\pi(g)$ is in the centre of $\pi(S) = \hat{G}/V \cong S \cong S_n$. However, S_n has trivial centre when $n \geq 3$, which means that $\pi(g) = 1$, that is, $g \in V$. Let $g = (\delta_1, \delta_2, \ldots, \delta_n)$. Again, since g centralises S, we must have that $g^\tau = g$ for all $\tau \in S_n$, and this can only happen when all entries δ_i are the same. Thus, $g = 1$ or $g = z$. We have shown that $C_{\hat{G}}(G) \leq Z$. On the other hand, clearly, $Z \leq Z(\hat{G}) \leq C_{\hat{G}}(G)$, and so we have the desired equality. □

If $n = 2$ then G is of order 4 and hence Abelian, i.e., it centralises itself. This shows that $n = 2$ is a true exception.

Recall that by an automorphism of a 3-transposition group (G, C) we mean an automorphism ϕ of G such that $C^\phi = C$. All automorphisms of (G, C) form a subgroup of $\mathrm{Aut}(G)$, which we denote by $\mathrm{Aut}(G, C)$.

Taking Proposition 7.2.1 into account, we now formulate the following key result.

Proposition 7.2.2. Let G be the extended symmetric group $2^{n-1} : S_n$ and $C = (1, 2)^G$. If $n \geq 3$ then every automorphism of (G, C) is induced by an element of $\hat{G}$. That is, $\mathrm{Aut}(G, C) \cong \hat{G}/Z$.

Proof. Recall that $G = WS$, where $W = \{(\delta_1, \delta_2, \ldots, \delta_n) \in V \mid \delta_1\delta_2 \cdots \delta_n = 1\}$, and W is a subgroup of V of index two. Consider an arbitrary automorphism $\phi \in \mathrm{Aut}(G, C)$.

Note that $W = O_2(G)$ char G, and so $W^\phi = W$. This means that ϕ also acts on $\bar{G} = G/W \cong S_n$. Furthermore, since $C^\phi = C$ and since $\bar{C}$ is the conjugacy class of transpositions in $\bar{G}$, we conclude that ϕ induces an inner automorphism of $\bar{G}$. (Indeed, if $n \neq 6$ then every automorphism of S_n is inner. The group S_6, on the other hand, does have outer automorphisms, but they do not preserve the class of transpositions, switching it with the class of triple 2-cycles.)

This means that correcting ϕ by an inner factor (certainly induced by an element of G), we can assume that ϕ acts on $\bar{G}$ as the identity. Then for every pair $\{i, j\}$, ϕ either fixes both (i, j) and $v_{i,j}(i, j)$ or it switches them. We will call the *height* of ϕ the largest i such that ϕ does not fix $(i-1, i)$. Note that the height is defined for every $\phi \neq 1$. Indeed, if ϕ fixes all $(i-1, i)$, $i = 2, 3, \ldots, n$, then ϕ centralises the subgroup of S_n that these elements generate. It follows that every (i, j) is fixed, and this means that all elements of C are fixed. However, C generates G and so $\phi = 1$. Thus every nonidentity automorphism ϕ has some height ≥ 2. To cover the case $\phi = 1$, we will say this identity automorphism has height 1.

We now prove that ϕ is induced by an element of $\hat{G}$ and we will use induction on height. If the height is 1 then $\phi = 1$ and there is nothing to prove. So suppose that the claim holds true for heights at most $k - 1 \geq 1$ and suppose that ϕ has height k. Let $\phi' = \phi c_{v_{k-1}}$ (as usual, c_g denotes conjugation by g). We note that $c_{v_{k-1}}$ switches $(k-1, k)$ and $v_{k-1,k}(k-1, k)$. Similarly, it switches $(k-2, k-1)$ and $v_{k-2,k-1}(k-2, k-1)$, but it fixes all the other elements $(i-1, i)$ and $v_{i-1,i}(i-1, i)$. From this it immediately follows that ϕ' also acts as identity on $\bar{G}$ and it has height smaller than the height of ϕ. By induction ϕ' coincides with c_g for some $g \in \hat{G}$ and therefore $\phi = c_{gv_{k-1}}$ is induced by some element of $\hat{G}$. This completes the proof. □

7.3 Classes of flips

We aim to find all conjugacy classes of flips (automorphisms of order 2) of (G, C).

In view of Proposition 7.2.2, we just need to find the conjugacy classes of involutions in $\hat{G}/Z$. Suppose that $g \in \hat{G}$ is such that gZ is an involution. Equivalently, $Z = (gZ)^2 = g^2 Z$, i.e., $g^2 \in Z = \{1, z\}$. The elements g with $g^2 = 1$ are simply the involutions of $\hat{G}$, and this is the first case we are going to consider (leaving the second case, $g^2 = z$, for later).

Suppose that $g = (\delta_1, \delta_2, \ldots, \delta_n : \sigma)$ is an involution, that is, it has order 2. Recall that

$\pi : \hat{G} \to S$ is the homomorphism sending g to σ. It follows that $\sigma = \pi(g)$ has order dividing 2, that is, it has cycle type 2^k, where $0 \le k \le \lfloor \frac{n}{2} \rfloor$. This includes the case $\sigma = 1$ arising for $k = 0$. We will call k the *rank* of the involution g. Clearly, the rank is preserved under conjugation.

Theorem 7.3.1. Every involution $g \in \hat{G}$ is conjugate to a unique element of the form

$$g_{k,r} = (1, 1, \dots, 1, -1, \dots, -1 : (1, 2)(3, 4) \cdots (2k - 1, 2k)),$$

where r is the number of -1s in the tail of the V-part of $g_{k,r}$ and it satisfies $0 \le r \le n - 2k$ (and also $k + r > 0$, to avoid the identity element).

Proof. Note that the V-part $w_r = (1, 1, \dots, 1, -1, \dots, -1)$ of $g_{k,r}$ has all 1s in positions $1, 2, \dots, 2k$. In particular, this means that w_r and $\sigma_k = \pi(g_{k,r}) = (1, 2)(3, 4) \cdots (2k - 1, 2k)$ commute, and so $g_{k,r} = w_r \sigma_k$ is indeed an involution.

We first argue that no two distinct elements $g_{k,r}$ are conjugate. Indeed, suppose $g_{k,r}$ and $g_{k',r'}$ are conjugate. Clearly, they have rank k and k', respectively, and so $k = k'$. It remains to show that $r = r'$. For this, we can use the realisation of $\hat{G}$ as the group of all monomial $n \times n$ matrices over $\mathbb{F}_3$. Here V becomes the group of diagonal matrices and S becomes the group of permutation matrices. Let $U := \mathbb{F}_3^n$ be the row space on which the matrices act. Let $\{u_1, u_2, \dots, u_n\}$ be the standard basis of U. Then it is easy to see that $[U, g_{k,r}]$ coincides with $\langle u_1 + u_2, u_3 + u_4, \dots, u_{2k-1} + u_{2k}, u_{n-r+1}, \dots, u_n \rangle$, and so it has dimension $k + r$. Similarly, $[U, g_{k',r'}]$ has dimension $k' + r'$. Since $g_{k,r}$ and $g_{k',r'}$ are conjugate, we must have $k + r = k' + r'$. We also have that $k = k'$ and hence $r = r'$. Thus, indeed no two different elements $g_{k,r}$ are conjugate.

We now need to show that every involution $g = v\sigma$ is conjugate to one of the elements $g_{k,r}$. Clearly, we must choose k to be equal to the rank of g, i.e., the number of 2-cycles in σ. Since two elements of S are conjugate if and only if they have the same cycle type, there exists an element of S conjugating $g = v\sigma$ to $g' = v'\sigma'$, where $\sigma' = \sigma_k$. Hence, without loss of generality, we may assume that $g = g'$ and so $\sigma = \sigma_k = (1, 2)(3, 4) \cdots (2k - 1, 2k)$.

From this point on, we are conjugating by the elements centralising $\sigma = \sigma_k$, so this part of g stays the same, and only v changes. Let us decompose $V = V' \times V''$ into a direct product of $V' = \langle v_1, \dots, v_{2k} \rangle$ and $V'' = \langle v_{2k+1}, \dots, v_n \rangle$. Correspondingly, $v = v'v''$, where $v' \in V'$ and $v'' \in V''$. Let us notice the following property of v': for $i = 1, 2, \dots, k$, we have that $\delta_{2i-1} = \delta_{2i}$. This is because $g = v\sigma$ is an involution, which implies that v and σ commute. The element v^σ has the entries δ_{2i-1} and δ_{2i} swapped. However, $v^\sigma = v$, since v and σ commute, and so, indeed, $\delta_{2i-1} = \delta_{2i}$.

Consider the element $w = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n) \in V$, where:

(a) For $i = 1, 2, \ldots, k$, we have that $\epsilon_{2i-1} = 1$ and $\epsilon_{2i} = \delta_{2i}$,

(b) For all $j > 2k$, we have that $\epsilon_j = 1$.

Clearly, $w \in V'$. Furthermore, $[w, \sigma] = v'$ and hence $g^w = (v\sigma)^w = v^w \sigma^w = vv'\sigma = (v'v'')v'\sigma = v''\sigma$. This means that we managed to conjugate g in such a way that the v' part of it becomes trivial. Thus, we can now assume that $v' = 1$ and $v = v'' \in V''$.

In particular, all the -1s in v are now located in positions $\{2k+1, 2k+2, \ldots, n\}$. Let r be the number of -1s. Conjugating by an element of $\mathrm{Sym}(2k+1, \ldots, n) \leq S$ permutes the last $n - 2k$ entries of v. Clearly, we can select a permutation in $\mathrm{Sym}(2k+1, \ldots, n)$ moving all -1s to the end of v, hence after this final conjugation, we obtain $g_{k,r}$ and this proves that every involution $g \in \hat{G}$ is conjugate to some $g_{k,r}$, completing the proof. □

Thus, the elements $g_{k,r}$ represent all classes of involutions in $\hat{G}$ and each class is identified uniquely by the pair (k, r) satisfying $k + r > 0$ and $2k + r \leq n$. For example, for $n = 4$, there are eight classes of involutions corresponding to pairs $(0,1)$, $(0,2)$, $(0,3)$, $(0,4)$, $(1,0)$, $(1,1)$, $(1,2)$, and $(2,0)$.

Now recall that g and gz induce the same automorphism of G. If g is conjugate to $g_{k,r}$ then it is easy to see that gz is conjugate to $g_{k,r'}$, where $r' = n - 2k - r$. In particular, in the above example for $n = 4$, the pairs $(0,1)$ and $(0,3)$ lead to the same class of involutions in $\mathrm{Aut}(G)$. In total, we only need here to take the five pairs $(0,1)$, $(0,2)$, $(1,0)$, $(1,1)$, and $(2,0)$, giving five conjugacy classes of flips of first kind in $\mathrm{Aut}(G)$.

Now we turn to the second kind, namely, we consider $g = v\sigma \in \hat{G}$ such that $g^2 = z$. Clearly, this means that $\sigma^2 = \pi(g)^2 = \pi(g^2) = \pi(z) = 1$, and so $\sigma^2 = 1$. If $\sigma = 1$ then $g = v$ is an involution, which contradicts that $g^2 = z$. Thus, σ is an involution in S.

Theorem 7.3.2. The group $\hat{G}$ contains elements g satisfying $g^2 = z$ if and only if $n = 2k$ is even. Furthermore, all such elements are conjugate to $h_k := (-1, 1, -1, 1, \ldots, -1, 1)\sigma_k$.

Proof. It easy to check that h_k and its conjugates square to z.

Conversely, conjugating by an element of S if necessary, we may assume that $\sigma = \sigma_k = (1,2)(3,4)\cdots(2k-1, 2k)$ for some $k \geq 1$. Let $v = (\delta_1, \delta_2, \ldots, \delta_n)$. Then $g^2 = (v\sigma_k)^2 = vv^{\sigma_k} = (\delta_1\delta_2, \delta_1\delta_2, \delta_3\delta_4, \delta_3\delta_4, \ldots, \delta_{2k-1}\delta_{2k}, 1, \ldots, 1)$. Since this must be equal to $z = (-1, -1, \ldots, -1)$, we conclude that (a) $n = 2k$ (and so n is even); and (b) for each $j = 1, 2, \ldots, k$, we have that $\delta_{2j-1} = -1$ and $\delta_{2j} = 1$, or $\delta_{2j-1} = 1$ and $\delta_{2j} = -1$.

Now notice that conjugating g by v_{2j-1} switches the values of δ_{2j-1} and δ_{2j}. Therefore, we conclude that every element g such that $g^2 = z$ is conjugate to the element $h_k = (-1, 1, -1, 1, \ldots, -1, 1)\sigma_k$, where $n = 2k$. $\square$

We have now completed the classification of all classes of flips for the extended symmetric case. Next, we determine, for each class of flips, the principal parameters of the corresponding flip subalgebras: the number of singles, doubles, and extras, as well as the total dimension of the flip subalgebra.

We have two cases: either $\sigma = g_{k,r}$ for $r \leq n - 2k$, or $n = 2k$ and $\sigma = h_k$.

7.4 Flip subalgebras for $\sigma = g_{k,r}$

Before we discuss the general case $\sigma = g_{k,r}$, let us focus on two particular situations.

7.4.1 Case $n = 2k$ and $\sigma = g_{k,0}$

Suppose that $n = 2k$ and let $\sigma = g_{k,0} = \sigma_k = (1, 2)(3, 4) \ldots (2k - 1, 2k)$. Let us calculate how many singles, doubles, and extras we have in this situation.

Proposition 7.4.1. The fixed subalgebra M_σ of the Matsuo algebra $M = M_\eta(2^{2k-1} : S_{2k})$ contains $2k$ singles, $2k(k - 1)$ doubles, and no extras. Therefore, the dimension of the flip subalgebra $A_\sigma = M_\sigma$ is $2k + 2k(k - 1) = 2k^2$.

Proof. Again, we will use the notation $\bar{i}$, meaning that $\{i, \bar{i}\} = \{2s - 1, 2s\}$ for some s with $1 \leq s \leq k$. Let $a = e_{i,j}$ or $f_{i,j}$. Then $a^\sigma = e_{\bar{i},\bar{j}}$ or $f_{\bar{i},\bar{j}}$, correspondingly. In particular, a is a single if and only if $\{i, j\} = \{\bar{i}, \bar{j}\}$, i.e., $j = \bar{i}$. It follows that there are exactly $2k = n$ singles: $e_{1,2}, f_{1,2}, e_{3,4}, f_{3,4}, \ldots, e_{2k-1,2k}, f_{2k-1,2k}$.

Now suppose that $j \neq \bar{i}$. Then $\{i, j\}$ and $\{\bar{i}, \bar{j}\}$ are disjoint, which implies that a and a^σ are orthogonal. Hence, all of such sums $a + a^\sigma$ are doubles. It follows that there are no extras and the number of doubles is $\frac{1}{2}(n(n - 1) - n) = 2k(k - 1)$, as claimed. $\square$

This flip subalgebra has already appeared in [14], where it was denoted $2Q_k(\eta)$.

7.4.2 Case $\sigma = g_{0,r}$

Now, consider σ is the conjugation by $w_r := g_{0,r} = (1, 1, \cdots, 1, -1, \cdots, -1)$. (For simplicity, we will write $\sigma = w_r$.) Here $k = 0$ and $r \leq \frac{n}{2}$.

Proposition 7.4.2. The fixed subalgebra M_σ of the Matsuo algebra $M = M_\eta(2^{n-1} : S_n)$ contains $r(r-1) + (n-r)(n-r-1)$ singles, $r(n-r)$ doubles, and no extras. Therefore, the dimension of $A_\sigma = M_\sigma$ is $r^2 + n(n-r-1)$.

Proof. Suppose $a = e_{i,j}$ or $f_{i,j}$, where $1 \leq i < j \leq n$. We have three different cases: (1) $j \leq n-r$; (2) $i \leq n-r < j$; and (3) $n-r < i$. In cases (1) and (3), we have that $w_r^a = w_r$, which implies that $a^\sigma = a^{w_r} = a$, so a is a single. This gives us $2\binom{n-r}{2}$ singles in case (1) and $2\binom{r}{2}$ in case (3). Finally, in case (2), the commutator $a^{-1}a^\sigma = [a, w_r] = v_{i,j}$, which means that σ switches $e_{i,j}$ and $f_{i,j}$. Note that $e_{i,j}$ and $f_{i,j}$ commute in G, and hence they are orthogonal in M. It follows that $a + a^\sigma = e_{i,j} + f_{i,j}$ is a double. So we have $(n-r)r$ doubles in case (2) and no extras.

So, in total, we have $(n-r)(n-r-1) + r(r-1)$ singles, $(n-r)r$ doubles, and no extras, which means that $A_\sigma = M_\sigma$ and the claims follow. □

In the special case $r = 1$, we have that A_σ has dimension $n(n-1-1) + 1^2 = (n-1)^2$. This algebra has already appeared in this thesis in Chapter 6, where it was denoted $Q^k(\eta)$, where $k = n-1$. For the general r, let us denote A_σ as $R_{n,r}$.

7.4.3 General $\sigma = g_{k,r}$

Suppose finally that n, k and r are arbitrary satisfying $n \geq 2k + r$. Let $\sigma = g_{k,r}$, where the latter acts on the basis C of $M = M_\eta(G, C)$ by conjugation. Let again $a = e_{i,j}$ or $f_{i,j}$ be a Matsuo axis, $1 \leq i < j \leq n$.

Lemma 7.4.3. If $i \leq 2k$ and $j > 2k$ then $a \neq a^\sigma$ and $a + a^\sigma$ is an extra.

Proof. Recall that $\bar{i}$ is defined by $\{i, \bar{i}\} = \{2s+1, 2s\}$ for some $1 \leq s \leq k$. Then $\pi(a^\sigma) = ((i,j)^{\sigma_k} = (\bar{i}, j)$. Therefore, the order of aa^σ in G is 3, since $\pi(aa^\sigma) = (i,j)(\bar{i},j) = (i,j,\bar{i})$ is of order 3. This means that a and a^σ are not equal and not orthogonal as elements of $M = M_\eta(G, C)$. □

Consider the subgroup H of $\hat{G}$, that contains V and such that $H \cap S$ is the stabilizer of the partition $\{\{1,2,\ldots,2k\},\{2k+1,2k+2,\ldots,n\}\}$ of $\{1,2,\ldots,n\}$. Then $H = K \times L$, where $K \cong \hat{G}_{2k}$ and $L \cong \hat{G}_{n-2k}$. It follows from the above lemma that singles and doubles can only be obtained when $i,j \leq 2k$ or when $i,j > 2k$. That is, all such axes a lie in K or L. Note that σ preserves both K and L, and furthermore, it acts on $K \cong \hat{G}_{2k}$ as $g_{k,0}$ and it acts on $L \cong \hat{G}_{n-2k}$ as $g_{0,r}$. This immediately yields the following result.

Proposition 7.4.4. The flip subalgebra A_σ is the direct sum of two subalgebras isomorphic to $2Q_k(\eta)$ and $R_{n-2k,r}$.

So in this case we don't obtain any new simple flip subalgebras.

7.5 Flip subalgebras for $\sigma = h_k$ and $n = 2k$

Assume that $n = 2k$ and let $\sigma = h_k = (-1,1,-1,1,\ldots,-1,1) : (1,2)(3,4)\cdots(2k-1,2k)$. We will again use the notation $\bar{i}$ defined by: $\{i,\bar{i}\} = \{2s-1,2s\}$ for some s.

Proposition 7.5.1. The fixed subalgebra M_σ of the Matsuo algebra $M = M_\eta(2^{n-1} : S_n)$ contains no singles, $\frac{n(n-1)}{2}$ doubles, and no extras. Therefore, the dimension of the flip subalgebra $A_\sigma = M_\sigma$ is $\frac{n(n-1)}{2} = k(2k-1)$.

Proof. Suppose that $a = e_{i,j}$ or $f_{i,j}$, where $1 \leq i < j \leq n$. We have two cases: either $j = \bar{i}$; or $j \neq \bar{i}$. In the first case, $a + a^\sigma = e_{i,j} + f_{i,j}$ is a double. In the second case, a^σ is equal to either $e_{\bar{i},\bar{j}}$ or $f_{\bar{i},\bar{j}}$ and, in each of these two subcases, $a + a^\sigma$ is again a double.

So there are no single or extras, and the number of doubles is $\frac{1}{2}(2\binom{n}{2}) = \binom{n}{2} = \frac{n(n-1)}{2}$. $\square$

We will denote this flip subalgebra of dimension $\frac{n(n-1)}{2}$ by H_k.

7.6 Connectivity

In this section we investigate the question of whether the flip subalgebras constructed above are connected. Recall that an axial algebra is called *connected* if its projection graph, introduced in Chapter 4, is strongly connected. We note that in the context of this chapter, every flip subalgebra inherits from the corresponding Matsuo algebra a Frobenius form, which respect to which all axes are non-singular. This means that the projection graph is a simple (undirected) graph and strong connectivity is the same as just connectivity.

Recall that the support set of an axis $a = e_{i,j}$ or $f_{i,j}$ is $\{i, j\}$.

Proposition 7.6.1. Two axes, singles or doubles are orthogonal if and only if their support sets are disjoint.

In the projection graph, two axes are adjacent if their supports are not disjoint.

We analyse the flip subalgebras case by case. Let first $n = 2k$ and

$$\sigma = \sigma_k = (1,2)(3,4)\dots(2k-1,2k).$$

Proposition 7.6.2. The flip subalgebra $A = 2Q_k(\eta)$ is connected whenever $k \geq 2$.

Proof. Recall that A contains $2k$ single axes, $e_{i,\bar{i}}$ and $f_{i,\bar{i}}$, $i = 1, 3, \dots, 2k-1$, and $2k(k-1)$ doubles axes, $e_{i,j} + e_{\bar{i},\bar{j}}$ and $f_{i,j} + f_{\bar{i},\bar{j}}$, for $1 \leq i < j \leq k$ with $j \neq \bar{i}$. Note that all singles here are pairwise non-adjacent, i.e., they form a coclique in the projection graph. However, when $k \geq 2$, each double $e_{i,j} + e_{\bar{i},\bar{j}}$ (and similarly, $f_{i,j} + f_{\bar{i},\bar{j}}$) is adjacent to $e_{i,\bar{i}}$, $f_{i,\bar{i}}$, $e_{j,\bar{j}}$, and $f_{j,\bar{j}}$. It follows that all doubles are adjacent to some singles and all singles are at mutual distance 2 in the projection graph. That is, the projection graph is connected. □

When $k = 1$, there are only two singles $e_{1,2}$ and $f_{1,2}$ and no doubles, and so the projection graph is disconnected, having two vertices and no edges. The corresponding disconnected algebra $2Q_1(\eta)$ is isomorphic to $2B = \mathbb{F} \oplus \mathbb{F}$. Next, let us consider the case where n is arbitrary and $\sigma = w_r$ for $r = 1, 2, \dots, \lfloor \frac{n}{2} \rfloor$.

Proposition 7.6.3. The flip subalgebra $A = R_{n,r}$ is connected.

Proof. The flip subalgebra A in this case contains two sets of single axes: $S_1 = \{e_{i,j}, f_{i,j} \mid 1 \leq i < j \leq n-r\}$ and $S_2 = \{e_{i,j}, f_{i,j} \mid n-r+1 \leq i < j \leq n\}$. It also contains $r(n-r)$ double axes $d = e_{i,j} + f_{i,j}$, where $1 \leq i \leq n-r < j \leq n$. Note that every single, $e_{i,j}$ of $f_{i,j}$ from S_1, and every single, $e_{i',j'}$ or $f_{i',j'}$, from S_2 are both adjacent to the double $e_{i,i'} + f_{i,i'}$. So all singles from S_1 are at distance two from all the singles in S_2. In particular, all singles are in the same connected component. On the other hand, it is immediate from above that every double is adjacent to some singles in S_1 and S_2 and so the entire projection graph is connected. □

In the general case, $\sigma = g_{k,r}$ we have the flip subalgebra A that is a direct sum of two sublagebras $2Q_k(\eta)$ and $R_{n,r}$. Note that no axis from the first factor is adjacent to any axis from the second factor, i.e., the projection graph here is disconnected unless one of the factors is trivial (which leads to the two cases that we have already considered).

In the last case, consider the flip

$$\sigma = h_k = (-1, 1, -1, 1, \ldots, -1, 1) : (1, 2)(3, 4) \cdots (2k-1, 2k),$$

where $n = 2k$.

Proposition 7.6.4. The flip subalgebra $A = H_k$ is connected.

Proof. This flip subalgebra A contains $\binom{n}{2} = \frac{n(n-1)}{2}$ double axes, split into two sets: $D_1 = \{e_{i,\bar{i}} + fi, \bar{\bar{i}} \mid i = 1, 3, \ldots, 2k-1\}$ and $D_2 = \{e_{i,j} + e_{\bar{i},\bar{j}}, f_{i,j} + f_{\bar{i},\bar{j}} \mid 1 \leq i < j \leq n \text{ and } j \neq \bar{i}\}$. Here D_1 is a co-clique, as it contains no edges. However, two doubles, $e_{i,\bar{i}} + f_{i,\bar{i}}$ and $e_{j,\bar{j}} + f_{j,\bar{j}}$ from D_1 are both adjacent to the double $e_{i,j} + e_{\bar{i},\bar{j}}$ (or $f_{i,j} + f_{\bar{i},\bar{j}}$) from D_2. So all doubles from D_1 are at pairwise distance 2 from each other, and in particular, they are in the same connected component. It is also clear from the above that all doubles from D_2 are adjacent to some doubles from D_1, and so the entire projection graph of A is connected. □

Note that here we do not have a special case when $k = 1$, as then there is only one double axis in A and so the projection graph is trivially connected.

This completes the determination of connected flip subalgebras in our cases. In all of these connected algebras $2Q_k(\eta)$, $R_{n,r}$ and H_k, there are no proper ideals containing axes. That is, any proper ideal would have to be contained in the radical.

7.7 Ambient subalgebra

From the previous section, we know that the connected flip subalgebras $2Q_k(\eta)$, $k \geq 2$, $R_{n,r}$, $r \leq \lfloor \frac{n}{2} \rfloor$, and H_k have all proper ideals contained in the radical. Therefore, there are simple if and only if their radical is zero. As it turns out, in each of the above cases, the radical is non-trivial only for a finite number of special values of η, called *critical values*, depending on the other parameters, k, n, and r. In this section we aim to find the critical values of η explicitly in all the above cases.

We start with the concept of the ambient Matsuo algebra.

Definition 7.7.1. Suppose (G, C) is a 3-transposition group and $M = M_\eta(G, C)$ is the corresponding Matsuo algebra. Let A be a subalgebra of M generated by a set X of single and double axes.

1. For a single axis $x = a$ or a double axis $x = a + b$ in M, we will say that the Matsuo axis a or the Matsuo axes a and b are *involved* in x.
2. The *ambient subalgebra* of A is the subalgebra $\hat{A}$ of M generated by all the Matsuo axes involved in the axes $x \in X$.

Note that, since $\hat{A}$ is generated by Matsuo axes, it is itself a Matsuo algebra corresponding to the 3-transposition subgroup of (G, C) generated by the involutions corresponding to the Matsuo axes involved in A. Often, $\hat{A}$ is all of M, but this is not always the case.

Proposition 7.7.2. For the flip subalgebras $A = 2Q_k(\eta)$, $R_{n,r}$, and H_k of $M = M_\eta(2^{n-1} : S_n)$, the corresponding ambient algebra $\hat{A}$ is equal to the entire M. Furthermore, $A = M_\sigma$.

Proof. Indeed, looking at Propositions 7.4.1, 7.4.2, and 7.5.1, we see that in all these cases there are no extras, i.e., singles and doubles involve all Matsuo axes. Furthermore, they form a basis of M_σ and they are contained in A. So $A = M_\sigma$. □

An opposite example comes from the general flip $\sigma = g_{k,r}$, where $k \notin \{0, \frac{n}{2}\}$. We see in Lemma 7.4.3 and Proposition 7.4.4 that the Matsuo axes involved in singles and doubles are exactly those elements $e_{i,j}$ and $f_{i,j}$, for which both i and j lie in $\{1, 2, \ldots, 2k\}$, or they both lie in $\{2k + 1, \ldots, n\}$. This means that the ambient subalgebra $\hat{A}$ of the flip subalgebra $A = A_\sigma$ in this cases is a proper subalgebra of M, isomorphic to $M_\eta(2^{2k-1} : S_{2k}) \oplus M_\eta(2^{n-2k-1} : S_{n-2k})$.

7.8 Critical values

Suppose A is a Matsuo algebra or a flip subalgebra of a Matsuo algebra. Then the multiplication on A depends on the parameter η, which appears in the fusion law for A.

Recall the definition of a critical value 5.4.1. We note that the algebra A in all our cases carries a Frobenius form (in case of the flip subalgebras, it is the form inherited from the ambient Matsuo algebra). With respect to this form, all Matsuo/single axes have length 1 and all double axes have length 2. In particular, all of them are non-singular, and so it follows from Theorem 4.3.4 that the radical of the algebra A coincides with the radical of its Frobenius form. Choose a basis of A (and in all of our cases A has a standard basis consisting of singles and doubles) and let F be the Gram matrix of the Frobenius form with respect to this basis. Then the radical of the Frobenius form is non-zero if and only if 0 is an eigenvalue of F, i.e., if and only if the determinant of F is zero.

We note that this determinant is a non-zero polynomial $d(\eta)$ in η and so $\eta = \eta_0$ is critical if and only if η_0 is a root of the polynomial $d(\eta)$. This implies that A has only finitely many critical values. Our goal is to find the critical values for all the new flip subalgebras we constructed in this chapter. A further related question is the multiplicity of the critical value $\eta = \eta_0$, that is, the highest power $(\eta - \eta_0)^m$ dividing $d(\eta)$.

Recall that in all our cases the flip subalgebra coincides with the fixed subalgebra M_σ, where $M = \hat{A}$ is the ambient Matsuo algebra.

Proposition 7.8.1. The fixed subalgebra M_σ corresponding to a flip σ has the same critical values as its ambient Matsuo algebra $M = \hat{A}$ but with different, smaller multiplicities.

Proof. Let $a_1, a_2, \ldots, a_s$ be the singles, i.e., the Matsuo axes fixed by σ, and let

$$\{b_1, c_1\}, \{b_2, c_2\}, \ldots, \{b_t, c_t\}$$

be the length 2 orbits of $\langle\sigma\rangle$ on the set of Matsuo axes of M. Then $B = \{a_1, a_2, \ldots, a_s, b_1 + c_1, b_2 + c_2, \ldots, b_t + c_t\}$ is a basis of M_σ and $C = \{b_1 - c_1, b_2 - c_2, \ldots, b_t - c_t\}$ is a basis of the commutator subspace $[M, \sigma]$. (Indeed, $b_i - c_i = c_i^\sigma - c_i = [c_i, \sigma]$.) It is easy to see that $M = M_\sigma \oplus [M, \sigma]$, and furthermore, this direct sum decomposition is orthogonal with respect to the Frobenius form.

It follows that $B \cup C$ is a second basis of M and let E be the Gram matrix with respect to this basis. We have already introduced the polynomial $d(\eta)$ as the determinant of the matrix F. Let us see how the same $d(\eta)$ can be expressed in terms of E.

Lemma 7.8.2. The determinant of E coincides with $2^{2t}d(\eta)$, so it differs from the determinant of F by a constant factor only.

Indeed, first of all, the determinant of a matrix does not change if we apply the same permutation to the rows and columns of this matrix. In particular, for this lemma, we may assume that F is written for the basis $\{a_1, a_2, \ldots, a_s, b_1, c_1, b_2, c_2, \ldots, b_t, c_t\}$ and E is written for the basis $\{a_1, a_2, \ldots, a_s, b_1 - c_1, b_1 + c_1, b_2 - c_2, b_2 + c_2, \ldots, b_t - c_t, b_t + c_t\}$. Then $E = T^t F T$, where the transition matrix T is block diagonal with s ones on the diagonal followed by t blocks $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. The latter blocks have determinant 2, and so the claim follows.

We have established Lemma 7.8.2, and so, as expected, the determinant of E has exactly the same roots as $d(\eta)$ and they have the same multiplicities. (Recall that the characteristic of $\mathbb{F}$ is not 2 by assumption, and so 2^{2t} is a non-zero constant.)

Now we return to the basis $B \cup C$. When E is written with respect to this basis, it is block diagonal with the blocks being the Gram matrix R of M_σ with respect to the basis B and the Gram matrix S of $[M, \sigma]$ with respect to the basis C. It follows that $2^{2t}d(\eta) = \det E = \det R \det S = d_{M_\sigma}(\eta) \det S$. In particular, if $d_{M_\sigma}(\eta)$ is zero then also $d_M(\eta)$ is zero. This means that the critical values of M_σ are critical for M as well. □

It remains to find the multiplicities of the critical values of M_σ. Let $\eta_1, \eta_2, \ldots, \eta_c$ be the critical values of M and let n_i be the multiplicity of η_i as a root of $d_M(\eta)$. Then n_i is the dimension of the radical of M (the 0-eigenspace of the Gram matrix F on M) when $\eta = \eta_i$. Similarly, let m_i be the multiplicity of η_i as a root of $d_{M_\sigma}(\eta)$. Note that when η_i is not critical for M_σ, it just means that $m_i = 0$.

Let us observe that the difference $n_i - m_i$ is equal to the multiplicity of η_i as a root of the polynomial $d_M(\eta)/d_{M_\sigma}(\eta)$, which is the determinant of the Gram matrix S on $[M, \sigma]$.

The critical values η_i and the corresponding multiplicities n_i for all finite 3-transposition groups (G, C) can be found from [21]. There the eigenvalues λ_i of the adjacency matrix D of the collinearity graph of the Fischer space of (G, C) are given together with their multiplicities n_i. The Gram matrix F of $M = M_\eta(G, C)$ with respect to its basis of Matsuo axes coincides with $I + \frac{\eta}{2}D$, and so the eigenvalues of F are equal to $1 + \frac{\eta\lambda_i}{2}$. The radical of M coincides with the 0-eigenspace of the Gram matrix F, so it is non-zero whenever $0 = 1 + \frac{\eta\lambda_i}{2}$, i.e., $\eta = -\frac{2}{\lambda_i}$ for some i, and its dimension then is n_i. This gives us the formula for the critical values η_i:

$$\eta_i = -\frac{2}{\lambda_i}$$

for all $i \in \{1, 2, \ldots, c\}$, for which $\lambda_i \neq 0$.

In the case of the extended symmetric group $2^{n-1} : S_n$, the eigenvalues λ_i, critical values η_i, and the corresponding multiplicities n_i are shown in Table 7.1. In particular, $c = 4$. Note that there is no a critical value corresponding to $\lambda_4 = 0$ and so M has only three critical values, namely the ones shown in the second column of the table.

Since by Proposition 7.7.2, $M = M_\eta(2^{n-1} : S_n)$ is the ambient Matsuo algebra for

Eigenvalues λ_i	Critical values η_i	Multiplicity n_i
$4(n-2)$	$-\frac{1}{2(n-2)}$	1
$2(n-4)$	$-\frac{1}{n-4}$	$n-1$
-4	$\frac{1}{2}$	$\frac{n(n-3)}{2}$
0		$\frac{n(n-1)}{2}$

Table 7.1: Critical values for the extended symmetric group $2^{n-1} : S_n$

the flip subalgebras $2Q_k(\eta)$, for $n = 2k$, $R_{n,r}$, for $r = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$, and H_k, for $n = 2k$. Furthermore, by the same proposition these flip subalgebras coincide with the corresponding fixed subalgebra M_σ. Thus, we conclude, using Proposition 7.8.1, that for all these algebras the critical values are the same as for $M = M_\eta(2^{n-1} : S_n)$. That is, we have the following:

Theorem 7.8.3. The critical values of the flip subalgebras $A = 2Q_k(\eta)$, $R_{n,r}$, and H_k are $\eta_1 = -\frac{1}{2(n-2)}$, $\eta_2 = -\frac{1}{n-4}$, and $\eta_3 = \frac{1}{2}$, where $n = 2k$ for $A = 2Q_k$ and H_k.

Turning to the multiplicities of these critical values, since the n_i in M are known and given in Table 7.1, we need to find the differences $n_i - m_i$, which we already identified as the multiplicities of λ_i within $W = [M, \sigma]$.

Proposition 7.8.4. The 0-eigenspace of F, denoted V^-, coincides with $\langle e_{ij} - f_{ij} \mid 1 \leq i < j \leq n \rangle$.

Proof. Note that the points collinear (in the Fischer space of $G = 2^{n-1} : S_n$) to e_{ij} are also collinear to f_{ij}, i.e., e_{ij} and f_{ij} have exactly the same set of neighbours in the collinearity graph Γ of the Fischer space of the extended symmetric group G. Since the linear map corresponding to F sends $e_{ij} - f_{ij}$ to the sum of all neighbours of e_{ij} minus the sum of all neighbours of f_{ij}, each vector $e_{ij} - f_{ij}$ is mapped to zero, and so it is in the 0-eigenspace. Clearly, the vectors $e_{ij} - f_{ij}$ are all linearly independent, and the number of them is exactly the dimension $n_4 = \frac{n(n-1)}{2}$ of the 0-eigenspace. Thus, they form a basis of that eigenspace. □

Proposition 7.8.5. The sum of the remaining three eigenspaces of F coincides with the subspace $V^+ = \langle e_{ij} + f_{ij} \mid 1 \leq i < j \leq n \rangle$.

Proof. Similarly to the previous proof, the linear map corresponding to F sends $e_{ij} + f_{ij}$ to the sum of all neighbours of e_{ij} plus the sum of all neighbours of f_{ij}. So this is twice the same sum, and it clearly is contained in V, meaning that V is invariant under the linear map. Since V^+ is obviously complementary to the 0-eigenspace V^-, the claim follows. □

We are ready to determine the values of $n_i - m_i$.

We will note that the length 2 orbits of $\langle\sigma\rangle$ are of two kinds. If $e_{ij}^\sigma = f_{ij}$, we will say that the orbit $\{e_{ij}, f_{ij}\}$ is of *cross type*. All other length 2 orbits split into *parallel pairs*: either $\{e_{ij}, e_{st}\}$ and $\{f_{ij}, f_{st}\}$ is for $\{e_{ij}, f_{st}\}$ and $\{f_{ij}, e_{st}\}$ for some $\{s,t\} \neq \{i,j\}$.

Proposition 7.8.6. The number $n_4 - m_4$ is equal to the number of cross type orbits plus the number of parallel pairs.

Proof. Clearly, every cross orbit contributes one dimension (vector $e_{ij} - f_{ij}$) to the intersection of V^- with $[M,\sigma]$. Similarly, every parallel pair contributes one dimension (either $e_{ij} - f_{ij} - e_{st} + f_{st}$ or $e_{ij} - f_{ij} + e_{st} - f_{st}$). □

Corollary 7.8.7.

1. For $\sigma = g_{k,0}$ and $n = 2k$, we have that $m_4 = k(2k-1) - k(k-1) = k^2$,
2. For $\sigma = g_{0,r}$ and $n \geq 2r$, we have that $m_4 = n(n-1)/2 - (n-r)r$.
3. For $\sigma = h_k$ and $n = 2k$, we have that $m_4 = k(2k-1) - k - k(k-1) = k(k-1)$.

Proof. This follows from Proposition 7.8.6 and the description of doubles in the three respective cases. □

We note that $m_1 = n_1 = 1$, so we only need to determine m_2 and m_3.

Clearly, we have that $\dim M_\sigma = m_1 + m_2 + m_3 + m_4 = 1 + m_2 + m_3 + m_4$, which gives us one equation on m_2 and m_3. The second equation is obtained from finding the trace of the map corresponding to D on $[M,\sigma]$.

Proposition 7.8.8. For the three cases above, the trace of the map corresponding to D on $[M,\sigma]$ is zero.

Proof. The subspace $[M,\sigma]$ has a basis consisting of differences of vectors from all length 2 orbits of σ. It is immediate to see that since all these orbits are orthogonal (as there are no extras) the matrix of the map D with respect to such basis only has zeroes on the main diagonal. □

Corollary 7.8.9. We have the following equation:

$$2(n-4)(n_2 - m_2) - 4(n_3 - m_3) = 0.$$

Using our two equations we can now find m_2 and m_3 in all cases.

From the three cases above, we have the corresponding first equation respectively:

1. $m_2 + m_3 = k^2 - 1$,
2. $m_2 + m_3 = \frac{n^2-n-2}{2} = \frac{(n-2)(n+1)}{2}$,
3. $m_2 + m_3 = k^2 - 1$.

Then by using the equation $2(n-4)(n_2 - m_2) - 4(n_3 - m_3) = 0$, and the multiplicities n_2 and n_3 from Table 7.1, we have that $(n-4)m_2 - 2m_3 = 2n - 4$. Hence, for the first case ($\sigma = g_{k,0}$ and $n = 2k$), we have the system of linear equation (where we use $n = 2k$ and cancel the factor of 2):

$$m_2 + m_3 = k^2 - 1, \tag{7.1}$$

$$(k-2)m_2 - m_3 = 2(k-1). \tag{7.2}$$

Solving this system we find the multiplicities m_2 and m_3: From the equation (7.1), we have that $m_3 = k^2 - 1 - m_2$. Then

$$\begin{aligned}
(k-2)m_2 - m_3 &= 2(k-1) \\
(k-2)m_2 - k^2 + 1 + m_2 &= 2(k-1) \\
(k-2+1)m_2 &= 2(k-1) + k^2 - 1 \\
(k-1)m_2 &= k^2 + 2k - 3 \\
(k-1)m_2 &= (k+3)(k-1) \\
m_2 &= k + 3.
\end{aligned}$$

Hence, $m_3 = k^2 - 1 - m_2 = k^2 - 1 - k - 3 = k^2 - k - 4$.

Similarly, for the class $\sigma = g_{0,r}, n \geq 2r$, the system of linear equation is as follows:

$$m_2 + m_3 = \frac{(n-2)(n+1)}{2}, \tag{7.3}$$

$$(n-4)m_2 - 2m_3 = 2n - 4. \tag{7.4}$$

By solving this system, we find the multiplicities m_2 and m_3:

$$m_2 = n + 3,$$
$$m_3 = \frac{(n-2)(n+1)}{2} - (n+3).$$

In the last case, $\sigma = h_k$ and $n = 2k$, we have the same system of linear equations as in the first case. So the values of m_2 and m_3 are also the same.

The following table summarises the multiplicities m_i corresponding to each class of flips:

The classes of flips	Multiplicity m_1	Multiplicity m_2	Multiplicity m_3	Multiplicity m_4
$\sigma = g_{k,0}, n = 2k$	1	$k + 3$	$k^2 - k - 4$	k^2
$\sigma = g_{0,r}, n \geq 2r$	1	$n + 3$	$\frac{(n-2)(n+1)}{2} - (n+3)$	$\frac{n(n-1)}{2} - (n-r)r$
$\sigma = h_k, n = 2k$	1	$k + 3$	$k^2 - k - 4$	$k(k-1)$

Table 7.2: The multiplicity m_i for the extended symmetric group $2^{n-1} : S_n$

CONCLUSION AND FUTURE STUDY

In this thesis, in particular, in Chapters 6 and 7 we constructed new series of axial algebras of Monster type $(2\eta, \eta)$. Subalgebras of the Matsuo algebras on the orthogonal group over a field of characteristic 3, $M_\eta(O^+_{k+1}(3))$, and subalgebras of the Matsuo algebras on the extended symmetric group $M_\eta(2^{n-1} : S_n)$.

In the latter case, we determined all the classes of flips $\sigma = g_{k,r}, h_k = g_{k,0}, g_{0,r}, h_k$ and their corresponding flip subalgebras $A = 2Q_k(\eta), R_{n,r}$, and H_k of dimension $2k^2, r^2 + n(n-r-1)$, and $\frac{n(n-1)}{2}$, respectively.

Furthermore, we investigated the question that when these subalgebras are simple. By computing the critical values η_i we deduced that flip subalgebras of $M_\eta(2^{n-1} : S_n)$ are simple unless $\eta = \frac{-1}{2(n-2)}, \frac{-1}{n-4}\ \frac{1}{2}$.

In the future study, we will extend this research. Also, new families of axial algebras of Monster type $(2\eta, \eta)$ of different groups (groups differ from symmetric and orthogonal groups) will be constructed.

APPENDIX A

GRAM MATRIX OF FROBENIUS FORM CODE

```
n:=  ;;
F:=FiniteField(3);;
V:=F^(n+1);;
e:=Basis(V);;
basis:=[];;
for i in [1..n] do
 for j in [i+1..n+1] do
  Append(basis,[e[i]+e[j],e[i]-e[j]]);
 od;
od;
flip:=List([1..Length(basis)],
     function(i)
      local v;
      v:=ShallowCopy(basis[i]);
      v[n+1]:=-v[n+1];
      return Position(basis,v);
    end);;
flip:=PermList(flip);;
orbs:=Orbits(Group(flip),[1..Length(basis)]);;
singles:=Filtered(orbs,o->Length(o)=1);;
doubles:=Filtered(orbs,o->Length(o)=2);;
k:=Length(singles);;
m:=Length(doubles);;
eta:=Indeterminate(Rationals,"eta");;
#eta:=1/2;;
```

```
K:=Field(eta);;
Gr:=List([1..k+m],i->[]);;
for i in [1..k] do
Gr[i][i]:=One(K);
od;
for i in [k+1..k+m] do
 Gr[i][i]:=2*One(K);
od;
for i in [1..k-1] do
  for j in [i+1..k] do
   s:=Filtered([1..n],u->basis[singles[i][1]][u]<>Zero(F));
   t:=Filtered([1..n],u->basis[singles[j][1]][u]<>Zero(F));
   r:=Length(Intersection(s,t));
   if r in [0,2] then
    Gr[i][j]:=Zero(K);
   else
     Gr[i][j]:=eta/2;
    fi;
  od;
od;
for i in [1..k] do
  for j in [1..m] do
    s:=Filtered([1..n],u->basis[singles[i][1]][u]<>Zero(F));
    t:=Filtered([1..n],u->basis[doubles[j][1]][u]<>Zero(F));
    r:=Length(Intersection(s,t));
    if r=0 then
     Gr[i][k+j]:=Zero(K);
   else
     Gr[i][k+j]:=eta;
    fi;
 od;
od;
for i in [1..m-1] do
for j in [i+1..m] do
Gr[k+i][k+j]:=2*eta;
od;
od;
```

```
for i in [1..k+m-1] do
  for j in [i+1..k+m] do
    Gr[j][i]:=Gr[i][j];
  od;
od;
poly:=DeterminantMat(Gr);;
```

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