# IRREGULAR ISOMONODROMIC DEFORMATIONS: HAMILTONIAN ASPECTS AND ALGEBRAIC DESCRIPTION OF THE PHASE SPACE

by

ILIA GAIUR

A thesis submitted to
The University of Birmingham
for the degree of
DOCTOR OF PHILOSOPHY

School of Mathematics
College of Engineering and Physical Sciences
The University of Birmingham
March 2020

# CONTENTS

# CHAPTER 1

# AKNOWLEDGMENTS

I want to express my gratitude to Marta Mazzocco for the great amount of stimulating mathematical discussions, her advice and support during the years I spent in the University of Birmingham.

I want to thank Vladimir Rubtsov for being my friend and collaborator during these years, for his patience and permanent wish to investigate and study new fields.

I also want to thank Tyler Kelly for his patient attitude to explain me things I don't understand.

I am grateful to the London Mathmatical Society and the Cecil King foundation who awarded the Cecil King Scholarship to me to visit IHÉS, France. I also thank IHÉS and prof. Maxim Kontsevich for inviting me. This support gave opportunity to gain invaluable experience in mathematics.

I thank the members of the International Groupe de Travail on differential equations in Paris, especially I want to thank Vasily Golyshev and Duco Van Straten from whom I learned lot in the last year.

Last but not the least, I want to thank my family who always supported me. Very special thanks to my beloved wife and to my mom and dad.

# CHAPTER 2

# INTRODUCTION

The isomonodromic deformations of a meromorphic connections over the Riemann surfaces provide a natural deautonomization of classical integrable systems. By deautonomization we mean that the system of commuting Hamiltonians depends explicitly on the time coordinates, so the integrability condition takes the form

$$\{H_i, H_j\} + \frac{\partial H_i}{\partial t_j} - \frac{\partial H_j}{\partial t_i} = 0.$$

In the case of the punctured Riemann sphere, the isomonodromic equations for a logarithmic connection (also known as the Schlesinger equations) are related to the classical Gaudin model (see [5]). On the quantum level, such correspondence provides a link between the quantum Gaudin system and Knizhnik–Zamolodchikov equations. While the quantum Gaudin system is a joint eigenproblem for a set of commuting Hamiltonians, the Knizhnik–Zamolodchikov system is a flat connection over the space of parameters for the Gaudin model. Such slogan may be extended to the general isomonodromic problem (not only for the logarithmic one) - the classical Gaudin system deals with a commuting family of quadratic functions in the universal enveloping algebra, while the isomonodromic systems provide a flat deautonomization of this family.

This thesis is dedicated to the theory of isomonodromic deformations for systems of differential equations with poles of any order on the Riemann sphere. The initial motivation was to generalise an observation by N.Reshetikhin that the quasi–classical solution of the standard Knizhnik–Zamolodchikov equations (i.e. with simple poles) is expressed via the isomonodromic $\tau$-function arising in the case of Fuchsian systems [80]. Along the way of pursuing the project of extending this results to poles of any order, we have found a number of interesting results, some of which were already known as folklore (i.e. either done as very specific examples or not really proved in detail), others completely original. For example we introduce a new algebraic description for all Painlevé equations which represents these equations as flows on an algebraic surface with linear Hamiltonian and non-linear Poisson bracket. In contrast with the local description which is given by the Painlevé equations, such algebraic description reflects non-trivial topology of the phase space for the isomonodromic deformation equations. For the Painlevé VI, such description was firstly obtained by Hitchin [49], however it was not known for the all other Painlevé equations.

The Knizhnik–Zamolodchikov (KZ) equations emerged in theoretical physics as the system of linear differential equations satisfied by the correlation functions in the two–dimensional Wess–Zumino–Witten model of conformal field theory associated to a genus 0 curve [62, 9]. In the case of $\mathfrak{g} = \mathfrak{gl}_m$, where $\mathfrak{gl}_m$ stands for the Lie algebra of the $m \times m$ matrices , the KZ equations can be represented as a system of linear differential equations for a local section $\psi$ of the trivial bundle $B \times U(\mathfrak{gl}_\mathfrak{m}(\mathbb{C}))^{\otimes n} \to B$ over the base $B$, where $U(\mathfrak{gl}_\mathfrak{m}(\mathbb{C}))^{\otimes n}$ is a $n$-th tensor power of the $\mathfrak{gl}_\mathfrak{m}(\mathbb{C})$ universal enveloping algebra and $B$ is given by the configuration space of ordered $n$-uples of points in $\mathbb{C}$, namely $B := \{(u_1, \dots, u_n) \in \mathbb{C}^n | u_i \neq u_j \text{ for } i \neq j\}$:

$$d\psi = \sum_{i \neq j} \Pi^{ij} \frac{du_i - du_j}{u_i - u_j} \psi, \tag{2.0.1}$$

where $\Pi^{ij} \in \mathrm{End}(U(\mathfrak{gl}_m(\mathbb{C})^{\otimes n})$ is the extension of the non-degenerated symmetric tensor

$$\Pi \in \mathfrak{gl}_m(\mathbb{C}) \times \mathfrak{gl}_m(\mathbb{C}) = \mathrm{End}(\mathfrak{gl}_\mathrm{m}(\mathbb{C}))$$

acting by left multiplication on the $i-$th and $j-$th components of the tensor product $U(\mathfrak{gl}_m(\mathbb{C})^{\otimes n})$, and reads

$$\Pi^{ij} = \sum_{\alpha\beta} 1 \otimes 1 \otimes \dots \otimes \underset{i-\text{th place}}{E_{\alpha\beta}} \otimes \dots \underset{j-\text{th place}}{E_{\beta\alpha}} \otimes 1 \otimes \dots 1,$$

where $E_{\alpha\beta} \in \mathfrak{gl}_m(\mathbb{C})$ are given by

$$(E_{\alpha\beta})_{ab} = \delta_{\alpha a}\delta_{\beta b}.$$

Geometrically one can think about (2.0.1) as a flat Hitchin connection in geometric quantisation

[48].

As proved by N. Reshetikhin in [80] (see also [43] where this result was explained in terms of passing from Schrödinger to Heisenberg representation), the KZ equations can be also viewed as deformation quantisation of the Schlesinger system [82] of non-linear differential equations

$$dA^{(i)} = \sum_{i\neq j} [A^{(i)}, A^{(j)}] \frac{du_i - du_j}{u_i - u_j}, \tag{2.0.2}$$

controlling the isomonodromic deformation of a Fuchsian system on $\mathbb{P}^1$,

$$\frac{dY}{d\lambda} = \sum_{i=1}^{n} \frac{A^{(i)}}{\lambda - u_i} Y, \tag{2.0.3}$$

with $n+1$ simple poles $u_1, \ldots, u_n, \infty$. These equations are multi–time non-autonomous Hamiltonian systems with Hamiltonians

$$H_i : B \times \mathfrak{gl}_m(\mathbb{C})^n \to \mathbb{C} \tag{2.0.4}$$

given by

$$H_i := \sum_{i\neq j} \frac{\mathrm{Tr}(A^{(i)} A^{(j)})}{u_i - u_j}.$$

Interestingly, if we treat the quantities $u_1 \ldots, u_n$ in the Hamiltonian as parameters rather than times, these Hamiltonians form a family of autonomous Poisson commuting Hamiltonians called *Gaudin Hamiltonians.* This simple observation has been key to several efforts to introduce specific examples of confluent analogues of KZ: by first introducing confluent analogues of Gaudin, then quantising them and finally generating the non-autonomous versions. Let us give a summary of our understanding of these results here below.

The main idea for the quantisation of the Gaudin Hamiltonians was based on the standard point of view that for any finite dimensional Lie algebra $\mathfrak{g}$, the universal enveloping algebra $U(\mathfrak{g})$ can be considered as a deformation of the symmetric algebra $S(\mathfrak{g})$ via the Poincaré-Birkhoff-Witt map. One then defines the quantum enveloping algebra as

$$U_\hbar(\mathfrak{g}) = T(\mathfrak{g})/(X \otimes Y - Y \otimes X - \hbar[X, Y]), \quad X, Y \in \mathfrak{g}, \quad \hbar \in \mathbb{C}$$

by naturally extending the symmetrisation map to the map $S(\mathfrak{g})^{\otimes n} \to U_\hbar(\mathfrak{g})^{\otimes n}$, and then the functions $\mathrm{Tr}(A^{(i)}A^{(j)})$ on $\mathfrak{g}^{\otimes n}$ are transformed to $\Pi^{ij}$, which is given by

$$\Pi^{ij} = \sum_\alpha 1 \otimes 1 \otimes \dots \otimes \underbrace{\hbar e_\alpha^\star}_{i-\text{th place}} \otimes \dots \underbrace{\hbar e_\alpha}_{j-\text{th place}} \otimes 1 \otimes \dots 1,,$$

where $e_\alpha$ is a basis of $\mathfrak{g}$ and $e_\alpha^\star$ is a dual basis. To define a Dirac quantisation of the Gaudin Hamiltonians it is necessary to describe the Hilbert space of the quantum model as tensor product of some representations of $U_\hbar(\mathfrak{g}^{\oplus n})$. The quantised Hamiltonians $\widehat{H}_i$ act on this Hilbert space and the quantum problem consists in finding their spectrum, matrix elements and so on. Formulated rigorously, the quantum Gaudin Hamiltonians generate a large commutative subalgebra in $U(\mathfrak{g})^{\otimes n}$ which can be easily completed to a maximal commutative subalgebra. This subalgebra is usually called *Gaudin* or *Bethe subalgebra*. The explicit formulae for the generators (namely the quantum Hamiltonians) were obtained in [71, 85].

In the case of $\mathfrak{g} = \mathfrak{gl}_\mathfrak{m}$, one can fix an element of the *dual space* $\mu \in \mathfrak{g}^*$ and using the standard basis of $\mathfrak{gl}_\mathfrak{m}$ one can re-write the quantised Gaudin Hamiltonians as

$$\widehat{H}_i = \sum_{j\neq i} \sum_{r,s=1}^{m} \frac{E_{rs}^{(i)} E_{sr}^{(j)}}{u_i - u_j} + \sum_{r,s=1}^{m} \mu(E_{rs}) E_{sr}^{(i)}, \tag{2.0.5}$$

where $E_{rs}^{(i)}$ means $E_{rs}$ (as the element of standard basis in $\mathfrak{gl}_\mathfrak{m}$) considering in the $i-$th tensor factor. We observe that even the case of *regular* $\mu \in \mathfrak{g}^*$ (i.e. semi-simple, when $\mu(E_{rs}) = \mu_r \delta_{rs}$ with distinct $\mu_r \in \mathbb{C}$), the point $\infty$ is an order two pole. The case of semi–simple but not regular $\mu$ was treated in [33].

The autonomous Gaudin model (2.0.5) can be generalised in two directions: by allowing higher order singularities at the marked points $u_i \in \mathbb{C}$ thus giving rise to Gaudin models with irregular singularities in [34] or by taking an element $\mu \in \mathfrak{g}^*$ that is not semi-simple (i.e. has non-trivial Jordan blocks). These two approaches were unified in the classical and in the quantum cases in [88] where an analogue of the *bispectral dynamical duality* of [30] between the models was proved.

The next important step consisted in deforming the quantum Gaudin Hamiltonian to obtain KZ. This was done in the case of the $A_n$ root system by de Concini and Procesi [26] and generalised to any Lie algebra in [69, 30]. More precisely, for any complex simple Lie algebra

$\mathfrak{g}$ with a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and a corresponding root system $\Delta \subset \mathfrak{h}^*$, Millson and Toledano-Laredo [69] introduced the following *Casimir connection:*

$$\nabla_C \psi := d\psi - \frac{\hbar}{2\pi i} \sum_{\alpha \in \Delta} C_\alpha \frac{d\alpha}{\alpha} \psi, \tag{2.0.6}$$

where for every $\alpha$ one takes the principal embedding of $\mathfrak{sl}_2$ so that $C_\alpha = \frac{\langle \alpha,\alpha \rangle}{2}(e_\alpha f_\alpha + f_\alpha e_\alpha + \frac{1}{2}h_\alpha^2)$ is the Casimir in 3-dimensional subalgebra $\mathfrak{sl}_{2,\alpha}$ with respect to the restriction of the fixed non-degenerated $ad-$invariant bilinear form $\langle -,- \rangle$ on $\mathfrak{sl}_{2,\alpha}$ and $\hbar \in \mathbb{C}$. A special class of quantum connections with one irregular singularity of Poincaré rank 2 and several other simple poles appeared in [30] as dual to the standard KZ connection, and in [13] was re–obtained as quantisation of Dubrovin's system (without the skew-symmetry condition). Dubrovin system was then generalised to simply laced Dynkin diagrams in [12] and quantised in [79].

Confluent versions of the KZ equation, or in other words, KZ equations with irregular singular points of arbitrary Poincaré rank were obtained for $\mathfrak{sl}_2$ by Jimbo, Nagoya and Sun [54]. In [34] a class of quantum integrable systems generalising the Gaudin model was introduced by considering non-highest weight representations of any simple Lie algebra. These *Gaudin models with irregular singularities* are expected to give rise to *confluent KZ equations* as the corresponding differential equations on conformal blocks. Such KZ equations have not been explicitly written and in this thesis we provide these irregular analogues of the quantum KZ Hamiltonians.

In order to achieve our aim, we first needed to find explicit formulae for the isomonodromic Hamiltonians and to introduce a good set of Darboux coordinates. We have succeeded in doing this for a class of isomonodromic connections which can be obtained via a confluence procedure. Let us describe this class in some details here. It is well known that the isomonodromic deformation equations in the case of higher order poles have a co-adjoint orbit interpretation on a loop algebra. In the case of the Painlevé equations, Harnad and Routhier [45] produced finite–dimensional parameterisations by introducing suitable truncations of the loop algebra. Korotkin and Samtleben [63] then conjectured the standard Lie–Poisson bracket on Takiff algebras (i.e. truncated current algebras, see Section 4.1 for the definition) and later Boalch proved that indeed these brackets are preserved by the Jimbo-Miwa isomonodromic deformations [15]. In this thesis, we unify these two approaches to study connections as elements of the product of co-adjoint orbits in the Takiff algebra. More precisely, we consider linear systems of ODEs with poles at $u_1, u_2, \dots, u_n, \infty$ of Poincaré rank $r_1, r_2, \dots, r_n, r_\infty$ respectively, in the form

$$\frac{dY}{d\lambda} = A(\lambda)Y, \quad A(\lambda) = \sum_{i=1}^{n}\sum_{k=0}^{r_i} \frac{A_k^{(i)}}{(\lambda - u_i)^{k+1}} + \sum_{k=1}^{r_\infty} A_k^{(\infty)} z^{k-1}, \tag{2.0.7}$$

where $A(\lambda)$ is an element of the phase space

$$M \simeq \hat{\mathcal{O}}^\star_{r_1} \times \hat{\mathcal{O}}^\star_{r_2} \times \dots \hat{\mathcal{O}}^\star_{r_n} \times \hat{\mathcal{O}}^\star_{r_\infty}, \tag{2.0.8}$$

where $\hat{\mathcal{O}}^\star_{r_i}$ stands for the co-adjoint orbit of the complex Lie group $\widehat{G}_{r_i}$ corresponding to the Takiff algebra of degree $r_i$, for $r_i > 0$, and for the standard Lie algebra $\mathfrak{g}$ co-adjoint orbit for $r_i = 0$.

Following the ideas of [3], in Theorem 5.4.5, we show how to obtain the standard Lie–Poisson bracket

$$\{A_k^{(i)} \underset{,}{\otimes} A_l^{(j)}\} = \begin{cases} -\delta_{ij}[\Pi, A_{k+l}^{(i)} \otimes \mathbb{I}] & k + l \le r_i \\ 0 & k + l > r_i. \end{cases} \tag{2.0.9}$$

on our phase space (2.0.8) as the Marsden–Weinstein reduction of the Poisson structure on

$$\oplus_{i=1}^{n+1} (T^\star \mathfrak{gl}_m)^{r_i+1} = \oplus_{k=1}^{d} T^\star \mathfrak{gl}_m,$$

obtained by endowing each copy of the cotangent bundle $T^\star \mathfrak{gl}_m$ with the canonical symplectic structure $\mathrm{d}P \wedge \mathrm{d}Q$. Here $d = \sum_{i=1}^{n+1} r_i + n + 1$ denotes the degree of the divisor $D$ of the connection (2.0.7). The Marsden–Weinstein reduction is obtained by the additional first integrals given by the moment maps of the inner group action by $\widehat{G}_{r_i}$ as in formulae (5.4.3).

These coordinates $(Q_1, P_1, \dots, Q_d, P_d)$, that we call *lifted Darboux coordinates,* were first introduced by Jimbo, Miwa, Mori and Sato in the case of linear systems of ODEs with $n$ simple poles and possibly a Poincaré rank one pole at $\infty$ [53]. Harnad generalised these coordinates to allow rectangular $m_1 \times m_2$ matrices and used them to generalise Dubrovin duality [27] between two systems of linear ODEs: one of dimension $m_1$ and the other of dimension $m_2$ [44] and [91]. Similar coordinates were also introduced and partly used in the context of non-autonomous Hamiltonian description of Garnier-Painlevé differential systems by M. Babich and Derkachov [6, 7]. However in these latter works, the authors restricted to the case of rational parametrisation of co–adjoint orbits of $Gl_n(\mathbb{C})$ and other semi-simple Lie groups and did not consider loop algebras.

Interestingly, using the lifted Darboux coordinates, we can describe all possible isomonodromic systems with a fixed degree $d$ of the divisor of the connection (2.0.7) as Marsden–Weinstein reductions of different inner group actions on the universal phase space $\oplus_{k=1}^{d} T^{\star}\mathfrak{gl}_m$. These reductions give rise to symplectic leaves of dimension $(r_1 + \cdots + r_n + r_\infty + n)(m^2 - m)$. We explain how to produce the Darboux coordinates, which we call *intermediate Darboux coordinates,* on such symplectic leaves. In the case of the Jimbo-Miwa isomonodromic problems associated to the fifth, fourth, third and second Painlevé equations the degree is always $d = 4$, the intermediate symplectic leaves have always dimension 6 and are determined by the choice of 3 spectral invariants giving a total dimension 9 for the Poisson manifold. This is the dimension of the moduli space of $SL_2(\mathbb{C})$ connections with a given divisor $D$ of degree 4 [64].

**Remark 2.0.1.** *The problem of extending the Riemann-Hilbert symplectomorphism between the de Rahm moduli space of meromorphic connections on a Riemann surface* $\Sigma$ *with non-simple divisor (a divisor of points that can have multiplicity* $> 1$*) and the analogous of the Betti moduli space of representations of the fundamental group of* $\Sigma$*, namely with the cusped character variety introduced in [16, 17] is still open and is beyond the scope of the current thesis. However, the Darboux coordinate description of the de Rahm moduli space achieved in this thesis constitues an important first step towards that goal.*

**Remark 2.0.2.** *It is worth mentioning here that the phase space* (2.0.8) *is not a moduli space per se, however K. Hiroe and D. Yamakawa [46] showed that the sub–space of stable connections admits a nice quotient with respect to the diagonal action of* $GL_m(\mathbb{C})$ *on* $M$*:*

$$M' = \{A(\lambda) \in M | \sum_{i=1}^{n+1} \pi(A_0^{(i)}) = 0, \text{“stable”} \}/GL_m(\mathbb{C}),$$

*where*

$$\pi : \widehat{\mathfrak{g}}_{r_i}^{*} \to \mathfrak{gl}_m^{*}$$

*is the moment map under the diagonal action of* $GL_m(\mathbb{C})$ *on* $M$*, thus assuring that* $M'$ *is a smooth complex symplectic variety. The space* $M'$ *can be regarded as a certain moduli space for meromorphic connections on* $\mathcal{O}_{\mathbb{P}^1}^{\oplus m}$*. Fix* $n$ *distinct points* $u_1, \ldots, u_n \in \mathbb{P}^1$*, and endow* $\mathbb{P}^1$

*with a coordinate $z$ for which $z(u_i) \neq \infty$. The variable $z_i$ can be identified with $\lambda - u_i$ and $\widehat{\mathfrak{g}}^*_{r_i}$ can be embedded in $\mathfrak{gl}_m(\mathbb{C}[z_i^{-1}])\frac{dz_i}{z_i}$ via trace-residue pairing. Then each $A(\lambda) \in M$ determines a meromorphic connection $d - A(\lambda)$ on $\mathcal{O}_{\mathbb{P}^1}^{\oplus m}$, having poles at $u_1, \dots, u_n, \infty$. The condition $\sum_{i=1}^{n+1} \pi(A_0^{(i)}) = 0$ singles out the connections which have no residue at infinity.*

The next result of the thesis is the classification of all linear Takiff algebra automorphisms that preserve the standard Lie–Poisson structure (2.0.9) on the phase space (2.0.8) (see Theorem 5.3.1 for a more articulated statement).

**Theorem 2.0.3.** *Consider two elements $A(\lambda)$ and $B(\lambda)$ of the phase space* (2.0.8)*, so that they both have the form* (2.0.7)*:*

$$A(\lambda) = \sum_{i=1}^{n}\sum_{k=0}^{r_i} \frac{A_k^{(i)}}{(\lambda - u_i)^{k+1}} + \sum_{k=1}^{r_\infty} A_k^{(\infty)}\lambda^{k-1}, \quad B(\lambda) = \sum_{i=1}^{n}\sum_{k=0}^{r_i} \frac{B_k^{(i)}}{(\lambda - u_i)^{k+1}} + \sum_{k=1}^{r_\infty} B_k^{(\infty)}\lambda^{k-1}.$$

*Then there exist parameters $t_1^{(i)}, t_2^{(i)}, \dots t_{r_i}^{(i)}$ for all $i = 1 \dots n$, such that*

$$B_k^{(i)} = \sum_{j=k}^{r_i} A_j^{(i)} \mathcal{M}_{k,j}^{(r_i)}(t_1^{(i)}, t_2^{(i)}, \dots t_{r_i}^{(i)}), \tag{2.0.10}$$

*where*

$$\mathcal{M}_{k,j}^{(r_i)} = \sum_{w(\alpha)=j}^{|\alpha|=k} \frac{k!}{\alpha_1!\alpha_2!\dots\alpha_{r_i}!}\left(\prod_{l=1}^{r_i}(t_l^{(i)})^{\alpha_l}\right), \quad |\alpha| = \sum_{l=1}^{r_i}\alpha_l, \quad w(\alpha) = \sum_{l=1}^{r_i} l\cdot\alpha_l, \tag{2.0.11}$$

*so the coefficients of $A(\lambda)$ and $B(\lambda)$ have the same Poisson bracket*

$$\{B_k^{(i)} \underset{,}{\otimes} B_l^{(j)}\} = \begin{cases} \delta_{ij}[B_{k+l}^{(i)} \otimes \mathbb{I}, \Pi] & k+l \leq r_i \\ 0 & k+l > r_i, \end{cases} \qquad \{A_k^{(i)} \underset{,}{\otimes} A_l^{(j)}\} = \begin{cases} \delta_{ij}[A_{k+l}^{(i)} \otimes \mathbb{I}, \Pi], & k+l \leq r_i \\ 0 & k+l > r_i, \end{cases}$$

This result allows us to introduce extra (i.e. in addition to the positions of poles) deformation parameters $t_1^{(i)}, \dots, t_{r_i}^{(i)}$, $i = 1, \dots, n, \infty$ for any connection belonging to the phase space (2.0.8). In other words, we consider families of the form

$$A(\lambda) = \sum_{i=1}^{n}\sum_{k=0}^{r_i} \frac{B_k^{(i)}}{(\lambda - u_i)^{k+1}} + \sum_{k=1}^{r_\infty} B_k^{(\infty)}\lambda^{k-1}$$

where the elements $B_k^{(i)}$ contain explicitly the deformation parameters $t_1^{(i)}, \dots, t_{r_i}^{(i)}$ as prescribed by formulae (2.0.10) and (2.0.11). The isomonodromic deformation equations will then impose a further implicit dependence of the matrices $A_k^{(i)}$ on the deformation parameters $t_1^{(i)}, \dots, t_{r_i}^{(i)}$ and on the position of the poles $u_1, \dots, u_n$.

**Remark 2.0.4.** *Let us stress that the class of connections we consider in this thesis are elements of the space (2.0.8). This class excludes some of the Jimbo-Miwa-Ueno connections. Indeed, our deformation parameters correspond to a subset of the Jimbo-Miwa-Ueno ones and this correspondence is* $1:1$ *only in the case of rank* $m=2$*. For example, the famous Dubrovin's system*

$$\frac{dY}{d\lambda} = \left(U + \frac{V}{\lambda}\right) Y,$$

*where* $U$ *is a diagonal* $n \times n$ *matrix and* $V \in \mathfrak{so}_n$*, is not an element of* $\hat{\mathcal{O}}^\star_{r_1} \times \hat{\mathcal{O}}^\star_{r_\infty}$ *for some* $r_1, r_\infty$ *because the diagonal elements of* $U$ *are independent deformation parameters. Of course the isomonodromic deformation equations for* $V$ *as a function of* $u_1, \dots, u_n$ *can be written as a flow on a co–adjoint orbit* $\mathcal{O}^\star$ *of the Lie algebra* $\mathfrak{so}_n$*, but not as equations for the whole connection* $U + \frac{V}{z}$ *on the product of two co–adjoint orbits as our theory dictates. To include the Dubrovin system (and indeed all of Jimbo-Miwa-Ueno deformation parameters) in our theory, one should either consider the extended coadjoint orbits introduced in [14, 15] or exploit the Laplace transform to transform the Dubrovin system to the Fuchsian one. In the latter setting, the confluence procedure destroys semi-simplicity, therefore it is a different process from the one considered by Cotti, Dubrovin and Guzzetti [20, 21].*

This is the correct framework to study confluence of two or more poles. Indeed, we show that the confluence cascade of $r+1$ simple poles at certain positions depending on $t_1^{(i)}, \dots, t_{r_i}^{(i)}$ gives rise to an element of the phase space (2.0.8) which has a singularity of Poincaré rank $r$ and depends on $t_1^{(i)}, \dots, t_{r_i}^{(i)}$, $i = 1, \dots, n, \infty$, as prescribed by formulae (2.0.10) and (2.0.11). The following theorem provides the inductive step to create the confluence cascade (we drop the index $^{(i)}$ for convenience).

**Theorem 2.0.5.** *Consider an* $r$*–parameter family of connections of the following form:*

$$A = \sum_{k=0}^{r} \frac{B_k(t_1, t_2 \dots t_r)}{(\lambda - u)^{k+1}} + \frac{C}{\lambda - v} + \text{holomorphic terms}, \tag{2.0.12}$$

*where by holomorphic terms we mean terms holomorphic in* $\lambda - u$ *and* $\lambda - v$*, and each* $B_k$

*depends on the parameters* $t_1, \dots, t_r$ *as specified by* (2.0.10), (2.0.11). *Assume*

$$v = u + \sum_{i=1}^{r} t_i \varepsilon^i = u + P_r(t, \varepsilon), \tag{2.0.13}$$

*and that we have the following asymptotic expansions as* $\varepsilon \to 0$

$$C \simeq \sum_{j=-r}^{\infty} W^{[j]} \varepsilon^j, \quad A_k \simeq -\sum_{l=1}^{r-k} \frac{W^{[-k-l]}}{\varepsilon^l} + A^{[k,0]} + \sum_{l=1}^{\infty} A^{[k,l]} \varepsilon^l, \tag{2.0.14}$$

*for some matrices* $W^{[-k-l]}, A^{[k,l]}$. *Then the limit* $\varepsilon \to 0$ *the connection exists and is equal to*

$$\tilde{A} = \sum_{i=0}^{r+1} \frac{\tilde{B}_i(t_1, t_2 \dots t_r, t_{r+1})}{(\lambda - u)^{i+1}} + \textit{holomorphic terms},$$

*where* $\tilde{B}_i$*'s are given by*

$$\tilde{B}_i(t_1 \dots, t_{r+1}) = \sum_{k=i}^{r} \tilde{A}_k \mathcal{M}^{(r+1)}_{i,k}(t_1 \dots t_{r+1}), \quad \tilde{A}_k = \begin{cases} W^{[-k]} + A^{[k,0]}, & k < r+1. \\ W^{[-r-1]}, & k = r+1. \end{cases} \tag{2.0.15}$$

We prove that the confluence procedure gives a Poisson morphism on the product of co-adjoint orbits and we calculate explicitly the confluent Hamiltonians, which define the correct isomonodromic deformations.

**Theorem 2.0.6.** *Let* $u$ *be a pole of a connection* $A$ *with Poincaré rank* $r$, *which is the result of confluence of* $r$ *simple poles with the simple pole* $u$. *Then the confluent Hamiltonians* $H_1, \dots, H_r$ *which correspond to the times* $t_1, \dots t_r$ *are defined as follows:*

$$\begin{pmatrix} H_1 \\ H_2 \\ \dots \\ H_r \end{pmatrix} = \left(\mathcal{M}^{(r)}\right)^{-1} \begin{pmatrix} S^{(u)}_1 \\ S^{(u)}_2 \\ \dots \\ S^{(u)}_r \end{pmatrix}, \tag{2.0.16}$$

*where*

$$S^{(u)}_k = \frac{1}{2} \oint_{\Gamma_u} (\lambda - u)^k \mathrm{Tr} A^2 \mathrm{d}\lambda \tag{2.0.17}$$

*are spectral invariants of order* $i$ *in* $u$ *and the matrix* $\mathcal{M}^{(r)}$ *is given by (2.0.11). The Hamiltonian*

*$H_u$ corresponding to the time $u$ is instead given by the standard formula*

$$H_u = \frac{1}{2}\operatorname*{res}_{\lambda=u} \mathrm{Tr} A(\lambda)^2.$$

**Remark 2.0.7.** *It is well known that the isomonodromic deformation equations are Hamiltonian, namely that the flow is Hamiltonian with respect to the Jimbo-Miwa-Ueno deformation parameters, see for example [32, 51, 90]. In [32], the isomonodromy equations have been described as integrable non-autonomous Hamiltonian systems. A symplectic fibre bundle whose base is the Jimbo-Miwa-Ueno deformation parameters space and the fibers are certain moduli spaces of unramified meromorphic connections was introduced in [15]. This approach was extended by D. Yamakawa for any reductive Lie algebra $\mathfrak{g}$ [92] who removed some multiplicity restrictions and introduced a symplectic two-form on the fibration. Following the same geometric approach and Jimbo-Miwa-Ueno isomonodromic tau-function D. Yamakawa [93] has proven that the isomonodromy equations of Jimbo–Miwa–Ueno is a completely integrable non-autonomous Hamiltonian systems. He was also motivated by the quantisation theorem of Reshetikhin but he did not try to consider the quantisation of general isomonodromy equations. Recently, Bertola and Korotkin have derived a new Hamiltonian formulation of the Schlesinger equations (i.e. for the Fuchsian case) in terms of the dynamical $r$–matrix structure.*

**Remark 2.0.8.** *The results of the theorem 2.0.5 still hold true for the autonomous systems which are obtained by the confluence procedure from the Gaudin system. It was shown by Yu. Chernyakov in [18] that the Poisson algebra which arises in the confluent elliptic and rational Gaudin systems coincides with the dual Takiff algebra equipped with the standard Lie–Poisson bracket (in [18] the author use the word"fission" instead of "confluence"). It also should be noted that the same result appeared in the paper by Chervov, Falqui and Rybnikov on the limits in the Gaudin system [25]*

One of the main quantum theorems of this thesis gives a general formula for the confluent KZ Hamiltonians with singularities of arbitrary Poincaré rank in any dimension.

**Theorem 2.0.9.** *Consider the differential operators:*

$$\nabla_{u_j} := \frac{\partial}{\partial u_j} - \widehat{H}_{u_j}, \quad j = 1, \ldots, n \tag{2.0.18}$$

*and*

$$\nabla_k^{(i)} := \frac{\partial}{\partial t_k^{(i)}} - \widehat{H}_k^{(i)}, \quad i = 1, \dots, n, \infty, \quad k = 1, \dots, r_i \tag{2.0.19}$$

*where the Hamiltonians $\widehat{H}_{u_j}$ which correspond to the positions of the poles $u_j$, $j = 1 \dots, n$, and $\widehat{H}_1^{(i)}, \dots, \widehat{H}_r^{(i)}$ which correspond to the times $t_1^{(i)}, \dots t_{r_i}^{(i)}$, for $i = 1, \dots, n, \infty$, are given by the following elements of the universal enveloping algebra $U\left(\hat{\mathfrak{g}}_{r_1} \oplus \dots \oplus \hat{\mathfrak{g}}_{r_\infty}\right)$:*

$$\hat{H}_{u_j} = \frac{1}{2} \operatorname*{res}_{\lambda = u_j} \mathrm{Tr}_0 \hat{A}(\lambda)^2,$$

*and*

$$\mathcal{M}^{(r_i)} \begin{pmatrix} \hat{H}_1^{(i)} \\ \hat{H}_2^{(i)} \\ \dots \\ \hat{H}_{r_i}^{(i)} \end{pmatrix} = \begin{pmatrix} \hat{S}_1^{(u_i)} \\ \hat{S}_2^{(u_i)} \\ \dots \\ \hat{S}_{r_i}^{(u_i)} \end{pmatrix}, \quad \hat{S}_k^{(u_i)} = \frac{1}{2} \oint_{\Gamma_{u_i}} (\lambda - u_i)^k \mathrm{Tr}_0 \hat{A}(\lambda)^2 \mathrm{d}\lambda,$$

*where*

$$\hat{A}(\lambda) = \sum_i^n \left( \sum_{j=0}^{r_i} \frac{\hat{B}_j^{(i)} \left( t_1^{(i)}, t_2^{(i)} \dots t_{r_i}^{(i)} \right)}{(\lambda - u_i)^{j+1}} \right),$$

*with $\hat{B}^{(i)}$'s given by*

$$\hat{B}_j^{(i)}(t_1^{(i)}, \dots t_{r_i}^{(i)}) = \sum_{k=j}^{r} \hat{A}_k^{(i)} \mathcal{M}_{j,k}^{(r_i)}(t_1^{(i)}, t_2^{(i)} \dots t_{r_i}^{(i)}), \quad \hat{A}_k = \sum_\alpha e_\alpha^{(0)} \otimes e_\alpha^{(i)} \otimes z_i^k,$$

*and $e_\alpha^{(0)}$ corresponds to the quantisation of $\mathfrak{g}^*$ to $\mathfrak{g}$ while*

$$e_\alpha^{(i)} = 1 \otimes \dots \otimes \underset{i}{e_\alpha} \otimes \dots \otimes 1.$$

*where $e_\alpha$ is chosen in some representation, with corresponding vector space $V$. Then the differential operators commute*

$$[\nabla_{u_j}, \nabla_{u_s}] = [\nabla_k^{(i)}, \nabla_{u_s}] = [\nabla_k^{(i)}, \nabla_l^{(a)}] = 0,$$

*$\forall\, j, s = 1, \dots, n$, $i, a = 1, \dots, n, \infty$, $k = 1, \dots, r_i$, $l = 1, \dots, r_a$. We call the system of differen-*

*tial equations*

$$\nabla_{u_j}\Psi = 0, \quad \nabla_k^{(i)}\Psi = 0, \qquad j = 1, \ldots, n,\ i = 1, \ldots, n, \infty,\ k = 0, \ldots, r_i, \quad \Psi \in V^{\otimes n},$$

*confluent KZ equations.*

Moreover, we express the isomonodromic Hamiltonians in terms of the lifted Darboux coordinates and show that the quasiclassical solutions of the confluent KZ equations is expressed via the isomonodromic $\tau$-function.

**Theorem 2.0.10.** *Given a solution $(P_1, \ldots, P_d, Q_1, \ldots, Q_d)$ of the classical isomonodromic deformation equations, the corresponding semi-classical solution $\Psi_{sc}$ of the confluent KZ equations*

$$\hbar \frac{\partial \Psi}{\partial u_j} = \widehat{H}_{u_j}\Psi, \quad j = 1, \ldots, n$$

*and*

$$\hbar \frac{\partial \Psi}{\partial t_k^{(i)}} = \widehat{H}_k^{(i)}\Psi, \quad i = 1, \ldots, n, \infty, \quad k = 1, \ldots, r_i$$

*evaluated along the solution $(P_1, \ldots, P_d, Q_1, \ldots, Q_d)$, admits the following WKB expansion*

$$\Psi_{sc}(Q(t), t) \sim \tau^{\frac{i}{\hbar}} \left(1 + O(\hbar)\right), \quad \hbar \to 0. \tag{2.0.20}$$

*in terms of the classical isomonodromic $\tau$-function*

$$\mathrm{d}\ln(\tau) := \sum_i \left( H_{u_i}^{(i)} \mathrm{d}u_i + \sum_{k=1}^{r_i} H_k^{(i)} \mathrm{d}t_k^{(i)} \right).$$

*The asymptotic expansion* (2.0.20) *is valid for $u_1, \ldots, u_n,$, $t_k^{(i)}$, $i = 1, \ldots, n, \infty$, $k = 1, \ldots, r_i$ in a polydisk that does not contain the poles of the solution $(P_1, \ldots, P_d, Q_1, \ldots, Q_d)$.*

This statement was mentioned in [80] for the case of the standard KZ, namely with simple poles. We also discuss the quantisation of the reduced Darboux coordinates and provide the quantised reduced systems in some examples.

Now we list the most of the results obtained in this thesis:

1. It was shown via a confluence procedure that the natural phase space for the irregu-

lar isomonodromic system is given by the direct product of the co-adjoint orbits of the corresponding Takiff algebras.

2. We prove that the moduli of special Poisson automorphisms for the Takiff co-algebras provides isomonodromic times for the irregular singularities.

3. The Hamiltonians for the corresponding irregular isomonodromic deformation equations have been written in an explicit closed form in terms of the spectral invariants.

4. We provided a symplectic embedding of the co-adjoint orbit of the degree $r$ Takiff algebra over $\mathfrak{sl}_m$ into the cotangent bundle of $r$ copies of $\mathfrak{gl}_m$.

5. In the $\mathfrak{sl}_2$ case, we established a correspondence between ramification in the linear systems of ODEs and the analogous of the nilpotent cone in the Takiff algebras.

6. Using the obtained irregular Hamiltonians, we provided a candidate for the irregular Knizhnik–Zamolodchikov system.

7. We extended Reshetikhin's theorem about the semi-classical solution of the Knizhnik–Zamolodchikov equations to the irregular case.

8. For all Painlevé equations, we provide an explicit reduction procedure starting from the co-adjoint orbits and finishing with the Darboux coordinates.

9. Moreover we provide an algebraic description of the phase spaces for all Painlevé equations using symplectic reduction.

10. We show that in all cases, the phase space for the Painlevé equations can be written as a double covering of $\mathbb{A}^2$ ramified along a (possibly rational) elliptic curve.

The results from 1 to 4 and from 6 to 8 are presented in a preprint [37] (collaboration with M.Mazzocco and V.Rubtsov). The initial interest to the theory of the isomonodromic deformations came from previous work on the nonlinear phenomena and asymptotical solutions to the Painlevé equations [39, 40, 41]. The author of this thesis also studied integrability [84]. The experience in the integrable systems give rise to the work of the author on the multi-particle Painlevé systems [38] (collaboration with V.Rubtsov) and a lot of results presented in this thesis. The ramified case (results 5 and 8 for the Painlevé III D7, III D8 and I) will be part of the single author paper by the candidate. The parts 9 and 10 are work in progress in collaboration with M. Mazzocco, V.Rubtsov and D.van Straten.

# CHAPTER 3

# BACKGROUND MATERIAL

## 3.1 Systems of Linear differential equations, Monodromy and Stokes phenomenon

The singularities of the solution to the system of the linear ODE's which location depends only on the singularities of the differential equation, and not on the Cauchy data (initial conditions), are called *fixed*. The fundamental solution can be easily defined away from the singularities of the differential operator or system via the Frobenius method. The question is what happens to the fundamental solution around the fixed singularities. Depending on the local form of the differential equation there are few possibilities - the solution has an apparent singularity, a regular singularity or an irregular singularity. The fundamental solution is not a single-valued object, and defines a so-called *local system*. The aim of this section is to give a review of the theory of linear differential equations on the Riemann sphere with the focus on the local and global behaviour of the fundamental solution. The results of this section may be found in [36, 83]

### 3.1.1 Linear systems of ODE's, gauge equivalence and principal G-bundles

Consider linear system of the form

$$\frac{d}{d\lambda}\Psi = A(\lambda)\Psi, \tag{3.1.1}$$

where $\Psi$ is a $m$-vector function on $\mathbb{C}$ and $A(\lambda)$ is a $m \times m$ matrix function in the complex variable $\lambda$ holomorphic apart from a finite set of isolated poles $u_1, \ldots, u_n, u_\infty = \infty$ such that

$$A(\lambda) = \sum_{i=1}^{n} \sum_{l=0}^{r_i} \frac{A^{(i)}_{r_i - l}}{(\lambda - u_i)^{l+1}} + \sum_{k=1}^{r_\infty} A^{(\infty)}_{r_\infty - l} \lambda^{l-1}, \tag{3.1.2}$$

where $A^{(i)}_0 \neq 0$, for $i = 1, \ldots, n, \infty$. The number $r_i$ is called *the Poincaré rank* at $\lambda = u_i$.

**Definition 1.** *A fundamental system of solutions of (3.1.1) is a $n \times n$ matrix $\Psi(\lambda)$ such that its columns form a basis of linearly independent solutions of (3.1.1).*

Given a fundamental matrix $\Psi(\lambda)$ of (3.1.1), the general solution can be represented as

$$\Psi(\lambda) C$$

where $C$ is a constant column vector.

**Example 2.** *Suppose $A(\lambda) = A_0$ is a constant matrix. Then the fundamental matrix is given by $\Psi(\lambda) = \exp(A_0 \lambda)$.*

**Example 3.** *Suppose $A(\lambda) = \frac{A_0}{\lambda}$, and $A_0$ is a constant matrix. Then the fundamental matrix is given by $\Psi(\lambda) = \lambda^{A_0}$. If $A_0$ is diagonalisable and $\Theta$ is the matrix of its eigenvalues, then another fundamental matrix is $\Psi(\lambda) = G\lambda^{\Theta}$, where $A_0 = G\Theta G^{-1}$.*

The general solution of the linear system (3.1.1) gives rise to the $G$-bundle with connection over the Riemann sphere with punctures at $u_1, u_2 \ldots u_n, u_\infty$. Then $\Psi(\lambda) \in G$ is a section of $G$-bundle over $\mathbb{P}^1 \setminus \{u_1, u_2 \ldots u_n, u_\infty\}$ while

$$\mathcal{A} = A(\lambda) d\lambda + 0 d\bar{\lambda} \in \Omega^{(1,0)}(\mathbb{P}^1 \setminus \{u_1, u_2 \ldots u_n, u_\infty\}, \mathfrak{g}) \subset \Omega^1(\mathbb{P}^1 \setminus \{u_1, u_2 \ldots u_n, u_\infty\}, \mathfrak{g})$$

is a connection. Here $\Omega^1(\mathbb{P}^1 \setminus \{u_1, u_2 \ldots u_n, u_\infty\})$ is a set of the globally defined $\mathfrak{g}$-valued one-forms, while $\Omega^(1,0)(\mathbb{P}^1 \setminus \{u_1, u_2 \ldots u_n, u_\infty\})$ stands for the globally defined *holomorphic* $\mathfrak{g}$-valued one-forms. The connection $\mathcal{A}$ is holomorphic on the punctured Riemann sphere. Two principal $G$-bundles are equivalent if there exists a gauge transformation which sends connection of one bundle to another. For the differential equations of the for (3.1.1) the local equivalence may be formulated in the following way

**Definition 4.** *Two systems of the above form (3.1.1)*

$$\frac{d}{d\lambda}\Psi = A(\lambda)\,\Psi, \qquad \frac{d}{d\lambda}\tilde{\Psi} = \tilde{A}(\lambda)\,\tilde{\Psi},$$

*having a pole of order $r+1$ zero at $0$, are formally equivalent near $0$ if there exists a formal series*

$$G(\lambda) = G_0 + G_1\lambda + \dots$$

*such that the transformation $\tilde{\Psi} = G(\lambda)\Psi$ maps the first system into the second one, i.e.*

$$\tilde{A}(\lambda)G(\lambda) = G(\lambda)A(\lambda) + \frac{d}{d\lambda}G(\lambda) \tag{3.1.3}$$

### 3.1.2 Logarithmic Singularities and Monodromy matrices

Consider a connection with simple poles only

$$\frac{d}{d\lambda}\Psi = \sum_{i=0}^{n} \frac{A^{(i)}}{\lambda - u_i}\Psi, \quad A^{(i)} \in \mathfrak{gl}_m, \quad \Psi \in GL_m. \tag{3.1.4}$$

Such systems are called *Fuchsian*. Near the singular point $\lambda = u_j$ the connection reads

$$A(\lambda) \underset{\lambda\sim u_j}{\sim} \frac{A^{(j)}}{\lambda - u_j} + \sum_{l=0}^{\infty}\sum_{k\neq j} \frac{(-1)^l A^{(k)}}{(u_i - u_k)^{l+1}}(\lambda - u_j)^{l+1} = \frac{A^{(j)}}{\lambda - u_j} + \sum_{l=0}^{\infty} R_l(\lambda - u_j)^{l+1}.$$

Let us do a technical assumption that the eigenvalues of all residues $A^{(i)}$ do not differ by a non-zero integer. Near the singularity $\lambda = u_i$ the fundamental solution may be written in the following form

$$\Psi \sim \Psi_j = P_j\left(1 + \sum \phi_k^{(j)}(\lambda - u_j)^k\right)\exp\left[\Lambda^{(j)}\ln(\lambda - u_j)\right], \quad A^{(j)} = P_j\Lambda^{(j)}P_j^{-1},$$

where $\Lambda^{(j)}$ is a Jordan normal form of $A^{(j)}$. Coefficients $\phi_k^{(j)}$ are defined via recursive relation. Locally, solution $\Psi_i$ is not a single-valued function. Indeed, turning around singularity $u_j$ due to logarithmic behaviour of the argument of the exponent we have

$$\Psi_j(ze^{2\pi i}) = \Psi_j(z)e^{2\pi i\Lambda^{(j)}}, \quad z = \lambda - u_i.$$

The matrix $e^{2\pi i\Lambda^{(j)}}$ is called a *local monodromy.* Globally multi-valuedness has less explicit description in the terms of the local data. Let $\pi_1(\mathbb{P}^1\backslash\{u_1, \dots u_n, \infty\}) = \langle\gamma_1, \gamma_2 \dots \gamma_n, \gamma_\infty \,|\, \gamma_1\gamma_2 \dots \gamma_n\gamma_\infty = 1\rangle$, then

$$\Psi|_{\gamma_k} = \Psi M_k \quad M_k = G_k e^{2\pi i\Lambda^{(k)}} G_k^{-1},$$

where $G_k$ is a transition matrix from the base-point to the neigbourhood of singularity $u_k$. In particular, $G_k$ is given by the prolognation of solution for the (3.1.4) along the interval $\mathcal{I}$ which connects base-point with $u_i + \varepsilon$, where $\varepsilon$ is chosen in such a way, that $\mathcal{I}$ do not intersect any other singularities. Monodromy matrices provide a representation of the fundamental group in the group $G$, since the following relation holds

$$M_1M_2 \dots M_nM_\infty = 1, \quad M_i \in G.$$

Moreover, the choice of the base-point provides an action of the group $G$ on the monodromy matrices $M_i$ via simultaneous conjugation. Finally the space of the monodromy data reads as

$$\mathcal{M} = \left\{ \begin{array}{c|c} M_1, M_2, \dots M_\infty & M_1, M_2 \dots M_nM_\infty = 1 \\ M_i \in G & M_i \sim e^{2\pi i\Lambda^{(i)}} \end{array} \right\} \Big/ G \tag{3.1.5}$$

This space is also called *G-character variety* and may be seen as a representation space of the fundamental group of $\mathbb{P}^1 \setminus \{u_1, \dots u_n, \infty\}$

### 3.1.3 Irregular singularities and Stokes phenomenon

We consider a principal $G$-bundle and a meromorphic connection with poles of arbitrary order over $\mathbb{P}^1$. In other words, we consider a $G$-bundle with a holomorphic connection over punctured Riemann sphere. Using the local coordinate for a pole $z = \lambda - u_i$, we write the system of linear differential equations in the following form

$$\frac{d}{dz}\Psi = A(z)\Psi, \quad A(z) = \frac{A_r}{z^{r+1}} + \frac{A_{r-1}}{z^r} + \dots + \frac{A_0}{z} + O(1), \quad A_i \in \mathfrak{g} = T_eG. \tag{3.1.6}$$

The number $r \in \mathbb{Z}$ is called a Poincaré rank of singularity.

**Theorem 3.1.1.** *Let $z$ be a local coordinate in a neigbourhood of singularity and consider the system of differential equations* (3.1.6). *In case when $A_r$ is a semi-simple, formal asymptotical solution takes form*

$$\Psi \sim P(z) \exp\left[-\Lambda(z)\right] z^{\Lambda_0}, \quad P(z) = \sum_{i=0}^{\infty} P_i z^i, \quad \Lambda(z) = \frac{1}{r}\frac{\Lambda_r}{z^r} + \ldots \frac{1}{j}\frac{\Lambda_j}{z^j} + \cdots + \frac{\Lambda_1}{z}, \tag{3.1.7}$$

*where $\Lambda_i$ are diagonal matrices for $i = 0 \ldots r$. In the case, when $A_r$ is not a semi-simple matrix, local asymptotic solution may be written in a form*

$$\Psi \sim P(z) \exp\left[-\Lambda(z)\right] z^{\Lambda_0}, \quad P(z) = \sum_{i=0}^{\infty} P_i z^{i/d},$$

*where $d \in \mathbb{Z}$ is a least common multiple of the eigenspace dimensions for $A_r$ and $\Lambda(z)$ is a diagonal matrix, with a pole at $z = 0$.*

**Definition 3.1.2.** The case when $A_r$ is not a semi-simple matrix called ramified.

The monodromy data nicely describes the fundamental solution of the system with regular singularities. However, when the connection has a higher order pole, the monodromy data is not enough to describe the behaviour of the fundamental solution. The main difficulty is that in the case of irregular singularities there is no way to define a convergent in a disk local solution like in the Fuchsian case. More precisely, there is only a way to define a formal local solution, which has a zero radius of convergence, and can be used to define the asymptotic behaviour of the solution. The main difficulty is that such asymptotic define unique solution only in some sectors near irregular singularity, which are called *Stokes sectors.*

### 3.1.4 Geometry of the Stokes rays and Stokes sectors

The Stokes rays are oriented half lines from zero to infinity. Let $\theta_\alpha$ be the eigenvalues of $\Lambda_r$ in (3.1.7). Let us study the equation $\mathrm{Re}[\lambda^{r_\infty}(\theta_\alpha - \theta_\beta)] = 0$, where $r_\infty \in \mathbb{Z}_+$. For $n$ distinct eigenvalues there are $n(n-1)$ differences (we consider $\theta_\alpha - \theta_\beta$ to be different from $\theta_\beta - \theta_\alpha$)

$$\theta_\alpha - \theta_\beta = \rho \exp(i\pi\phi_{\alpha\beta}),$$

let $\lambda = \sigma \exp(\pi i \psi)$ then

$$\lambda^{r_\infty}(\theta_\alpha - \theta_\beta) = \sigma\rho \exp\left(i\pi\phi_{\alpha\beta} + ir_\infty \pi\psi\right)$$

thus why we impose

$$i\pi\phi_{\alpha\beta} + ir_\infty \pi\psi_{\alpha\beta} = \frac{2\mathbb{Z}+1}{2} i\pi$$

that is

$$\psi_{\alpha\beta} = \frac{1}{r_\infty}\left(\frac{1}{2} - \phi_{\alpha\beta} + m\right), \qquad m = 0, \ldots, 2r_\infty - 1.$$

Imposing the condition on the Imaginary part we see that $m$ can only be odd, thus we have $n(n-1)r_\infty$ Stokes rays. For each $j$ they are distant $\frac{1}{r_\infty}$. In fact we can take $m$ even and we have the Stokes rays corresponding to the opposite difference. In a sector wider than $\frac{\pi}{r_\infty}$, we have at least two Stokes rays for each couple of eigenvalues. Crossing a Stokes ray determines a change of *dominance relations*

The following lemma states essentially that the sectors of validity of our actual fundamental matrices $Y$ can be extended to the adjacent Stokes rays (without including them). The uniqueness of actual fundamental matrix $Y$ having asymptotic behavior $\Psi_f$ in a sector of opening $> \frac{\pi}{r_\infty}$ is due to the fact that such a sector contains $r_\infty + 1$ Stokes rays.

**Lemma 5.** *Let $\Psi(\lambda)$ be an actual fundamental matrix of the system (3.1.1) at $\infty$ such that $\Psi(\lambda) \sim \Psi_f(\lambda)$ as $\lambda \to \infty$, $\lambda \in \Sigma \cap \{|\lambda| > N\}$, $N \in I\!R$ where $\Sigma$ is some sector of opening $< 2\pi$. Suppose that $\tilde{\Sigma}$ is another sector of opening $< \frac{\pi}{r_\infty}$ such that $\Sigma \cap \tilde{\Sigma}$ is simply connected and non–empty and $\tilde{\Sigma}$ does not contain any Stokes ray, then $\Psi(\lambda) \sim \Psi_f(\lambda)$ in as $\lambda \to \infty$, $\lambda \in \Sigma \cup \tilde{\Sigma}$.*

*Proof.* Let $\tilde{\Psi}(\lambda)$ be an actual fundamental matrix of the system (3.1.1) at $\infty$ such that $\tilde{\Psi}(\lambda) \sim \Psi_f(\lambda)$ as $\lambda \to \infty$, $\lambda \in \tilde{\Sigma} \cap \{|\lambda| > N\}$. Since $\Sigma \cap \tilde{\Sigma} \neq \emptyset$, and it is simply connected, there exists a constant matrix $C$ such that $\Psi(\lambda) = \tilde{\Psi}(\lambda)C$ for $\lambda \in \Sigma \cap \tilde{\Sigma}$. Thus $\tilde{\Psi}(\lambda)C$ is the analytic continuation of $\Psi(\lambda)$ in $\tilde{\Sigma}$. Since $\tilde{\Psi}(\lambda)$ and $\Psi(\lambda)$ have the same asymptotic behavior, we have

$$\lambda^{R^{(\infty)}} \exp(Q^{(\infty)}(\lambda))C \exp(-Q^{(\infty)}(\lambda))\lambda^{-R^{(\infty)}} \sim \mathbb{I}$$

as $\lambda \to \infty$, $\lambda \in \Sigma \cap \tilde{\Sigma}$. Since $\tilde{\Sigma}$ doesn't contain any Stokes ray, then the dominance relations in

$\tilde{\Sigma}$ are the same as in $\lambda \in \Sigma \cap \tilde{\Sigma}$, i.e. the above relation is valid in all $\tilde{\Sigma}$. Thus

$$\begin{aligned}
&\Psi(\lambda) \exp(-Q^{(\infty)}(\lambda))\lambda^{-R^{(\infty)}} = \tilde{\Psi}(\lambda) C \exp(-Q^{(\infty)}(\lambda))\lambda^{-R^{(\infty)}} = \\
&= \tilde{\Psi}(\lambda) \exp(-Q^{(\infty)}(\lambda))\lambda^{-R^{(\infty)}}\lambda^{R^{(\infty)}} \exp(Q^{(\infty)}(\lambda)) C \exp(-Q^{(\infty)}(\lambda))\lambda^{-R^{(\infty)}} = \\
&= \tilde{\Psi}(\lambda) \exp(-Q^{(\infty)}(\lambda))\lambda^{-R^{(\infty)}}
\end{aligned}$$

i.e. $\Psi(\lambda) \sim \tilde{\Psi}(\lambda) \sim \Psi_f(\lambda)$ in all $\tilde{\Sigma}$.

Let us now prove uniqueness. Suppose that $\Psi(\lambda)$ and $\tilde{\Psi}(\lambda)$ are two actual fundamental matrices of the system (3.1.1) at $\infty$ such that $\Psi(\lambda)$, $\tilde{\Psi}(\lambda) \sim \Psi_f(\lambda)$ as $\lambda \to \infty$, $\lambda \in \tilde{\Sigma} \cap \{|\lambda| > N\}$. Then there exists a constant matrix $C$ such that $\Psi(\lambda) = \tilde{\Psi}(\lambda) C$, i.e.

$$\lambda^R \exp(Q(\lambda)) C^{-1} \exp(-Q(\lambda))\lambda^{-R} \sim \mathbb{I}$$

as $\lambda \to \infty$, $\lambda \in \Sigma$ as above. Now let us call $q_i(\lambda)$ the entries of the diagonal matrix $Q(\lambda)$ and $r_i$ the ones of $R$. On the matrix elements we have

$$\exp(q_i(\lambda) - q_j(\lambda)) z^{r_i - r_j} C_{ij} \sim \mathbb{I}_{ij}.$$

Thus $C_{ii} = 1$ and since the sector is big enough, for each dominance relation it contains also the opposite one. Thus $C_{ij} = 0$ for $i \neq j$. This proves that $C = \mathbb{I}$ and the proof of our theorem is concluded. □

If the sector is narrow, we have ambiguity in the choice of the true fundamental solution. Such ambiguity is described by the so-called *Stokes multipliers* defined in the following theorem.

**Theorem 6.** *Consider the system (3.1.1) under our basic assumption that $A_0^{\infty}$ has a simple spectrum and let $\Psi_l^{(\infty)}(\lambda)$,$l = 1, \dots, 2r_\infty$, denote the unique $2r_\infty$ actual fundamental matrices in $\Sigma_l$ satisfying (3.1.7). Let the $2r_\infty$ matrices $S_1, \dots, S_{2r_\infty}$ be defined corresponding to the intersections*

$$\Sigma_{l,l+1} = \Sigma_l \cap \Sigma_{l+1}$$

*by means of*

$$\Psi_l^{(\infty)}(\lambda) = \Psi_{l+1}^{(\infty)}(\lambda) S_l, \qquad , \lambda \in \Sigma_{l,l+1},$$

*where $\Psi_{2r_\infty+1}^{(\infty)} = \Psi_1^{(\infty)}$ and $\Sigma_{2r_\infty+1} = \Sigma_1$. The matrices $S_1, \dots, S_{2r_\infty}$ are constant invertible* *matrices such that $S_{\alpha\alpha} = 1$ and $S_{\alpha\beta} = 0$ for all $\alpha \neq \beta$ such that*

$$\left| e^{\lambda^{r_\infty}(\theta_\alpha - \theta_\beta)} \right| \to \infty$$

*as $\lambda \to \infty$ along some ray in the chosen sector.*

**Definition 7.** *The matrices $S_1, \ldots, S_{2r_\infty}$ introduced in Theorem 6 are called the Stokes multipliers associated to the system (3.1.1) at $\infty$.*

In such a way we may introduce additionaly a Stokes data for each irregular singularity. This data extends the notion of the monodromy data for the case of the irregular singularities.

## 3.2 Isomonodromic deformations

The aim of this section is to provide a review of isomonodromic deformations. The corresponding material can be found in [36].

### 3.2.1 Fuchsian systems and Schlesinger equations

Consider the Fuchsian system (3.1.4), generalized for an arbitrary Lie group $G$, i.e. the system of the equations

$$\frac{d}{dz}\Psi = \sum_{i=1}^{n} \frac{A^{(i)}}{z - u_i}\Psi, \quad \Psi \in G, \quad A^{(i)} \in \mathfrak{g}, \quad z \in \mathbb{P}^1 \setminus \{u_1, u_2, \ldots, u_n\}, \quad \forall i \neq j : u_i \neq u_j,$$

where $\mathfrak{g}$ is the Lie algebra of $G$. Such system may be seen as a connection on the $G$-bundle over $\Sigma_{0,n} = \mathbb{P}^1 \setminus \{u_1, u_2, \ldots, u_n\}$. The monodromy map, introduced in the subsection 3.1.2, provides a representation of the fundamental group into the group $G$, by sending each generator of $\pi_1(\Sigma_{0,n})$ to the corresponding monodromy matrix $M_i$.

The isomonodromic deformation theory studies the following question: *How should elements $A^{(i)}$ depend on the position of the poles $u_i$, such that the monodromy matrices $M_i$ do not change during the local variation of the poles $u_i$?*

The answer to this question was given by L. Schlesinger [82], which may be formulated in the following theorem.

**Theorem 3.2.1.** *The monodromy matrices are constant under the local variations of the position of the poles $u_i$, if the elements $A^{(i)}$ solve the following system of the non-linear differential equations*

$$\begin{aligned} \frac{\partial A^{(i)}}{\partial u_j} &= \frac{[A^{(j)}, A^{(j)}]}{u_j - u_i}, \quad j \neq i \\ \frac{\partial A^{(i)}}{\partial u_i} &= -\sum_{j \neq i} \frac{[A^{(j)}, A^{(j)}]}{u_j - u_i}. \end{aligned} \tag{3.2.1}$$

*The equations* (3.2.1) *are called Schlesinger equations. We call the system* (3.2.1) *isomonodromic deformation, because it preserves the monodromy matrices of the corresponding local system.*

### 3.2.2 Isomonodromic deformations as a compatibility condition

The results of the theorem 3.2.1 may be reformulated in the following way:

**Theorem 3.2.2.** *Consider the following overdetermined system of the linear differential equations*

$$\begin{cases} \frac{d}{dz}\Psi = A(z)\Psi \\ \mathrm{d}_u\Psi = B(z)\Psi, \end{cases} \qquad \Psi \in G$$

*where*

$$\mathrm{d}_u\Psi = \sum_{i=1}^{n} \frac{\partial \Psi}{\partial u_i}\mathrm{d}u_i, \quad A(z) = \sum_{i=1}^{n} \frac{A^{(i)}}{z - u_i}, \quad B(z) = -\sum_{i=1}^{n} A^{(i)} \frac{\mathrm{d}u_i}{z - u_i}.$$

*Then, compatibility condition*

$$\mathrm{d}_u A - \frac{d}{dz}B + [A, B] = 0,$$

*is equivalent to the Schlesinger equations* (2.0.2)*.*

In a nutshell, this means that isomonodromic deformation can be seen as a flatness condition on the extended space which is a product of the base curve $\Sigma_{0,n}$ and the configuration space of the deformation parameters (in the Fuchsian case this extended space may be associated with the product of the base curve and its moduli of the complex structures). For the Fuchsian system, such flat connection may be written as

$$\Omega = \sum_{i=1}^{n} A^{(i)} \mathrm{d}\ln(z - u_i), \quad \mathrm{d}f = \frac{\partial}{\partial z} f \mathrm{d}z + \sum_{i=1}^{n} \frac{\partial}{\partial u_i} f \mathrm{d}u_i.$$

Indeed, the flatness conditions for $\Omega$ coincides with the compatibility condition from the theorem 3.2.2 and, as a consequence, with the Schlesinger equations (3.2.1).

In the case of irregular singularities the notion of the isomonodromic deformation has to be extended also to the Stokes data described in the subsection 3.1.3. Still such deformations can be formulated as flatness conditions on the extended space which is a product of the base curve $\Sigma_{0,n}$ and the configuration space of the deformation parameters. We give a description of the irregular isomonodromic deformations later in the thesis, see Section 5.1.

### 3.2.3 Painlevé equations as the isomonodromic deformations of $\mathfrak{sl}_2$ connections

In this subsection we provide a brief introduction to the Painlevé equations as the isomonodromic systems for the $\mathfrak{sl}_2$ connections with possibly irregular singularities. The Painlevé equations form a list of the six non-linear second order differential equations, whose general solution defines a transcendental function. Here is the list of the Painlevé equations:

$$\begin{aligned}
P_{\mathrm{VI}}: \quad \tfrac{d^2y}{dt^2} &= \frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2-\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{y-t}\right)\frac{dy}{dt}\\
&\quad+\frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha+\beta\frac{t}{y^2}+\gamma\frac{t-1}{(y-1)^2}+\delta\frac{t(t-1)}{(y-t)^2}\right),\\
P_{\mathrm{V}}: \quad \tfrac{d^2y}{dt^2} &= \left(\frac{1}{2y}+\frac{1}{y-1}\right)\left(\frac{dy}{dt}\right)^2-\frac{1}{t}\frac{dy}{dt}++\frac{(y-1)^2}{t^2}\left(\alpha y+\frac{\beta}{y}\right)\\
&\quad+\gamma\frac{y}{t}+\delta\frac{y(y+1)}{y-1},\\
P_{\mathrm{IV}}: \quad y\tfrac{d^2y}{dt^2} &= \tfrac{1}{2}\left(\frac{dy}{dt}\right)^2+\beta+2(t^2-\alpha)y^2+4ty^3+\tfrac{3}{2}y^4,\\
P_{\mathrm{III}}: \quad y\tfrac{d^2y}{dt^2} &= t\left(\frac{dy}{dt}\right)^2-y\frac{dy}{dt}+\delta t+\beta y+\alpha y^3+\gamma t y^4,\\
P_{\mathrm{II}}: \quad \tfrac{d^2y}{dt^2} &= 2y^3+ty+\alpha,\\
P_{\mathrm{I}}: \quad \tfrac{d^2y}{dt^2} &= 6y^2+t,
\end{aligned} \tag{3.2.2}$$

where $\alpha, \beta, \gamma$ and $\delta$ are the complex numbers. All these equations may be obtained as an isomonodromic deformation of the $\mathfrak{sl}_2$ connection on $\mathbb{P}^1$ with sum of pole orders equal to 4. In such a way, the Painlevé VI equation is an isomonodromic deformation of the $\mathfrak{sl}_2$ Fuchsian system with 4 poles - at $0, 1, t$ and $\infty$. The other Painlevé equations are related to the connections with higher order poles. In particular, divisors of the singularities of the connections for all Painlevé equations are given in the following table:

| | |
|---|---|
| Painlevé VI | $\mathbf{0}+\mathbf{1}+\mathbf{t}+\infty$ |
| Painlevé V | $\mathbf{0}+\mathbf{1}+2\cdot\infty$ |
| Painlevé IV | $\mathbf{0}+3\cdot\infty$ |
| Painlevé III | $2\cdot\mathbf{0}+2\cdot\infty$ |
| Painlevé II | $4\cdot\infty$ |
| Painlevé I | $4\cdot\infty$ |

In Section 5.6 we provide a systematic derivation of the all Painlevé equations except the Painlevé VI, using a Hamiltonian description of the isomonodromic deformations for irregular connections. The derivation of the Painlevé VI equation as a reduction of the Schlesinger equations may be found in [6].

## 3.3 Symplectic and Poisson geometry, Hamiltonian group actions and Lie algebra co-adjoint orbits

The aim of this section is to provide a brief review of symplectic geometry with focus on the moment map theory. Raised from Hamiltonian mechanics, symplectic geometry became an important tool of the modern mathematical physics.

In this section we use following notation

- $\mathbb{K}$ is a base field, $\operatorname{char}\mathbb{K}=0$.
- $M$ stands for smooth (algebraic, holomorphic) variety,
- $\mathcal{C}(M)$ denotes the algebra of $\mathcal{C}^\infty$ (resp. algebraic, holomorphic) functions on $M$
- $\mathcal{O}$ stands for an adjoint orbit of a Lie algebra, $\mathcal{O}^\star$ for a co-adjoint orbit

We do not put references during the text of this section. Most of the material presented in this section can be found in [5, 19].

### 3.3.1 Adjoint and co-adjoint orbits

Consider the Lie group $G$ with the Lie algebra $\mathfrak{g} \simeq T_e G$ and its dual $\mathfrak{g}^\star$. We define by

$$\langle \Theta, X \rangle, \quad X \in \mathfrak{g}, \quad \Theta \in \mathfrak{g}^\star$$

the bi-linear pairing between the Lie algebra $\mathfrak{g}$ and its dual.

**Definition 3.3.1.** The adjoint action of the group $G$ on itself is the map $\mathrm{Ad} : G \times G \to G$ defined by

$$\mathrm{Ad}(g, h) := \mathrm{Ad}_g\, h = g h g^{-1} \tag{3.3.1}$$

**Definition 3.3.2.** The adjoint action of the group $G$ on its Lie algebra $\mathfrak{g}$ is the map $\mathrm{ad} : G \times \mathfrak{g} \to \mathfrak{g}$ defined by

$$\mathrm{ad}(g, X) := \mathrm{ad}_g(X) = \left[ \frac{d}{dt} \mathrm{Ad}_g \exp(tX) \right]_{t=0} = \left[ \frac{d}{dt} (g \exp(tX) g^{-1}) \right]_{t=0}, \quad X \in \mathfrak{g}, \quad g \in G, \tag{3.3.2}$$

here exp is an exponential map from the Lie algebra to the Lie group.

**Definition 3.3.3.** The co-adjoint action of the group $G$ on the dual of its Lie algebra is the map $\mathrm{ad}^\star : G \times \mathfrak{g}^\star \to \mathfrak{g}^\star$ defined in such way, that

$$\langle \mathrm{ad}^\star_g \Theta, X \rangle = \langle \Theta, \mathrm{ad}_g X \rangle. \tag{3.3.3}$$

The maps defined above are related in the following way - ad is a differential of the map Ad, $\mathrm{ad}^\star$ is the categorical dual to ad. The orbits of the adjoint and co-adjoint actions are quite important objects which defined as

**Definition 3.3.4.** The adjoint orbit of the element $X \in \mathfrak{g}$ is defined as

$$\mathcal{O}(X) = \{ Y \in \mathfrak{g} |\, \exists g \in G : \, \mathrm{ad}_g(X) = Y \} \tag{3.3.4}$$

The co-adjoint orbit of the element $\Lambda \in \mathfrak{g}^\star$ is defined as

$$\mathcal{O}^\star(\Lambda) = \{ \Theta \in \mathfrak{g}^\star |\, \exists g \in G : \, \mathrm{ad}^\star_g(\Theta) = \Lambda \}$$

One of the most important example is the case of the Lie group $GL_n$. The Lie algebra $\mathfrak{gl}_n$ is just an algebra of the matrices of size $n$. Moreover, there exists a Killing form, given by the trace of matrix product

$$K(X,Y) = \mathrm{Tr}(XY), \quad X, Y \in \mathfrak{gl}_n,$$

which allows to identify $\mathfrak{gl}_n^\star$ with $\mathfrak{gl}_n$. Then both adjoint and co-adjoin action are given by conjugation

$$\mathrm{ad}_g X = gXg^{-1}, \quad \mathrm{ad}_g^\star \Theta = g^{-1}\Theta g.$$

Orbits in $\mathfrak{gl}_n$ are classified by the Jordan normal form. The orbit is called *semi-simple* if it contains a diagonal matrix with distinct eigenvalues $\theta_i \neq \theta_j$.

### 3.3.2 Poisson Algebras

The notion of a Poisson algebra originates from Hamiltonian mechanics. In Hamiltonian mechanics such algebras arise as a coordinate ring of the cotangent bundle to $\mathbb{R}^n$ and in fact are given by the inversion of the canonical symplectic structure. However, a Poisson algebra is an algebraic object which can be formulated intrinsically.

**Definition 3.3.5.** A commutative algebra $(A, \cdot)$ equipped with skew-symmetric bi-linear pairing $\{,\} : A \times A \to A$ is called Poisson algebra if

1. $(A, \{,\})$ is a Lie algebra
2. $\{,\}$ is a derivation satisfying Leibniz rule $\{x \cdot y, z\} = \{x, z\} \cdot y + x \cdot \{y, z\}$

The bracket $\{,\}$ is called a *Poisson* bracket on $A$. In other words, the bracket $\{,\}$ defines a Poisson structure on the commutative algebra $A$.

**Example 3.3.6.** Consider the algebra $A = C^\infty(\mathbb{R}^2)$ of smooth functions of two variables $p$ and $q$ in $\mathbb{R}^2$. Define a Poisson bracket as

$$\{f, g\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}.$$

This is an example of the $\infty$-dimensional Poisson algebra.

**Example 3.3.7.** More general, consider the algebra of smooth functions of $n$ variables $x_1, x_2, \dots x_n$. Consider $(2,0)$ skew-tensor $\pi^{ij}$, satisfying

$$\sum_l \left( \pi^{lj} \frac{\partial \pi^{ik}}{\partial x_l} + \pi^{lk} \frac{\partial \pi^{ji}}{\partial x_l} + \pi^{li} \frac{\partial \pi^{kj}}{\partial x_l} \right) = 0,$$

defines a Poisson bracket

$$\{f, g\} = \sum_{ij} \pi^{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}.$$

This example gives rise to the notion of the Poisson manifold.

**Definition 3.3.8.** A Poisson manifold $M$ is a (smooth, algebraic, etc.) manifold with a bivector $(2,0$ skew-tensor field) $\pi \in \wedge^2 \Gamma(TM)$, s.t.

$$[[\pi, \pi]] = 0,$$

where $[[\cdot, \cdot]]$ is a superalgebra structure on the exterior algebra of polyvector fields on $M$. The Poisson bracket is given by

$$\{f, g\} = \langle df \wedge dg, \pi \rangle$$

**Example 3.3.9.** For the ring of polynomials $\mathbb{C}[x, y, z]$ the following bracket

$$\{x, y\} = z, \quad \{y, z\} = x, \quad \{z, x\} = y,$$

defines a Poisson structure.

**Example 3.3.10.** Consider an affine surface in $\mathbb{C}^3$ given by the equation $P(x, y, z) = 0$ for $x, y, z \in \mathbb{C}^3$. Then the following bracket

$$\{x, y\} = \frac{\partial P}{\partial z}, \quad \{y, z\} = \frac{\partial P}{\partial x}, \quad \{z, x\} = \frac{\partial P}{\partial y},$$

defines a Poisson bracket. Example 3.3.9 corresponds to $P(x, y, z) = \frac{1}{2}(x^2 + y^2 + z^2) + \text{const}$. Introduced Poisson structure is the particular example of the more general structure called Nambu bracket. Nambu bracket can be defined on any algebraic variety and gives a Poisson bracket for the algebraic surfaces.

**Example 3.3.11.** The algebra of functions $\mathcal{C}(M)$ of any symplectic manifold is a Poisson algebra. We consider this example in more details in the next section.

### 3.3.3 Symplectic geometry

In this subsection we provide a basic definitions of symplectic geometry.

**Definition 3.3.12.** A symplectic manifold is a pair $(M, \omega)$ where $M$ is an even dimensional (smooth, algebraic etc.) manifold and $\omega$ is a globally defined (nowhere vanishing) closed differential two form.

**Example 3.3.13.** In classical mechanics the most natural symplectic manifold is the cotangent bundle of $\mathbb{R}^n$, which provides a basic example of canonical coordinates. In general, the cotangent bundle to any smooth manifold is a symplectic manifold. The symplectic form in that case is given by differential of the so called Liouville form (or tautological one-form) $\rho \in \Omega^1(T^\star M)$ : $\omega = d\rho$. The tautological one-form is defined in the following way

$$\langle \rho, v\rangle(m) = \langle p, d\pi_m(v)\rangle, \quad m = (p, x) \in T^\star M, \quad x \in M, \quad v \in T_m(T^\star M),$$

and $d\pi_m$ is a pushforward of the projection map $\pi : T^\star M \to M$ at the point $m$.

**Example 3.3.14.** Consider $\mathbb{C}^n$ with coordinates $z_1, z_2, \dots z_n$, then the standard symplectic form on $\mathbb{C}^n$ reads

$$\omega = \sum_{i=1}^{n} dz_i \wedge d\bar{z}_i.$$

**Example 3.3.15.** Consider a non-singular closed algebraic surface in $\mathbb{R}^3$ given by the equation $P(x, y, z) = 0$, then symplectic form is given by

$$\omega = \frac{\mathrm{d}y \wedge \mathrm{d}z}{\partial P/\partial x} + \frac{\mathrm{d}x \wedge \mathrm{d}y}{\partial P/\partial z} + \frac{\mathrm{d}z \wedge \mathrm{d}x}{\partial P/\partial y}$$

In case when $P(x, y, z) = x^2 + y^2 + z^2 - R^2$, where $R$ is a constant non-zero real number, we have a family of the symplectic manifolds parametrized by $R$. Each member of this family is a symplectic leaf of the Poisson structure from example 3.3.9

The existence of symplectic form allows to construct vector fields from the globally defined functions on the manifold. Indeed, for each globally defined function $f \in \mathcal{C}(M)$ we may canonically associate a vector field $X_f$, such that

$$\omega(X_f, \cdot)(p) = -df(p), \quad p \in M$$

In general, the converse is not true - there is no canonical way to associate a function to an arbitrary vector field. The set of vector fields which are generated by a function are called *Hamiltonian* vector fields, while the associated function is called *Hamiltonian.* As a consequence, we get a Poisson structure on $\mathcal{C}(M)$ given by

$$\{f, g\}(p) = \omega(X_f, X_g)(p), \quad p \in M$$

### 3.3.4 Haniltonian group actions

Let $G$ be a Lie group which acts on $(M, \omega)$ by symplectomorphisms - such that that symplectic form is invariant under this action

$$\begin{array}{ccc} G \times M & \to & M \\ (g, m) & \to & \phi_g(m) \end{array}, \quad \phi_g^\star(\omega) = \omega$$

For each element $\xi$ of the Lie algebra $\mathfrak{g}$ of $G$ we may associate a vector field $X_\xi$, which can be defined at point $p \in M$ by the following rule

$$X_\xi(p) = \frac{d}{dt}\left(e^{t\xi} \circ p\right)\Big|_{t=0}$$

The group action is called Hamiltonian if for each $\xi \in \mathfrak{g}$ exists $h_\xi \in \mathcal{C}(M)$ such that

$$\omega(X_\xi, \cdot) = -dh_\xi$$

In the case of the Hamiltonian action we have a map

$$\begin{array}{cccc} h: & \mathfrak{g} & \to & \mathcal{C}(M) \\ & \xi & \to & h_\xi \end{array} \tag{3.3.5}$$

The map $h$ is called *co-moment map.* The symplectic manifold $(M,\omega)$ with Hamiltonian $G$ action is called Hamiltonian $G$-space $(M,\omega,G)$. In further text, we will use the brief notation $M$ for Hamiltonian $G$-space $(M,\omega,G,h)$

**Proposition 3.3.16.** *If $f$ is a $G$-invariant function, i.e. $f(g\circ m)=f(m)$, then, for each $\xi\in\mathfrak{g}$ $h_\xi$, Poisson commutes with $f$.*

*Proof.*

$$\{h_\xi,f\}=\omega(X_\xi,X_f)=X_\xi df=0$$

since $f$ is invariant under the $G$ action, so it is invariant under the infinetisimal action. □

### 3.3.5 Moment map

Consider the dual map to $h$

$$h^\star:\ (\mathcal{C}(M))^\star\to\mathfrak{g}^\star$$

The dual space $(\mathcal{C}(M))^\star$ is not nicely defined, but we may think about $M$ as a subspace of $(\mathcal{C}(M))^\star$. Indeed, we have an evaluation map

$$\begin{aligned}\text{eval}:\ \mathcal{C}(M)\times M &\to \mathbb{K},\\ (f,m) &\to f(m).\end{aligned}$$

The evaluation map is a linear map on the $\mathcal{C}(M)$, so it defines a pairing between $\mathcal{C}(M)$ and $M$. This means that $M\subset(\mathcal{C}(M))^\star$. Restricting $h^*$ to $M$ we obtain a map

$$\mu:\ M\to\mathfrak{g}^\star$$

such that

$$\langle\mu(m),\xi\rangle=h_\xi(m),\quad\forall\xi\in\mathfrak{g}$$

The map $\mu$ is called a **moment map**.

**Theorem 3.3.17.** *Let $H$ be a $G$-invariant function, then the moment map $\mu$ is the constant of motion for the system of ODEs defined by the Hamiltonian vector field with Hamiltonian $H$*

*Proof.* Since $H$ is $G$-invariant, we have

$$\frac{d}{dt}h_\xi = \{h_\xi, H\} = 0,$$

for an arbitrary $\xi \in \mathfrak{g}$. On the other hand

$$\frac{d}{dt}h_\xi = \frac{d}{dt}\langle \mu, \xi \rangle = \langle \dot{\mu}, \xi \rangle = 0.$$

Since $\xi$ is arbitrary we finally have

$$\frac{d}{dt}\mu = 0.$$

□

The moment map $\mu$ is called *equivariant* if the following diagram is commutative

$$\begin{array}{ccc} M & \xrightarrow{\ \mu\ } & \mathfrak{g}^\star \\ \big\downarrow {\scriptstyle g\circ} & & \big\downarrow {\scriptstyle \mathrm{Ad}^\star_{g^{-1}}\circ} \\ M & \xrightarrow{\ \mu\ } & \mathfrak{g}^\star \end{array}$$

which means that $\mu(g \circ m) = \mathrm{Ad}^\star_{g^{-1}} \circ \mu(m)$, where Ad stands for the coadjoint action of Lie group $G$ on the $\mathfrak{g}^\star$. The thing is that in general case, the map $h : \mathfrak{g} \to \mathcal{C}(M)$ is not a Lie algebra homomorphism, but the equivariance of the moment map gives us this property.

**Theorem 3.3.18.** *If $\mu$ is an equivariant moment map from Hamiltonian $G$-space $M$ to Lie coalgebra $\mathfrak{g}^\star$ then the comoment map ($\mu^\star = h : \mathfrak{g} \to \mathcal{C}(M)$) is a Lie algebra homomorphism, i.e.*

$$h([X, Y]) = \{h(X), h(Y)\}.$$

**Remark 3.3.19.** *The map $h$ may be defined up to the addition of constant to the Hamiltonian $h_\xi$. This choice is not arbitrary since we want the Jacobi identity for the image of $h$, which requires this constant to be a 2-cocycle.*

**Example 3.3.20.** Consider $\mathbb{C}^n$ and with $U(1)^{\times n}$ action

$$(t_1, t_2, \ldots t_n) \cdot (z_1, z_2, \ldots z_n) = (t_1 z_1, t_2 z_2, \ldots t_n z_n), \quad t_i \in U(1).$$

This action is symplectic with the following moment map

$$\mu(z) = \sum_{i=1}^{n} |z_i|^2.$$

### 3.3.6 Lie–Poisson bracket

There is a way to define a Poisson structure on $\mathfrak{g}^\star$, which means that the space of functions on $\mathfrak{g}^\star$ is equipped with Poisson bracket. This Poisson bracket is a tautological lift of the initial Lie algebra structure to the coordinatee ring of $\mathfrak{g}^\star$. Consider a Taylor expansion of a function

$$f : \mathfrak{g}^\star \to \mathbb{C}, \quad \Theta \in \mathfrak{g}^\star, \quad f|_\Theta = f(\Theta) + df(\Theta) + \dots$$

Since $df$ is a linear function on $\mathfrak{g}^\star$, there is a canonical way to embed it into $\mathfrak{g}$. Then Lie–Poisson bracket defined as follows

$$\begin{gathered} \{\} : C^\infty(\mathfrak{g}^\star) \times C^\infty(\mathfrak{g}^\star) \to C^\infty(\mathfrak{g}^\star) \\ \{f, g\}(\Theta) = \langle \Theta, [df, dg]|_\Theta \rangle \end{gathered} \tag{3.3.6}$$

Here $|_\Theta$ underlines that differentials are taken from the expansion at $\Theta$.

**Example 3.3.21.** Consider $\mathfrak{gl}_n^\star$. For each basis element $E_{ij}$ we associate a linear function

$$f_{ij}(A) = \langle E_{ij} A \rangle = \mathrm{Tr}(E_{ij} A).$$

The Lie–Poisson bracket is given by

$$\{f_{ij}, f_{lk}\} = \delta_{jl} f_{ik} - \delta_{ik} f_{lj}$$

This bracket is degenerated since for any function $g$ we have

$$\{g, \det ||f_{ij}||\} = 0.$$

More generally, let $\mathfrak{g}$ be spanned by $X_1, X_2 \dots X_n$ as a vector space. The Lie algebra structure is given by the structure constants

$$[X_\alpha, X_\beta] = \sum_{i=1}^{n} C^{\gamma}_{\alpha\beta} X_\gamma.$$

Now let $F_\alpha \in C^\infty(\mathfrak{g}^\star)$ is given by

$$F_\alpha(\Theta) = \langle \Theta, X_\alpha \rangle$$

Then the Lie-Poisson bracket given by

$$\{F_\alpha, F_\beta\} = \sum_{i=1}^{n} C^{\gamma}_{\alpha\beta} F_\gamma.$$

Therefore it define a Lie algebra homomorphism.

This Poisson structure for the dual of the Lie algebra is always degenerated (at least because not every Lie Algebra is even dimensional). The question is what are the Casimirs of the Lie-Poisson bracket? The answer is contained in the following theorem by Kostant, Kirillov and Souriau.

**Theorem 3.3.22.** *Invariant functions of the co-adjoint action* $\mathrm{ad}^\star$ *of the group* $G$ *are Casimirs of the Lie-Poisson bracket.*

**Corollary 8.** *The co-adjoint orbits are symplectic leaves of the Lie-Poisson bracket.*

**Example 3.3.23.** For the Lie algebra $\mathfrak{sl}_2$, the dual $\mathfrak{sl}_2^\star$ is isomorphic to algebra itself. We choose following basis

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

for $\mathfrak{sl}_2$ so each element $A$ of $\mathfrak{sl}_2^\star$ is given as a linear combination - $A = A_h h + A_e e + A_f f$. And we have that

$$F_h(A) = 2A_h, \quad F_e(A) = A_f, \quad F_h(A) = A_e.$$

This gives an opportunity to endow sturucture ring of $\mathfrak{g}^\star$ which is $\mathbb{C}[A_h, A_f, A_e]$ with the stucture of the Poisson algebra. Indeed, we treat coefficients $A_h, A_f$ and $A_e$ as a coordinates in $\mathfrak{g}^\star$, i.e. we associate these variables with the generators of the structure ring of $\mathfrak{g}^\star$ which is

$\mathbb{C}[A_h, A_f, A_e]$. On the other hand, we introduce Poisson algebra structure via

$$\begin{aligned}
\{A_h, A_e\} =& \{F_h/2, F_f\}(A) = F_f(A) = A_e \\
\{A_h, A_f\} =& \{F_h/2, F_e\}(A) = -F_e(A) = -A_f \\
\{A_f, A_e\} =& \{F_e, F_f\}(A) = F_h(A) = 2A_h.
\end{aligned} \tag{3.3.7}$$

The invariant functions are traces of powers, the only non-trivial independent one is $\mathrm{Tr}(A^2)$. Since that the co-adjoint orbits are affine conics given by the equation

$$\frac{1}{2}\,\mathrm{Tr}(A^2) = A_h^2 - A_e A_f = \theta,$$

where $\theta$ is a constant. For a generic non-zero value of $\theta$, the co-adjoint orbits are smooth affine varieties. In the case when $\theta = 0$, there is a singularity which corresponds to trivial element of $\mathfrak{sl}_2^\star$. The obtained symplectic manifold allows an explicit bi-rational Darboux parametrization (since any smooth conic is a rational curve), which is given by

$$A_h = pq - \theta, \quad A_e = -p(pq - 2\theta), \quad A_f = q, \quad \omega = dp \wedge dq.$$

There is another useful definition of the Lie-Poisson bracket that will be used in this thesis

### 3.3.7 Symplectic reduction

The moment map allows to use a symmetry to reduce the degrees of freedom of the initial dynamical system. This procedure is called *symplectic reduction.* One the central theorems regarding symplectic reduction is the following Marsden-Weinstein theorem

**Theorem 3.3.24.** *(Marsden-Weinstein [67]) Let $G$ be a compact Lie group. Let $(M, \omega, G)$ be a Hamiltonian $G$-space with moment map $\mu$. Define $i : \mu^{-1}(0) \to M$ to be the inclusion map. Assume that $G$ acts freely and properly on $\mu^{-1}(0)$. Then*

- *the orbit space $M_{\mathrm{red}} = \mu^{-1}(0)/G$ is a manifold,*
- *$\pi : \mu^{-1}(0) \to M_{\mathrm{red}}$ is a principal $G$-bundle, where $\pi$ is a quotient map,*
- *there is a symplectic form $\omega_{\mathrm{red}}$ on $M_{\mathrm{red}}$ satisfying $i_\star\omega = \pi_\star\omega_{\mathrm{red}}$.*

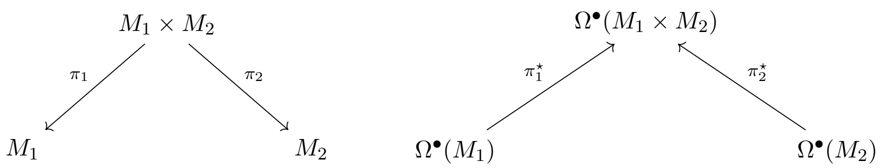
**Definition 3.3.25.** The symplectic manifold $(M_{\text{red}}, \omega_{\text{red}})$ is called a Marsden-Weinstein reduction of $(M, \omega)$ with the group $G$ and moment map $\mu$. We use the following notation for symplectic quotient

$$(M_{\text{red}}, \omega_{\text{red}}) = M /\!\!/ G := \mu^{-1}(0)/G. \tag{3.3.8}$$

The choice of zero-level of the moment map is crucial for the Marsden-Weinstein quotient. Indeed, since the stabilizer of zero coincides with the whole Lie group, the reduced space dimension is minimal. However, a lot of different interesting examples requires to consider a non-zero level set of the moment map. There is a way to fix this problem and to consider a non-trivial level set of the moment map. To put this case into the Marsden-Weinstein theory we have to extend the phase space in the following way. Let $(M_1, \omega_1, G)$ and $(M_2, \omega_2, G)$ are Hamiltonian $G$-spaces with the moment maps $\mu_1$ and $\mu_2$. Consider the direct product $M_1 \times M_2$ as a symplectic space with the diagonal $G$-action, i.e.

$$\begin{array}{rcl} G \times (M_1 \times M_2) & \to & M_1 \times M_2 \\ \phi_g \circ (m_1, m_2) & = & (\phi_g^{(1)}(m_1), \phi_g^{(2)}(m_2)) \end{array}$$

where $\phi_g^{(i)}$ is action of $G$ on $M_i$. From that point we use the following notation for any group action $\phi_g(p) = g \circ p$. Now we want to construct symplectic form on the space $M_1 \times M_2 = M$ using pullbacks of symplectic forms $\omega_1$ and $\omega_2$. Let $\Omega^\bullet(M_i)$ be a set of the all globally defined differential forms on the manifold $M_i$, then




Consider the following 2-form on $M$

$$\omega = \lambda_1 \pi_1^\star \omega_1 + \lambda_2 \pi_2^\star \omega_2 = \lambda_1 \widehat{\omega}_1 + \lambda_2 \widehat{\omega}_2 \tag{3.3.9}$$

where $\widehat{\omega}_i$ is just a notation for the pull-back.

**Proposition 3.3.26.** *The differential form (3.3.9) defines a symplectic form on $M$ if* $(\lambda_1, \lambda_2) \in (\mathbb{R}^\star)^2$

*Proof.* The closedness of (3.3.9) is obvious, the condition on $\lambda$ provides the non-degeneracy. $\square$

It is also easy to see, that $(M, \omega)$ is symplectic $G$-space with moment map given by

$$\mu(m_1, m_2) = \lambda_1 \mu_1(m_1) + \lambda_2 \mu_2(m_2).$$

The Marsden-Weinstein reduction in that case takes form

$$M /\!\!/ G = \mu^{-1}(0)/G$$

and $\mu^{-1}(0)$ is the locus in $M$ given by equation

$$\mu_1(m_1) = -\frac{\lambda_2}{\lambda_1}\mu_2(m_2).$$

Let us consider the following diagrams

$$M \xleftarrow{\ i\ } \mu^{-1}(0) \xrightarrow{\ p\ } M /\!\!/ G \qquad \Omega^\bullet(M) \xrightarrow{\ i^\star\ } \Omega^\bullet \mu^{-1}(0) \xleftarrow{\ p^\star\ } \Omega^\bullet(M /\!\!/ G),$$

where $p$ is the quotient map and $i$ is the inclusion map. According to the Marsden-Weinstein theorem symplectic form $\omega_{\mathrm{red}}$ on the space $M /\!\!/ G$ is uniquely defined and the following relation holds

$$i^\star \omega = p^\star \omega_{\mathrm{red}}.$$

Using this construction we now handle the non-zero level reduction for a symplectic $G$-manifold $M$ and the co-adjoint orbit $\mathcal{O}$ of $G$-action on it's Lie coalgebra equipped with Kirillov-Konstant-Souriau symplectic structure. If we consider $M \times \mathcal{O}$ with symplectic form

$$\omega = \widehat{\omega}_1 + \widehat{\omega}_{KKS},$$

but the diagonal action is twisted in the following way

$$g \circ (x, \xi) = (g \circ x, g^{-1} \circ \xi) = (g \circ x, Ad_{g^{-1}}\xi),$$

in order to satisfy the equivariance property of the moment map. In such situation the Marsden-

Weinstenin reduction leads to

$$M \times \mathcal{O} /\!\!/ G = \mu^{-1}(0)/G = \{(x,\xi) \in M \times \mathcal{O} :\, \mu_1(x) = \xi\}/G, \tag{3.3.10}$$

because the moment map on the co-adjoint orbit is the identity map. The reduced space may be viewed as $M /\!\!/_{\mathcal{O}} G = \mu_1^{-1}(\mathcal{O})/G$. Here we provide an example of such reduction.

**Example 3.3.27.** Let $\mathfrak{g} \cong T_eG$ be a Lie algebra of the group $G$, consider $M = T^\star\mathfrak{g}$ which we identify with $\mathfrak{g} \oplus \mathfrak{g}$ using a Killing form on $\mathfrak{g}$. The natural symplectic form is given by

$$\omega = \mathrm{Tr}(\mathrm{d}P \wedge \mathrm{d}Q),$$

where $P$ and $Q$ are coordinates on the direct sum $\mathfrak{g} \oplus \mathfrak{g}$. If we consider the $Ad$-action of $G$ on $M$, the equivariant moment map takes form

$$\mu = [P, Q].$$

Let's choose the (co)-adjoint orbit $\mathcal{O}$ of the element $\mu_0 \in \mathfrak{g}^\star \cong \mathfrak{g}$. Let's consider symplectic quotient (3.3.10) at the point, where $Q$ is diagonal (this point is always on the orbit if $Q$ is the matrix of full rank with distinct eigenvalues; the transformation is just conjugation by the transition matrix $C$ of eigenbasis for $Q$). Resolving the moment map relation we get that

$$P_{ij} = p_i\delta_{ij} + (1-\delta_{ij})\frac{S_{ij}}{q_j - q_i}, \quad S = Ad_{C^{-1}} \circ \mu_0.$$

The symplectic form on the reduced space writes as

$$\omega = \sum_i dp_i \wedge dq_i + \mathrm{Tr}(\mu_0 dCC^{-1} \wedge dCC^{-1}).$$

The last term is nothing but the Kirillov-Konstant-Souriau symplectic form. If $\mathfrak{g} \cong \mathfrak{gl}_n$ then the Poisson brackets on the reduced space take form

$$\{p_i, p_j\} = \{q_i, q_j\} = \{p_i, S_{kl}\} = \{q_i, S_{kl}\} = 0, \quad \{p_i, q_j\} = \delta_{ij}, \quad \{S_{ij}, S_{kl}\} = \delta_{jk}S_{il} - \delta_{il}S_{kj}.$$

For example, such coordinates gives spin Calogero-Moser system [42] as a Hamiltonian reduction of the following free Hamiltonian

$$H = \mathrm{Tr}(P^2/2),$$

which writes in reduced coordinates as

$$H = \sum_{i=1}^{n} \frac{p_i^2}{2} - \sum_{i<j} \frac{S_{ij}S_{ji}}{(q_i - q_j)^2}.$$

We also have to mention that the requirement for the Lie group to be compact may be also omitted. Such generalization is known as Marsden-Weinstein-Meyer theorem

**Theorem 3.3.28.** *(Marsden-Weinstein-Meyer [68]) Let $G$ be a Lie group which acts on a symplectic manifold $(M, \omega)$ in a Hamiltonian way with the moment map $\mu : M \to \mathfrak{g}^\star$. Let $\eta \in \mathfrak{g}^\star$ be a fixed point of the co-adjoint action. If the action of $G$ on $\mu^{-1}(\eta)$ is free and proper, then*

- *The symplectic reduction $M_{\mathrm{red}} = M /\!\!/_\eta G := \mu^{-1}(\eta)/G$ is a smooth manifold,*
- *$\pi : \mu^{-1}(\eta) \to M_{\mathrm{red}}$ is a principal $G$ bundle,*
- *There exists unique symplectic form $\omega_{red}$ on $M_{\mathrm{red}}$ satisfying $i_\star \omega = \pi_\star \omega_{red}$, where $i : \mu^{-1}(\mu) \to M$ is embedding of the level set of the moment map to initial symplectic manifold.*

In this thesis we mostly will deal with reduction with respect to the non-compact Lie group (usually $SL_n(\mathbb{C})$). During the further text we assume that we use Marsden-Weinstein-Meyer theorem to justify obtained results.

CHAPTER 4

# HAMILTONIAN DESCRIPTION OF THE SCHLESINGER EQUATIONS

## 4.1 Takiff Algebras

The Takiff algebra $\hat{\mathfrak{g}}_r$ of the Lie algebra $\mathfrak{g}$ is the Lie algebra of polynomials of fixed degree $r$ in the auxiliary variable $z \in \mathbb{C}$ with the following Lie bracket

$$\left[\sum_{i=0}^{r} A_i z^i, \sum_{j=0}^{r} B_j z^j\right] = \sum_{i=0}^{r}\left(\sum_{j=0}^{i}[A_i, B_{i-j}]\right) z^i. \tag{4.1.1}$$

**Definition 4.1.1.** [87, 77] The Takiff algebra $\hat{\mathfrak{g}}_r$ is a double quotient of the loop algebra $\mathfrak{g}[z, z^{-1}]$ by the Lie algebra ideals $\mathfrak{g}[z^{-1}]$ and $z^{r+1}\mathfrak{g}[z]$, i.e

$$\hat{\mathfrak{g}}_r = (\mathfrak{g}[z, z^{-1}]/\mathfrak{g}[z^{-1}])/z^{r+1}\mathfrak{g}[z].$$

Because the Takiff algebras are deeply connected with the loop algebras, we will also call such algebras as the truncated loop algebras or truncated current algebras. The variable $z$ is usually called the *spectral parameter* and, as we will illustrate here below, it's degrees induce a grading on the Takiff algebra. In the case when $\mathfrak{g}$ has an invariant non-degenerate bi-linear form, we may define the co-algebra $\hat{\mathfrak{g}}_r^\star$ in the following way

$$\hat{\mathfrak{g}}_r^\star = \mathfrak{g}[z]^-/z^{-(r+1)-1}\mathfrak{g}[z]^- = \left\{A(z) \,\middle|\, A(z) = \frac{A_r}{z^{r+1}} \cdots + \frac{A_0}{z},\ A_i \in \mathfrak{g}\right\}.$$

The pairing between $\hat{\mathfrak{g}}_r$ and $\hat{\mathfrak{g}}_r^\star$ is given by the residue formula

$$\langle A, B\rangle = \frac{1}{2\pi i}\oint\limits_{S^1} \mathrm{Tr}(AB)dz = \sum_{i=0}^{r} \mathrm{Tr}A_i B_i. \tag{4.1.2}$$

Let us assume that the Lie algebra $\mathfrak{g}$ has a basis $X_1, X_2, \dots X_n$ such that

$$[X_i, X_j] = C_{ij}^k X_k, \quad \langle X_i, X_j\rangle = \delta_{ij},$$

then for the truncated loop algebra $\hat{\mathfrak{g}}_r$ we have the following basis and structure equations

$$X_{\alpha,i} = X_i z^\alpha, \quad [X_{i,\alpha}, X_{j,\beta}] = \begin{cases} C_{ij}^k X_{k,\alpha+\beta}, & \alpha+\beta \le r \\ 0 & \alpha+\beta > r. \end{cases}$$

For the dual algebra $\mathfrak{g}_r^\star$ we use the following basis

$$X^{\alpha,i} = X^i z^{-\alpha-1}, \quad \langle X^i, X_j\rangle = \delta_{ij},$$

so the pairing for the truncated loop algebra is given by

$$\langle X^{i,\alpha}, X_{j,\beta}\rangle = \delta_{\alpha\beta}\langle X^i, X_j\rangle = \delta_{\alpha\beta}\delta_{ij}.$$

The details about Takiff algebras or truncated current algebras and it's standard Lie–Poisson bracket may be found in [31] (see part 2, chap. 4 §1). In the next sub-section we recall the essentials of this construction.

### 4.1.1 Standard Lie–Poisson bracket for the Takiff algebras

The coadjoint orbits $\mathcal{O}^\star$ are symplectic leaves of the standard Lie–Poisson structure on $\mathfrak{g}^\star$. The vector fields on $\mathcal{O}^\star$ may be identified with the elements of Lie algebra $\mathfrak{g}$ and the symplectic form takes form

$$\omega_{\mathrm{KKS}}(X, Y)(L) = -\langle L, [X, Y]\rangle.$$

Following [31], we now describe the standard Lie–Poisson structure on the dual $\hat{\mathfrak{g}}_n^\star$ of the Takiff algebra. Let's consider the following element of the coadjoint orbit

$$A = \sum_{\alpha=1}^{r} \sum_{i} A_{\alpha,i} X^{\alpha,i} \in \hat{\mathfrak{g}}_n^\star,$$

The coefficients $A_{\alpha,i}$ are functions on the coadjoint orbit, with $\mathrm{d}A_{\alpha,i} = X_{i,\alpha}$ so that the standard Lie–Poisson bracket is given by

$$\{A_{\alpha,i}, A_{\beta,j}\} = -\langle A, [X_{i,\alpha}, X_{j,\beta}]\rangle = -\langle A, C_{ij}^k X_k z^{\alpha+\beta}\rangle = \begin{cases} -C_{ij}^k A_{\alpha+\beta,k}, & \alpha+\beta \leq r \\ 0 & \alpha+\beta > r. \end{cases} \tag{4.1.3}$$

This is a graded Poisson structure of degree 1, and the Takiff co-algebra inherits the grading:

$$\hat{\mathfrak{g}}_r^\star := \bigoplus_{i=0}^{r} \hat{\mathfrak{g}}_r^{\star,i}, \quad \{\hat{\mathfrak{g}}_r^{\star,i}, \hat{\mathfrak{g}}_r^{\star,j}\} \subseteq \hat{\mathfrak{g}}_r^{\star,i+j},$$

where $\hat{\mathfrak{g}}_r^{\star,i} = \left\{ A = \frac{A_i}{z^{i+1}} \,\middle|\, A_i \in \mathfrak{g}^\star \right\}$. The same grading is induced to the co-adjoint orbit $\hat{\mathcal{O}}_r^\star$.

**Example 4.1.2.** In the case when $\mathfrak{g}$ is $\mathfrak{gl}_\mathfrak{m}$ we have the following Poisson structure

$$\{(A_\alpha)_{ij}, (A_\beta)_{kl}\} = \begin{cases} (A_{\alpha+\beta})_{il}\delta_{jk} - (A_{\alpha+\beta})_{kj}\delta_{il} & \alpha+\beta \leq r \\ 0 & \alpha+\beta > r, \end{cases} \tag{4.1.4}$$

which may be written in the $r$-matrix form

$$\{A_\alpha \underset{,}{\otimes} A_\beta\} = \begin{cases} -[\Pi, A_{\alpha+\beta} \otimes \mathbb{I}] & \alpha+\beta \leq r \\ 0 & \alpha+\beta > r, \end{cases} \tag{4.1.5}$$

where $\{\underset{,}{\otimes}\}$ stands for Leningrad bracket, which is given by

$$\{F \underset{,}{\otimes} G\}^\alpha_\beta = \{F_\alpha, G_\beta\}.$$

Takiff algebras, in a nutshell, are very special finite graded Lie algebras. The main feature of such Lie algebras is that there is a way to do computations without using any representation, but working with Lie algebra valued polynomials. Such constructions arise widely in the theory of integrable systems with spectral parameter. Instead of using quite universal, but extremely rough approach which treats Lax matrix as an element of the loop algebra, sometimes it is useful to consider the Lax matrix as evaluation morphism from a direct product of the co-adjoint orbits of Takiff algebras to the loop algebra. Moreover, Takiff algebras also include the Lie algebras as a particular case. Here we mean that $\hat{\mathfrak{g}}_0$ is isomorphic to $\mathfrak{g}$ (isomorphism is obvious).

## 4.2 Lifted Darboux coordinates

One of the first attempts to study the general isomonodromic and isospectral systems with spectral parameter as the Hamiltonian systems was made by Adams, Harnad, Hurtubise and Previato in the series of works [1, 2, 3]. In these papers the authors heavily use moment map theory and introduced a way to write down isomonodromic and isospectral systems as unreduced Hamiltonian systems on the cotangent bundle to $\oplus_{i=1}^{n}\mathfrak{gl}_m$.Here *unreduced* means that such description inherits a great number of symmetries. However, their approach allows to work with Darboux coordinates initially, which gives an opportunity to do explicit computations in a sense of the classical mechanics.

The concept introduced by Adams, Harnad, Hurtubise and Previato may be formulated disregarding integrable systems theory. In a nutshell, the concept of *lifted Darboux coordinates* gives a Darboux parametrisation of the Lie-Poisson bracket. The approach introduced in original papers [3] doesn't cover a case of Takiff algebras, so it is valid only for Fuchsian/Gaudin systems. In order to cover irregular systems, authors used Laplace transform from Dubrovin type systems

$$\frac{d}{dz}Y = \left(U + \frac{V}{z}\right)Y,$$

which deformation may be formulated in terms of the co-adjoint orbits of Lie algebra $\mathfrak{so}_n$. In this thesis we expand the lifted Darboux coordinates to the case of Takiff algebras in order to give full description for irregular isomonodromic deformations obtained as a confluence of the simple poles. In this section we do a review of the results by Adams, Harnad, Hurtubise and Previato for the case of the Lie algebra $\mathfrak{gl}_n$. One of the key results of this theory may be formulated in the following proposition

**Proposition 4.2.1.** *[3] Consider the canonical symplectic structure on* $T^\star\mathfrak{gl}_m$*:*

$$\omega = \operatorname{Tr}(\mathrm{d}P \wedge \mathrm{d}Q) = \sum_{i,j=1}^{m} \mathrm{d}P_{ij} \wedge \mathrm{d}Q_{ji}. \tag{4.2.1}$$

*Let*

$$A = QP, \tag{4.2.2}$$

*where we use the ring structure of* $\mathfrak{gl}_m$ *to justify the multiplication of the* $Q$ *and* $P$*. Then* $A$ *satisfies the standard Lie–Poisson bracket for* $\mathfrak{gl}_m$*. Moreover, the the entries of the matrix*

$$\Lambda = PQ \tag{4.2.3}$$

*commutes with the entries of* (4.2.2) *with respect to the Poisson bracket induced by* (4.2.1)

*Proof.* The Poisson bracket which corresponds to the symplectic form in (4.2.1) may be written in the following way

$$\{P \underset{,}{\otimes} Q\} = -\Pi, \quad \{P \underset{,}{\otimes} P\} = \{Q \underset{,}{\otimes} Q\} = 0,$$

where $\Pi$ is a permutation matrix. Computing the Leningrad bracket for $A$ with itself, we get

$$\{A \underset{,}{\otimes} A\} = \{QP \underset{,}{\otimes} QP\} = (Q \otimes \mathbb{I})\{P \underset{,}{\otimes} Q\}(\mathbb{I} \otimes P) + (\mathbb{I} \otimes Q)\{Q \underset{,}{\otimes} P\}(P \otimes \mathbb{I}) =$$
$$= (Q \otimes \mathbb{I})\Pi(\mathbb{I} \otimes P) - (\mathbb{I} \otimes Q)\Pi(P \otimes \mathbb{I}) = [\Pi, \mathbb{I} \otimes QP] = [\Pi, \mathbb{I} \otimes A]$$

As we wanted to prove. Now let us compute bracket between $\Lambda$ and $A$

$$\{A \underset{,}{\otimes} \Lambda\} = \{QP \underset{,}{\otimes} PQ\} = \{Q \underset{,}{\otimes} P\}(P \otimes \mathbb{I})(\mathbb{I} \otimes Q) + (Q \otimes \mathbb{I})(\mathbb{I} \otimes P)\{P \underset{,}{\otimes} Q\}$$
$$= \Pi(P \otimes \mathbb{I})(\mathbb{I} \otimes Q) - (Q \otimes \mathbb{I})(\mathbb{I} \otimes P)\Pi = \Pi(P \otimes \mathbb{I})(\mathbb{I} \otimes Q) - \Pi(P \otimes \mathbb{I})(\mathbb{I} \otimes Q) = 0.$$

□

**Definition 4.2.2.** We call $T^\star\mathfrak{gl}_m$ *extended phase space* and the canonical coordinates $P, Q$ *lifted Darboux coordinates.*

The provided straightforward proof has a very nice interpretation via commuting Hamiltonian group actions. The space $T^\star\mathfrak{gl}_m \simeq \mathfrak{gl}_m \times \mathfrak{gl}_m$ carries two natural commuting symplectic actions of $GL_m$ which we call *inner and outer*:

$$g \underset{\text{inner}}{\times} (P,Q) = (gP, Qg^{-1}), \qquad h \underset{\text{outer}}{\times} (P,Q) = (Ph, h^{-1}Q), \qquad h, g \in GL_m. \tag{4.2.4}$$

**Lemma 4.2.3.** *These inner and outer actions are Hamiltonian with equivariant moment maps given by*

$$\begin{array}{rlrl} \mu_{\text{inner}}: & T^\star\mathfrak{gl}_m \to \mathcal{O}^\star(\mathfrak{gl}_m) & \mu_{\text{outer}}: & T^\star\mathfrak{gl}_m \to \mathcal{O}^\star(\mathfrak{gl}_m) \\ & (P,Q) \mapsto \Lambda = PQ & & (P,Q) \mapsto A = QP \end{array}. \tag{4.2.5}$$

This lemma makes computations trivial. Indeed, equivariant moment map is a symplectomorphism to the co-adjoint orbit, which proves that symplectic form on $T^\star\mathfrak{gl}_m$ induces Lie-Poisson structure on $A$ and $\Lambda$. Moreover, moment maps of the commuting actions Poisson commute, which proves that $\{\Lambda \underset{,}{\otimes} A\} = 0$.

Let us restrict to the open affine subset of $T^\star\mathfrak{gl}_m$ where at least one of the two matrices $Q$ and $P$ is invertible. For example $Q$. Then, resolving the moment map for $\Lambda$ we obtain

$$P = \Lambda Q^{-1}, \quad A = QP = Q\Lambda Q^{-1}.$$

As a consequence, $A$ and $\Lambda$ belong to the same co-adjoint orbit. Since the inner and outer actions commute, $A$ is invariant under the inner action, while $\Lambda$ is invariant under the outer action. Therefore we use the inner group action to fix $\Lambda$ in a Jordan normal form without changing $A$. In other words, we take the Jordan normal form $\Lambda_0$ of $A$ and select $\Lambda = \Lambda_0$. This gives

$$T^\star\mathfrak{gl}_m \underset{\Lambda_0}{/\!/} G = \mu^{-1}_{\text{inner}}(\Lambda_0)/G,$$

here we denote by $\underset{\Lambda_0}{/\!/}$ the quotient with respect to the inner action of $GL_m$ on $T^\star\mathfrak{gl}_m$. We may resume these results in the following:

**Lemma 4.2.4.** *The map*

$$\begin{array}{ccc} T^\star\mathfrak{gl}_m \underset{\Lambda_0}{/\!/} G_{\text{inner}} & \to & \mathcal{O}^\star \\ (Q,P) & \mapsto & A := QP \end{array}$$

*is a rational symplectomorphism and the Jordan normal form $\Lambda_0$ of $A$ is given by*

$$\Lambda_0 = PQ.$$

**Remark 4.2.5.** *When $A$ is a full-rank matrix, both $P$ and $Q$ must be invertible. So we may embed $(P, Q)$ into the group $GL_m$ and $P$ and $Q$ can be seen as left and right eigenvector matrices for the matrix $A$. In the case when $A$ may be diagonilized, the action of the Cartan torus (i.e. the stabilizer of $\Lambda$) leads to a well known fact from linear algebra - the eigenvectors are defined up to multiplication by non-zero constant. When $A$ is not a full-rank matrix, we may choose $Q$ to be an invertible matrix (so it may be viewed as an element of $GL_n$). Then the rank of $P$ must equal to the rank of $A$. Then the moment map $\Lambda$ will inherit the rank of $A$ automatically. Since $P$ in this case not invertible, the reduced coordinates take the form*

$$P = \Lambda Q^{-1}, \quad A = Q\Lambda Q^{-1}, \quad \det \Lambda = \det A = \det P = 0.$$

*This means that instead of considering $T^\star \mathfrak{gl}_m$ as lifted space, we could take $T^\star GL_m \ni (Q, \Lambda Q^{-1})$. Such consideration is closely related to the approach introduced in [11]. However, this approach is not very useful for our purposes, since we wish to work with polynomial unreduced parametrisation, rather then rational.*

**Remark 4.2.6.** *In the case when we consider $\mathfrak{g}$ to be any reductive Lie algebra and $A \in \mathfrak{g}^*$, then we expect that Lemma 4.3.3 is still valid if we fix the value $\Lambda$ of the moment map in $\mathfrak{g}^*$ and $Q$ and $P$ (or just $Q$ in the case of degenerate orbit) as the elements from the corresponding Lie group $G$.*

## 4.3 Hamiltonian approach to the isomonodromic deformations of Fuchsian systems

In this section we summarize known facts about Fuchsian systems. The aim of this section is to review the Poisson and symplectic aspects of the deformation equations for connections over the $n+1$-holed sphere with simple poles at punctures. Starting from the linear system with simple poles at $\lambda = u_1, \ldots, u_n, \infty$,

$$\frac{d}{d\lambda}\Psi = \sum_{i=1}^{n} \frac{A^{(i)}}{\lambda - u_i}\Psi, \quad \lambda \in \Sigma_{0,n+1}, \quad u_i \neq u_j, \quad \Psi \in G, \quad A^{(i)} \in \mathfrak{g} := T_e G, \tag{4.3.1}$$

we consider the following matrix-valued 1-form

$$\Omega = (\mathrm{d}_u \Psi)\Psi^{-1}, \quad \mathrm{d}_u \Psi := \sum_i \partial_{u_i} \Psi \mathrm{d}u_i. \tag{4.3.2}$$

Since we consider only isomonodromic deformations, i.e. $\mathrm{d}M_i = 0$, the form $\Omega$ is a single-valued meromorphic 1-form with possible singularities at $u_i$'s. Using the local solutions of (4.3.1) in the neighbourhood of the poles $u_i$'s and applying Liouville theorem this form may be written as

$$\Omega = -\sum_i \frac{A^{(i)}}{\lambda - u_i} \mathrm{d}u_i. \tag{4.3.3}$$

The compatibility condition for (4.3.1) and (4.3.2) (zero-curvature equation)

$$\mathrm{d}_u A - \frac{d}{d\lambda}\Omega + [A, \Omega] = 0, \tag{4.3.4}$$

gives the Schlesinger equations

$$d_u A^{(i)} = \sum_{j \neq i} [A^{(i)}, A^{(j)}] \frac{du_i - du_j}{u_i - u_j}. \tag{4.3.5}$$

Schlesinger equations have

### 4.3.1 Phase space

The Schlesinger equations are Hamiltonian, with the natural phase space given by the direct product of co-adjoint orbits which are symplectic leaves of the standard Lie–Poisson bracket:

$$\left(A^{(1)}, A^{(2)}, \ldots A^{(n)}\right) \in \mathcal{O}_1^\star \times \mathcal{O}_2^\star \times \cdots \times \mathcal{O}_n^\star.$$

In case when $\mathfrak{g}$ is a Lie algebra with a non-degenerate bi-linear form (i.e. Killing form), we may identify the co-adjoint orbits with the adjoint orbits. The Poisson brackets then may be written as

$$\left\{A^{(i)} \underset{,}{\otimes} A^{(j)}\right\} = \delta_{ij}[\Pi, 1 \otimes A^{(i)}] \quad \Longleftrightarrow \quad \left\{A^{(i)}_\alpha, A^{(j)}_\beta\right\} = -\delta_{ij} \sum_\gamma \chi^\gamma_{\alpha\beta} A^{(i)}_\gamma \tag{4.3.6}$$

where the lower indices $\alpha, \beta$ and $\gamma$ correspond to the Lie co-algebra basis, $\chi^\gamma_{\alpha\beta}$ are the structure constants of the Lie algebra and $\Omega$ is a quadratic Casimir element. In the case of $\mathfrak{gl}_m$ it acts as a permutation operator, i.e.

$$\Pi(A \otimes B)\Pi = B \otimes A.$$

In the case of a Lie algebra which allows orthogonal with respect to Killing form basis $e_\alpha$, quadratic Casimir $\Pi$ writes as

$$\Pi = \sum_{\alpha \in I} e_\alpha \otimes e_\alpha, \quad \mathfrak{g} := \operatorname{span}\langle e_\alpha, \alpha \in I \rangle.$$

Such bracket may be rewritten as an $r$-matrix bracket for the connection, i.e.

$$\{A(\lambda) \underset{,}{\otimes} A(\mu)\} = \left[\frac{\Pi}{\lambda - \mu}, A(z) \otimes \mathbb{I} + \mathbb{I} \otimes A(\mu)\right]. \tag{4.3.7}$$

The isomonodromic Hamiltonians for the Schlesinger equations are

$$H_i = \operatorname*{Res}_{\lambda = u_i} \operatorname{Tr} \frac{A(\lambda)^2}{2} = \sum_{j \neq i} \frac{\operatorname{Tr}(A^{(i)} A^{(j)})}{u_i - u_j}. \tag{4.3.8}$$

The Schlesinger equations can be reduced, for example in the case of $n = 3$ $\mathfrak{sl}_2$ co-adjoint orbits, their reduction is the Painlevé VI equation which is non-autonomous Hamiltonian system with 1 degree of freedom.

In the case of any number $n$ of co-adjoint orbits, the fully reduced dimension can be computed using the spectral type technique introduced by Katz [59]. In case when $A^{(i)}$'s and $A^{(\infty)}$ are semi-simple, the spectral type approach gives the formula for the dimension of the fully reduced phase space as a function of the eigenvalues multiplicities of the residues [59]

$$N = 2 - (1 - n)m^2 - \sum_{i=1}^{n} \sum_{j=1}^{l_i} (m^i_j)^2 - \sum_{j=1}^{l_\infty} (m^\infty_j)^2, \tag{4.3.9}$$

where $l_i$ is a cardinality of the set of eigenvalues for the residue $A^{(i)}$ and $m^i_j$ is the multiplicity of the $j$-th eigenvalue of the residue $A^{(i)}$. The fully reduced systems may be seen as a reduction with respect to the additional Fuchs condition:

$$\sum_{i=1}^{n} A^{(i)} = -A^{(\infty)}. \tag{4.3.10}$$

On the other hand this relation may be viewed as a moment map of the gauge group action via constant matrix (i.e. gauge doesn't depend in $z$) and $A_\infty$ is a constant of motion for the Schlesinger equations.

From the point of view the symplectic reduction Katz formula may be rewritten in the following way

$$N = \sum_{i=1}^{n} \dim \mathcal{O}_i^\star - \dim G - \operatorname{stab} \mathcal{O}_\infty^\star, \tag{4.3.11}$$

where $\operatorname{stab} \mathcal{O}_\infty^\star$ is the dimension of the stabilizer for the Jordan form of the residue at $\infty$. When $A^{(\infty)}$ is the element of the co-adjoint orbit of the general form (regular), we have that $\operatorname{stab} \mathcal{O}_\infty^\star = \dim \mathfrak{h}$, so the formula simplifies to

$$N = \sum_{i=1}^{n} \dim \mathcal{O}_i^\star - \dim G - \dim \mathfrak{h}.$$

For example, in the case of the Painlevé VI equation we deal with the coadjoint orbits of the $\mathfrak{sl}_2(\mathbb{C})$. In the general situation formula (4.3.11) gives

$$N = 3 \cdot \dim \mathcal{O}_{\mathfrak{sl}_2} - \dim(SL_2) - \dim \mathfrak{h}_{\mathfrak{sl}_2} = 3 \cdot 2 - 3 - 1 = 2,$$

which is exactly the dimension of the phase space for the Painlevé VI equation. In some sense the multiplicity of the eigenvalues tells us that the Jordan form may be written as the tensor product of identity matrices of sizes corresponding to the the multiplicities. The stabilizer of such matrix is a set of the block diagonal matrices, so the dimension is greater then the dimension of the Cartan torus and finally we obtain the smaller phase space.

Our first goal is to describe this full reduction as a Hamiltonian reduction and a Marsden-Weinstein quotient. To this aim, we will need first to extend the phase space to $T^\star\mathfrak{gl}_m$ and show that the Darboux coordinates on this cotangent bundle reduced to the Kirillov-Kostant-

Souriau form on the co-adjoint orbits. We will then discuss how the invariants of the co-adjoint orbits correspond to moment maps with respect to different Hamiltonian group actions on the extended phase space.

### 4.3.2 Extended phase space and its Darboux coordinates

In this subsection, we start by working locally, namely we restrict to the case of a single co-adjoint orbit $\mathcal{O}^\star$ of $\mathfrak{gl}_m$ and identify $\mathfrak{gl}_m^\star$ with $\mathfrak{gl}_m$ via Killing form. In the last part of this subsection we extend to the product of $n$ co-adjoint orbits.

We consider $T^\star\mathfrak{gl}_m$ with the standard Darboux coordinates $(Q, P)$ and the canonical symplectic structure:

$$\omega = \mathrm{Tr}\left(\mathrm{d}P \wedge \mathrm{d}Q\right) = \sum_{i,j} \mathrm{d}P_{ij} \wedge \mathrm{d}Q_{ji}. \tag{4.3.12}$$

Following [3, 1, 2], we explain how to obtain the standard Lie–Poisson bracket (4.3.6) on $\mathfrak{g}^\star$ as Marsden–Weinstein reduction of the Poisson structure on $T^\star\mathfrak{gl}_m$. There is a direct way to see this reduction by a straightforward computation (see [53]), that, in a nutshell, coincides with the proposition 4.2.1.

**Definition 4.3.1.** We call $T^\star\mathfrak{gl}_m$ *extended phase space* and the canonical coordinates $P, Q$ *lifted Darboux coordinates.*

To restrict ourself to the co-adjoint orbit we have to fix invariants of the co–adjoint actions, i.e. the Jordan form of matrix $QP = A$. Such procedure leads to some additional non-linear equations for the entries of $Q$ and $P$, and there is no hope to derive the explicit symplectic structure on the co-adjoint orbit from such a perspective. Therefore, we follow the construction of [3] to obtain the co-adjoint orbits via Hamiltonian reduction.

The space $T^\star\mathfrak{gl}_m \simeq \mathfrak{gl}_m \times \mathfrak{gl}_m$ carries two natural commuting symplectic actions of $GL_m$ which we call *inner and outer*:

$$g \underset{\text{inner}}{\times} (P, Q) = (gP, Qg^{-1}), \qquad h \underset{\text{outer}}{\times} (P, Q) = (Ph, h^{-1}Q), \qquad h, g \in GL_m. \tag{4.3.13}$$

**Lemma 4.3.2.** *These inner and outer actions are Hamiltonian with equivariant moment maps* *given by*

$$\begin{array}{llll} \mu_{\text{inner}}: & T^\star\mathfrak{gl}_m \to \mathfrak{gl}_m^\star & \mu_{\text{outer}}: & T^\star\mathfrak{gl}_m \to \mathfrak{gl}_m^\star \\ & (P,Q) \mapsto \Lambda = PQ & & (P,Q) \mapsto A = QP \end{array}. \tag{4.3.14}$$

Let us restrict to the open affine subset of $T^\star\mathfrak{gl}_m$ where at least one of the two matrices $Q$ and $P$ is invertible. For example $Q$. Then, resolving the moment map for $\Lambda$ we obtain

$$P = \Lambda Q^{-1}, \quad A = QP = Q\Lambda Q^{-1}.$$

As a consequence, $A$ and $\Lambda$ belong to the same co-adjoint orbit.

Since the inner and outer actions commute, $A$ is invariant under the inner action, while $\Lambda$ is invariant under the outer action. Therefore we use the inner group action to fix $\Lambda$ in a Jordan normal form without changing $A$. In other words, we take the Jordan normal form $\Lambda_0$ of $A$ and select $\Lambda = \Lambda_0$. This gives

$$T^\star\mathfrak{gl}_m \underset{\Lambda_0}{/\!/} G = \mu_{\text{inner}}^{-1}(\Lambda_0)/G,$$

here we denote by $\underset{\Lambda_0}{/\!/}$ the quotient with respect to the inner action of $GL_m$ on $T^\star\mathfrak{gl}_m$. We may resume these results in the following:

**Lemma 4.3.3.** *The map*

$$\begin{array}{ccc} T^\star\mathfrak{gl}_m \underset{\Lambda_0}{/\!/} G_{\text{inner}} & \to & \mathcal{O}^\star \\ (Q,P) & \mapsto & A := QP \end{array}$$

*is a rational symplectomorphism and the Jordan normal form $\Lambda_0$ of $A$ is given by*

$$\Lambda_0 = PQ.$$

**Remark 4.3.4.** *When $A$ is a full-rank matrix, both $P$ and $Q$ must be invertible. So we may embed $(P,Q)$ into the group $GL_m$ and $P$ and $Q$ can be seen as left and right eigenvector matrices for the matrix $A$. In the case when $A$ may be diagonilized, the action of the Cartan torus (i.e. the stabilizer of $\Lambda$) leads to a well known fact from linear algebra - the eigenvectors are defined up to multiplication by non-zero constant. When $A$ is not a full-rank matrix, we may choose $Q$ to be an invertible matrix (so it may be viewed as an element of $GL_n$). Then the rank of $P$ must equal to the rank of $A$. The the moment map $\Lambda$ will inherit the rank of $A$ automatically.*

*Since $P$ in this case not invertible, the reduced coordinates take the form*

$$P = \Lambda Q^{-1}, \quad A = Q\Lambda Q^{-1}, \quad \det\Lambda = \det A = \det P = 0.$$

*This means that instead of considering $T^\star\mathfrak{gl}_m$ as lifted space, we could take $T^\star GL_m \ni (Q, \Lambda Q^{-1})$. Such consideration is closely related to the approach introduced in [11]. However, this approach is not very useful for our purposes, since we wish to work with polynomial unreduced parametrisation, rather then rational.*

**Remark 4.3.5.** *In the case when we consider $\mathfrak{g}$ to be any reductive Lie algebra and $A \in \mathfrak{g}^*$, then we expect that Lemma 4.3.3 is still valid if we fix the value $\Lambda$ of the moment map in $\mathfrak{g}^*$ and $Q$ and $P$ (or just $Q$ in the case of degenerate orbit) as the elements from the corresponding Lie group $G$.*

Let us now consider the case of the product of many co-adjoint orbits. Since the Poisson brackets (4.3.6) are *local,* namely the residues at different marked points commute, the facts we summarised so far easily extend to this case. Indeed, we can apply the above construction to the co-adjoint orbit at each pole of the Fuchsian system (except $\infty$) and define:

$$A^{(i)} = Q_i P_i.$$

In this case we have that inner and outer actions can be lifted to the direct sum of n copies $T^\star\mathfrak{gl}_m$ in a natural way

$$g \underset{\text{inner}}{\times} (P_1, P_2, \dots P_n, Q_1, Q_2, \dots Q_n) = (g_1 P_1, \dots g_n P_n, Q_1 g_1^{-1}, \dots Q_i g_i^{-1}, \dots Q_n g_n^{-1}), \quad g \in \underset{n}{\times} GL_m$$

$$h \underset{\text{outer}}{\times} (P_1, P_2, \dots P_n, Q_1, Q_2, \dots Q_n) = (P_1 h_1, \dots P_n h_n, h_1^{-1} Q_1, \dots h_i^{-1} Q_i, \dots h_n^{-1} Q_n), \quad h \in \underset{n}{\times} GL_m$$

and the lemma 4.3.2 repeats

**Lemma 4.3.6.** *These inner and outer actions are Hamiltonian with equivariant moment maps given by*

$$\mu_{\text{inner}}: \begin{array}{ccc} \bigoplus_n T^\star\mathfrak{gl}_m & \to & \bigoplus_n \mathfrak{gl}_m^\star \\ (P_1, \dots P_n; Q_1, \dots Q_n) & \to & (P_1 Q_1, P_2 Q_2, \dots P_n Q_n) \end{array}$$

$$\mu_{\text{outer}}: \quad \begin{array}{ccc} \bigoplus_n T^\star \mathfrak{gl}_m & \to & \bigoplus_n \mathfrak{gl}_m^\star \\ (P_1, \dots P_n; Q_1, \dots Q_n) & \to & (Q_1 P_1, Q_2 P_2, \dots Q_n P_n) \end{array}$$

*Proof.* Let us prove it for the inner action only. The vector field generated by the group action (via element $\xi = (\xi_1, \xi_2, ...\xi_n) \in \oplus_n \mathfrak{gl}_m$ is given by

$$X_\xi(P_i, Q_i) = \frac{d}{dt}(e^{-t\xi_i} P_i, Q_i e^{t\xi_i})\Big|_{t=0} = (-\xi_i P_i, Q_i \xi_i) = \sum_{i=1}^n \Big( \sum_{k,j} -(\xi_i P_i)_{kj} \frac{\partial}{\partial P_{i_{kj}}} + (Q_i \xi_i)_{kj} \frac{\partial}{\partial Q_{i_{kj}}} \Big).$$

Inserting $X_\xi$ into the symplectic form we obtain

$$\omega(X_\xi, \circ) = \sum_i^n \sum_{k,j} \Big[ -(\xi_i P_i)_{kj} dQ_{i_{jk}} - (Q_i \xi_i)_{kj} dP_{i_{jk}} \Big] = -\sum_i^n \mathrm{Tr}\left(\xi_i P_i dQ_i + Q_i \xi_i dP_i\right) = -\sum_i^n d\, \mathrm{Tr}\left(\xi_i P_i Q_i\right),$$

so the corresponding Hamiltonian is

$$h_\xi(m) = \langle \mu(m), \xi \rangle = \sum_i^n \mathrm{Tr}\left(\xi_i \mu(m)_i\right) = \mathrm{Tr}\left(\xi_i P_i Q_i\right)$$

where $m = (P_1, P_2, \dots P_n, Q_1, \dots Q_n)$. So the moment map is given by

$$\mu(m) = (P_1 Q_1, P_2 Q_2, \dots P_i Q_i, \dots P_n Q_n),$$

which is equivariant

$$\mu(g \circ m) = \left(g_1^{-1} P_1 Q_1 g_1, g_2^{-1} P_2 Q_2 g_2, \dots g_i^{-1} P_i Q_i g_i, \dots g_n^{-1} P_n Q_n g_n\right) = g^{-1} \mu(m) g = \mathrm{Ad}^\star_{g^{-1}}(\mu(m))$$

□

Then the following result is a straightforward computation

**Lemma 4.3.7.** *A Hamiltonian system on the phase space*

$$\mathcal{O}_1^\star \times \mathcal{O}_2^\star \times \dots \times \mathcal{O}_n^\star \ni \left(A^{(1)}, A^{(2)}, \dots A^{(n)}\right)$$

*can be lifted up to the extended phase space*

$$T^\star \mathfrak{gl}_m \times T^\star \mathfrak{gl}_m \times \dots \times T^\star \mathfrak{gl}_m \ni (Q_1, P_1, Q_2, P_2 \dots Q_n, P_n)$$

*with additional first integrals given by the moment maps of the inner group action*

$$\mu_{\text{inner}} := P_i Q_i = \Lambda^{(i)},$$

*where the inner group action is given by*

$$(g_1, g_2, \dots g_n) \underset{\text{inner}}{\times} (P_1, Q_1, P_2, Q_2, \dots P_n, Q_n) = (g_1 P_1, Q_1 g_1^{-1}, \dots g_i P_i, Q_i g_i^{-1}, \dots g_n P_n, Q_n g_n^{-1}).$$

*Moreover, if* $\Lambda_0^{(i)}$ *is the Jordan normal form of* $A^{(i)}$*, we can fix* $\Lambda^{(i)} = \Lambda_0^{(i)}$*.*

In particular, the Schlesinger Hamiltonians (4.3.8) can be lifted to the extended phase space $T^\star \mathfrak{gl}_m$ as follows

$$H_i = \sum_{j \neq i} \frac{\operatorname{Tr}(Q_i P_i Q_j P_j)}{u_i - u_j}, \tag{4.3.15}$$

and it can be checked directly that they Poisson commute with the moment maps of the inner group action.

### 4.3.3 Outer group action and the gauge group

We have seen that the inner group action allows us to restrict from $T^\star \mathfrak{gl}_m$ to $\mathcal{O}_1^\star \times \mathcal{O}_2^\star \times \cdots \times \mathcal{O}_n^\star$. Now we consider the *outer group action* that will allow us to reduce further. This is given by

$$(g_1, g_2, \dots g_n) \underset{\text{outer}}{\times} (P_1, Q_1, P_2, Q_2, \dots P_n, Q_n) = (P_1 g_1, g_1^{-1} Q_1, \dots P_i g_i, g_i^{-1} Q_i, \dots P_n g_n, g_n^{-1} Q_n)$$

and is also Hamiltonian (see Lemma 4.3.2).

Because inner and outer group actions commute, their moment maps Poisson commute too. However, the Schlesinger Hamiltonians are generally not invariant under outer action, unless the outer action is restricted to be a diagonal action, i.e.

$$g_1 = g_2 = \cdots = g_n = g.$$

In this case, the outer action reduces to the standard $GL_m$-action on $\mathcal{O}_1^\star \times \mathcal{O}_2^\star \times \cdots \times \mathcal{O}_n^\star$, or equivalently to the constant gauge group action:

$$g \underset{\text{outer}}{\times} A = \sum \frac{g^{-1} A^{(i)} g}{z - u_i}.$$

The moment map of such diagonal action is

$$\sum_{i=1}^{n} Q_i P_i = \sum_{i=1}^{n} A^{(i)} = -A^{(\infty)}, \tag{4.3.16}$$

which is the Fuchs relation.

In order to describe the reduction procedure induced by the outer diagonal action in terms of the Marsden-Weinstein reduction, following Proposition 2.2.7 of [4] (see also [47]) we further extend the phase space by adding another copy of $T^{\star}\mathfrak{gl}_m$:

$$(P_1, Q_1 \dots P_n, Q_n; P_\infty, Q_\infty) \in \bigoplus_{i=1}^{n+1} T^{\star}\mathfrak{gl}_m, \quad \omega = \sum_{i=1}^{n} \mathrm{Trd}P_i \wedge \mathrm{d}Q_i + \mathrm{Trd}P_\infty \wedge \mathrm{d}Q_\infty, \tag{4.3.17}$$

with the outer group action of the form

$$g \underset{\text{outer}}{\times} (P_1, Q_1 \dots P_n, Q_n; P_\infty, Q_\infty) = (P_1 g, g^{-1} Q_1, \dots P_i g, g^{-1} Q_i, \dots P_n g, g^{-1} Q_n; P_\infty g, g^{-1} Q_\infty).$$

The corresponded extended space which is given by the reduction with respect to the inner group action takes form

$$\left(A^{(1)}, A^{(2)}, \dots A^{(n)}; A^{(\infty)}\right) \in \mathcal{O}_1^{\star} \times \mathcal{O}_2^{\star} \times \dots \mathcal{O}_n^{\star} \times \mathcal{O}_\infty^{\star}.$$

The reduction with respect to the relation (4.3.16) on the extended phase space may be viewed as the Marsden-Weinstein quotient

$$\bigoplus_{i=1}^{n+1} T^{\star}\mathfrak{gl}_m /\!\!/ G = \mu^{-1}(0)/G, \quad \mu = \sum_{i=1}^{n} Q_i P_i + Q_\infty P_\infty.$$

which corresponds to the Fuchsian relation on the reduced with respect to the inner group action phase space.

Finally, the fully reduced phase space then has form

$$M \simeq \mathcal{O}_1^\star \times \mathcal{O}_2^\star \times \ldots \mathcal{O}_n^\star \times \mathcal{O}_\infty^\star /\!\!/ G \simeq \bigoplus_{i=1}^{n+1} \left( T^\star \mathfrak{gl}_m \underset{\Lambda^{(i)}}{/\!\!/} G \right) /\!\!/ G.$$

Moreover, the Hamiltonians are the homogeneous polynomials in the lifted Darboux coordinates. Such dependence plays a crucial role in the quantisation of the isomonodromic systems.

# CHAPTER 5

# IRREGULAR ISOMONOROMIC DEFORMATIONS

## 5.1 Isomonodromic deformations

Suppose we consider a connection on the Riemann sphere with $n+1$ poles of Poincaré ranks $r_1, \ldots, r_n, r_\infty$ and ask about how to deform it by keeping the monodromy data constant. To answer, we have to choose some independent deformation variables and then impose that all other quantities depend on those according to the isomonodromicity condition. When all poles are simple, their positions give us enough independent variables for generic isomonodromic deformations, because the number of the isomonodromic Hamiltonians equals half of the dimension of the space of accessory parameters. When higher order poles are present, their positions don't give enough independent variables. Theorem 5.3.1 allows us to introduce further $r-1$ independent variables for every singularity of Poincaré rank $r$, or in other words we have the following

**Corollary 5.1.1.** The general element in the Takiff algebra co-adjoint orbit $\widehat{\mathcal{O}}_r^\star$ has the form

$$A \sim \sum_{i=0}^{r} \frac{B_i(t_1, t_2 \ldots t_r)}{(\lambda - u)^{i+1}} + \ldots, \tag{5.1.1}$$

with

$$B_i(t_1, t_2, \ldots t_r) = \sum_{j=i}^{r} A_j \mathcal{M}_{i,j}^{(r)}(t_1, t_2, \ldots t_r), \quad \mathcal{M}_{i,j}^{(r)} = \frac{1}{j!} \frac{d^j}{d\varepsilon^j} P_r(t,\varepsilon)^i \Big|_{\varepsilon=0}, \quad P_r(t,\varepsilon) = \sum_{i=1}^{r} \varepsilon^i t_i,$$

and the coefficients $A_j$ satisfy the Takiff algebra Poisson bracket (4.1.5).

In this paper, we therefore consider the isomondoromic deformations of connections of the form

$$\frac{d}{d\lambda}\Psi=\sum_{i=0}^{n}\left(\sum_{j=0}^{r_i}\frac{B_j^{(i)}\left(t_1^{(i)},t_2^{(i)}\dots t_{r_i-1}^{(i)}\right)}{(\lambda-u_i)^{j+1}}-\sum_{i=1}^{r_\infty}\lambda^{i-1}B_i^{(\infty)}\left(t_1^{(\infty)},t_2^{(\infty)}\dots t_{r_\infty-1}^{(\infty)}\right)\right)\Psi, \quad (5.1.2)$$

where the deformation parameters are the locations of the poles $u_1\dots u_n$ and the coefficients of the Poisson Takiff algebra automorphisms $t_j^{(i)}$. The isomonodromic deformation condition means that the matrix differential one from

$$\Omega=\mathrm{d}_{u,t}\Psi\Psi^{-1}=\sum_{i=1}^{n}\left[\Omega_i^{(0)}\mathrm{d}u_i+\sum_{j=1}^{r_i-1}\Omega_i^{(j)}\mathrm{d}t_j^{(i)}\right], \quad (5.1.3)$$

is a single valued holomorphic one form on $\mathbb{CP}^1\setminus\{u_1\dots u_n\}$. In general, the explicit form of $\Omega$ may be obtained by studying the local solutions of the equation (5.1.2) as in the celebrated papers by Jimbo, Miwa [55] and by Flaschka and Newell [35].

In this paper we consider the general isomonodromic problem as a non-autonomous Hamiltonian system written on a suitable set of the co-adjoint orbits. Therefore, the zero curvature condition splits into a Lax equation that defines the dynamics on the co-adjoint orbits, and an additional relation between the partial derivative of $\Omega$ w.r.t. $\lambda$ and the partial derivative of the connection with respect to deformation parameters

$$\frac{d}{dt_j^{(i)}}A-\frac{\partial}{\partial\lambda}\Omega_j^{(i)}+\left[A,\Omega_j^{(i)}\right]=\underbrace{\left(\frac{\partial}{\partial t_j^{(i)}}A-\frac{\partial}{\partial\lambda}\Omega_j^{(i)}\right)}_{0}+\underbrace{\left(\left(\frac{d}{dt_j^{(i)}}-\frac{\partial}{\partial t_j^{(i)}}\right)A+\left[A,\Omega_j^{(i)}\right]\right)}_{0}=0.$$

Thanks to this, we may define deformation the one form $\Omega$ through the following formula:

$$\Omega_j^{(i)}=\int\frac{\partial A}{\partial t_j^{(i)}}d\lambda. \quad (5.1.4)$$

The matrix $\Omega_j^{(i)}$ is defined up to the addition of a matrix which does not depend on $\lambda$. Different choices of the gauge result in different constant terms - we will see how to fix this constant term in the examples (see for example Section 5.6.8).

As mentioned before, the deformation parameters $t_1^{(i)},\dots t_{r_i}^{(i)},\ i=1,\dots,n,\infty$ appear as the result of confluence and may be seen as avatars of the Schlesinger system deformation parameters we start with. If we consider the divisor of singularities (where we denote $\infty$ by $u_{n+1}$)

$$D := \sum_{i+1}^{n+1} (r_i + 1) u_i,$$

we see that the total number of deformation parameters we introduce is given via the degree of such divisor, i.e.

$$d = \underset{\#\text{ of singularities}}{n+1} + \underset{\#\text{ irregular times}}{\sum r_i} .$$

In this paper, the idea is that the number of deformation parameters doesn't change during the confluence procedure, or in other words $d$ is fixed.

Here we want to answer an important question raised by Bertola and Harnad: what is the relation between our deformation parameters and the Jimbo-Miwa-Ueno ones? In [55], the number of deformation parameters depends on the degree of singularity divisor *as well as on the rank of the connection.* The number of Jimbo-Miwa deformation parameters is not preserved during the confluence cascade. Each coalescence leads to the appearance of additional $m - 1$ parameters, where $m$ is a rank of isomonodromic problem. Here we refer to the rank of Lie algebra which is dimension of the Cartan subalgebra $\mathfrak{h}$. Obviously in the case of $\mathfrak{sl}_2$ connection, this number equals to zero and the number of Jimbo-Miwa-Ueno coincides with ours.

Let's dwell on this case in more details to explain the relation. Consider a $\mathfrak{sl}_2$ connection with a pole of the Poincaré rank $r$, i.e.

$$A \underset{\lambda \simeq u}{\sim} \frac{B_r}{z^{r+1}} + \frac{B_{r-1}}{z^r} + \dots \frac{B_0}{z} + O(1) \quad \in \mathfrak{sl}_2,$$

where $z = \lambda - u$ is the local coordinate and the matrices $B_k$ are linear combinations of the bare co-adjoint orbit coordinates $A_j$ and contain our deformation parameters as specified in formula (2.0.10).

The Jimbo-Miwa-Ueno deformation parameters $w_j$ are the exponents of asymptotic behaviour of the formal solution at the irregular pole:

$$\Psi \underset{\lambda \simeq u}{\sim} P(z) \left(\mathbb{I} + o(z)\right) z^{\Lambda_0} \exp\left[ -\sum_{j=1}^{r} \frac{w_j}{j z^j} \sigma_3 \right], \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These $w_j$ can in fact be seen as the spectral invariants associated to the matrices $B_k$. Thanks to this fact, in the case of $\mathfrak{sl}_2$ there is a rational map which sends Jimbo-Miwa deformation parameters to the parameters obtained via coalescence.To obtain this map explicitly, we perform local diagonalisation at the pole $\lambda \sim u$ and we obtain the following correspondence between Jimbo-Miwa deformation parameters $w_i$ and our $t_j$ via

$$\begin{aligned}
&w_r = \theta_r t_1^r \\
&w_{r-1} = \theta_{r-1} t_1^{r-1} + (r-1)\theta_r t_1^{r-2} t_2 \\
&\dots \\
&w_k = \sum_{j=k}^{r} \theta_j \mathcal{M}_{k,j}^{(r)}(t_1, t_2 \dots, t_r) \\
&\dots \\
&w_1 = \sum_{i=1}^{r} \theta_i t_i.
\end{aligned}$$

Here $\theta$'s can be seen as the spectral invariants of matrices $A_j$, so we separate non-autonomous part (dependence on deformation parameters) and phase space symplectic leaf. Roughly speaking, this map is a map between 2 phase spaces

$$\hat{\mathfrak{g}}_r \rightarrow \hat{\mathcal{O}}_r \times \mathbb{C}^r,$$

which is not bi-rational - starting from the irregular point of Poincaré rank 2 we have to deal with square roots if when we write $t_1 \dots t_r$ via Jimbo-Miwa parameters $w_j$'s.

For higher rank, we may think about our times as a special subfamily of the Jimbo-Miwa isomonodromic deformations. The local solution writes as

$$\Psi \underset{\lambda \simeq u}{\sim} P(z)\left(\mathbb{I} + o(z)\right) z^{\Lambda_0} \exp\left[-\sum_{j=1}^{r} \frac{1}{j z^j} \begin{pmatrix} w_1^{(j)} & 0 & \dots & 0 \\ 0 & w_2^{(j)} & \dots & 0 \\ & \dots & \dots & \\ 0 & \dots & 0 & w_m^{(j)} \end{pmatrix}\right]$$

and $w_k^{(j)}$ are the deformation parameters. Then our deformation parameters are given by the special trajectory into the Jimbo-Miwa parameters which may be written as

$$\frac{w_k^{(j)}}{w_l^{(j)}} = \text{const},$$

and may be considered as the deformation along the projective line in a space of Jimbo-Miwa parameters.

In the next section we will see how the general form (5.1.2) of the isomonodromic problem with irregular singularities naturally arises during the confluence procedure.

## 5.2 Confluence procedure

### 5.2.1 Coalescence of two simple poles

Without loss of generality, we consider confluence of $u_n := v_1$ and $u_{n-1} := w$, which is given by the following change of deformation parameters

$$u_i = u_i, \quad i = 1 \dots n-1, \quad v_1 = w + \varepsilon t_1. \tag{5.2.1}$$

Taking the limit $\varepsilon \to 0$ the deformation parameter $v_1$ tends to $w$ which is a coalescence. We rewrite matrix $A(\lambda)$ as

$$A(\lambda) = \sum_{i=1}^{n-2} \frac{A^{(i)}}{\lambda - u_i} + \frac{B}{\lambda - w} + \frac{C}{\lambda - w - \varepsilon t_1}, \quad B = A^{(n-1)}, \quad C = A^{(n)},$$

where $B$ and $C$ are introduced as a convenient notation to avoid too many indices. We want to assume some $\varepsilon$ expansions for the matrices $B$ and $C$ in order that the limit of $A(\lambda)$ as $\varepsilon \mapsto 0$ is well defined and the resulting system has a double pole at $w$. To this aim, observe that by rewriting the last two terms in $A(\lambda)$ as

$$\frac{B}{\lambda - w} + \frac{C}{\lambda - w - \varepsilon t_1} = \frac{B}{\lambda - w} + \frac{1}{\lambda - w} C \left(1 - \frac{\varepsilon t_1}{\lambda - w}\right)^{-1}$$

and expanding $\left(1-\frac{\varepsilon t_1}{\lambda-w}\right)^{-1}$ in $\varepsilon$ we obtain

$$\frac{B}{\lambda-w}+\frac{C}{\lambda-w-\varepsilon t_1}\sim\frac{C+B}{\lambda-w}+\frac{\varepsilon t_1}{(\lambda-w)^2}C+O(\varepsilon^2).$$

In order to produce the second order pole we need two limits to exist

$$\lim_{\varepsilon\to 0}(\varepsilon C):=A_1^{(n-1)}\neq 0,\quad \lim_{\varepsilon\to 0}(C+B):=A_0^{(n-1)},$$

Assuming that $A^{(i)}$'s, $B$ and $C$ may be expanded in the Laurent series in $\varepsilon$ we obtain expansions

$$A^{(i)}=\tilde{A}^{(i)}+O(\varepsilon),\quad C=\frac{1}{\varepsilon}A_1^{(n-1)}+C_0+O(\varepsilon),\quad B=-\frac{1}{\varepsilon}A_1^{(n-1)}+B_0+O(\varepsilon),\quad C_0+B_0=A_0^{(n-1)}. \tag{5.2.2}$$

Note that we have called these limits $A_0^{(n-1)}$ and $A_1^{(n-1)}$ respectively to adhere to the notation of section 3.

In these hypotheses, we can take the limit as $\varepsilon\to 0$ and define

$$\tilde{A}(\lambda):=\lim_{\varepsilon\to 0}A(\lambda)=\sum_{i=1}^{n-2}\frac{\tilde{A}^{(i)}}{\lambda-\tilde{u}_i}+t_1\frac{A_1^{(n-1)}}{(\lambda-w)^2}+\frac{A_0^{(n-1)}}{\lambda-w} \tag{5.2.3}$$

**Remark 5.2.1.** *Observe that the number of deformation parameters has not changed after the confluence, $n-1$ of them have remained as positions of poles, but one of them has become part of the leading term at the second order pole - this is compatible with Theorem 5.3.1. Indeed, in the next Proposition 5.2.3 we will prove that the matrices $A_1^{(n-1)}$ and $A_0^{(n-1)}$ satisfy the Takiff algebra Poisson brackets. We will see that as we increase the Poincaré rank of the poles in the confluence procedure, more and more deformation parameters will appear in the numerators of pole expansions exactly in the way predicted by Theorem 5.3.1.*

Now let us focus on the deformation equations. The change of variables (5.2.1) transforms the deformation 1-form (4.3.3) to

$$\Omega=-\sum_{i=1}^{n-2}\frac{A^{(i)}}{\lambda-u_i}\mathrm{d}u_i-\frac{A^{(n-1)}}{\lambda-w}\mathrm{d}w-\frac{A^{(n)}}{\lambda-w-\varepsilon t_1}(\mathrm{d}w+\varepsilon\mathrm{d}t_1).$$

Applying the expansion (5.2.2) we obtain

$$\tilde{\Omega} = \lim_{\varepsilon \to 0} \Omega = -\sum_{i=1}^{n-2} \frac{\tilde{A}^{(i)}}{\lambda - u_i} \mathrm{d}u_i - \left( t_1 \frac{A_1^{(n-1)}}{(\lambda - w)^2} + \frac{A_0^{(n-1)}}{\lambda - w} \right) \mathrm{d}w - \frac{A_1^{(n-1)}}{\lambda - w} \mathrm{d}t_1. \tag{5.2.4}$$

The obtained deformation 1-form coincides with the deformation form which can be constructed by considering the local expansions. The deformation one form $\Omega$ satisfies equation (5.1.4).

**Definition 5.2.2.** We call the process of taking the expansions (5.2.2) and the limits (5.2.3), (5.2.4), *1+1 confluence procedure.*

The considered structures - the connection $A$ and the deformation one form $\Omega$ are linear in $A^{(i)}$'s so the $O(\varepsilon)$ terms vanish during the limiting procedure. Since the Poisson structure and the Schlesinger Hamiltonians are quadratic structures the limiting procedure becomes more complicated.

**Proposition 5.2.3.** *The 1+1 confluence procedure gives a Poisson morphism between the direct product of the co-adjoint orbits to the Lie algebra and the co-adjoint orbit of the Takiff algebra:*

$$\mathcal{O}_1^\star \times \mathcal{O}_2^\star \times \dots \mathcal{O}_n^\star \times \mathcal{O}_\infty^\star \xrightarrow{\text{confluence}} \mathcal{O}_1^\star \times \mathcal{O}_2^\star \times \dots \mathcal{O}_{n-2}^\star \times \hat{\mathcal{O}}_{2,n-1}^\star \times \mathcal{O}_\infty^\star.$$

*Namely, if the matrices $A^{(i)}, B, C$ satisfy the standard Lie–Poisson brackets* (4.3.6)*, then the matrices $\tilde{A}^{(i)}, A_0^{(n-1)}, A_1^{(n-1)}$ satisfy the Poisson algebra of the coefficients for the Takiff algebra* (4.1.5)*, i.e.*

$$\left\{ \tilde{A}_\alpha^{(i)}, \tilde{A}_\beta^{(j)} \right\} = -\delta_{ij} \sum_\gamma \chi_{\alpha\beta}^\gamma \tilde{A}_\gamma^{(i)}, \quad \left\{ \tilde{A}_\alpha^{(i)}, A_{0,\beta}^{(n-2)} \right\} = \left\{ \tilde{A}_\alpha^{(i)}, A_{1,\beta}^{(n-2)} \right\} = 0 \quad i,j = 1, \dots n-2,$$

$$\left\{ A_{1,\alpha}^{(n-2)}, A_{1,\beta}^{(n-2)} \right\} = 0, \quad \left\{ A_{1,\alpha}^{(n-2)}, A_{0,\beta}^{(n-2)} \right\} = -\chi_{\alpha\beta}^\gamma A_{1,\gamma}^{(n-2)}, \quad \left\{ A_{0,\alpha}^{(n-2)}, A_{0,\beta}^{(n-2)} \right\} = -\chi_{\alpha\beta}^\gamma \left( A_{0,\gamma}^{(n-2)} \right), \tag{5.2.5}$$

*Proof.* That Poisson structure (5.2.5) for the coefficients of the connection near the irregular singularity is given by Kirillov-Kostant-Souriau form for the co-adjoint orbit $\tilde{\mathcal{O}}_2^\star$ of the Takiff algebra $\mathfrak{g}_2 \backsimeq \mathfrak{g}[z]/(z^2\mathfrak{g}[z])$, where $\mathfrak{g}[z]$ is a Lie algebra of the polynomials with coefficients in $\mathfrak{g}$. Therefore, if we prove that (5.2.5), then the 1+1 confluence procedure gives a Poisson morphism between the direct product of the co-adjoint orbits to the Lie algebra and the co-adjoint orbit of the Takiff algebra.

Let us prove (5.2.5). The first row relations are straightforward and we omit the proof. To prove the relations in the second row of (5.2.5), let us consider the Poisson relations (4.3.6) for $B$ and $C$

$$\{C_\alpha, C_\beta\} = -\sum_\gamma \chi_{\alpha\beta}^\gamma C_\gamma, \quad \{B_\alpha, B_\beta\} = -\sum_\gamma \chi_{\alpha\beta}^\gamma B_\gamma, \quad \{C_\alpha, B_\beta\} = 0.$$

Inserting the expansion (5.2.2) and expanding the Poisson relations in $\varepsilon$, we obtain

$$\frac{1}{\varepsilon^2}\left\{A_{1,\alpha}^{(n-1)}, A_{1,\beta}^{(n-1)}\right\} + \frac{1}{\varepsilon}\left(\left\{A_{1,\alpha}^{(n-1)}, C_{0,\beta}\right\} + \left\{C_{0,\alpha}, A_{1,\beta}^{(n-1)}\right\}\right) +$$
$$+ \{C_{0,\alpha}, C_{0,\beta}\} + \left\{A_{1,\alpha}^{(n-1)}, C_{1,\beta}\right\} + \left\{C_{1,\alpha}, A_{1,\beta}^{(n-1)}\right\} = -\chi_{\alpha\beta}^\gamma\left(\frac{1}{\varepsilon}A_{1,\gamma}^{(n-1)} + C_{0,\gamma}\right) + o(\varepsilon)$$

$$\frac{1}{\varepsilon^2}\left\{A_{1,\alpha}^{(n-1)}, A_{1,\beta}^{(n-1)}\right\} - \frac{1}{\varepsilon}\left(\left\{A_{1,\alpha}^{(n-1)}, B_{0,\beta}\right\} + \left\{B_{0,\alpha}, A_{1,\beta}^{(n-1)}\right\}\right) +$$
$$+ \{B_{0,\alpha}, B_{0,\beta}\} - \left\{A_{1,\alpha}^{(n-1)}, B_{1,\beta}\right\} - \left\{B_{1,\alpha}, A_{1,\beta}^{(n-1)}\right\} = \chi_{\alpha\beta}^\gamma\left(\frac{1}{\varepsilon}A_{1,\gamma}^{(n-1)} - B_{0,\gamma}\right) + o(\varepsilon)$$

$$-\frac{1}{\varepsilon^2}\left\{A_{1,\alpha}^{(n-1)}, A_{1,\beta}^{(n-1)}\right\} + \frac{1}{\varepsilon}\left(\left\{A_{1,\alpha}^{(n-1)}, B_{0,\beta}\right\} - \left\{C_{0,\alpha}, A_{1,\beta}^{(n-1)}\right\}\right) +$$
$$+ \{C_{0,\alpha}, B_{0,\beta}\} + \left\{A_{1,\alpha}^{(n-1)}, B_{1,\beta}\right\} - \left\{C_{1,\alpha}, A_{1,\beta}^{(n-1)}\right\} = o(\varepsilon).$$

Collecting different terms in $\varepsilon$, we obtain

$$
\begin{aligned}
\varepsilon^{-2}: &\quad \left\{A_{1,\alpha}^{(n-1)}, A_{1,\beta}^{(n-1)}\right\} = 0, \\
\varepsilon^{-1}: &\quad \left\{A_{1,\alpha}^{(n-1)}, C_{0,\beta}\right\} + \left\{C_{0,\alpha}, A_{1,\beta}^{(n-1)}\right\} = -\chi_{\alpha\beta}^{\gamma} A_{1,\gamma}^{(n-1)}, \\
\varepsilon^{-1}: &\quad \left\{A_{1,\alpha}^{(n-1)}, B_{0,\beta}\right\} + \left\{B_{0,\alpha}, A_{1,\beta}^{(n-1)}\right\} = -\chi_{\alpha\beta}^{\gamma} A_{1,\gamma}^{(n-1)}, \\
\varepsilon^{-1}: &\quad \left\{A_{1,\alpha}^{(n-1)}, B_{0,\beta}\right\} - \left\{C_{0,\alpha}, A_{1,\beta}^{(n-1)}\right\} = 0, \qquad (5.2.6) \\
\varepsilon^{0}: &\quad \{C_{0,\alpha}, C_{0,\beta}\} + \left\{A_{1,\alpha}^{(n-1)}, C_{1,\beta}\right\} + \left\{C_{1,\alpha}, A_{1,\beta}^{(n-1)}\right\} = -\chi_{\alpha\beta}^{\gamma} C_{0,\gamma}, \\
\varepsilon^{0}: &\quad \{B_{0,\alpha}, B_{0,\beta}\} - \left\{A_{1,\alpha}^{(n-1)}, B_{1,\beta}\right\} - \left\{B_{1,\alpha}, A_{1,\beta}^{(n-1)}\right\} = -\chi_{\alpha\beta}^{\gamma} B_{0,\gamma} \\
\varepsilon^{0}: &\quad \{C_{0,\alpha}, B_{0,\beta}\} + \left\{A_{1,\alpha}^{(n-1)}, B_{1,\beta}\right\} - \left\{C_{1,\alpha}, A_{1,\beta}^{(n-1)}\right\} = 0.
\end{aligned}
$$

The term of order $\varepsilon^{-2}$ in (5.2.6) proves the first relation in the second row of (5.2.5). Let us prove the second relation. Take the $1/\varepsilon$ term

$$\left\{A_{1,\alpha}^{(n-2)}, B_{0,\beta}\right\} - \left\{C_{0,\alpha}, A_{1,\beta}^{(n-2)}\right\} = 0 \quad \Longleftrightarrow \quad \left\{C_{0,\alpha}, A_{1,\beta}^{(n-2)}\right\} = \left\{A_{1,\alpha}^{(n-2)}, B_{0,\beta}\right\}$$

and put it in the Poisson relation between $A_1^{(n-1)}$ and $C_0$. We get

$$
\begin{aligned}
-\chi_{\alpha\beta}^{\gamma} A_{1,\gamma}^{(n-2)} &= \left\{A_{1,\alpha}^{(n-2)}, C_{0,\beta}\right\} + \left\{C_{0,\alpha}, A_{1,\beta}^{(n-2)}\right\} = \left\{A_{1,\alpha}^{(n-2)}, C_{0,\beta}\right\} + \left\{A_{1,\alpha}^{(n-2)}, B_{0,\beta}\right\} = \\
&= \boxed{\left\{A_{1,\alpha}^{(n-2)}, C_{0,\beta} + B_{0,\beta}\right\} = -\chi_{\alpha\beta}^{\gamma} A_{1,\gamma}^{(n-2)}}
\end{aligned}
$$

which proves the second relation. Now let us compute the last Poisson bracket

$$\{C_{0,\alpha} + B_{0,\alpha}, C_{0,\beta} + B_{0,\beta}\} = \{C_{0,\alpha}, C_{0,\beta}\} + \{C_{0,\alpha}, B_{0,\beta}\} + \{B_{0,\alpha}, C_{0,\beta}\} + \{B_{0,\alpha}, B_{0,\beta}\}.$$

Using $\varepsilon^0$-terms from (5.2.6) for $\{C_{0,\alpha}, C_{0,\beta}\}$ and $\{B_{0,\alpha}, B_{0,\beta}\}$ we obtain

$$
\begin{aligned}
\{C_{0,\alpha} + B_{0,\alpha}, C_{0,\beta} + B_{0,\beta}\} &= -\chi_{\alpha\beta}^{\gamma}(C_{0,\beta} + B_{0,\beta}) - \left\{A_{1,\alpha}^{(n-2)}, C_{1,\beta}\right\} - \left\{C_{1,\alpha}, A_{1,\beta}^{(n-2)}\right\} + \\
&+ \left\{A_{1,\alpha}^{(n-2)}, B_{1,\beta}\right\} + \left\{B_{1,\alpha}, A_{1,\beta}^{(n-2)}\right\} + \{C_{0,\alpha}, B_{0,\beta}\} + \{B_{0,\alpha}, C_{0,\beta}\} \qquad (5.2.7)
\end{aligned}
$$

The last $\varepsilon^0$-term in (5.2.6) leads to the following relations

$$\{C_{0,\alpha}, B_{0,\beta}\} = \left\{C_{1,\alpha}, A_{1,\beta}^{(n-2)}\right\} - \left\{A_{1,\alpha}^{(n-2)}, B_{1,\beta}\right\}$$

$$\{B_{0,\alpha}, C_{0,\beta}\} = \left\{A_{1,\alpha}^{(n-2)}, C_{1,\beta}\right\} - \left\{B_{1,\alpha}, A_{1,\beta}^{(n-2)}\right\}$$

which cancel all terms in the right-hand side of (5.2.7) except the first term, so we obtain

$$\{C_{0,\alpha} + B_{0,\alpha}, C_{0,\beta} + B_{0,\beta}\} = -\chi_{\alpha\beta}^{\gamma}(C_{0,\gamma} + B_{0,\gamma}),$$

which concludes proof. □

Observe that the relations (5.2.6) contain more information than we need, and that one could actually try to come up with a Poisson algebra involving all coefficients $B_k$, $C_k$ in the expansion (5.2.2). However we are only interested in the Poisson subalgebra generated by $A_1^{(n-1)}$, $A_0^{(n-1)} = C_0 + B_0$ and $\tilde{A}^{(i)}$ for $i = 1, \dots, n-2$. The main feature of this subalgebra is that it does not depend on a choice of a Poisson algebra for the coefficients $B_k$ and $C_k$. We call this subalgebra Isomonodromic Poisson Algebra (IPA), since these are the only elements which survive in the isomonodromic problem after the confluence procedure.

**Proposition 5.2.4.** *The $1+1$ confluence procedure produces the isomonodromic Hamiltonians giving the zero curvature condition*

$$\mathrm{d}_u \tilde{A} - \frac{d}{d\lambda}\tilde{\Omega} + [\tilde{A}, \tilde{\Omega}] = 0$$

*as equation of motion.*

*Proof.* To prove this, we start from the extended symplectic form for the Schlesinger equations:

$$\omega_{\mathrm{KKS}} + \sum_{i=1}^{n} \mathrm{d}u_i \wedge \mathrm{d}H_i.$$

Here $\omega_{\mathrm{KKS}}$ is the symplectic form which corresponds to the standard Lie–Poisson structure on the direct product of the co-adjoint orbits. Thanks to Proposition 5.2.3, the standard Lie–Poisson bracket tends to the Takiff algebra Poisson bracket, therefore $\omega_{\mathrm{KKS}}$ tends to the corresponding symplectic form. Let us concentrate on the $\sum_{i=1}^{n} \mathrm{d}u_i \wedge \mathrm{d}H_i$ part. This part transforms to

$$\sum_{i=1}^{n} \mathrm{d}u_i \wedge \mathrm{d}H_i \to \sum_{i=1}^{n-2} \mathrm{d}u_i \wedge \mathrm{d}H_i + \mathrm{d}w \wedge \mathrm{d}\left(H_{n-1} + H_n\right) + \mathrm{d}t_1 \wedge \mathrm{d}\left(\varepsilon H_n\right).$$

Since we are working on a symplectic leaf of the standard Lie–Poisson bracket, the central elements, or Casimirs, can be considered as fixed scalars, i.e. the differential d acts on them as a zero. To find the Hamiltonians of the confluent dynamic we have to calculate the limit of the "time-dependent" part of the symplectic structure as $\varepsilon$ goes to zero. In other words, we have to find

$$\mathrm{d}\tilde{H}_i := \lim_{\varepsilon \to 0} \mathrm{d}H_i, \quad \mathrm{d}\tilde{H}_{n-1} := \lim_{\varepsilon \to 0} \mathrm{d}(H_{n-1} + H_n), \quad \mathrm{d}\tilde{H}_n := \lim_{\varepsilon \to 0} \varepsilon \mathrm{d}H_n. \tag{5.2.8}$$

To compute these limits, we can treat the Hamiltonians up to addition of Casimirs. This allows us to use the Casimirs for regularizing parts of the Hamiltonains that are singular in $\varepsilon$ parts of Hamiltonians. Therefore all = signs in the rest of the proof are intended as equal up to Casimirs. For $i < n-2$ we have

$$\tilde{H}_i := \lim_{\varepsilon \to 0} H_i = \sum_{j \neq i}^{n-2} \frac{\mathrm{Tr}\left(\tilde{A}^{(i)} \tilde{A}^{(j)}\right)}{u_i - u_j} + t_1 \frac{\mathrm{Tr}\left(\tilde{A}_1^{(n-1)} \tilde{A}^{(i)}\right)}{(u_i - w)^2} + \frac{\mathrm{Tr}\left(\tilde{A}_0^{(n-1)} \tilde{A}^{(i)}\right)}{u_i - w}, \tag{5.2.9}$$

for $i = n-1$ we have

$$\begin{aligned} \tilde{H}_{n-1} = \lim_{\varepsilon \to 0} \left(H_{n-1} + H_n\right) = \lim_{\varepsilon \to 0} \sum_{j < n-2} \mathrm{Tr}\tilde{A}^{(j)} \left( \frac{A^{(n-1)}}{w - u_j} + \frac{A^{(n)}}{w + \varepsilon t_1 - u_j} \right) = \\ = \sum_{j < n-1} \mathrm{Tr}\tilde{A}^{(j)} \left( \frac{\tilde{A}_0^{(n-1)}}{w - u_j} - t_1 \frac{\tilde{A}_1^{(n-1)}}{(w - u_j)^2} \right). \end{aligned} \tag{5.2.10}$$

For $i = n$

$$\tilde{H}_n = \lim_{\varepsilon \to 0} \varepsilon H_n$$

Substituting coalescence expansions we get

$$\varepsilon H_n = \left[ \sum_{j < n-2} \frac{\mathrm{Tr}\tilde{A}^{(j)} A_1^{(n-2)}}{w - u_j} + O(\varepsilon) \right] + \frac{\mathrm{Tr}A^{(n)} A^{(n-1)}}{t_1}. \tag{5.2.11}$$

The last term in (5.2.11) contains the $1/\varepsilon$ terms

$$\frac{\mathrm{Tr}A^{(n)}A^{(n-1)}}{\tilde{u}_n} = \frac{1}{\tilde{u}_n}\left(-\frac{1}{\varepsilon^2}\mathrm{Tr}\left(\tilde{A}_1^{(n-1)}\right)^2 + \frac{1}{\varepsilon}\mathrm{Tr}\left(\tilde{A}_1^{(n-1)}B_0 - C_0\tilde{A}_1^{(n-1)}\right) + \mathrm{Tr}(B_0C_0)\right) + \\ + \frac{1}{\tilde{u}_n}\mathrm{Tr}\left(\tilde{A}_1^{(n-1)}B_1 - C_1\tilde{A}_1^{(n-1)}\right)$$

The $1/\varepsilon^2$ term is a Casimir of the Poisson structure associated with truncated loop algebra, so we may drop it.

Let us show that also the $1/\varepsilon$-term is a Casimir and that, after eliminating the Casimirs, $\epsilon H_n \to \tilde{H}_n + O(\varepsilon)$ where

$$\tilde{H}_n = \sum_{j<n-2} \frac{\mathrm{Tr}\tilde{A}^{(j)}A_1^{(n-2)}}{w-u_j} + \frac{1}{t_1}\frac{\mathrm{Tr}\left(\tilde{A}_0^{(n-1)}\right)^2}{2}. \tag{5.2.12}$$

To see this, let us remind that the Casimirs of the Poisson algebra in a Fuchsian case are $\mathrm{Tr}\left(A^{(i)}\right)^k$, so the function

$$\frac{1}{2}\mathrm{Tr}\left(A^{(n)} + A^{(n-1)}\right)^2$$

differs from the last term of (5.2.11)

$$\mathrm{Tr}A^{(n)}A^{(n-1)}.$$

by a Casimir. Since the Hamiltonians are defined up to the addition of a Casimir, we obtain

$$\varepsilon H_n = \sum_{j<n-2} \frac{\mathrm{Tr}\tilde{A}^{(j)}A_1^{(n-2)}}{w-u_j} + \frac{\mathrm{Tr}\left(A^{(n)}+A^{(n-1)}\right)^2}{2t_1} + O(\varepsilon) = \sum_{j<n-2} \frac{\mathrm{Tr}\tilde{A}^{(j)}A_1^{(n-2)}}{w-u_j} + \frac{1}{t_1}\frac{\mathrm{Tr}\left(\tilde{A}_0^{(n-1)}\right)^2}{2} + O(\varepsilon).$$

Taking the limit as $\varepsilon \to 0$ we obtain the Hamiltonian (5.2.12).

### 5.2.2 Irregular singularities arising as confluence cascades.

In this section we consider an irregular singularity of arbitrary Poincaré rank $r$ as the result of a confluence cascade of $r$ simple poles $v_1, v_2 \dots v_r$ with some chosen simple pole $u$ on the Riemann sphere. At the first step, we send $v_1$ to $u$ and create second order pole as in the previous subsection. Then we do the same for $v_2$ - we collide it with the second order pole at $u$ and create a pole of order 3. In such a way, we continue this procedure, so at the $l$-th step we collide simple pole $v_l$ with the pole of order $l$ at $u$ to create a new pole of order $l+1$. On the final $r$-th step, we obtain a pole of order $r+1$, i.e. of Poincaré rank $r$. During this procedure we expect that the poles $v_l$'s that disappear give rise to deformation parameters $t_l$'s for the irregular isomonodromic problem[1]. Since the number of poles decreases during the confluence procedure, these deformation parameters appear explicitly in the coefficients of the local expansion of the connection near the singularity $u$. In the sub-section 5.2.3, we prove Theorem 2.0.5 that tells us that this dependence is the one described in Corollary 5.1.1. Before attacking that proof, let us formalise the definition of confluence:

**Definition 5.2.5.** The limiting procedure described in the hypotheses of Theorem 2.0.5 is called $r+1$ *confluence.*

Observe that as a result of the $1+1$ confluence in subsection 5.2.1 we obtain a connection of the form (2.0.12) with $r=2$. We can then apply the $1+2$ confluence to this and again obtain a connection of the form (2.0.12) with $r=3$ and so on. Therefore we can give the following recursive definition:

**Definition 5.2.6.** The procedure of applying Theorem 2.0.5 recursively $r$ times is called *confluence cascade of* r + 1 *simple poles* on the Riemann sphere.

As mentioned at the beginning of this section, the inductive hypothesis on the local form of the connection (2.0.12) is not restrictive. Indeed, we expect the local form of a connection with a pole of order $r$ at $u$ to be given by an element in the Takiff algebra co-adjoint orbit $\widehat{\mathcal{O}}_r^\star$ with some spectral parameter $z=\lambda-u$. However, if we want to keep the number of independent variables to be maximal, we need to introduce some extra variables $t_i$ by hand in such a way that they can be treated as independent variables. In Corollary 5.1.1, we proved that the only way to do this is by taking precisely the form (2.0.12). Indeed, formula (2.0.11) corresponds to (5.3.5). Therefore, we obtain theorem 2.0.5 which we remind here:

**Theorem 5.2.7.** *Assume that $u$ is a singularity of Poincaré rank $r$ obtained by the confluence*

[1]The confluence procedure is not symmetric in $v_i$, however different choices of the order of the coalescence cascade will lead to the action of the permutation group on $t_l$'s, so we fix this ordering once for all.

*of $r+1$ simple poles. Then the coefficients of the local expansion*

$$A(\lambda) \sim \sum_{i=0}^{r} \frac{B_i(t_1,\dots t_r)}{(\lambda-u)^{i+1}} + \dots,$$

*take the form*

$$B_i(t_1,t_2,\dots t_r) = \sum_{j=i}^{r} B^{[j]} \mathcal{M}^{(r)}_{i,j}(t_1,t_2,\dots t_r),$$

*where*

$$\mathcal{M}^{(r)}_{k,j} = \sum_{w(\alpha)=j}^{|\alpha|=k} \frac{k!}{\alpha_1!\alpha_2!\dots\alpha_r!} \left(\prod_{i=1}^{r} t_i^{\alpha_i}\right), \quad |\alpha| = \sum_{i=1}^{r} \alpha_i, \quad w(\alpha) = \sum_{i=1}^{r} i\cdot\alpha_i, \quad \mathcal{M}^{(r)}_{k>j} := 0,$$

*and $B_i^{[j]}$'s hold the following Poisson relations*

$$\left\{ \left(B^{[k]}\right)_\alpha, \left(B^{[p]}\right)_\beta \right\} = \begin{cases} -\chi^\gamma_{\alpha\beta}\left(B^{[k+p]}\right)_\gamma & k+p\le r \\ 0 & k+p>r, \end{cases} \tag{5.2.13}$$

*where $\chi^\gamma_{\alpha\beta}$ are the structure constants of the corresponding Lie algebra.*

*Proof.* Will be presented in Section 5.2.4 □

In this section we give an explicit description of the local coefficients $B_i$ for the connection near irregular singularity of Poincaré rank $r$ in terms of the generators of the Takiff co-algebra of degree $r$. It turns out that the $B_i(t_1,\dots t_{r-1})$ are the special linear combinations of the generators of the Takiff co-algebra $A_0,\dots A_r$ with coefficients in $\mathbb{C}[t_1,t_2,\dots t_r]$. Now we state a theorem we are going to prove in this subsection.

We want to underline here that the Poisson structure (5.2.13) gives rise to the Takiff co-algebra Poisson structure on the coefficients of the local expansion, i.e

$$\{B_i(t_1,\dots t_r)_\alpha, B_j(t_1,\dots t_r)_\beta\} = -\sum_\gamma \chi^\gamma_{\alpha\beta} B_{i+j}(t_1,\dots t_r)_\gamma. \tag{5.2.14}$$

However, such Poisson structure on the coefficients of the local expansion is quite rough, the coefficients depend on the deformation parameters explicitly and it is important to understand this dependence when we do the deformation with respect to $t_i$'s. On the other hand, Poisson structure (5.2.13) contains the information about explicit dependence on $t_i$'s, moreover this

Poisson structure compatible with more rough Poisson structure.

To motivate the constructions which appear in the statement of this theorem we introduce some preliminaries on the confluence procedure and algebraic structures which appear during coalescence before the proof.

### 5.2.3 The algebra of the weighted monomials and associated polynomials.

The aim of this subsection is to collect some useful algebraic relations involving the parameters $t_1, \ldots, t_r$ that arise during the confluence procedure and describe the general elements of the Takiff co-algebra with respect to the Poisson automorphisms.

In order to prove Theorem 2.0.5, in the of a simple pole $v_r$ with a pole $w$ of Poincaré rank $r$ we take the following expansion

$$v_r = w + P_r(t, \varepsilon) = w + \sum_{i=1}^{r} t_i \varepsilon^i.$$

The powers of the polynomial $P_r(t, \varepsilon)$ play a significant role since they appear in the following expansion

$$\frac{C}{\lambda - v_r} = \frac{C}{\lambda - w}\left(1 - \frac{P_r(t,\varepsilon)}{\lambda - w}\right)^{-1} =$$
$$= \frac{C}{\lambda - w}\left(1 + \frac{P_r(t,\varepsilon)}{\lambda - w} + \cdots + \frac{P_r(t,\varepsilon)^j}{(\lambda - w)^j} + \cdots + \frac{P_r(t,\varepsilon)^r}{(\lambda - w)^r}\right) + O(\varepsilon^{r+1}).$$

Each power of $P_r(t, \varepsilon)$ may be seen as a polynomial in $\varepsilon$ with coefficients in $\mathbb{C}[t_1, t_2 \ldots t_r]$

$$P_r(t, \varepsilon)^r = t_1^r \varepsilon^r + O(\varepsilon^{r+1}).$$

Because the aim of the confluence is to create a pole of Poinceré rank $r + 1$, we need the coefficeints $(\lambda - w)^{-r-2}$ to survive, therefore, we have to require $C$ to be a Laurent polynomial in $\varepsilon$ starting from the power $-r$. Taking this fact into account, it is important to understand how each power of $P_r(t, \varepsilon)$ expands via $\varepsilon$

$$P_r(t,\varepsilon)^i \mod \varepsilon^{r+1} = \sum_{j=i}^{r} \mathcal{M}^{(r)}_{i,k}(t_1, \ldots t_r)\varepsilon^k = \mathcal{M}^{(r)}_{i,i}(t_1, \ldots t_r)\varepsilon^i + \cdots + \mathcal{M}^{(r)}_{i,r}(t_1, \ldots t_r)\varepsilon^r,$$

where

$$\mathcal{M}^{(r)}_{i,k}(t_1,\dots t_r) = \frac{1}{k!}\frac{d}{d\varepsilon}P_r(t,\varepsilon)^i\Big|_{\varepsilon=0}.$$

The following simple Lemma calculates an explicit formula for $\mathcal{M}^{(r)}_{i,j}(t_1,\dots t_r)$ and gives some useful identites.

**Lemma 5.2.8.** *$\mathcal{M}^{(r)}_{i,k}$ is a homogeneous polynomial in $t_1 \dots t_r$ of degree $i$ for any $k$ given by*

$$\mathcal{M}^{(r)}_{i,k}(t_1,\dots t_r) = \sum_{w(\alpha)=k}^{|\alpha|=i} \frac{i!}{\alpha_1!\alpha_2!\dots\alpha_r!}\left(\prod_{j=1}^{r} t_j^{\alpha_j}\right), \quad w(\alpha) = \sum_{j=1}^{r} j\alpha_j, \quad |\alpha| = \sum_{j=1}^{r}\alpha_j.$$

*The polynomials $\mathcal{M}^{(r)}_{i,k}$ satisfy the following identities*

$$\mathcal{M}^{(r+1)}_{i,k} = \mathcal{M}^{(r)}_{i,k}, \quad \forall k \le r \tag{5.2.15}$$

*and*

$$\mathcal{M}^{(r)}_{j,k} = \sum_{p=0}^{k}\left[\mathcal{M}^{(r)}_{j-i,p}\cdot\mathcal{M}^{(r)}_{i,k-p} + \mathcal{M}^{(r)}_{j-i,k-p}\cdot\mathcal{M}^{(r)}_{i,p}\right], \quad \forall i \le j. \tag{5.2.16}$$

*Proof.* Consider a ring of polynomials $\mathbb{C}[\varepsilon]$ and the quotient ring $\mathbb{C}[\varepsilon]/\varepsilon^{r+1}$. The ring homomorphism $T_r$ which sends the elements of $\mathbb{C}[\varepsilon]$ to the elements $\mathbb{C}[\varepsilon]/\varepsilon^{r+1}$ is given by truncation of polynomials

$$T_r(Q(\varepsilon)) = Q(\varepsilon) \mod \varepsilon^{r+1}, \quad Q(\varepsilon) \in \mathbb{C}[\varepsilon].$$

Since $T$ is a ring homomorphism, it respects multiplication operation and, as a consequence, commutes with the exponentiation, i.e.

$$T_r(Q(\varepsilon)^n) = T_r(Q(\varepsilon))^n, \quad T_r(Q(\varepsilon))^n = \prod_{i=1}^{n} T_r(Q(\varepsilon)) \mod \varepsilon^{r+1}.$$

Now consider polynomial $P_{r+1}(t,\varepsilon)$. By definition $\mathcal{M}^{(r+1)}_{i,k}$ is a coefficient of $P_{r+1}(t,\varepsilon)^i$ in front of $\varepsilon^k$ for $k \le r+1$. Since that the ring homomorphism $T_r$ gives a generating polynomial for $\mathcal{M}^{(r+1)}_{i,k}$ fro $k < r+1$. On the other hand, we have that

$$T_r(P_{r+1}(t,\varepsilon)) = P_r(t,\varepsilon),$$

and since that

$$T_r(P_{r+1}(t,\varepsilon)^i) = T_r(P_{r+1}(t,\varepsilon))^i = (P_r(t,\varepsilon))^i \mod \varepsilon^{r+1},$$

which proves formula (5.2.15). Formula (5.2.16) also uses the fact that $T_r$ is a ring homomorphism. In particular, $\mathcal{M}^r_{j,k}$ are coefficients of the polynomial

$$T_r(P_r(t,\varepsilon)^j) = \sum_{k=1}^{r} \mathcal{M}^r_{j,k}\varepsilon^k.$$

On the other hand, for any $i \leq j$ we have

$$T_r(P_r(t,\varepsilon)^j) = T_r(P_r(t,\varepsilon)^{j-i}P_r(t,\varepsilon)^i) = T_r(P_r(t,\varepsilon)^{j-i})T_r(P_r(t,\varepsilon)^i) \mod \varepsilon^{r+1}.$$

Collecting the coefficients in front of $\varepsilon^k$ in the right side of the formula above we obtain (5.2.16).

□

Note that the function $w(\alpha)$, that we call *weight,* can be given by the following formula

$$w\left(\prod_{i=1}^{n} t_i^{\alpha_i}\right) = (\alpha_1, \alpha_2, \ldots \alpha_n)\begin{pmatrix} 1 \\ 2 \\ .. \\ n \end{pmatrix} = \sum_{k=1}^{n} k\alpha_k. \tag{5.2.17}$$

The weights may be seen as elements in the semi-group of homomorphism from the semi-group of monomials in the variables $t_1, \ldots t_r$ to the $(\mathbb{Z}_{\geq 0}, +)$, in fact:

$$w(\theta \cdot \eta) = w(\theta) + w(\eta).$$

**Remark 5.2.9.** *Instead of considering the polynomials $P_r(t_1, \ldots t_r)$, we might consider formal power series*

$$P^{(\infty)}(\varepsilon, t) = \sum_{i=1}^{\infty} t_i\varepsilon^i,$$

*and truncate all expansions at $\varepsilon^{r+1}$. The result will be the same, but such approach probably makes more clear the recursive nature of the confluence procedure. In similar way, the matrix $\mathcal{M}^{(r)}$ with entries $\mathcal{M}^{(r)}_{i,k}$ given in (2.0.11) can be considered as as a sub-matrix of size $r \times r$ in* *the upper left corner of some infinite dimensional upper triangular "master" matrix $\mathcal{M}^{(\infty)}$ with entries given by*

$$\mathcal{M}^{(\infty)}_{j,r} = \frac{1}{r!}\frac{d^r}{d\varepsilon^r}P^{(\infty)}(\varepsilon,t)^j\Big|_{\varepsilon=0}.$$

### 5.2.4 Proof of the Theorems 5.2.7 and 2.0.5

We use induction here to prove the theorem. Here we will start with the proof of the explicit dependence of the local expansion on $t_i$'s and then we will handle the Poisson structure. The statement of the theorem is true for the $r = 0, 1$ (Fuchsian case and the pole of order 2), this was proven via examples we consider before. Now let the statement be true for the irregular singularity of Pincaré rank $r - 1$. Adding another simple pole $v_r$, we consider the following connection

$$A = \sum_{i=0}^{r-1} \frac{B_i(t_1, t_2 \dots t_{r-1})}{(\lambda - w)^{i+1}} + \frac{C}{\lambda - v_r} + \dots, \quad B_i = \sum_{j=i}^{r-1} B^{[j]}\mathcal{M}^{(r-1)}_{i,j},$$

where the dots denote regular terms in $(\lambda - w)$ and $(\lambda - v_1)$, with the following behaviour of $v_r$

$$v_r = w + \sum_{j=1}^{r} t_j\varepsilon^j, \quad C \simeq \sum_{j=-r}^{\infty} C^{[i]}\varepsilon^i.$$

Expanding $A$ with respect to $\varepsilon$ at $r$'th order we obtain

$$A = \frac{C^{[-r]}t_1^r}{(\lambda - w)^{r+1}} + \sum_{i=0}^{r-1} \frac{B_i + CP_r(t,\varepsilon)^i}{(\lambda - w)^{i+1}} + \dots.$$

Using the formula (5.2.15) the coefficients $B_i$ expands via polynomials $\mathcal{M}^{(r)}_{i,j}$ which gives the following

$$B_i + CP_r(t,\varepsilon)^i = C^{[-r]}\mathcal{M}^{(r)}_{i,r} + \sum_{j=i}^{r-1}\left(B^{[j]} + \varepsilon^j C\right)\mathcal{M}^{(r)}_{i,j}.$$

Since the confluence procedure requires existence of the limit $\varepsilon \to 0$, the negative powers of $\varepsilon$ should vanish, so we obtain expansions for the coefficients $A^{(r)}_j$ in the form

$$B^{[k]} \simeq -\sum_{m=-r}^{-(k+1)} \frac{C^{[m]}}{\varepsilon^{m+k}} + B^{[k,0]} + \sum_{l=1}^{\infty} B^{[k,l]}\varepsilon^l.$$

Using these expansions and taking the $\varepsilon \to 0$ limit we obtain

$$A = \sum_{i=1}^{r+1} \frac{\tilde{B}_i(t_1 \dots t_r)}{(\lambda - u)^i} + \dots,$$

where $\tilde{B}_i$'s are given by (2.0.15), which finishes the proof of the first part of the theorem.

Finally we want to prove that the Poisson structure for the coefficient of the local expansion of the connection near irregular singularity, which arises after confluence procedure is a Poisson structure given by (5.2.13). Our IPA here is given by the formula (2.0.15).

Again we use the induction here. The statement is obvious in case when $r = 0$, and we have previously proved that it holds for $r = 1$. Using the previous results we consider the same coalescence

$$A = \sum_{i=0}^{r-1} \frac{B_i(t_1, t_2 \dots t_{r-1})}{(\lambda - w)^{i+1}} + \frac{C}{\lambda - v_r} + \dots \sim \sum_{i=0}^{r} \frac{\tilde{B}_i}{(\lambda - w)^{i+1}}$$

where $\tilde{B}_i$ is given by (2.0.15).The expansions take the same form

$$C \sim \sum_{j=-r}^{\infty} C^{[i]} \varepsilon^i, \quad B^{[k]} \simeq - \sum_{m=-r}^{-(k+1)} \frac{C^{[m]}}{\varepsilon^{m+k}} + B^{[k,0]} + \sum_{l=1}^{\infty} B^{[k,l]} \varepsilon^l.$$

In order to get rid of the indices let us use the following notation

$$V = B^{[k]}, \quad W = B^{[l]}, \quad U = B^{[k+l]}.$$

In case if indices on the right hand sides are out of bound we assume that the values are zero. The Poisson relations are

$$\{V_\alpha, W_\beta\} = -\chi_{\alpha\beta}^{\gamma} U_\gamma, \quad \{C_\alpha, C_\beta\} = -\chi_{\alpha\beta}^{\gamma} C_\gamma, \quad \{C_\alpha, V_\beta\} = \{C_\alpha, W_\beta\} = \{C_\alpha, U_\beta\} = 0$$

and the expansions are

$$C = \frac{C^{[-r]}}{\varepsilon^r} + \frac{C^{[-r+1]}}{\varepsilon^{r-1}} + \cdots + \frac{C^{[-1]}}{\varepsilon} + C^{[0]} + \sum_{i=1}^{\infty} C^{[i]}\varepsilon^i$$

$$V = -\frac{C^{[-r]}}{\varepsilon^{r-k}} - \frac{C^{[-r+1]}}{\varepsilon^{r-k-1}} - \cdots - \frac{C^{[-k+1]}}{\varepsilon} + V^{[0]} + \sum_{i=1}^{\infty} V^{[i]}\varepsilon^i \ldots$$

$$W = -\frac{C^{[-r]}}{\varepsilon^{r-l}} - \frac{C^{[-r+1]}}{\varepsilon^{r-l-1}} - \cdots - \frac{C^{[-l+1]}}{\varepsilon} + W^{[0]} + \sum_{i=1}^{\infty} W^{[i]}\varepsilon^i \ldots$$

$$U = -\frac{C^{[-r]}}{\varepsilon^{r-k-l}} - \frac{C^{[-r+1]}}{\varepsilon^{r-k-l-1}} - \cdots - \frac{C^{[-k-l+1]}}{\varepsilon} + U^{[0]} + \sum_{i=1}^{\infty} U^{[i]}\varepsilon^i.$$

Due to the confluence formula we want to prove that the following Poisson relation holds

$$\left\{V_\alpha^{[0]} + C_\alpha^{[-k]}, W_\beta^{[0]} + C_\beta^{[-l]}\right\} = \left\{V_\alpha^{[0]}, W_\beta^{[0]}\right\} + \left\{V_\alpha^{[0]}, C_\beta^{[-l]}\right\} + \left\{C_\alpha^{[-k]}, W_\beta^{[0]}\right\} + \left\{C_\alpha^{[-k]}, C_\beta^{[-l]}\right\} =$$
$$= -\chi_{\alpha\beta}^{\gamma}\left(U_\gamma^{[0]} + C_\gamma^{[-k-l]}\right).$$

Taking the corresponding $\varepsilon$-terms of the expansions of the Poisson relations we get

$$\operatorname*{Res}_{\varepsilon=0} \varepsilon^{0-1}\left\{V_\alpha, W_\beta\right\}: \quad \left\{V_\alpha^{[0]}, W_\beta^{[0]}\right\} - \underline{\sum_{i=1}^{r-l}\left\{V_\alpha^{[i]}, C_\beta^{[-i-l]}\right\}} - \underline{\sum_{i=1}^{r-k}\left\{C_\alpha^{[-i-k]}, W_\beta^{[i]}\right\}} = -\chi_{\alpha\beta}^{\gamma} U_\gamma^{[0]},$$

$$\operatorname*{Res}_{\varepsilon=0} \varepsilon^{l-1}\left\{V_\alpha, C_\beta\right\}: \quad \left\{V_\alpha^{[0]}, C_\beta^{[-l]}\right\} + \underline{\sum_{i=1}^{r-l}\left\{V_\alpha^{[i]}, C_\beta^{[-i-l]}\right\}} - \underline{\sum_{i=1}^{r-k}\left\{C_\alpha^{[-i-k]}, C_\beta^{[i-l]}\right\}} = 0,$$

$$\operatorname*{Res}_{\varepsilon=0} \varepsilon^{k-1}\left\{C_\alpha, W_\beta\right\}: \quad \left\{C_\alpha^{[-k]}, W_\beta^{[0]}\right\} + \underline{\sum_{i=1}^{r-k}\left\{C_\alpha^{[-i-k]}, W_\beta^{[i]}\right\}} - \underline{\sum_{i=1}^{r-l}\left\{C_\alpha^{[i-k]}, C_\beta^{[-i-l]}\right\}} = 0$$

$$\operatorname*{Res}_{\varepsilon=0} \varepsilon^{k+l}\frac{\left\{C_\alpha, C_\beta\right\}}{\varepsilon}: \quad \left\{C_\alpha^{[-k]}, C_\beta^{[-l]}\right\} + \underline{\sum_{i=1}^{r-k}\left\{C_\alpha^{[-k-i]}, C_\beta^{[i-l]}\right\}} + \underline{\sum_{i=1}^{r-l}\left\{C_\alpha^{[-k+i]}, C_\beta^{[-i-l]}\right\}} =$$
$$= -\chi_{\alpha\beta}^{\gamma} C_\gamma^{[-l-k]}. \tag{5.2.18}$$

Finally, taking a sum of the relations in (5.2.18) we get the desired Poisson structure

$$\left\{V_\alpha^{[0]}, W_\beta^{[0]}\right\} + \left\{V_\alpha^{[0]}, C_\beta^{[-l]}\right\} + \left\{C_\alpha^{[-k]}, W_\beta^{[0]}\right\} + \left\{C_\alpha^{[-k]}, C_\beta^{[-l]}\right\} = -\chi_{\alpha\beta}^{\gamma}\left(U_\gamma^{[0]} + C_\gamma^{[-l-k]}\right)$$

### 5.2.5 Confluent Hamiltonians

As we saw before, in the case of the 1+1 confluence, the confluent Hamiltonians can be obtained as the limits of some functions on a phase space - linear combinations of the initial Hamiltonians with coefficients depending on a small parameter $\varepsilon$. Moreover the procedure of taking this limit requires the introduction of some shifts by Casimirs, since the Hamiltonians are defined up to Casimir element of the Poisson algebra. Such Casimir normalisation may be exploited in the case of the confluence for the higher order poles, however, this procedure becomes very heavy. In this section, we calculate these limits using residue calculus. Let us start by explaining these limits of the Hamiltonians in the $1+1$ confluence procedure; we want to calculate limits in (5.2.8):

$$\mathrm{d}\tilde{H}_i := \lim_{\varepsilon\to 0} \mathrm{d}H_i, \quad \mathrm{d}\tilde{H}_{n-1} := \lim_{\varepsilon\to 0} \mathrm{d}(H_{n-1}+H_n), \quad \mathrm{d}\tilde{H}_n := \lim_{\varepsilon\to 0} \varepsilon \mathrm{d}H_n.$$

The Fuchsian Hamiltonians (4.3.8) can be written in the following form

$$H_{u_i} = \frac{1}{2} \oint\limits_{\Gamma_{u_i}} \mathrm{Tr}\left(A^2\right) \mathrm{d}\lambda, \tag{5.2.19}$$

where $\Gamma_{u_i}$ is a such contour that no singularities except $u_i$ are inside of it. Since the matrix $A$ admits finite limit as $\varepsilon \to 0$, the integrand has a finite limit. When $u_i \neq v_1, u$, we can always deform $\Gamma_{u_i}$ in such a way that the coalescence of $v_1$ and $u$ does not affect the contour of integration. This gives us opportunity to switch the limit and the integration operations, which gives the formula for the confluent Hamiltonian

$$\tilde{H}_{u_i} = \frac{1}{2} \oint\limits_{\Gamma_{u_i}} \mathrm{Tr}\left(\tilde{A}^2\right) \mathrm{d}\lambda, \qquad u_i \neq t_1, u. \tag{5.2.20}$$

Let us now deal with the limit of $H_w + H_1$. Because both contours $\Gamma_w$ and $\Gamma_1$ will depend on $\varepsilon$, we cannot calculate the limits of $H_w$ and $H_1$ separately. However, we can calculate the limit of the sum due to

$$H_u + H_{v_1} = \frac{1}{2}\oint\limits_{\Gamma_w} \mathrm{Tr}\left(A^2\right) \mathrm{d}\lambda + \frac{1}{2}\oint\limits_{\Gamma_{v_1}} \mathrm{Tr}\left(A^2\right) \mathrm{d}\lambda = \frac{1}{2}\oint\limits_{\Gamma_w \cup \Gamma_{v_1}} \mathrm{Tr}\left(A^2\right) \mathrm{d}\lambda,$$

where the last equality holds since the integrands in both integrals are the same and $\Gamma_u \cup \Gamma_{v_1}$

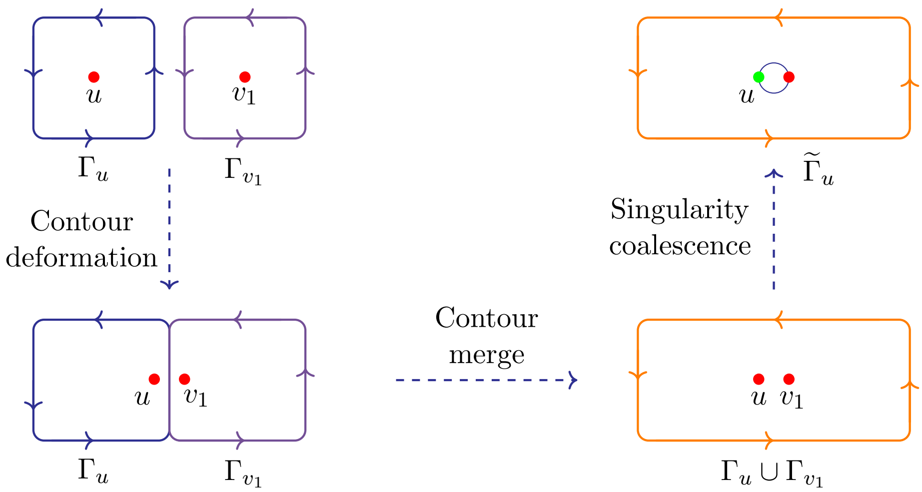



Figure 5.1: Poles and contours confluence procedure

denotes the contour obtained by merging $\Gamma_u$ and $\Gamma_1$ like in Fig. 5.1. Such contour may be deformed to the contour $\tilde{\Gamma}_u$, such that the coalescent singularities are located inside this contour and the confluence doesn't affect the contour itself. Using the same arguments as before we obtain that

$$\tilde{H}_u = \lim_{\varepsilon\to 0} \frac{1}{2} \oint\limits_{\Gamma_w\cup\Gamma_{v_1}} \mathrm{Tr}\left(A^2\right) \mathrm{d}\lambda, = \frac{1}{2}\oint\limits_{\Gamma_w} \mathrm{Tr}\left(\tilde{A}^2\right) \mathrm{d}\lambda. \tag{5.2.21}$$

In order to obtain $\tilde{H}_1$ we consider the following sum of Casimirs

$$\frac{1}{2}\oint\limits_{\Gamma_u} (\lambda-u)\mathrm{Tr}\left(A^2\right) \mathrm{d}\lambda + \frac{1}{2}\oint\limits_{\Gamma_{v_1}} (\lambda-v_1)\mathrm{Tr}\left(A^2\right) \mathrm{d}\lambda$$

which may be put to zero since the Hamiltonians are defined up to Casimirs. Expanding $v_1$ in $\varepsilon$ we obtain

$$\begin{aligned}\frac{1}{2}\oint\limits_{\Gamma_u} (\lambda-u)\mathrm{Tr}\left(A^2\right) \mathrm{d}\lambda + \frac{1}{2}\oint\limits_{\Gamma_{v_1}} (\lambda-u)\mathrm{Tr}\left(A^2\right) \mathrm{d}\lambda - t_1\varepsilon\frac{1}{2}\oint\limits_{\Gamma_{v_1}} \mathrm{Tr}\left(A^2\right) \mathrm{d}\lambda = \\ = \frac{1}{2}\oint\limits_{\Gamma_u\cup\Gamma_{v_1}} (\lambda-u)\mathrm{Tr}\left(A^2\right) \mathrm{d}\lambda - t_1\varepsilon H_1 = 0.\end{aligned}$$

The relation written above finally gives us

$$\tilde{H}_1 = \frac{1}{2\tilde{t}_1}\oint\limits_{\Gamma_u} (\lambda-u)\mathrm{Tr}\left(\tilde{A}^2\right) \mathrm{d}\lambda = \operatorname*{Res}_{\lambda=u}\left[\frac{(\lambda-u)}{t_1}\mathrm{Tr}\left(\frac{\tilde{A}^2}{2}\right)\right]. \tag{5.2.22}$$

Let us now add one more simple pole to $w$ using the confluence, by a similar computation as above, we obtain the following isomonodromic Hamiltonians

$$H_{u_i} = \operatorname*{Res}_{\lambda=u_i} \mathrm{Tr}\left(\frac{A^2}{2}\right), \quad H_u = \operatorname*{Res}_{\lambda=u} \mathrm{Tr}\left(\frac{A^2}{2}\right)$$

$$H_1 = \operatorname*{Res}_{\lambda=u}\left(\left[\frac{(\lambda-u)}{t_1} - \frac{t_2(\lambda-u)^2}{t_1^3}\right]\mathrm{Tr}\left(\frac{A^2}{2}\right)\right), \quad H_2 = \operatorname*{Res}_{\lambda=u}\left[\frac{(\lambda-u)^2}{t_1^2}\mathrm{Tr}\left(\frac{A^2}{2}\right)\right].$$

The form of the Hamiltonians for $t_1$ and $t_2$ looks quite bizarre, but we may obtain them by solving the following linear system

$$\mathcal{M}^{(2)}\begin{pmatrix} H_1 \\ H_2 \end{pmatrix} = \begin{pmatrix} t_1 & t_2 \\ 0 & t_1^2 \end{pmatrix}\begin{pmatrix} H_1 \\ H_2 \end{pmatrix} = \begin{pmatrix} S_1 \\ S_2 \end{pmatrix}, \quad S_i = \frac{1}{2}\oint_{\Gamma_u}(\lambda-u)^i \mathrm{Tr}A^2 \mathrm{d}\lambda.$$

Here $\mathcal{M}^{(2)}$ is a matrix which entries were already introduced in (2.0.11).We now prove Theorem 2.0.6.

**Theorem 5.2.10.** *Let $u$ be a pole of a connection $A$ with Poincaré rank $r$, which is the result of confluence of $r$ simple poles with the simple pole $u$. Then the confluent Hamiltonians $H_1, \dots, H_r$ which correspond to the times $t_1, \dots t_r$ are defined as follows:*

$$\begin{pmatrix} H_1 \\ H_2 \\ \dots \\ H_r \end{pmatrix} = \left(\mathcal{M}^{(r)}\right)^{-1}\begin{pmatrix} S_1^{(u)} \\ S_2^{(u)} \\ \dots \\ S_r^{(u)} \end{pmatrix}, \tag{5.2.23}$$

*where*

$$S_k^{(u)} = \frac{1}{2}\oint_{\Gamma_u}(\lambda-u)^k \mathrm{Tr}A^2 \mathrm{d}\lambda \tag{5.2.24}$$

*are spectral invariants of order $i$ in $u$ and the matrix $\mathcal{M}^{(r)}$ is given by (2.0.11). The Hamiltonian $H_u$ corresponding to the time $u$ is instead given by the standard formula*

$$H_{u_i} = \frac{1}{2}\operatorname*{res}_{\lambda=u_i} \mathrm{Tr}A(\lambda)^2.$$

*Proof.* To prove this theorem we again use induction. We showed that it holds for $r = 2$ and it is trivial in case when $r=1$. Now we want to prove that if the statement of the theorem holds for rank $r$ it is also true for rank $r+1$. The confluence expansion

$$v_{r+1} = u + P_{r+1}(t,\varepsilon) = u + \sum_{i=1}^{r+1} \varepsilon^i t_i$$

sends the extended symplectic form

$$\omega = \mathrm{d}t_1 \wedge \mathrm{d}H_1 + \dots + \mathrm{d}t_r \wedge \mathrm{d}H_r + \mathrm{d}u \wedge \mathrm{d}H_u + \mathrm{d}v_{r+1} \wedge \mathrm{d}H_{r+1} + \dots$$

where the last set of dots corresponds to the terms that are not changed in the confluence procedure, to

$$\mathrm{d}t_1 \wedge \mathrm{d}(H_1 + \varepsilon H_{r+1}) + \dots + \mathrm{d}t_r \wedge \mathrm{d}(H_r + \varepsilon^r H_{r+1}) + \mathrm{d}t_{r+1} \wedge \mathrm{d}(\varepsilon^{r+1} H_{r+1}) + \mathrm{d}u \wedge \mathrm{d}(H_u + H_{r+1}) + \dots =$$

that must be equal to

$$\mathrm{d}t_1 \wedge \mathrm{d}\tilde{H}_1 \dots \mathrm{d}t_r \wedge \mathrm{d}\tilde{H}_r + \mathrm{d}t_{r+1} \wedge \mathrm{d}\tilde{H}_{r+1} + \mathrm{d}u \wedge \mathrm{d}\tilde{H}_u + \dots.$$

Therefore, we must find the limits

$$\tilde{H}_u = \lim_{\varepsilon \to 0} \left(H_u + H_{r+1}\right), \quad \tilde{H}_k = \lim_{\varepsilon \to 0} \left(H_k + \varepsilon^k H_{r+1}\right), \quad k = 1 \dots r, \quad \tilde{H}_{r+1} = \lim_{\varepsilon \to 0} \varepsilon^{r+1} H_{r+1}$$

The first limit is quite simple and may be obtained via the union of contours which we already described before. To find the other limits, let us consider the relation

$$\oint_{\Gamma_u} (\lambda - u)^i \mathrm{Tr} \frac{A^2}{2} \mathrm{d}\lambda + \oint_{\Gamma_{r+1}} (\lambda - v_{r+1})^i \mathrm{Tr} \frac{A^2}{2} \mathrm{d}\lambda = S_i^{(u)} \mod (\text{Casimirs})$$

expanding $v_{n+1}$ we obtain

$$\oint_{\Gamma_u \cup \Gamma_{r+1}} (\lambda - u)^i \mathrm{Tr} \frac{A^2}{2} \mathrm{d}\lambda - \oint_{\Gamma_{r+1}} \phi(\lambda) \mathrm{Tr} \frac{A^2}{2} = S_i^{(u)} \mod (\text{Casimirs}),$$

where $\phi(\lambda)$ is a holomorphic function inside $\Gamma_{r+1}$ which is given by

$$\phi(\lambda) = (\lambda - u)^i - (\lambda - u - P_{r+1}(t,\varepsilon))^i.$$

Since $\phi(\lambda)$ has no zeros at $v_{r+1}$ we have

$$\oint_{\Gamma_{r+1}} \phi(\lambda)\mathrm{Tr}\frac{A^2}{2} = \phi(u + P_{r+1}(t,\varepsilon)) \oint_{\Gamma_{r+1}} \mathrm{Tr}\frac{A^2}{2} = P_{r+1}(t,\varepsilon)^i \oint_{\Gamma_{r+1}} \mathrm{Tr}\frac{A^2}{2} = P_{r+1}(t,\varepsilon)^i H_{r+1}.$$

Finally, we obtain the following identity:

$$\oint_{\Gamma_u \cup \Gamma_{r+1}} (\lambda - u)^i \mathrm{Tr}\frac{A^2}{2}\mathrm{d}\lambda - P_{r+1}(t,\varepsilon)^i H_{r+1} = S_i^{(u)} \mod (\mathrm{Casimirs})\,. \tag{5.2.25}$$

In the case $i = r+1$, $S_{r+1}^{(u)}$ is a Casimir due to the formula (5.4.7), therefore we have

$$\oint_{\Gamma_u \cup \Gamma_{r+1}} (\lambda - u)^{r+1} \mathrm{Tr}\frac{A^2}{2}\mathrm{d}\lambda = P_{r+1}(t,\varepsilon)^{r+1} H_{r+1} \mod (\mathrm{Casimirs})\,.$$

The left hand side of this identity has a finite limit when $\varepsilon$ goes to 0. Indeed, since the contour contains both $u$ and $v_{r+1}$ the confluence procedure doesn't affect it and the only dependence in $\varepsilon$ is in $A$. According to Theorem 2.0.5, $A$ has a finite limit, the same has $\mathrm{Tr}A^2$, so we have that

$$\lim_{\varepsilon\to 0} \oint_{\Gamma_u \cup \Gamma_{r+1}} (\lambda - u)^{r+1} \mathrm{Tr}\frac{A^2}{2}\mathrm{d}\lambda = \oint_{\Gamma_u \cup \Gamma_{r+1}} (\lambda - u)^{r+1} \lim_{\varepsilon\to 0}\mathrm{Tr}\frac{A^2}{2}\mathrm{d}\lambda = \oint_{\Gamma_u} (\lambda - u)^{r+1} \mathrm{Tr}\frac{\tilde{A}^2}{2}\mathrm{d}\lambda = \tilde{S}_{r+1}^{(u)},$$

where $\tilde{S}_{r+1}^{(u)}$ is a spectral invariant of the confluent system with connection $\tilde{A}$. Since after the confluence, the order of pole increases to $r+2$, such spectral invariant is not a Casimir for the confluent system. This means, that the limit of $P_{r+1}(t,\varepsilon)^{r+1} H_{r+1}$ up to Casimirs exists and equals to $\tilde{S}_{r+1}^{(u)}$. On the other hand we have

$$P_{r+1}(t,\varepsilon)^{r+1} H_{r+1} = \left(t_1^{r+1}\varepsilon^{r+1} + O(\varepsilon^{r+2})\right) H_{r+1},$$

and since the limit exists up to Casimirs we get that

$$H_{r+1} \mod (\mathrm{Casimirs}) = \frac{\tilde{S}_{r+1}^{(u)}}{\varepsilon^{r+1}} + \sum_{i=-r}^{\infty} H_{r+1}^{[i]} \varepsilon^i,$$

so in principle $H_{r+1}$ may have terms of lower order than $1/\varepsilon^{r+1}$, but these terms has to be Casimirs. Considering relations (5.2.25) for $i = 1 \dots r+1$ as a linear system we obtain

$$\begin{pmatrix} \tilde{S}_1 \\ \tilde{S}_2 \\ \tilde{S}_3 \\ \dots \\ \tilde{S}_{r+1} \end{pmatrix} - \mathcal{M}^{(r+1)} \begin{pmatrix} \varepsilon \\ \varepsilon^2 \\ \varepsilon^3 \\ \dots \\ \varepsilon^{r+1} \end{pmatrix} H_{r+1} = \begin{pmatrix} S_1 \\ S_2 \\ \dots \\ S_r \\ 0 \end{pmatrix} \quad \text{mod (Casimirs)} \tag{5.2.26}$$

where

$$\tilde{S}_i = \oint_{\Gamma_u \cup \Gamma_{r+1}} (\lambda - u)^i \mathrm{Tr} \frac{A^2}{2} \mathrm{d}\lambda, \quad S_i = \oint_{\Gamma_u} (\lambda - u)^i \mathrm{Tr} \frac{A^2}{2} \mathrm{d}\lambda$$

Note that the contours in the definition of $\tilde{S}_i$ are not affected by the confluence procedure. Using the same arguments as above, we compute the limits of these integrals, which are

$$\lim_{\varepsilon \to 0} \tilde{S}_i = \tilde{S}_i^{(u)} = \oint_{\Gamma_u} (\lambda - u)^i \mathrm{Tr} \frac{\tilde{A}^2}{2} \mathrm{d}\lambda$$

where $\tilde{S}_i^{(u)}$ denote the spectral invariants of the confluent system with connection $\tilde{A}$.

The crucial point here is that the matrix $\mathcal{M}^{(r+1)}$ contains $\mathcal{M}^{(r)}$ as $r+1, r+1$ minor, i.e.

$$\mathcal{M}^{(r+1)} = \begin{pmatrix} & & & & t_{r+1} \\ & & & & \vdots \\ & & \mathcal{M}^{(r)} & & \vdots \\ & & & & \\ 0 & 0 & \dots & 0 & t_1^{r+1} \end{pmatrix}$$

Now let us consider the following matrix

$$\mathcal{C} = \begin{pmatrix} & & & & 0 \\ & & \left(\mathcal{M}^{(r)}\right)^{-1} & & \vdots \\ & & & & \vdots \\ & & & & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

and let us act via $\mathcal{C}$ on the equation (5.2.26) from the left. Then we get

$$\mathcal{C}\tilde{S} = \begin{pmatrix} & & & & \vdots \\ & \mathbb{I}_r & & & \vdots \\ & & & & \\ 0 & 0 & \dots & 0 & t_1^{r+1} \end{pmatrix} \begin{pmatrix} \varepsilon H_{r+1} \\ \varepsilon^2 H_{r+1} \\ \dots \\ \varepsilon^n H_{r+1} \\ \varepsilon^{n+1} H_{r+1} \end{pmatrix} + \begin{pmatrix} H_1 \\ H_2 \\ \dots \\ H_r \\ 0 \end{pmatrix} =$$

$$= \begin{pmatrix} & & & & \vdots \\ & \mathbb{I}_r & & & \vdots \\ & & & & \\ 0 & 0 & \dots & 0 & t_1^{r+1} \end{pmatrix} \begin{pmatrix} \varepsilon H_{r+1} + H_1 \\ \varepsilon^2 H_{r+1} + H_2 \\ \dots \\ \varepsilon^r H_{r+1} + H_r \\ \varepsilon^{r+1} H_{r+1} \end{pmatrix} = \mathcal{C}\mathcal{M}^{(r+1)} \begin{pmatrix} \varepsilon H_{r+1} + H_1 \\ \varepsilon^2 H_{r+1} + H_2 \\ \dots \\ \varepsilon^r H_{r+1} + H_r \\ \varepsilon^{n+1} H_{r+1} \end{pmatrix}.$$

In this way we have arranged the entries of equation (5.2.26) in such a way that the left hand side has a nice limit as $\varepsilon$ goes to zero (the confluence of points is inside the contour of integration for $\tilde{S}_i$). On the right hand side we have the functions whose limits we want to find. Finally, multiplying by $\mathcal{C}^{-1}$ from the left we obtain

$$\mathcal{M}^{(r+1)} \begin{pmatrix} \tilde{H}_1 \\ \tilde{H}_2 \\ \dots \\ \tilde{H}_r \\ \tilde{H}_{r+1} \end{pmatrix} = \mathcal{M}^{(r+1)} \lim_{\varepsilon \to 0} \begin{pmatrix} \varepsilon H_{r+1} + H_1 \\ \varepsilon^2 H_{r+1} + H_2 \\ \dots \\ \varepsilon^r H_{r+1} + H_n \\ \varepsilon^{r+1} H_{r+1} \end{pmatrix} = \lim_{\varepsilon \to 0} \begin{pmatrix} \tilde{S}_1 \\ \tilde{S}_2 \\ \tilde{S}_3 \\ \dots \\ \tilde{S}_{r+1} \end{pmatrix} = \begin{pmatrix} \tilde{S}_1 \\ \tilde{S}_2 \\ \tilde{S}_3 \\ \dots \\ \tilde{S}_{r+1} \end{pmatrix}$$

which finishes the proof. □

**Remark 5.2.11.** *The matrix $\mathcal{M}^{(r)}$ is automatically upper triangular matrix, so it is quite easy to solve such a system for any reasonable $n$.*

In general we may consider Hamiltonians which are related to poles locations as a spectral invariant $S_0^{(u)}$. Then it is easy to extend formula from (5.2.23) as follows

$$\mathcal{N}^{(r)}\begin{pmatrix} H_0 \\ H_1 \\ \dots \\ H_r \end{pmatrix} = \begin{pmatrix} S_0^{(u)} \\ S_1^{(u)} \\ \dots \\ S_r^{(u)} \end{pmatrix}, \quad \mathcal{N}^{(r)} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & \mathcal{M}^{(r)} & \\ 0 & & & \end{pmatrix}$$

### 5.2.6 Examples of Hamiltonians

We want to demonstrate obtained Hamiltonians in the previous section on a couple of examples. In order to see all the features of obtained Hamiltonians we consider connection with 3 simple poles at $0, 1$ and $\infty$, and one irregular singularity at some point $u$. The minimal example is

$$A(\lambda) = \frac{A^{(0)}}{\lambda} + \frac{A^{(1)}}{\lambda - 1} + \frac{A_0^{(u)}}{\lambda - u} + \frac{t_1 A_1^{(u)}}{(\lambda - u)^2}. \tag{5.2.27}$$

Explicit formulas for Hamiltonians are

$$H_u = \mathrm{Tr}\left[A_0^{(u)}\left(\frac{A^{(0)}}{u} + \frac{A^{(1)}}{u-1}\right) - t_1 A_1^{(u)}\left(\frac{A^{(1)}}{(u-1)^2} + \frac{A^{(0)}}{u^2}\right)\right] \tag{5.2.28}$$

$$H_1 = \mathrm{Tr}\left[A_1^{(u)}\left(\frac{A^{(0)}}{u} + \frac{A^{(1)}}{u-1}\right) + \frac{A_0^{(u)}A_0^{(u)}}{2t_1}\right] \tag{5.2.29}$$

In lifted Darboux coordinates, Hamiltonians take form

$$H_u = \mathrm{Tr}\left(Q_0^{(u)}P_0^{(u)} + Q_1^{(u)}P_1^{(u)}\right)\left(\frac{Q^{(0)}P^{(0)}}{u} + \frac{Q^{(1)}P^{(1)}}{u-1}\right) - \\ - t_1 Q_0^{(u)}P_1^{(u)}\left(\frac{Q^{(1)}P^{(1)}}{(u-1)^2} - \frac{Q^{(0)}P^{(0)}}{u^2}\right) \tag{5.2.30}$$

$$H_1 = \mathrm{Tr}\frac{Q^{(0)}P^{(0)}Q_0^{(u)}P_1^{(u)}}{u} + \frac{Q^{(1)}P^{(1)}Q_0^{(u)}P_1^{(u)}}{u-1} + \frac{\left(Q_0^{(u)}P_0^{(u)} + Q_1^{(u)}P_1^{(u)}\right)^2}{2t_1}. \tag{5.2.31}$$

Hamiltonians obviously Poisson commute, moreover we may check that cross-derivative w.r.t. $u$ and $t_1$ is zero, i.e.

$$\frac{\partial}{\partial t_1}H_u - \frac{\partial}{\partial u}H_1 = 0,$$

which tells us that $\tau$-function

$$\mathrm{d}\ln\tau = \mathrm{Tr}\left(A_0^{(u)}\left(\frac{A^{(0)}}{u} + \frac{A^{(1)}}{u-1}\right) - t_1 A_1^{(u)}\left(\frac{A^{(1)}}{(u-1)^2} - \frac{A^{(0)}}{u^2}\right)\right)\mathrm{d}u + \\ + \mathrm{Tr}\left(A_1^{(u)}\left(\frac{A^{(0)}}{u} + \frac{A^{(1)}}{u-1}\right) + \frac{A_0^{(u)}A_0^{(u)}}{2t_1}\right)\mathrm{d}t_1$$

is defined correctly. If we go further and consider pole of Poincaré rank 2 at $u$ connection takes form

$$A(\lambda) = \frac{A^{(0)}}{\lambda} + \frac{A^{(1)}}{\lambda-1} + \frac{A_0^{(u)}}{\lambda-u} + \frac{t_1A_1^{(u)} + t_2A_2^{(u)}}{(\lambda-u)^2} + \frac{t_1^2A_2^{(u)}}{(\lambda-u)^3}. \tag{5.2.32}$$

Hamiltonians then write as

$$H_u = \mathrm{Tr}\Bigg[A_0^{(u)}\left(\frac{A^{(0)}}{u} + \frac{A^{(1)}}{u-1}\right) - t_1A_1^{(u)}\left(\frac{A^{(0)}}{u^2} + \frac{A^{(1)}}{(u-1)^2}\right) + \\ + A_2^{(u)}\left(\frac{t_1^2A^{(0)}}{u^3} + \frac{t_1^2A^{(1)}}{(u-1)^3} - \frac{t_2A^{(0)}}{u^2} - \frac{t_2A^{(1)}}{(u-1)^2} + \right)\Bigg]$$

$$H_1 = \mathrm{Tr}\left[A_1^{(u)}\left(\frac{A^{(0)}}{u} + \frac{A^{(1)}}{u-1}\right) - t_1A_2^{(u)}\left(\frac{A^{(0)}}{u^2} + \frac{A^{(1)}}{(u-1)^2}\right) + \frac{A_0^{(u)}A_0^{(u)}}{2t_1} - t_2\frac{A_0^{(u)}A_1^{(u)}}{t_1^2} - t_2^2\frac{A_0^{(u)}A_2^{(u)}}{t_1^3}\right]$$

$$H_2 = \mathrm{Tr}\left[A_2^{(u)}\left(\frac{A^{(0)}}{u} + \frac{A^{(1)}}{u-1}\right) + \frac{A_0^{(u)}A_1^{(u)}}{t_1} + t_2\frac{A_0^{(u)}A_2^{(u)}}{t_1^2}\right].$$

As in the previous example cross-derivatives are zeros

$$\frac{\partial}{\partial t_1}H_u - \frac{\partial}{\partial u}H_1 = \frac{\partial}{\partial t_2}H_u - \frac{\partial}{\partial u}H_2 = \frac{\partial}{\partial t_1}H_2 - \frac{\partial}{\partial t_2}H_1 = 0,$$

so the $\tau$-function is defined correctly.

The first example corresponds to the confluence of the Garnier system – isomonodromic deformation of the connection on the 5 punctured sphere. Reduction of this example for $\mathfrak{sl}_2$ has to give equation form the list introduced in [60, 61] and corresponds to so-called fourth-order Painlevé equations. The second example has to be more complicated – Hamiltonian reduction then gives Hamiltonian system with 3 degrees of freedom, which corresponds to the higher order Painlevé equations.

### 5.2.7 Confluence of the higher order poles.

The confluence of two poles $w_1$ and $w_2$ of Poincaré rank $r_1$ and $r_2$ respectively can be treated a the confluence of $r_1 + r_2 + 2$ simple poles. Indeed, we have seen at the beginning of section 5.2 that the generic connection with a Poincaré rank $r$ singularity can be obtained as confluence of $r + 1$ simple poles. Therefore, a connection with two poles $w_1$ and $w_2$ of Poincaré rank $r_1$ and $r_2$ respectively is obtained by coalescing $r_1 + 1$ and $r_2 + 1$ simple poles. To confluence $w_1$ with $w_2$ is therefore the same as coalescing $r_1 + r_2 + 2$ simple poles.

## 5.3 Poisson automorphisms of the Takiff algebra and independent deformation parameters.

In this subsectionwe describe the class of linear automorphisms of the Takiff algebra which preserve the Poisson bracket, namely linear maps

$$B_i = \sum_{j=0}^{r} T_{ij} A_j, \qquad T_{ij} \in \mathbb{C}, \quad i, j = 0, \dots r, \tag{5.3.1}$$

such that

$$\{B_i \underset{,}{\otimes} B_j\} = [\Pi, \mathbb{I} \otimes B_{i+j}] \quad \Longleftrightarrow \quad \{A_i \underset{,}{\otimes} A_j\} = [\Pi, \mathbb{I} \otimes A_{i+j}] \tag{5.3.2}$$

In the next theorem we describe explicitly the constraints on the coefficients $T_{ij}$ and show that they define an ideal in $\mathbb{C}[T_{00} \dots T_{rr}]$. We then give an explicit parameterisation of the quotient of $\mathbb{C}[T_{00} \dots T_{rr}]$ with respect to this ideal in terms of $r$ parameters. $t_1, \dots, t_r$.

**Theorem 5.3.1.** *The Poisson condition (5.3.2) generates the ideal $\mathcal{P}$ in the ring $\mathbb{C}[T_{11} \dots T_{rr}]$*

*given by the equations*

$$\mathcal{P} = \begin{cases} T_{00} = 1 & \\ T_{0k} = 0, & k > 0 \\ T_{k0} = 0, & k > 0 \\ T_{ik} = 0, & k<i \\ T_{sl} = \sum\limits_{i,j>0}^{i+j=l} T_{pi}T_{mj} & \forall p, m > 0 : \, p+m = s. \end{cases} \tag{5.3.3}$$

*Moreover we have the following ring isomorphism for the quotient*

$$\mathcal{Q}: \quad \mathbb{C}[T_{00} \dots T_{rr}]/\mathcal{P} \to \mathbb{C}[t_1 \dots t_r] \tag{5.3.4}$$

*such that*

$$T_{1i} = t_i, \quad T_{ki} = \frac{1}{i!}\frac{d^i}{d\varepsilon^i} P_r(t,\varepsilon)^k \Big|_{\varepsilon=0}, \quad P_r(t,\varepsilon) = \sum_{i=1}^{r} \varepsilon^i t_i, \tag{5.3.5}$$

*so that $T_{ki}$ is just the coefficient of the $\varepsilon^i$ term in the polynomial $P_r(t,\varepsilon)^k$.*

**Remark 5.3.2.** *The equations which define the ideal $\mathcal{P}$ do not depend on the specific form of* $\Pi$*, i.e. on the structure constants of a Poisson bracket. Therefore, the classification of the automorphisms is a consequence of the grading structure and not a property of the specific Lie co-algebra.*

*Proof.* Assume the matrices $A_i$ and $B_i$ satisfy the Poisson relations (5.3.2) and prove the relations for the coefficients $T_{ij}$. Let us start from the relation for $B_1$

$$\{B_0 \underset{,}{\otimes} B_0\} = [\Pi, \mathbb{I} \otimes B_0]. \tag{5.3.6}$$

Substituting (5.3.1) in (5.3.6) and expanding, we obtain

$$\{B_0 \underset{,}{\otimes} B_0\} = \sum_{i,j=0}^{r} T_{0i}T_{0j}\{A_i \underset{,}{\otimes} A_j\} = \sum_{k=0}^{r}\left(\sum_{i=0}^{k} T_{0i}T_{0,k-i}\right)[\Pi, \mathbb{I}\otimes A_k] =$$

$$= [\Pi, \mathbb{I}\otimes B_0] = \sum_{k=1}^{r} T_{0k}[\Pi, \mathbb{I}\otimes A_k]. \tag{5.3.7}$$

This equation defines the system for the coefficients $T_{0j}$, which takes form

$$T_{00}T_{00} = T_{00}, \quad 2T_{00}T_{0k} + \sum_{i=1}^{k-1} T_{0i}T_{0,k-i} = T_{0k},$$

that, by recursion, leads to the first set of equations which generate ideal $\mathcal{P}$:

$$T_{00} = 1, \quad T_{0k} = 0, \quad k > 0.$$

The next statement we want to prove is that $T_{k0} = 0$ for $k > 1$. We use

$$\{B_1 \underset{,}{\otimes} B_k\} = [\Pi, \mathbb{I} \otimes B_{k+1}] \qquad k = 1, \ldots, r. \tag{5.3.8}$$

Again, substituting (5.3.1) and expanding, we obtain

$$\sum_{i,j=0}^{r} T_{1i}T_{kj}[\Pi, \mathbb{I} \otimes A_{i+j}] = \sum_{j=0}^{r} T_{k+1,j}[\Pi, \mathbb{I} \otimes A_j],$$

and collecting all coefficients of $[\Pi, \mathbb{I} \otimes A_1]$, we have that

$$T_{k+1,0} = T_{10}T_{k0},$$

that is solved by

$$T_{k+1,0} = (T_{10})^k.$$

On the other hand substituting (5.3.1) in

$$\{B_1 \underset{,}{\otimes} B_r\} = 0, \tag{5.3.9}$$

we obtain

$$T_{10}T_{r0} = 0 = (T_{10})^r \quad \Rightarrow \quad T_{10} = 0,$$

as we wanted. Now to demonstrate the statement that $T_{ik} = 0$ for $k<i$ we use the relation

$$\{B_1 \underset{,}{\otimes} B_1\} = [\Pi, \mathbb{I} \otimes B_2]. \tag{5.3.10}$$

By substituting (5.3.1) we see that the left hand side of (5.3.10)

$$\{B_1 \underset{,}{\otimes} B_1\} = \sum_{i,j>2} T_{1i}T_{1j}[\Pi, A_{i+j}] = T_{11}T_{11}[\Pi, A_2] + \sum_{i=4} \kappa_i[\Pi, A_i], \tag{5.3.11}$$

does not contain terms in $A_0$, or $A_1$, it contains only one term that depends on $A_2$, given by $T_{11}T_{11}[\Pi, A_2]$ and all other terms depend on $A_3, \ldots, A_r$. Expanding right hand side of (5.3.10) we get

$$[\Pi, \mathbb{I} \otimes B_2] = T_{21}[\Pi, \mathbb{I} \otimes A_1] + \sum_{i=2} T_{2i}[\Pi, \mathbb{I} \otimes A_i]. \tag{5.3.12}$$

Therefore, we obtain that $T_{21} = 0$. Similarly, applying the $\{B_1 \otimes \circ\}$ to $B_2 \ldots B_r$ and using the same approach we obtain that $T_{ik} = 0$ for $k < i$. The last relation in (5.3.3) is obtained by imposing (5.3.2), substituting (5.3.1) and expanding as before, and then by imposing all other conditions we have obtained so far.

We now prove the second part of the Theorem. First of all, we observe that thanks to relations (5.3.3), the coefficients $t_j := T_{1j}$ for $j > 0$ form a basis in the quotient ring $\mathcal{Q}$ : $\mathbb{C}[T_{00} \ldots T_{rr}]/\mathcal{P}$. Then, because each $T_{ik}$ must be given by a polynomial $P_k^{(i)}$of $t_1, \ldots, t_r$, we just need to check the degree and the form of the coefficients. To this aim we use the last relation of (5.3.3) for $T_{ij}$ by induction on $j$ from $i$ to $r$. We omit this computation as it is straightforward. □

**Example 5.3.3.** In order to give a taste of how the general elements of the Takiff co-algebra depend on the Poisson automorphism parameters $t_i$, we provide a few examples of low degree. We consider an element of the Takiff co-algebra as a polynomial in $\frac{1}{z}$. In the case of $\hat{\mathfrak{g}}_1^\star$ we have

$$B(z) = \frac{t_1 A_1}{z^2} + \frac{A_0}{z}. \tag{5.3.13}$$

Here this automoprhism just establish that the invariant space of the action of $A_0$ is defined up to multiplication by a constant, so in a sense this example is quite trivial. For the $\hat{\mathfrak{g}}_2^\star$ the general element writes as

$$B(z) = \frac{t_1^2 A_2}{z^3} + \frac{t_1 A_1 + t_2 A_2}{z^2} + \frac{A_0}{z}. \tag{5.3.14}$$

The next example is the case of $\hat{\mathfrak{g}}_3^\star$ and the element of the co-algebra writes as

$$B(z) = \frac{t_1^3 A_3}{z^4} + \frac{t_1^2 A_2 + 2t_1 t_2 A_3}{z^3} + \frac{t_1 A_1 + t_2 A_2 + t_3 A_3}{z^2} + \frac{A_0}{z}. \tag{5.3.15}$$

In the previous section we saw how such dependence on the parameters $t_i$'s arises during the confluence procedure. In some sense, the irregular deformation parameters are just the deformation of the representation for the Takiff algebra.

## 5.4 Darboux coordinates for $\mathfrak{sl}_2$ Takiff algebra coadjoint orbits

In this section, we compute explicitly the reduced coordinates for the co-adjoint orbits of the quotient of Takiff algebras with respect to the inner group action on the lifted Darboux coordinates in the case of degrees 1, 2, 3 and 4 - this choice is motivated by the fact that in the Painlevé confluence scheme the maximal pole order we have is 4. However, the described procedure can be easily expanded for the Takiff algebra of any degree - we give a hint and some explanation in the discussion after the examples. In each example we give explicit results in the case of $\mathfrak{sl}_2$, since this is the case of the isomonodromic problems for the Painlevé equations. We also provide the coordinates in the diagonal gauge - the case when the leading term is diagonal. We do this because there is also the additional outer action of the group $G$ which can be used to put one orbit in such form.

### 5.4.1 Lifted Darboux coordinates

As seen in the previous section, the lifted Darboux coordinates for the co-adjoint orbits of an ordinary Lie algebra are given by a symplectic reduction from $T^\star \mathfrak{gl}_m$. We use the same idea for the truncated loop algebras. This follows ideas introduced in [24] to parametrize the space of the irregular Gaudin systems, which are autonomous version of the isomonodromic systems. However, Chervov and Talalaev didn't perform the explicit reduction and the parametrisation of the co-adjoint orbit of the truncated loop algebras. So in this section we provide detailed reduction procedure.

We start from the following space

$$\mathfrak{g} = \mathfrak{gl}_{\mathfrak{m}}, \quad T^\star\hat{\mathfrak{g}}_r = \left\{ (P,Q) \,\Big|\, P = \sum_{i=0}^{r} P_i z^i,\, Q = \sum_{i=0}^{r} Q_i z^{-i-1}, \quad P_i, Q_i \in \mathfrak{gl}_m \right\}.$$

The symplectic form on $T^\star\hat{\mathfrak{g}}_n$ is given by the differential of the Liouville form:

$$\omega = \mathrm{d}\langle P, \mathrm{d}Q\rangle = \oint_{S^1} \mathrm{Tr}\left(\mathrm{d}P \wedge \mathrm{d}Q\right) dz = \mathrm{d}\sum_{i=0}^{r} \mathrm{Tr}\left(P_i \wedge \mathrm{d}Q_i\right), \tag{5.4.1}$$

here $d$ is a differential on the space of spectral parameter $z$, while d is a differential on a phase space.

**Lemma 5.4.1.** *The map*

$$\begin{array}{ccc} \bigoplus_{i=0}^{r} T^\star\mathfrak{gl}_m & \to & T^\star\hat{\mathfrak{g}}_r \\ (P_0,\dots,P_r,Q_0,\dots,Q_r) & \mapsto & (P,Q) \end{array}$$

*is a symplectomorphism.*

The proof of this result is a straightforward consequence of the fact that $T^\star\hat{\mathfrak{g}}_n$ and $\bigoplus_{i=1}^{n} T^\star\mathfrak{gl}_m$ are isomorphic as vector spaces and formula (5.4.1) shows that they are symplectomorphic to each other. However, we have enphasised this simple fact into a Lemma because $\bigoplus_{i=1}^{n} T^\star\mathfrak{gl}_m$ provides the ambient space for the confluence procedure.

We now want to construct the Lie group $\hat{G}_r$ of the Takiff algebra. Its elements are given by:

$$g(z) = g_0 + \sum_{i=1}^{r} g_i z^i, \quad g_0 \in GL_m, \quad g_i \in \mathfrak{gl}_m,$$

where, in order to be able to multiply both on the left and on the right, $\mathfrak{gl}_m$ is considered as a bi-module of $GL_m$. The group structure of $\hat{G}_n$ is given by $GL_m$ multiplication mod $z^n$, i.e.

$$g(z)\cdot h(z) = g(z)h(z) \bmod z^{r+1} = g_0 h_0 + \sum_{i=1}^{r}\left(\sum_{j=0}^{i} g_{i-j}h_j\right) z^i.$$

The inverse is given by

$$g^{-1} = g_0^{-1}\left[1 + \sum_{i=1}^{r} g_0^{-1} g_{i+1} z^i\right]^{-1} = g_0^{-1}(1+\tilde{g}(z))^{-1} = g_0^{-1}\sum_{i=0}^{\infty}(-1)^i \tilde{g}(z)^i \bmod z^{r+1},$$

and the neutral element is given by the identity matrix. The induced inner and outer actions on $T^\star\hat{\mathfrak{g}}^r$ are given by

$$g \underset{\text{outer}}{\times} (P, Q) = \left([P \circ g] \,\mathrm{mod}\, z^{r+1}; \pi_- \left[g^{-1} \circ Q\right]\right) \tag{5.4.2}$$

$$g \underset{\text{inner}}{\times} (P, Q) = \left([g \circ P] \,\mathrm{mod}\, z^{r+1}; \pi_- \left[Q \circ g^{-1}\right]\right) \tag{5.4.3}$$

where $\pi_-$ is a projection to the Laurent part with respect to spectral parameter $z$, i.e.

$$\pi_- \left[\sum_{i=-\infty}^{\infty} T_i z^i\right] = \sum_{i=-\infty}^{-1} T_i z^i$$

**Lemma 5.4.2.** *Both inner and outer actions are Hamiltonian with the moment maps respectively*

$$\begin{array}{rl} \mu_{\text{inner}}: & T^\star\hat{\mathfrak{g}}_r \to \hat{\mathfrak{g}}_r^\star \\ & (P, Q) \mapsto \Lambda(z) = \pi_- [PQ] \end{array} \qquad \begin{array}{rl} \mu_{\text{outer}}: & T^\star\hat{\mathfrak{g}}_r \to \hat{\mathfrak{g}}_r^\star \\ & (P, Q) \mapsto A(z) = \pi_- [QP] \end{array}. \tag{5.4.4}$$

These two moment maps are dual in a sense of Adams–Harnad–Previato duality [3]. Since inner and outer group actions commute, $A(z)$ and $\Lambda(z)$ Poisson commute with respect to Poisson bracket induced by (5.4.1). As in the Fuchsian case, $A(z)$ is an element of the co-adjoint orbit for the truncated loop algebra. On the other hand, $\Lambda(z)$ becomes an invariant of the orbit after quotient via the inner group action.

This gives us the opportunity to generalise the statement of the Lemma 4.3.3 to the case of Takiff algebras:

**Lemma 5.4.3.** *The map*

$$\begin{array}{ccc} T^\star\hat{\mathfrak{g}}_r \underset{\Lambda_0}{/\!/} \hat{G}_r & \to & \hat{\mathcal{O}}_r^\star \\ (Q, P) & \mapsto & A(z) := \pi_- [QP] \end{array}$$

*where* $\underset{\Lambda_0}{/\!/}$ *denotes the Hamiltonian reduction w.r.t. the inner action in which the moment map has value* $\Lambda_0$*, is a rational symplectomorphism and the Jordan normal form* $\Lambda_0$ *of* $A$ *is given by*

$$\Lambda_0(z) = \pi_- [PQ] \,.$$

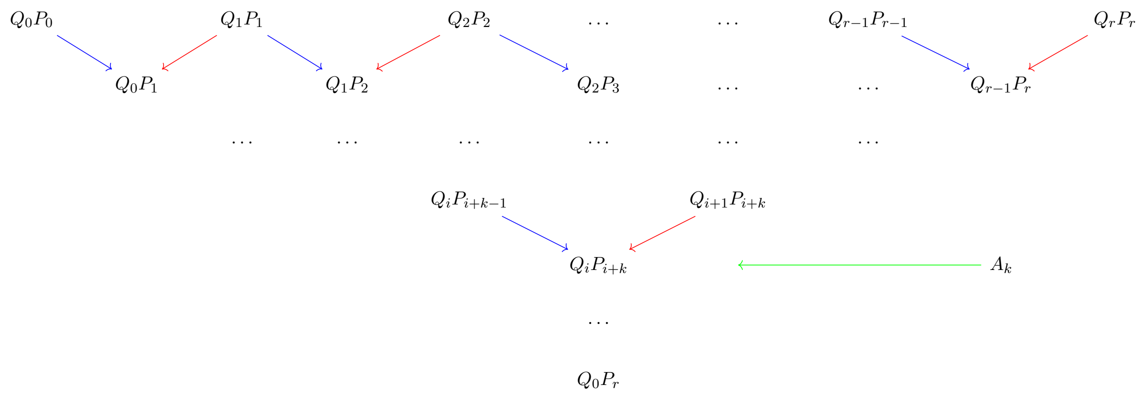



Figure 5.2: Lifted Darboux coordinates for the Takiff algebra of degree $r$. In this diagram we have $r+1$ rows, and we number them starting at the top with row 0, all the way down to row $r$. The sum of the elements in row $k$ gives the coefficient $A_k$ of the power of $z^{-k-1}$, the blue arrow follows each $Q_i$ matrix from the formula above to the one below, while the red one follows $P_i$.

*The explicit form of $A(z)$ is*

$$A(z) = \frac{A_r}{z^{r+1}} \cdots + \frac{A_0}{z}, \quad A_k = \sum_{i=0}^{r-k} \chi_{i,i+k}, \quad \chi_{i,j} = Q_i P_j. \tag{5.4.5}$$

*while $\Lambda_0(z)$ takes form*

$$\Lambda_0(z) = \frac{\Lambda_r}{z^{r+1}} \cdots + \frac{\Lambda_0}{z}, \quad \Lambda_k = \sum_{i=0}^{r-k} P_{i+k} Q_i, \tag{5.4.6}$$

**Remark 5.4.4.** *According to Lemma 5.4.1, all co-adjoint orbits, i.e the ones of for the ordinary Lie algebras and the one for the Takiff algebras, are reductions of the same phase space. Systems with different orders of poles are obtained by different choices of the group realising the reduction: in the Fuchsian case we considered the action of the direct product of $GL_m$, while in the case of the Takiff algebra we use the inner action of $\hat{G}^r_m := GL_m[z]/z^{r+1}GL_m[z]$.*

The parametrisation (5.4.5) allows a nice combinatorial description which is presented on the Fig. 5.6.1.

**Theorem 5.4.5.** *The Poisson bracket induced by the Darboux coordinates $Q_i, P_i$ to the space of matrices $A_k$, $k = 0, \dots, r$ coincides with the graded Poisson structure (4.1.5).*

*Proof.* This statement is a straightforward corollary of the Lemma 5.4.3. However, here we prove id directly for the sake of clarity. The Poisson bracket on the elements $\chi_{ij}$ in (5.4.5) is given by

$$\{\chi_{ij} \underset{,}{\otimes} \chi_{kl}\} = \{Q_i P_j \underset{,}{\otimes} Q_k P_l\} = \delta_{jk}(Q_i \otimes 1)\Omega(\mathbb{I} \otimes P_l) - \delta_{il}(\mathbb{I} \otimes Q_k)\Omega(P_j \otimes \mathbb{I}) =$$

$$= \delta_{jk}(Q_i P_l \otimes \mathbb{I})\Omega - \delta_{il}\Omega(Q_k P_j \otimes \mathbb{I}) = \delta_{jk}(\chi_{il} \otimes \mathbb{I})\Omega - \delta_{il}\Omega(\chi_{kj} \otimes \mathbb{I})$$

which is the same as

$$\left\{(\chi_{ij})_{\alpha\beta}, (\chi_{kl})_{\gamma\delta}\right\} = \delta_{jk}\delta_{\gamma\beta}(\chi_{il})_{\alpha\delta} - \delta_{il}\delta_{\alpha\delta}(\chi_{kj})_{\gamma\beta}.$$

By direct computation

$$\{A_k \underset{,}{\otimes} A_l\} = \sum_{i,j}\{\chi_{i,i+k} \underset{,}{\otimes} \chi_{j,j+l}\} = \sum_{i,j} \delta_{j,i+k}(\chi_{i,j+l} \otimes \mathbb{I})\Pi - \delta_{i,j+l}\Pi(\chi_{j,i+k} \otimes \mathbb{I}) =$$

$$= \sum_i (\chi_{i,i+k+l} \otimes \mathbb{I})\Pi - \sum_j \Pi(\chi_{j,j+k+l} \otimes \mathbb{I}) = -[\Pi, A_{k+l} \otimes \mathbb{I}]$$

we obtain the proof of the statement. In case if $k+l > r$ the Poisson bracket shall automatically be zero. □

In the next lemma, we show that the quadratic Casimirs for the Takiff algebra are given by functions of the spectral invariants of the co-adjoint orbit:

**Lemma 5.4.6.** *For the Takiff algebra of degree $r$, the following quantities are Casimirs*

$$I_k = \underset{z=0}{\text{res}}\left(z^{r+k}\text{Tr}A^2\right), \quad 0 < k < r. \tag{5.4.7}$$

*Proof.* The fact that $I_k$ are Casimirs may be checked by the direct computation, here we demonstrate it for $k = 1$, since we use it in the further text. Explicitly $I_1$ writes as follows

$$I_1 = \sum_{j=0}^{r} \text{Tr}A_j A_{r-j}.$$

The Poisson bracket with an arbitrary generator of the Poisson algebra defined via Lie-Poisson bracket for the Takiff algebra gives

$$\sum_{j=0}^{r}\{(A_i)_\alpha, \mathrm{Tr} A_j A_{r-j}\} = \sum_{j=0}^{r}[A_{i+j}, A_{r-j}]_\alpha + \sum_{j=i}^{r}[A_{r-j+i}, A_j]_\alpha =$$

$$= \sum_{l=i}^{r}[A_l, A_{r-l+i}]_\alpha + \sum_{j=i}^{r}[A_{r-j+i}, A_j]_\alpha = 0.$$

In the same way we may prove that $I_k$ are the Casimirs for $k > 1$. □

**First order pole. Takiff algebra of degree 1**

In this case Takiff algebra coincide with the ordinary Lie algebra. Parametrisation in such situation was obtained in works [6, 7]

**Second order pole. Takiff algebra of degree** 1

The Darboux parametrisation is given by

$$A(z) = \frac{Q_0 P_1}{z^2} + \frac{Q_0 P_0 + Q_1 P_1}{z}, \quad \omega = \mathrm{d}\Theta, \quad \Theta = \mathrm{Tr}\left(P_1 \mathrm{d}Q_1 + P_0 \mathrm{d}Q_0\right),$$

so that the extended phase space is of dimension $4m^2$. We now want to reduce this dimension by solving the moment map conditions

$$P_1 Q_0 = \Lambda_1, \quad P_0 Q_0 + P_1 Q_1 = \Lambda_0$$

w.r.t. $P_0$ and $P_1$. To do this, we only need to assume that $Q_0$ is invertible, namely $(Q_0, Q_1) \in \bigoplus Gl_m \times \mathfrak{gl}_m$. This inversion sends the Liouville form to

$$\theta = \mathrm{Tr}\left(\Lambda_1 Q_0^{-1} \mathrm{d}Q_1 + \Lambda_1 Q_0^{-1} \mathrm{d}Q_0 - \Lambda_1 Q_0^{-1} Q_1 Q_0^{-1} \mathrm{d}Q_0\right),$$

while $A$ goes to

$$A(z) = \frac{Q_0 \Lambda_1 Q_0^{-1}}{z^2} + \frac{Q_0 \Lambda_0 Q_0^{-1} + [Q_1 Q_0^{-1}, Q_0 \Lambda_1 Q_0^{-1}]}{z}.$$

We now want to reduce the dimension by $2m$ via the torus action $Q_i \to Q_i D_i$, where $D_i$ is a diagonal matrix, that fixes the invariants of the co-adjoint orbit $\Lambda_0, \Lambda_1$. To this aim, we find the Darboux coordinates $p_1, \dots p_{m(m-1)}, q_1, \dots q_{m(m-1)}$ explicitly in such a way that

$$\Theta = \mathrm{Tr}\left(\Lambda_1 Q_0^{-1}\mathrm{d}Q_1 + \Lambda_0 Q_0^{-1}\mathrm{d}Q_0 - \Lambda_1 Q_0^{-1}Q_1 Q_0^{-1}\mathrm{d}Q_0\right) = \sum_{i=1}^{m(m-1)} p_i \mathrm{d}q_i. \tag{5.4.8}$$

The number of unknown functions also equals to $2m(m-1)$, due to the factorisation of the torus action (this is the truncated current algebra analog of the statement that the eigenvectors are defined up to multiplication of the diagonal matrix). There are many possible choices for the Darboux coordinats in this situation, our aim to find one of them; it is convenient to use the following change

$$L_1 = Q_0^{-1}Q_1,$$

then Liouville form transforms to

$$\Theta = \mathrm{Tr}\left[\Lambda_1 \mathrm{d}L_1 + \left(\Lambda_0 + [L_1, \Lambda_1]\right) Q_0^{-1}\mathrm{d}Q_0\right].$$

The Liouville form is always defined up to a closed form. Since $\Lambda_1$ is an invariant of the co-adjoint orbit (i.e. is a constant) the term

$$\Lambda_1 \mathrm{d}L_1 = \mathrm{d}\left(\Lambda_1 L_1\right)$$

is exact, so we may drop it. The equation for the differential form therefore simplifies to

$$\mathrm{Tr}\left[\left(\Lambda_0 + [L_1, \Lambda_1]\right) Q_0^{-1}\mathrm{d}Q_0\right] = \sum_{i=1}^{m(m-1)} p_i \mathrm{d}q_i,$$

which allows us to pick our Darboux coordinates in such a way that $Q_0$ depends only on $q_1, \dots q_{m(m-1)}$ (i.e.$Q_0$ is a section of a principal bundle over the Lagrangian sub-manifold), while the entries of $L_1$ are given by solution of $m(m-1)$ linear equations. For example we can we may take off-diagonal entries of $Q_0$ as the coordinates on the Lagrangian sub-manifold. By using the torus action, we can make the following choice of $Q_0$:

$$Q_0 = \begin{pmatrix} 1 & q_1 & \dots & \dots & q_{m-1} \\ 0 & 1 & q_m & \dots & q_{2m-3} \\ \vdots & 0 & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & q_{\frac{m(m-1)}{2}} \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ q_{\frac{m(m-1)}{2}+1} & 1 & 0 & \dots & 0 \\ q_{\frac{m(m-1)}{2}+2} & q_{\frac{m(m-1)}{2}+3} & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ q_{(m-1)^2} & \dots & \dots & q_{m(m-1)} & 1 \end{pmatrix}.$$

For $\mathfrak{sl}_2$ we have

$$\Lambda_i = \begin{pmatrix} \theta_i & 0 \\ 0 & -\theta_i \end{pmatrix}, \quad Q_0 = \begin{pmatrix} 1 & q_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ q_2 & 1 \end{pmatrix}, \quad L_1 = \frac{1}{2\theta_1} \begin{pmatrix} 0 & -p_2 \\ p_2 q_2^2 - 2\theta_0 q_2 + p_1 & 0 \end{pmatrix}$$

and $A$ goes to

$$A(z) = \frac{2\theta_1}{z^2} \begin{pmatrix} q_1 q_2 + 1/2 & -(q_1 q_2 + 1)\, q_1 \\ q_2 & -q_1 q_2 - 1/2 \end{pmatrix} + \\ + \frac{1}{z} \begin{pmatrix} p_1 q_1 - q_2 p_2 + \theta_0 & -p_1 q_1^2 + (2 q_1 q_2 + 1)\, p_2 - 2\theta_0 q_1 \\ p_1 & -p_1 q_1 + q_2 p_2 - \theta_0 \end{pmatrix}. \quad (5.4.9)$$

If we take into account the outer action of $SL_2$, the leading term can be chosen in diagonal form and we have

$$Q_1^{-1} A(z) Q_1 = \frac{\theta_1}{z^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{z} \begin{pmatrix} \theta_0 & p_2 \\ p_2 q_2^2 - 2\,\theta_0 q_2 + p_1 & -\theta_0 \end{pmatrix}$$

In the case of degenerated orbit, when

$$\Lambda_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

we have

$$L_2 = \begin{pmatrix} 1+p_2 & 0 \\ -\frac{(1+p_2)q_2^2 - q_2^2 - 2\lambda_1 q_2 + p_1}{2q_2} & 1 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} \frac{q_1 q_2 p_2}{2} + q_1 q_2 + p_2 + 1 + \lambda_1 q_1 - \frac{q_1 p_1}{2q_2} & q_1 \\ \frac{p_2 q_2}{2} + q_2 + \lambda_1 - \frac{p_1}{2q_2} & 1 \end{pmatrix}$$

and $A$ goes to

$$A(z) = \frac{1}{z^2} \begin{pmatrix} -(q_1 q_2 + 1) q_2 & (q_1 q_2 + 1)^2 \\ -{q_2}^2 & (q_1 q_2 + 1) q_2 \end{pmatrix} + \\ + \frac{1}{z} \begin{pmatrix} \left(q_1 + \frac{1}{2q_2}\right) p_1 - \frac{p_2 q_2}{2} & \left(-q_1^2 - \frac{q_1}{q_2}\right) p_1 + p_2 (q_1 q_2 + 1) \\ p_1 & \left(-q_1 - \frac{1}{2q_2}\right) p_1 + \frac{p_2 q_2}{2} \end{pmatrix} \tag{5.4.10}$$

Diagonal gauge gives

$$Q_1^{-1} A(z) Q_1 = \frac{1}{z^2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \frac{1}{z} \begin{pmatrix} \frac{q_2^2 p_2 + p_1}{2q_2} & p_2 \\ 0 & -\frac{q_2^2 p_2 + p_1}{2q_2} \end{pmatrix}$$

**Third order pole. Takiff algebra of degree** 2

In this case, the parametrisation in terms of lifted Darboux coordinates is given by

$$A(z) = \frac{Q_0 P_2}{z^3} + \frac{Q_0 P_1 + Q_1 P_2}{z^2} + \frac{Q_0 P_0 + Q_1 P_1 + Q_2 P_2}{z},$$

so that the extended phase space is of dimension $6m^2$. The moment map is given by the equations

$$P_2 Q_0 = \Lambda_2, \quad P_1 Q_0 + P_2 Q_1 = \Lambda_1, \quad P_0 Q_0 + P_1 Q_1 + P_2 Q_2 = \Lambda_0.$$

Here we again use the following change of variables

$$L_1 = Q_0^{-1} Q_1, \quad L_2 = Q_0^{-1} Q_2$$

that maps the Liouville form to

$$\Theta = \mathrm{Tr}\left(\Lambda_2 \mathrm{d}L_2 + \Lambda_1 \mathrm{d}L_1 - \Lambda_2 L_1 \mathrm{d}L_1 + \left(\Lambda_0 + [L_2, \Lambda_2] + [L_1, \Lambda_1 - \Lambda_2 L_1]\right) Q_0^{-1} \mathrm{d}Q_0\right).$$

As in the previous case, the first 2 terms are closed differential forms, so we can drop them. The dimension of the reduced phase space equals to $3m(m-1) = 3N$ and we consider the following parametrisation

$$\mathrm{Tr}(-\Lambda_2 L_1 \mathrm{d}L_1) = \sum_{i=1}^{N/2} p_i \mathrm{d}q_i, \quad \mathrm{Tr}\left[\left(\Lambda_0 + [L_2, \Lambda_2] + [L_1, \Lambda_1 - \Lambda_2 L_1]\right) Q_0^{-1} \mathrm{d}Q_0\right] = \sum_{i=N/2+1}^{3N/2} p_i \mathrm{d}q_i.$$

For simplicity, let us denote

$$\Theta_1 = \mathrm{Tr}(-\Lambda_2 L_1 \mathrm{d}L_1), \quad \Theta_2 = \mathrm{Tr}\left[\left(\Lambda_0 + [L_2, \Lambda_2] + [L_1, \Lambda_1 - \Lambda_2 L_1]\right) Q_0^{-1} \mathrm{d}Q_0\right],$$

so that $\Theta = \Theta_1 + \Theta_2$. Now if we will find the right parametrisation of $L_1$, we may put $Q_0$ to be a matrix which depends only on $q_{N/2+1}, \dots q_{3N/2}$ (i.e. again $Q_0$ depends only on the coordinates of Lagrangian sub-manifold) and then obtain $L_2$ by solving a system of linear equations. In non-degenerate case when $\Lambda_2$ is a semi-simple matrix with distinct eigenvalues $\zeta_i$ we have

$$\Theta_1 = \sum_{i<j} -\zeta_i (L_1)_{ij} \mathrm{d}(L_1)_{ji} - \zeta_j (L_1)_{ji} \mathrm{d}(L_1)_{ij} =$$

$$\sum_{i<j} (\zeta_i - \zeta_j)(L_1)_{ji} \mathrm{d}(L_1)_{ij} - \mathrm{d}(\zeta_i (L_1)_{ij} (L_1)_{ji}) \simeq \sum_{i<j} (\zeta_i - \zeta_j)(L_1)_{ji} \mathrm{d}(L_2)_{ij}$$

and we see that a natural choice of the Darboux coordinates are the off-diagonal entries of $L_1$, such that

$$\{(L_1)_{ij}, (L_1)_{kl}\} = \mathrm{sgn}(j-i)\delta_{kj}\delta_{li}(\zeta_i - \zeta_j).$$

In the case of $\mathfrak{sl}_2$ we have

$$\Lambda_2 = \begin{pmatrix} \theta_2 & 0 \\ 0 & -\theta_2 \end{pmatrix}, \quad L_1 = \begin{pmatrix} \dots & q_1 \\ \frac{p_1}{2\theta_2} & \dots \end{pmatrix}.$$

Here the diagonal part of $L_1$ is irrelevant, since it does not contribute to $\Theta_1, \Theta_2$ and generally may be be chosen to be zero by a torus action. Solving the linear equations for the Cartan form

$\Theta_2$ we obtain

$$\Lambda_i = \begin{pmatrix} \theta_i & 0 \\ 0 & -\theta_i \end{pmatrix}, \quad Q_0 = \begin{pmatrix} 1 & q_1 \\ q_2 & 1 \end{pmatrix}$$

$$L_1 = \frac{1}{2\theta_2} \begin{pmatrix} 0 & (p_2q_2 + p_3q_3 - \theta_0)q_1 - p_2 + \frac{\theta_1}{\theta_2}p_3 \\ p_1 - p_1q_1q_2 + (p_3q_3 - \theta_0)q_2 - 2\theta_1q_3 & 0 \end{pmatrix}$$

Here we take in a slightly different form of $Q_0$ respect to the previous example for the sake of obtaining a neater final formula. The matrix $A(z)$ takes form

$$\begin{aligned} A(z) = {}& \frac{1}{z^3}\frac{1}{1-q_1q_2} \begin{pmatrix} \theta_2\,(q_1q_2+1) & -2\,\theta_2q_1 \\ 2\,q_2\theta_2 & -\theta_2\,(q_1q_2+1) \end{pmatrix} + \\ & + \frac{1}{z^2}\frac{1}{1-q_1q_2} \begin{pmatrix} \theta_1q_1q_2 + 2\,\theta_2q_1q_3 - q_2p_3 + \theta_1 & -2\,q_1^2q_3\theta_2 - 2\,\theta_1q_1 + p_3 \\ -q_2^2p_3 + 2\,\theta_1q_2 + 2\,\theta_2q_3 & -\theta_1q_1q_2 - 2\,\theta_2q_1q_3 + q_2p_3 - \theta_1 \end{pmatrix} + \\ & + \frac{1}{z} \begin{pmatrix} p_1q_1 - q_2p_2 - p_3q_3 + \theta_0 & -p_1{q_1}^2 + p_3q_1q_3 - \theta_0q_1 + p_2 \\ -p_2{q_2}^2 - p_3q_2q_3 + \theta_0q_2 + p_1 & -p_1q_1 + q_2p_2 + p_3q_3 - \theta_0 \end{pmatrix} \end{aligned} \tag{5.4.11}$$

The diagonal gauge gives

$$\begin{aligned} Q_0^{-1}A(z)Q_0 = {}& \frac{1}{z^3}\begin{pmatrix} \theta_3 & 0 \\ 0 & -\theta_3 \end{pmatrix} + \frac{1}{z^2}\begin{pmatrix} \theta_1 & p_3 \\ 2\,\theta_2q_3 & -\theta_1 \end{pmatrix} \\ & + \\ & \frac{1}{z}\begin{pmatrix} -p_3q_3 + \theta_0 & -p_2q_1q_2 - p_3q_1q_3 + \theta_0q_1 + p_2 \\ -p_1q_1q_2 + p_3q_2q_3 - \theta_0q_2 + p_1 & p_3q_3 - \theta_0 \end{pmatrix} \end{aligned} \tag{5.4.12}$$

Choosing another LU parametrisation for $Q_0$, i.e.

$$Q_0 = \begin{pmatrix} 1 & q_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ q_2 & 1 \end{pmatrix}$$

the diagonal gauged system takes form

$$Q_0^{-1}A(z)Q_0 = \frac{\theta_3}{z^3}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{z^2}\begin{pmatrix} \theta_2 & -2\theta_3 q_1 \\ p_1 & -\theta_2 \end{pmatrix} + \frac{1}{z}\begin{pmatrix} q_1p_1+\theta_1 & p_3 \\ p_3q_3^2 + (-2q_1p_1 - 2\,\theta_1)\,q_3 + p_2 & -q_1p_1 - \theta_1 \end{pmatrix}.$$

In the degenerate case, when

$$\Lambda_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

reduced Darboux coordinates take form

$$A(z) = \frac{1}{z^3}\begin{pmatrix} -(q_1q_2+1)\,q_2 & (q_1q_2+1)^2 \\ -{q_2}^2 & (q_1q_2+1)\,q_2 \end{pmatrix} +$$

$$+\frac{1}{z^2}\begin{pmatrix} -2q_1q_2^2q_3 + 2\lambda_2q_1q_2 + p_3q_1q_2 - 2q_2q_3 + \lambda_2 + \frac{1}{2}p_3 & -(-2q_1q_2q_3 + 2\lambda_2q_1 + q_1p_3 - 2q_3)(q_1q_2+1) \\ (-2q_2q_3 + 2\lambda_2 + p_3)\,q_2 & 2q_1q_2^2q_3 - 2\lambda_2q_1q_2 - p_3q_1q_2 + 2q_2q_3 - \lambda_2 - \frac{1}{2}p_3 \end{pmatrix} +$$

$$+\frac{1}{z}\frac{1}{8q_2}\begin{pmatrix} 8p_1q_1q_2 - 4p_2q_2^2 + 4p_3\lambda_2 + p_3^2 + 4p_1 & -8p_1q_1^2q_2 + 8p_2q_1q_2^2 - 8\lambda_2p_3q_1 - 2p_3^2q_1 - 8p_1q_1 + 8p_2q_2 \\ 8q_2p_1 & -8p_1q_1q_2 + 4p_2q_2^2 - 4p_3\lambda_2 - p_3^2 - 4p_1 \end{pmatrix}. \tag{5.4.13}$$

In the eigenbasis of the leading term we have

$$Q_1^{-1}A(z)Q_1 = \frac{1}{z^3}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \frac{1}{z^2}\begin{pmatrix} \lambda_2 + 1/2\,p_3 & 2\,q_3 \\ 0 & -\lambda_2 - 1/2\,p_3 \end{pmatrix} +$$

$$+\frac{1}{z}\frac{1}{8q_2}\begin{pmatrix} 4\,p_2q_2^2 + 4\,p_3\lambda_2 + p_3^2 + 4\,p_1 & 8\,p_2q_2 \\ -8\,q_2p_3\lambda_2 - 2\,q_2p_3^2 & -4\,p_2q_2^2 - 4\,p_3\lambda_2 - p_3^2 - 4\,p_1 \end{pmatrix} \tag{5.4.14}$$

**Fourth order pole. Takiff algebra of degree 3**

Here we provide only the result

$$Q_1^{-1}A(z)Q_1 = \frac{\theta_4}{z^4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{z^3}\begin{pmatrix} \theta_3 & -2\theta_4 q_3 \\ 2\theta_4 q_4 & -\theta_3 \end{pmatrix} +$$

$$+\frac{1}{z^2}\begin{pmatrix} 2\theta_4 q_3 q_4 + \theta_2 & -\theta_4 q_3^3 q_4^2 + (\theta_3 - 4\theta_4)\, q_4 q_3^2 - \theta_4 q_3 + p_4 \\ -\theta_4 q_3^2 {q_4}^3 + (\theta_3 - 4\theta_4)\, q_4^2 q_3 + (2\theta_3 - \theta_4)\, q_4 + p_3 & -2\theta_4 q_3 q_4 - \theta_2 \end{pmatrix} +$$

$$+\frac{1}{z}\begin{pmatrix} q_3 p_3 - q_4 p_4 + \theta_1 & p_2 \\ p_2 q_2^2 - 2\, p_3 q_2 q_3 + 2\, p_4 q_2 q_4 - 2\,\theta_1 q_2 + p_1 & -q_3 p_3 + q_4 p_4 - \theta_1 \end{pmatrix} \quad (5.4.15)$$

**Remark 5.4.7.** *There is an interesting difference between poles of odd or even order. Indeed, when the order of pole is even $r+1 = 2k$, then the reduced phase space dimension is divisible by 4, and we have a kind of polarisation. Indeed, for poles of order $2k$ we locally write the connection as*

$$\frac{A_0}{z} + \dots \frac{A_{2k-1}}{z^{2k}},$$

*and the matrices $A_k, \dots, A_{2k-1}$ form a Poisson commuting family of half the total dimension. Therefore they define a Lagrangian sub-manifold in the phase space. We can then assume that these matrices are parameterized by $Q_0, \dots, Q_{k-1}, P_k, \dots P_{2k-1}$ only. In the case pole of odd order, we will still have that $A_{k+1}, \dots, A_{2k-1}$ form a Poisson commuting family, but now this is not of half the dimension.*

## 5.5 Geometry of the $\mathfrak{sl}_2$ Takiff algebra co-adjoint orbits and ramification

The aim of this section is to describe the ramification phenomenon for the $\mathfrak{sl}_2$ linear systems with irregular singularities using the classification of the co-adjoint orbits for the corresponding Takiff algebras. All the results in this section are local by nature. Without loss of generality, assume that system has pole of the Poincaré rank $r$ at zero

$$\frac{d}{dz}\Psi = \left[\frac{A_r}{z^{r+1}} + \frac{A_{r-1}}{z^r} + \cdots + \frac{A_1}{z^2} + \frac{A_0}{z} + O(1)\right]\Psi, \quad \Psi \in SL_2, \quad \mathrm{Tr}(A_i) = 0. \tag{5.5.1}$$

Due to the results obtained by the confluence procedure, we may treat the Laurent part of the connection as an element of a co-adjoint orbit of $\hat{\mathfrak{sl}}_2^r$. Since the rank of $\mathfrak{sl}_2$ is one, all the invariants for the co-adjoint orbit of $\hat{\mathfrak{sl}}_2^r$ are given by quadratic Casimirs

$$I_k = \sum_{j=0}^{r} \mathrm{Tr} A_j A_{r+k-j-1}, \quad 0 < k \leq r+1,$$

due to the theorem by Molev [70]. For a generic choice of the values for the invariants $I_k$, the formal solution is defined as a divergent series in $z$ and the number of the Stokes rays is $2r$. However, when $I_{r+1} = 0$, i.e. when $A_r$ is conjugated to a Jordan block, the formal solution is no longer a series in $z$, but it may be written only as a series in $\sqrt{z}$. The geometry of the Stokes phenomenon also changes and the number of the Stokes sectors becomes less than $2r$. To illustrate such phenomenon let us give an example which demonstrates ramification for the classical special functions.

**Example 5.5.1.** Consider the $\mathfrak{sl}_2$ linear system with only one pole of order 3 at $Z = 0$. Choosing inverse coordinate $z = \frac{1}{Z}$ such system writes as

$$\frac{d}{dz}\Psi = (A_2 z + A_1)\Psi, \quad \mathrm{Tr}\, A_i = 0. \tag{5.5.2}$$

Now, let $A_2$ be a semi-simple matrix, then there exist a gauge transformation $\Phi = g\Psi$ which sends (5.5.2) to

$$\frac{d}{dz}\Phi = \left[\begin{pmatrix} \theta & 0 \\ 0 & -\theta \end{pmatrix} z + \begin{pmatrix} v & w \\ u & -v \end{pmatrix}\right]\Phi,$$

for some numbers $u, v, w$. Reducing the system to a second order ODE we get

$$\frac{d^2 f}{dz^2} = \left(\theta^2 z^2 + 2\theta v z + uw + v^2 - \theta\right) f, \quad \Phi = \frac{1}{u}\begin{pmatrix} \frac{df}{dz} + (\theta z + v) f \\ u f \end{pmatrix}$$

which can be solved via parabolic cylinder functions. The number of Stokes sectors in such case is 4, since solution behaves as $\exp(\pm z^2/2)$ at $\infty$. Now let us assume that $A_2$ is a non-zero nilpotent matrix. The there exists a gauge which sends (5.5.2) to

$$\frac{d}{dz}\Phi = \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} z + \begin{pmatrix} v & w \\ u & -v \end{pmatrix}\right]\Phi.$$

Rewriting the system as a second order ODE we get

$$\frac{d^2 f}{dz^2} = \left(uz + uw + v^2\right) f, \quad \Phi = \frac{1}{u}\begin{pmatrix} \frac{df}{dz} + vf \\ uf \end{pmatrix}$$

whose solution writes via general solution of the Airy equation. The asymptotic solution of the Airy equation is of the following form

$$y_{\pm}(z) \sim \exp\left(\mp\frac{2}{3}z^{-\frac{3}{2}}\right) z^{\frac{1}{4}} \sum_{m\geq 0} a_m z^m.$$

and defines 3 Stokes sectors. Computing the Takiff co-adjoint invariants in both cases we see that Airy case corresponds to the co-dimension one hyperplane

$$I_3 = \operatorname{Tr} A_2^2 = 0$$

in the space of parameters $(I_1, I_2, I_3)$, while for the parabolic cylinder case $I_3 = 2\theta^2 \neq 0$.

Ramification was studied before in the context of differential equations without refering to the co-adjoint orbits of the Takiff algebras. See [89] for the details. Connection with co-adjoint invariants of the Takiff algebra gives an opportunity to consider transition from the generic system to the ramified one as a transition from the generic co-adjoint orbit to the special one. Such special orbit can be seen as an analogue of the nilpotent cone in the Takiff algebra.

### 5.5.1 Normal forms of the element in the Takiff algebra $\hat{\mathfrak{sl}}_2^r$

The aim of this section is to provide classifications of the co-orbits in the Takiff algebra over $\mathfrak{sl}_2$.

**Theorem 5.5.2.** *Let $A(z)$ is an element of the Takiff co-algebra $\hat{\mathfrak{sl}}_2^{r\star}$. Then it can be sent by* *the co-adjoint action of the group to the one of the following forms*

$$A(z) = \hat{\mathrm{Ad}}^{\star}_{g(z)}\Lambda(z), \quad \Lambda(z) = \begin{cases} \frac{1}{z^{r+1}} \begin{pmatrix} \theta_r + \theta_{r-1}z + \dots + \theta_0 z^r & 0 \\ 0 & -(\theta_r + \theta_{r-1}z + \dots + \theta_0 z^r) \end{pmatrix} \\ \\ \frac{1}{z^{r+1}} \begin{pmatrix} 0 & 1 \\ \theta_{r-1}z + \dots + \theta_0 z^r & 0 \end{pmatrix} \end{cases} \tag{5.5.3}$$

*where $\theta_1, \theta_2, \dots \theta_r$ are the invariants on the orbit.*

*Proof.* First of all let us proof that $\theta_i$ are invariants of the orbit. Indeed, let $A(z) = \Lambda(z)$, then using formulas for the quadratic invariants (5.4.7) in the diagonal case we get

$$I_k = 2\sum_{j=0}^{r} \theta_j \theta_{r+k-j}.$$

In the case when the leading term is nilpotent we get

$$I_{r+1} = 0, \quad I_k = \theta_k.$$

Now let us proof the statement about the normal form. Consider co-adjoint action of the corresponding Takiff group.

**Definition 5.5.3.** The Takiff group is a Lie group given by the application of the exponential map to the corresponding Takiff algebra.

**Lemma 5.5.4.** *Let $v(z)$ is an element of the Takiff group corresponding to the Lie algebra $\hat{\mathfrak{gl}}^r_2$, then there exists such $g_0 \in GL_2$ and $g_i \in \mathfrak{gl}_2$ for $i = 1..n$, s.t.*

$$v(z) = (1 + g_n z^n)(1 + g_{n-1}z^{n-1})\dots(1 + g_1 z)g_0, \quad g_0 \in GL_2, \quad g_i \in \mathfrak{gl}_2. \tag{5.5.4}$$

*This decomposition is defined uniquely.*

*Proof.* Proof of this lemma is a straightforward computation - $v(z)$ depends linearly on each

$g_i$. On the other hand we may write $v(z)$ as

$$v(z) = v_0 + \sum_{i=1}^{n} v_i z^i, \quad v_0 \in GL_2, \quad v_i \in \mathrm{Mat}_{m\times m}\,. \tag{5.5.5}$$

with some requirements for $v_i$. Comparing the terms in front of each power of $z$ in (5.5.4) and (5.5.5), we obtain linear matrix equations for the coefficients $g_i$. The solution is

$$g_k = v_k g_0^{-1} - \sum_{\alpha\in\Pi(k)} g_{\alpha_m} g_{\alpha_{m-1}} \dots g_{\alpha_1}.$$

Here $\Pi(k)$ is a set of the ordered monotone partitions of the number $k$ and $\alpha$ is a vector of partition, i.e.

$$\alpha = (\alpha_1, \dots \alpha_m) \in \Pi(k) \quad \Rightarrow \quad \begin{array}{c} 0 < \alpha_1 < \alpha_2 < \dots < \alpha_m \\ \alpha_1 + \alpha_2 + \dots + \alpha_m = k \end{array}.$$

The explicit formulas can be seen as some summation over some special Young diagrams. However, here we are not interested in explicit formulas, but we want to emphasize that every element of the truncated loop group has unique decomposition of the form

$$T(z) = (1 + g_n z^n)(1 + g_{n-1} z^{n-1}) \dots (1 + g_1 z) g_0 = G_m \dots G_1 g_0. \tag{5.5.6}$$

Now let us show that $g_k \in \mathfrak{gl}_2$ for $k > 0$. Indeed $g_0 \in GL_2$ and $1 + z^k g_k$ are the generators of the Lie group, which tangent space is a $\hat{\mathfrak{gl}}_2^r$, then choosing entries of $g_k$ for $k > 0$ as a linear coordinates in the Lie group we get that $g_k$ are in $\mathfrak{gl}_2$. □

**Remark 5.5.5.** *Provided description of the Takiff group for $\mathfrak{gl}_2$ seems to be new, up to the knowledge of the author. It can be easily extended $\mathfrak{gl}_n$ case.*

**Remark 5.5.6.** *Since $\hat{\mathfrak{sl}}_2^r$ is a quotient of the $\hat{\mathfrak{gl}}_2^r$ by the commutative subalgebra, the co-adjoint orbits of $\hat{\mathfrak{sl}}_2^r$ coincide with the orbits of the conjugation by the Takiff group corresponding to the $\hat{\mathfrak{gl}}_2^r$. Because of that, we use introduced in the lemma 5.5.4 group to investigate normal forms in $\hat{\mathfrak{sl}}_2^r$.*

Due to the lemma 5.5.4, co-adjoint action of the Takiff algebra may be decomposed into a series of co-adjoint actions of the simplier form. Each of these actions has a very nice property - $\mathrm{Ad}^\star_{1+g_k z^k}$ leaves invariant the coefficients at the powers $z^{-n-1}, z^{-n}, \dots z^{-n+k+1}$ for any element of the Takiff co-algebra. If we associate element of the Takiff algebra with its coefficients at powers of $z$, then the action formally writes as

$$\begin{array}{rcl} \mathrm{Ad}^\star_{g_0} \quad (A_r, A_{r-1}, \dots A_k, \dots A_0) & = & (g_0^{-1} A_r g_0, g_0^{-1} A_{r-1} g_0, \dots, g_0^{-1} A_0 g_0) \\ \mathrm{Ad}^\star_{1+zg_1} \quad (A_r, A_{r-1}, \dots A_k, \dots A_0) & = & (A_r, A_{r-1} + [g_1, A_r], A_{n-2} \dots A_k, \dots A_0) \\ & \dots & \\ \mathrm{Ad}^\star_{1+z^k g_k} \quad (A_r, A_{r-1}, \dots A_k, \dots A_0) & = & (A_r, \dots, A_{n-k+1}, A_{n-k} + [g_k, A_r], \dots, \tilde{A}_0) \\ & \dots & \\ \mathrm{Ad}^\star_{1+z^n g_r} \quad (A_r, A_{r-1}, \dots A_k, \dots A_0) & = & (A_r, A_{r-1}, \dots, A_1, A_0 + [g_r, A_r]) \end{array} \tag{5.5.7}$$

Then, choosing $g_0$ to be a transition matrix to the basis in which $A_r$ is in it's Jordan form $\Lambda_r$, we find $g_i$ by solving linear equations

$$A_{r-k} + [g_k, \Lambda_r] \in \ker \mathrm{ad}_{\Lambda_r} .$$

In case when $\Lambda_r$ is diagonal $\ker \mathrm{ad}_{\Lambda_r} = \mathfrak{h}$, while in case when $\Lambda_r$ is nilpotent, i.e. $\Lambda_r \in \mathfrak{e}$, kernel $\ker \mathrm{ad}_{\Lambda_r}$ is anti-borel algebra $\mathfrak{f}$. □

### 5.5.2 Local solutions near irregular singularity. Gauge and unfold

Now we explain how the normal forms introduced in the previous subsection are related to the formal solutions near the irregular singularity. Firstly, let us demonstrate the link between the co-adjoint action in the Takiff algebras and the gauge transform. Let $G(z)$ be an element of the gauge group, such that $G(0) \in SL_2$. Then the gauge transform may be written as

$$\begin{aligned} G(z)^{-1} \left[ \frac{A_r}{z^{r+1}} + \frac{A_{r-1}}{z^r} + \dots + \frac{A_1}{z^2} + \frac{A_0}{z} + O(1) \right] G(z) - G(z)^{-1} \frac{d}{dz} G(z) = \\ = \hat{\mathrm{Ad}}_{\tilde{G}} \left[ \frac{A_r}{z^{r+1}} + \frac{A_{r-1}}{z^r} + \dots + \frac{A_1}{z^2} + \frac{A_0}{z} \right] + O(1), \end{aligned} \tag{5.5.8}$$

where $\tilde{G}$ is the $r$-th jet of $G(z)$, i.e.

$$G(z) = \sum_{i=0}^{\infty} g_i z^i \quad \to \quad \tilde{G}(z) = \sum_{i=0}^{r} g_i z^i.$$

Then there are two options - the first one is that there is a $G(z)$ such that the Laurent part after the gauge is diagonal. This is the non-ramified case and we may use formal asymptotic solution written as a divergent series in the local coordinate $z$. The second option is that the Laurent part has a non-trivial nilpotent leading term which allows us use a gauge transformation which sends system to the form

$$\frac{d}{dz}\Psi = \left[\begin{pmatrix} 0 & \frac{1}{z^{r+1}} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \frac{\theta_{r-1}}{z^r} & 0 \end{pmatrix} + \cdots + \begin{pmatrix} 0 & 0 \\ \frac{\theta_0}{z} & 0 \end{pmatrix} + O(1)\right]\Psi.$$

However, to write down a local solution we have to diagonilize the Laurent part. The general theory for rank $n$ systems is given in the book by Wasow, however in the $\mathfrak{sl}_2$ case there is a way to do it geometricaly and see the ramification explicitly. Let $k$ be the the integer such that

$$\theta_{k-1} \neq 0, \quad \forall l \geq k : \theta_l = 0,$$

then locally we have

$$\frac{d}{dz}\Psi = \left[\begin{pmatrix} 0 & \frac{1}{z^{r+1}} \\ \frac{\theta_k - 1}{z^k} & 0 \end{pmatrix} + O\left(z^{-k+1}\right)\right]\Psi. \tag{5.5.9}$$

**Lemma 5.5.7.** *The gauge transformation of the form*

$$\Psi = \begin{pmatrix} z^{\frac{k}{2}-\frac{1}{2}-\frac{r}{2}} & -z^{\frac{k}{2}-\frac{1}{2}-\frac{r}{2}} \\ \sqrt{\theta_{k-1}} & \sqrt{\theta_{k-1}} \end{pmatrix}\Phi,$$

*Transforms* (5.5.9) *to the following system*

$$\frac{d}{dz}\Phi = \left[\frac{1}{(\sqrt{z})^{k+r+1}}\begin{pmatrix} \sqrt{\theta_{k-1}} & 0 \\ 0 & -\sqrt{\theta_{k-1}} \end{pmatrix} + O\left(\frac{1}{(\sqrt{z})^{k+r}}\right)\right]\Phi.$$

*Proof.* Passing to the double covering coordinate $w^2 = z$ we get a local differetial equation in $w$ with diagonal leading term. The number of Stokes sectors is $2(r+k)$ on the double cover and $r+k$ in the initial local coordinate $z$. The theorem provide a scheme which connects the

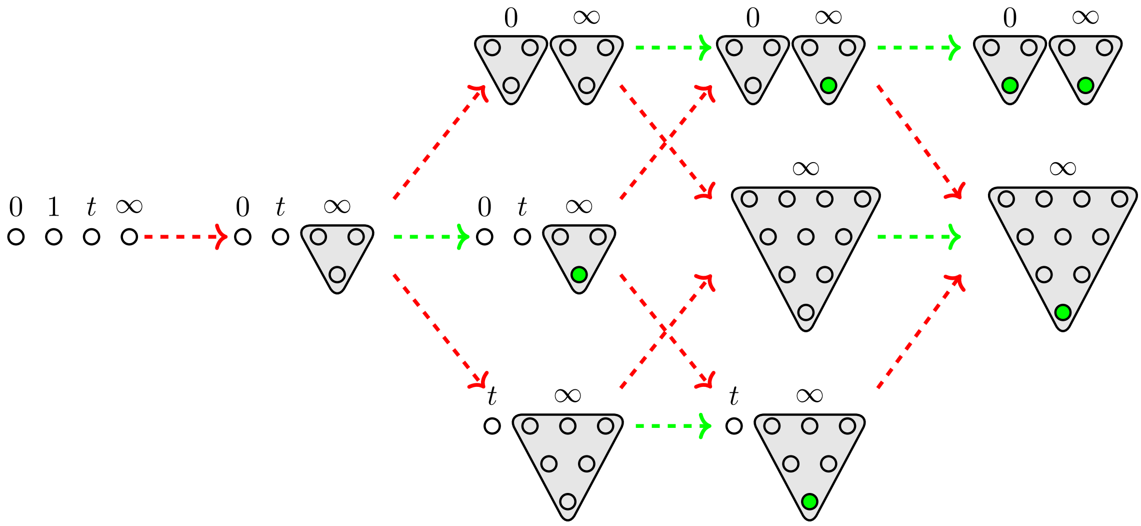



Figure 5.3: Confluence scheme for Painlevé equations. Each triangle at this diagram corresponds to the Takiff algebra Darboux coordinates which was introduced at the Fig. 5.6.1. Red arrows stand for confluence procedure, while green ones for remification.

degeneration in the Takiff algebra with the degeneration of the Stokes phenomenon

$$(I_{r+1}, I_r, I_{r-1}, \dots I_1) : 2r \text{ Stokes sectors}$$

$$(0, I_r, I_{r-1}, \dots I_1) : 2r - 1 \text{ Stokes sectors}$$

$$(0, 0, I_{r-1}, \dots I_1) : 2r - 2 \text{ Stokes sectors}$$

$$\dots$$

$$(0, 0, 0, \dots 0, I_1) : r + 1 \text{ Stokes sectors.}$$

From that perspective to study ramified systems like the isomonodromic problem for Painlevé I, degenerated Painlevé III (D7 and D8) or degenerated Painlevé V, we have to perform a reduction from the coadjoint orbit with nilpotent leading term. □

## 5.6 The Painlevé equations

### 5.6.1 General scheme

The confluence diagram of the Darboux parametrisations in the case of rank 2 non-ramified connections with 4 points is given at Fig. 5.3.

The Hamiltonians of the isomonodromic problem with irregular singularities of the Poincaré rank $r_i$ at point $u_i$ allow additional symmetries with respect to the inner action (choice of the spectral invariants at each singularity) and outer action (gauge group action). Using the Darboux parametrisation of the co-adjoint for the $\mathfrak{sl}_2$–Takiff algebras, we automatically fix spectral invariants, i.e. reduce with respect to the inner action. The only symmetry which still needs to be taken into account is the gauge freedom which leads to the fully reduced phase space. In all the examples of this section, we write down Darboux coordinates with partly resolved gauge group moment map by diagonalizing the leading terms at one of the irregular singularities. We do it automatically by writing diagonal gauge intermediate Darboux coordinates for the Takiff algebra co-adjoint orbit. Therefore, the number of the intermediate coordinates in all examples is 4 and not 6 (because we have eliminated 2 coordinates by diagonalisation). Such coordinates are in correspondence with the Darboux coordinates which were used in [45]. In order to reduce to the smallest dimension of the system (namely 2), we have to always reduce with respect to the action of the stabilizer of the leading term. The same scheme works for the ramified cases, which corresponds to the special choice of the moment map.

### 5.6.2 Painlevé V

The Isomonodromic problem takes form

$$\begin{aligned} &\tfrac{d}{d\lambda}\Psi = \left(\tfrac{A^{(0)}}{\lambda} + \tfrac{A^{(t)}}{\lambda-t} + B_1\right)\Psi \\ &\tfrac{d}{dt}\Psi = -\tfrac{A^{(t)}}{\lambda-t}\Psi. \end{aligned} \tag{5.6.1}$$

Deformation equations are

$$\frac{d}{dt}A^{(0)} = \frac{1}{t}[A^{(t)}, A^{(0)}], \quad \frac{d}{dt}A^{(t)} = \frac{1}{t}[A^{(0)}, A^{(t)}] + [B_1, A^{(t)}], \quad \frac{d}{dt}B_1 = 0. \tag{5.6.2}$$

The Poisson algebra is given by

$$\left\{A^{(i)} \underset{,}{\otimes} A^{(i)}\right\} = [\Pi, \mathbb{I}\otimes A^{(i)}], \quad \left\{A^{(0)} \underset{,}{\otimes} A^{(t)}\right\} = \left\{A^{(0)} \underset{,}{\otimes} B_1\right\} = \left\{A^{(t)} \underset{,}{\otimes} B_1\right\} = \left\{B_1 \underset{,}{\otimes} B_1\right\} = 0 \tag{5.6.3}$$

Isomonodromic Hamiltonian writes as

$$H_V = \underset{\lambda=t}{\text{res}}\text{Tr}\left(\frac{A(\lambda)^2}{2}\right) = \text{Tr}\left(A^{(t)}B_1 + \frac{1}{t}A^{(t)}A^{(0)}\right). \tag{5.6.4}$$

In the $\mathfrak{sl}_2$ case Darboux parametrisation of the elements of the coadjoint orbit takes form

$$A^{(0)} = \begin{pmatrix} p_0q_0 - \theta_0 & -(p_0q_0 - 2\theta_0)p_0 \\ q_0 & -p_0q_0 + \theta_0 \end{pmatrix}, \quad A^{(t)} = \begin{pmatrix} p_tq_t - \theta_t & -(p_tq_t - 2\theta_t)p_t \\ q_t & -p_tq_t + \theta_t \end{pmatrix},$$

with the symplectic form

$$\omega = \mathrm{d}p_t \wedge \mathrm{d}q_t + \mathrm{d}p_0 \wedge \mathrm{d}q_0.$$

Using gauge freedom, we set a constant matrix $B$ to be diagonal

$$B_1 = \begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix}.$$

In such parametrisation Hamiltonian takes form

$$H = \underset{\lambda=t}{\text{res}}\text{Tr}\left(\frac{A(\lambda)^2}{2}\right) = 4k(p_tq_t - \theta_t) - \frac{2}{t}(q_tq_0(p_t - p_0)^2 - 2(q_0\theta_t - q_t\theta_0)(p_t - p_0) - 2\theta_0\theta_t) =$$
$$= 4k(p_tq_t - \theta_t) - \frac{2q_tq_0}{t}\left(p_t - p_0 - \frac{\theta_t}{q_t} + \frac{\theta_0}{q_0}\right)^2 + \frac{2}{t}\left(\theta_t^2\frac{q_0}{q_t} + \theta_0^2\frac{q_t}{q_0}\right).$$

This Hamiltonian is invariant under the following rescaling

$$p_i \to p_i\alpha, \quad q_i \to \frac{q_i}{\alpha}$$

which is the same as the gauge $SL(2)$ action via diagonal matrix. The moment map is

$$q_0p_0 + q_tp_t.$$

The change of the coordinates

$$I = q_0p_0 + q_tp_t, \quad \phi = \ln(q_0), \quad u = -\frac{q_t}{q_0}, \quad v = p_tq_0,$$

is a canonical transformation. Resolving it with respect to the $q$'s and $p$'s we obtain

$$q_0 = e^{\varphi}, \quad q_t = -e^{\varphi}u, \quad p_0 = e^{-\varphi}(I + uv) \quad p_t = e^{-\varphi}v,$$

and the symplectic form goes to

$$\omega = \mathrm{d}p_t \wedge \mathrm{d}q_t + \mathrm{d}p_0 \wedge \mathrm{d}q_0 = \mathrm{d}I \wedge \mathrm{d}\varphi + \mathrm{d}v \wedge \mathrm{d}u.$$

The Hamiltonian in these coordinates writes as

$$H = -4k(uv + \theta_t) + 2\frac{u}{t}\left(v - I - uv + \frac{\theta_t}{u} + \theta_0\right)^2 - \frac{2}{t}\left(\theta_t^2\frac{1}{u} + \theta_0^2 u\right)$$

and it is obvious that $I$ and $\varphi$ are the part of the action-angle variables, so we may decrease degrees of freedom by 1 and consider the following Hamiltonian system

$$H = -4k(uv+\theta_t)+2\frac{u}{t}\left(v - a - uv + \frac{\theta_t}{u} + \theta_0\right)^2 - \frac{2}{t}\left(\theta_t^2\frac{1}{u} + \theta_0^2 u\right), \quad \omega = \mathrm{d}v\wedge\mathrm{d}u, \quad a = \mathrm{const}\,.$$

The equations of motion take form

$$\dot{u} = \frac{\partial H}{\partial v} = -4ku + \frac{4u}{t}(1-u)\left(v - a - uv + \frac{\theta_t}{u} + \theta_0\right)$$

$$\dot{v} = -\frac{\partial H}{\partial u} = 4kv - \frac{2}{t}\left((v - uv - a + \frac{\theta_t}{u} + \theta_0)^2 - 2u(v - uv - a + \frac{\theta_t}{u} + \theta_0)\left(v + \frac{\theta_t}{u^2}\right) + \frac{\theta_t^2}{u^2} - \theta_0^2\right).$$

Writing second order ODE for $u$ we obtain

$$\frac{d^2u}{dt^2} = \left(\frac{1}{u-1} + \frac{1}{2u}\right)\left(\frac{du}{dt}\right)^2 - \frac{1}{t}\frac{du}{dt} + 8\theta_0\frac{(u-1)^2}{t^2}\left(u - \left(\frac{\theta_t}{\theta_0}\right)^2\frac{1}{u}\right) + 4k(4(a-\theta_0-\theta_t)-1)\frac{u}{t} - 8k^2\frac{u(u+1)}{u-1}$$

which is the Gambier's form of the Painlevé V equations and the constants are given by

$$\theta_0 = \frac{\alpha}{8}, \quad \theta_t^2 = -\frac{\alpha\beta}{64}, \quad k^2 = -\frac{\delta}{8}, \quad 4k(4(a - \theta_0 - \theta_t) - 1) = \gamma.$$

The following canonical transformation

$$u = \frac{x}{x-1}, \quad v = -((x-1)y + a - 2\theta_0)(x-1), \quad \mathrm{d}v \wedge \mathrm{d}u = \mathrm{d}y \wedge \mathrm{d}x,$$

send $H$ to the following form

$$tH = 2x(x-1)y^2 + 4(ktx(x-1) + x(\theta_t - \theta_0) - \theta_t)y + 4(xkt(a-2\theta_0) - \theta_t(kt - \theta_0))$$

which was introduced in [58]. The example of the Painlevé V equation as a system written on the co-adjoint orbits of the Takiff algebra was recently studied by [57] in more details.

### 5.6.3 Painlevé IV

The connectionis

$$A(\lambda) = \frac{A^{(t)}}{\lambda - t} - B_1 - B_2\lambda \tag{5.6.5}$$

and the deformation one-form is

$$\Omega = -\frac{A^{(t)}}{\lambda - t}\mathrm{d}t. \tag{5.6.6}$$

Deformation equations are

$$\dot{A}^{(t)} = [A^{(t)}, B_1 + B_2 t], \quad \dot{B}_1 = [B_2, A^{(t)}], \quad \dot{B}_2 = 0.$$

The Poisson structure is

$$\{A^{(t)} \underset{,}{\otimes} A^{(t)}\} = [\Pi, 1 \otimes C], \quad \{B_1 \underset{,}{\otimes} B_1\} = [\Pi, 1 \otimes B_2], \quad \{B_1 \underset{,}{\otimes} B_2\} = \{B_2 \underset{,}{\otimes} B_2\} = 0.$$

Hamiltonian writes as

$$H = \operatorname*{res}_{\lambda=t} \mathrm{Tr}\frac{A^2}{2} = -\mathrm{Tr}\left(A^{(t)}B_1 + tA^{(t)}B_2\right). \tag{5.6.7}$$

Since $B_3$ is a constant of motion the same holds for the transition matrix to the eigenbasis to $B_3$. This allows us to consider the gauge, which is equal to this transition matrix without changing the Poisson structure of $A^{(t)}$. In the case of $\mathfrak{sl}_2$ we have

$$A^{(t)} = \begin{pmatrix} p_t q_t - \theta_t & -(p_t q_t - 2\theta_t)\, p_t \\ q_t & -p_t q_t + \theta_t \end{pmatrix}, \quad -B_2\lambda - B_1 = -\lambda\theta_3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} \theta_2 & -2\theta_3 q_3 \\ p_3 & -\theta_2 \end{pmatrix} \tag{5.6.8}$$

The Hamiltonian writes as

$$H = (p_t q_t - 2\theta_t)\, p_t p_3 - 2\,(p_t q_t - \theta_t)\,(t\theta_3 + \theta_2) + 2\theta_3 q_3 q_t. \tag{5.6.9}$$

Since $B_3$ is a diagonal matrix (has no Jordan blocks) the stabilizer is the Cartan torus of $SL_2$,i.e.

$$S = \begin{pmatrix} h & 0 \\ 0 & 1/h \end{pmatrix}$$

The additional action of the stabilizer of $B_3$ leads to the following action on the reduced phase space

$$q_t \to \frac{q_t}{h^2}, \quad p_t \to h^2 p_t, \quad q_3 \to h^2 q_3, \quad p_3 \to \frac{p_3}{h^2},$$

which is Hamiltonian with the following moment map

$$I = q_3 p_3 - q_t p_t.$$

Using the symplectic change of coordinates

$$q_3 = e^{\phi}, \quad q_t = e^{-\phi} u, \quad p_3 = e^{-\phi}(I + uv), \quad p_t = e^{\phi} v, \quad \mathrm{d}p_3 \wedge \mathrm{d}q_3 + \mathrm{d}p_t \wedge \mathrm{d}q_t = \mathrm{d}I \wedge \mathrm{d}\phi + \mathrm{d}v \wedge \mathrm{d}u \tag{5.6.10}$$

and fixing the level set of moment map $I = I_0 = \text{const}$ we reduce to the system with one degree of freedom

$$H = (uv - 2\theta_t)\, v\,(uv + I_0) - 2\,(uv - \theta_t)\,(t\theta_3 + \theta_2) + 2\theta_3 u. \tag{5.6.11}$$

Finally, using the change of variables

$$u = x(xy - I_0), \quad v = \frac{1}{x}, \quad \mathrm{d}v \wedge \mathrm{d}u = \mathrm{d}y \wedge \mathrm{d}x$$

sends Hamiltonian to the Okamoto form of $P_{\mathrm{IV}}$

$$H = 2yx^2 + \big(\theta_3 y^2 + (-2t\theta_3 - 2\,\theta_2)\, y - 2I_0\big)\, x + (-I_0\theta_3 - 2\theta_3\theta_t)\, y \tag{5.6.12}$$

Taking

$$\theta_3 = -1, \quad \theta_2 = 0, \quad I_0 = -\theta_0, \quad \theta_t = -\frac{1}{2}(\theta_\infty + \theta_0)$$

we obtain the $P_{\rm IV}$ Hamiltonian

$$H = 2yx^2 - (y^2 + 2ty + 2\theta_0)x + \theta_\infty y.$$

### 5.6.4 Degenerated Painlevé IV. Flashka-Newell Painlevé II.

The degenerated case corresponds to the situation when the co-adjoint orbit of Takiff algebra is the co-adjoint orbit of the Jordan form

$$B_3\lambda + B_2 + \frac{1}{\lambda}B_1 \sim \lambda \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \theta_2 & 0 \end{pmatrix} + \frac{1}{z}\begin{pmatrix} 0 & 0 \\ \theta_1 & 0 \end{pmatrix},$$

here conjugation is considered via adjoint action of Takiff group. The Darboux parametrization takes form

$$A^{(t)} = \begin{pmatrix} p_t q_t - \theta_t & -(p_t q_t - 2\theta_t)\, p_t \\ q_t & -p_t q_t + \theta_t \end{pmatrix}, \quad -B_3\lambda - B_2 = -\lambda \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} \frac{1}{2}p_3 & 2q_3 \\ \theta_2 & -\frac{1}{2}p_3 \end{pmatrix}. \tag{5.6.13}$$

The Hamiltonian (5.6.7) takes form

$$H = -\frac{1}{2}\left(p_t q_t - \theta_t\right) p_3 + \left(p_t q_t - 2\theta_t\right) p_t \theta_2 - q_t\left(t + 2q_3\right) + \frac{1}{2}\left(-p_t q_t + \theta_t\right) p_3. \tag{5.6.14}$$

Stabilizer of $B_3$ now takes values in Borel subgroup

$$S = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$$

which leads to the following action on the Darboux coordinates

$$q_t \to q_t, \quad q_3 \to q_3 - \frac{1}{2}(h^2\theta_2 + hp_3), \quad p_t \to p_t + h, \quad p_3 \to p_3 + h2\theta_2.$$

The moment map of this action is a constant of motion of the following form

$$I = \frac{1}{4}p_3^2 + 2\theta_2 q_3 + q_t.$$

After symplectic change of variables

$$p_3 = -2\theta_2(\phi + u), \quad p_t = -\phi, \quad q_3 = \frac{v - \theta_2^2(\phi+u)^2}{2\theta_2}, \quad q_t = I - v, \tag{5.6.15}$$

Hamiltonian writes as

$$H = \frac{v^2}{\theta_2} + \left(t - \frac{I}{\theta_2} - \theta_2 u^2\right) v + I\theta_2 u^2 - 2u\theta_2\theta_t.$$

Hamilton equations take form

$$\frac{\mathrm{d}}{\mathrm{d}t} v + 2\theta_2(1-I)uv + 2\theta_2\theta_t v = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t} u + t + \frac{2v}{\theta_2} - \frac{I}{\theta_2} - \theta_2 u^2 = 0$$

which are equivalent to

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} u = 2\theta_2^2 u^3 - (2\theta_2 t + 2I)u + 4\theta_t - 1.$$

After some shift and rescaling of time $t$ and solution $u$ this equation gives the Painlevé II equation.

### 5.6.5 Painlevé III

The connection takes form

$$A = \frac{B_0}{\lambda} + t\frac{B_1}{\lambda^2} + C \tag{5.6.16}$$

with deformation one-form

$$\Omega = -\frac{B_1}{\lambda}\mathrm{d}t. \tag{5.6.17}$$

Poisson algebra is

$$\{C \underset{,}{\otimes} C\} = \{C \underset{,}{\otimes} B_{0,1}\} = \{B_1 \underset{,}{\otimes} B_1\} = 0, \quad \{B_0 \underset{,}{\otimes} B_0\} = [\Pi, 1 \otimes B_0], \quad \{B_0 \underset{,}{\otimes} B_1\} = [\Pi, 1 \otimes B_1] \tag{5.6.18}$$

Hamiltonian is given by

$$H = \frac{1}{2}\underset{\lambda=0}{\mathrm{res}}\,\mathrm{Tr}\frac{\lambda}{t}A^2 = \mathrm{Tr}\left(CB_1 + \frac{B_0^2}{2t}\right). \tag{5.6.19}$$

In the case of $\mathfrak{sl}_2$, choosing the gauge such that $C$ is diagonal we have the following Darboux parametrisation

$$B_0 = \begin{pmatrix} p_1q_1 - p_2q_2 + \theta_1 & -p_1q_1^2 + (2q_1q_2+1)\,p_2 - 2\theta_1 q_1 \\ p_1 & -p_1q_1 + p_2q_2 - \theta_1 \end{pmatrix}$$

$$B_1 = \begin{pmatrix} 2q_1q_2\theta_2 + \theta_2 & -2\theta_2\,(q_1q_2+1)\,q_1 \\ 2\theta_2 q_2 & -2q_1q_2\theta_2 - \theta_2 \end{pmatrix}, \quad C = \begin{pmatrix} \theta_3 & 0 \\ 0 & -\theta_3 \end{pmatrix}. \quad (5.6.20)$$

Hamiltonian writes as

$$tH = p_2^2q_2^2 + 4t\theta_2\theta_3 q_1q_2 - 2\theta_1 p_2q_2 + p_1p_2. \quad (5.6.21)$$

The stabilizer of $C$ ($SL_2$ torus) action gives integral of motion

$$I = q_1p_1 - q_2p_2.$$

In order to reduce the degrees of freedom we use change of variables

$$q_1 = e^{\phi}, \quad q_2 = -e^{-\phi}u, \quad p_1 = e^{-\phi}(I+uv), \quad p_2 = -e^{\phi}v, \quad \mathrm{d}p_1\wedge\mathrm{d}q_1 + \mathrm{d}p_2\wedge\mathrm{d}q_2 = \mathrm{d}I\wedge\mathrm{d}\phi + \mathrm{d}v\wedge\mathrm{d}u$$

which leads to the following Hamiltonian

$$tH = v^2u^2 - \left(v^2 + 2\theta_1 v + 4t\theta_2\theta_3\right)u - I_0 v \quad (5.6.22)$$

where $I_0$ is a level set of the first integral $I$. Obtained Hamiltonian corresponds to the Painlevé III equation of type $D_6$ after some choice of constants. To obtain degenerations to $D_7$ and $D_8$ we have to consider nilpotent orbits.

### 5.6.6 Painlevé III D7

In this case we consider situation when $C$ is rank 1 matrix, i.e.

$$C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The Hamiltonian takes form

$$tH = p_2^2 q_2^2 + 2t\theta_2 q_2 - 2\theta_1 p_2 q_2 + p_1 p_2. \tag{5.6.23}$$

The Hamiltonian doesn't depend on $q_1$ so

$$p_1 = I_0 = \text{const},$$

which coinsides with the moment map for the additional action of the stabilizer for $C$. Changing $p_2$ to $v$ and $q_2$ to $u$ we obtain

$$tH = u^2v^2 + 2\theta_2 tu - 2\theta_1 uv + I_0 v, \quad \omega = \mathrm{d}v \wedge \mathrm{d}u,$$

which coinsides with the Hamiltonian of Painlevé III D7.

### 5.6.7 Painlevé III D6

Here we also consider the co-adjoint orbit at 0 to the nilpotent orbit. The Darboux parametrization takes form

$$B_1 = \begin{pmatrix} \left(q_1 + \frac{1}{2q_2}\right) p_1 - \frac{p_2 q_2}{2} - \frac{\theta_1}{2q_2} & \left(-q_1^2 - \frac{q_1}{q_2}\right) p_1 + p_2\left(q_1 q_2 + 1\right) + \theta_1 \frac{q_1}{q_2} \\ p_1 & \left(-q_1 - \frac{1}{2q_2}\right) p_1 + \frac{p_2 q_2}{2} + \frac{\theta_1}{2q_2} \end{pmatrix}$$

$$B_2 = \begin{pmatrix} -\left(q_1 q_2 + 1\right) q_2 & \left(q_1 q_2 + 1\right)^2 \\ -q_2^2 & \left(q_1 q_2 + 1\right) q_2 \end{pmatrix} \tag{5.6.24}$$

Hamiltonian takes form

$$tH = \frac{q_2^2 p_2^2}{4} - tq_2^2 + \frac{p_1 p_2}{2} + \theta_1 \frac{p_2}{2} + \frac{1}{4}\frac{{p_1}^2}{q_2^2} - \frac{1}{4}\frac{(p_1 - \theta_1)^2}{q_2^2}$$

Fixing the first integral

$$p_1 = I_0$$

we obtain

$$tH = \frac{u_2^2 v_2^2}{4} - tu_2^2 + \frac{1}{2}(I_0 + \theta_1)v + \frac{1}{4}\frac{{I_0}^2}{u^2} - \frac{1}{4}\frac{(I_0 - \theta_1)^2}{u^2}$$

In case when $I_0 = -\theta_1$, after the change of variables

$$u = \frac{1}{\sqrt{q}}, \quad v = -2q^{\frac{3}{2}}\left(p + \frac{1}{2q}\right)$$

we obtain Painlevé III D8 Hamiltonian

$$tH = q^2p^2 + qp + \theta_1^2 q - \frac{t}{q}.$$

### 5.6.8 Painlevé II. Jimbo-Miwa

The connection takes form

$$A(\lambda) = \frac{B_3}{\lambda^4} + \frac{B_2}{\lambda^3} + \frac{B_1 + tB_3}{\lambda^2} + \frac{B_0}{\lambda}. \tag{5.6.25}$$

Deformation one form is

$$\Omega = -\frac{B_3}{\lambda}\mathrm{d}t \tag{5.6.26}$$

Deformation equations are

$$\frac{d}{dt}B_3 = [B_2, B_3], \quad \frac{d}{dt}B_2 = [B_1, B_3], \quad \frac{d}{dt}B_1 = [B_0 - tB_2, B_3], \quad \frac{d}{dt}B_0 = 0. \tag{5.6.27}$$

The Poisson structure is given by

$$\left\{B_i \underset{,}{\otimes} B_j\right\} = [\Pi, \mathbb{I} \otimes B_{i+j-1}] \tag{5.6.28}$$

Hamiltonian takes form

$$H = \underset{\lambda=0}{\mathrm{res}}\,\mathrm{Tr}\lambda^3 \frac{A(\lambda)^2}{2} = \mathrm{Tr}\left(\frac{B_1^2}{2} + B_0B_2 + tB_1B_3\right), \tag{5.6.29}$$

here we drop $\mathrm{Tr}B_3^2$ part since it is a Casimir. Since we assume that for Painlevé II there is no singularity at $\infty$, the value of the gauge group moment map should be put to zero, i.e.

$$B_0 = 0. \tag{5.6.30}$$

Such reduction, has to be viewed as a Hamiltonian reduction written on the co-adjoint orbit of the Takiff algebra $\hat{\mathfrak{g}}_3$, so we have to change not only Hamiltonian, but also the Poisson structure. However, usually the second Painlevé equation isomonodromic problem writes in chart where the only singularity is at $\infty$. The connection takes form

$$A(\lambda) = B_3\lambda^2 + B_2\lambda + B_3 t + B_1. \tag{5.6.31}$$

Here we already resolve the gauge group moment map, by setting residue at $\infty$ to be zero. The deformation one-form then my be written as

$$\Omega = (B_3\lambda + B_2)\,\mathrm{d}t.$$

The deformation equations are

$$\dot{B}_3 = 0, \quad \dot{B}_2 = [B_3, B_1], \quad \dot{B}_1 = t[B_2, B_3] + [B_2, B_1].$$

The deformation equations are Hamiltonian, with Hamiltonian written as

$$H = \operatorname*{res}_{\lambda=0} \mathrm{Tr}\frac{A^2}{2\lambda} = \mathrm{Tr}\left(\frac{B_1^2}{2} + tB_1B_3\right). \tag{5.6.32}$$

To obtain Painlevé II equation we consider the $\mathfrak{sl}_2$ case. Darboux parametrisation is given by

$$B_3 = \begin{pmatrix} \theta_4 & 0 \\ 0 & -\theta_4 \end{pmatrix}, \quad B_2 = \begin{pmatrix} \theta_3 & -2\theta_4 q_3 \\ 2\theta_4 q_4 & -\theta_3 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 2\theta_4 q_3 q_4 + \theta_2 & -\theta_4 q_3^3 q_4^2 + (\theta_3 - 4\theta_4)\, q_4 q_3^2 - \theta_4 q_3 + p_4 \\ -\theta_4 q_3^2 {q_4}^3 + (\theta_3 - 4\theta_4)\, q_4^2 q_3 + (2\theta_3 - \theta_4)\, q_4 + p_3 & -2\theta_4 q_3 q_4 - \theta_2 \end{pmatrix}.$$

The Hamiltonian takes form

$$\begin{aligned} H = -(2\theta_4 q_3 q_4 + \theta_2)^2 - 2t(2\theta_4 q_3 q_4 + \theta_2)\theta_4 - ((\theta_3 - 4\theta_4)q_4 q_3^2 - \theta_4 q_3 + p_4 \\ - \theta_4 q_3^3 q_4^2)((\theta_3 - 4\theta_4)q_4^2 q_3 + (2\theta_3 - \theta_4)q_4 + p_3 - \theta_4 q_3^2 q_4^3). \end{aligned} \tag{5.6.33}$$

The action of stabilizer of $B_4$ gives us the moment map

$$I = p_3q_3 - p_4q_4$$

which gives us the following change of variables $(p_3, p_4, q_3, q_4) \to (I, v, \phi, u)$

$$p_3 = e^{-\phi}(I+uv), \quad p_4 = e^{\phi}v, \quad q_3 = e^{\phi}, \quad q_4 = e^{-\phi}u, \quad \mathrm{d}p_3\wedge\mathrm{d}q_3+\mathrm{d}p_4\wedge\mathrm{d}q_4 = \mathrm{d}v\wedge\mathrm{d}u+\mathrm{d}I\wedge\mathrm{d}\phi.$$

The Hamiltonian then writes as

$$H = -(2\theta_4u+\theta_2)^2-2t(2\theta_4u+\theta_2)\theta_4-(v-\theta_4u^2+(\theta_3-4\theta_4)u-\theta_4)(uv+(\theta_3-4\theta_4)u^2+(2\theta_3-\theta_4)u+I-\theta_4u^3)$$

The change of variable

$$v = w + \frac{1}{2u}(2\theta_4u^3 - 2u^2\theta_3 + 8\theta_4u^2 - 2u\theta_3 + 2\theta_4u - I), \quad w = -\frac{p}{q}, \quad u = -\frac{q^2}{2}$$

gives us

$$H = \frac{p^2}{2} - \theta_4^2q^4 + \left(2\theta_4^2t + 2\theta_2\theta_4 - \frac{\theta_3^2}{2}\right)q^2 - \frac{I^2}{2q^2}$$

which is the Hamiltonian of $P_{34}$ equation, which is equivalent to Painlevé II in case when $I = 0$.

**Remark 5.6.1.** *The isomonodromic problem with connection matrix* (5.6.31) *corresponds to the non-autonomous version of the famous Nahm top which first appeared in [73]. Treating the variable $t$ as a constant, we obtain an integrable system with Lax matrix* (5.6.31) *which is gauge equivalent to the Lax matrix for the Nahm equation. This gives the explicit Hamiltonian formulation of the Nahm equation in terms of the coadjoint orbits of the Takiff algebras. This should coincide with the Hamiltonian formalism for the Nahm equations introduced in [81].*

### 5.6.9 Painlevé I. Degenerated Jimbo-Miwa problem.

We use the same setup as in previous case of Painlevé II, but consider degenerated case when

$$B_4 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then Darboux parametrization takes form

$$B_3 = \begin{pmatrix} -q_4 & q_3 q_4 \\ \theta_1 & q_4 \end{pmatrix}, \quad B_2 = \begin{pmatrix} (2\theta_1 - q_4^2)q_3 - q_4 + \theta_2 + \frac{p_3}{q_4} & p_4 - 1 - \theta_1(q_3^2) - \left(q_4 + \frac{p_3}{q_4}\right) q_3 \\ -\theta_1 q_3 q_4 - q_4^2 & -(2\theta_1 - q_4^2)q_3 + q_4 - \theta_2 - \frac{p_3}{q_4} \end{pmatrix}.$$

In order to make the calculations more compact we use the following symplectic transformation

$$p_3 = -2\theta_1 q_2 - p_2 q_1 + q_1^2, \quad p_4 = \frac{2q_2 q_1^2 - p_1 q_1^2 - p_2 q_2 q_1 - \theta_1 q_2^2 + q_1^2}{q_1^2}, \quad q_3 = -\frac{q_2}{q_1}, \quad q_4 = -q_1,$$

and Darboux parametrization writes as

$$B_3 = \begin{pmatrix} q_1 & q_2 \\ \theta_1 & -q_1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} q_1 q_2 + z q_1 + p_2 + \theta_2 & z q_2 + z^2 - p_1 + t \\ z\theta_1 - \theta_1 q_2 - q_1^2 & -q_1 q_2 - z q_1 - p_2 - \theta_2 \end{pmatrix}.$$

Hamiltonian writes as

$$H = -q_1^2 q_2^2 - p_1 q_1^2 - p_1 q_2 \theta_1 + t q_1^2 - 2p_2 q_2 q_1 - 2q_1 q_2 \theta_2 + q_2 t \theta_1 - p_2^2 - 2p_2 \theta_2 - \theta_2^2.$$

The action of the stabilizer of $B_4$

$$C = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$$

takes the following form on a phase space

$$p_1 \to p_1 + 2c(q_1 q_2 + p_2 + \theta_2) - c^2(\theta_1 q_2 + q_1^2), \quad q_1 \to q_1 + c\theta_1,$$

$$p_2 \to p_2 + c(q_1^2 - 2q_2\theta_1) + c^2 3 q_1 \theta_1 + c^3 \theta_1^2, \quad q_2 \to q_2 - 2c q_1 - c^2 \theta_1.$$

the moment map is

$$I = q_2 q_1^2 + 2p_2 q_1 + 2\theta_2 q_1 - \theta_1 q_2^2 - \theta_1 p_1.$$

Applying transformation

$$p_1 = -\frac{1}{\theta_1}\left(\theta_1^2 \phi^2 u + 2\theta_1 \phi v + 2\theta_1 \theta_2 \phi + \theta_1 u^2 + I\right), \quad q_1 = -\theta_1 \phi,$$

$$p_2 = -\theta_1^2 \phi^3 + 2\theta_1 \phi u + v, \quad q_2 = -\theta_1 \phi^2 + u$$

we obtain Hamiltonian

$$H = \theta_1 u^3 + \theta_1 ut + Iu - (v + \theta_2)^2.$$

which is equivalent to the Painlevé I Hamiltonian for a general values of the constants $I, \theta_1$.

## 5.7 Quantum Isomonodromic Hamiltonians and Irregular Knizhnik–Zamolodchikov system

In this section, we give a general formula for the confluent KZ equations with singularities of arbitrary Poincaré rank in any dimension. Moreover, we use the lifted Darboux coordinates in order to generalise an observation by Reshetikhin that the quasiclassical solution of the standard KZ equations (i.e. with simple poles) is expressed via the isomonodromic $\tau$-function [80]. Here we propose an easy proof which is valid for any Poincaré configuration of the singularities on a Riemann sphere. Firstly we review a Reshetikhin approach for the quantum isomonodromic problems and then produce our proof which is based on the generalisation of an observation by Malgrange [65].

Throughout this section we work with the canonical quantisation of the linear Poisson brackets that prescribes the standard correspondence principle

$$\{f, g\} \longrightarrow [\hat{f}, \hat{g}] = i\hbar\widehat{\{f, g\}}, \tag{5.7.1}$$

where the symbol ˆ denotes the quantum operator, i.e. $\hat{f}$ is the quantum operator corresponding to the classical function $f$. In the case of a semi-simple Lie algebra, $\hbar$ can be written via the dual Coxeter number and the level. Here we ignore this fact and we focus on the $\mathfrak{gl}_m$ case, so we simply replace the Poisson bracket by the commutator.

More accurately, one can speak about the so–called *Rees deformation* that assigns to a filtered vector space $R = \cup_i R_i$ a canonical deformation of its associated graded algebra $\mathrm{gr}(R)$ over the affine line $\mathbb{A}_1$ considered as the spectrum $\mathrm{Spec}(\mathbb{C}[\hbar])$ of the polynomials $\mathbb{C}[\hbar]$. The fiber at the point $\hbar$ is isomorphic to $R$ if $\hbar \neq 0$ and to $\mathrm{gr}(R)$ for $\hbar = 0$. The corresponding $\mathbb{C}[\hbar]$–module here is the direct sum $\oplus_i R_i$ on which $\hbar$ acts by mapping each $R_i$ to $R_{i+1}$ [28]. In our case the Rees construction gives a one-parameter family of algebras $U_\hbar(\mathfrak{g})$, with the associated graded algebra $U_0(\mathfrak{g})$ being the symmetric algebra $S(\mathfrak{g})$. The $\hbar$ deformation re-scales the bracket by $\hbar$, so that the $\hbar$ linear terms define the standard Poisson bracket on $S(\mathfrak{g})$.

### 5.7.1 Finite-dimensional representation

In this sub-section, we recall the basic ideas at the basis of Reshetikhin's approach to quantum isomonodromic problems for Fuchsian systems and the adapt it to the irregular case. We fix $\hbar = 1$ for simplicity.

In the case of the Fuchsian system we are dealing with canonical quantisation of the direct product of the co-algebras $\mathfrak{g}^\star$. The quantisation functor sends the functions on the phase space of the classical system to the differential operators which act on some Hilbert space in a way that (5.7.1) holds. In principle, a choice of finite dimensional representation may be seen as a choice of the special subspace of the Hilbert space of functions on which the algebra of quantum operators acts. However, we may avoid such complicated construction of finite dimensional representation when the classical Poisson algebra is given by a linear Poisson bracket. Indeed, for $\mathfrak{g}^\star$ the standard Lie–Poisson bracket endows the space of functions with the structure of a Lie algebra so that the structure constants of this Poisson algebra are identified with the structure constants of the Lie algebra $\mathfrak{g}$.

In general, the quantisation procedure for the phase space of the Fuchsian system may be viewed as a map from

$$\underbrace{\mathfrak{g}^\star \times \mathfrak{g}^\star \times \cdots \times \mathfrak{g}^\star}_{n}$$

to the differential operators which act on the tensor product of Hilbert spaces $\mathcal{H}_i$:

$$\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n.$$

However, the isomonodromic nature of the Hamiltonian systems we consider gives additional information which may be used to define a quantum problem in a uniform way. Following [54], we quantise the connection that becomes the generating function for the quantum Hamiltonians. Considering the connection as a matrix whose entries are functions on the $\mathfrak{g}^\star \times \mathfrak{g}^\star \times \cdots \times \mathfrak{g}^\star$, we obtain a following quantisation for the Fuchsian case:

$$\hat{A}(\lambda) = \sum_{i=1}^{n} \frac{\hat{A}^{(i)}}{\lambda - u_i}, \quad \hat{A}^{(i)} = \sum_{\alpha} e_\alpha^{(0)} \otimes e_\alpha^{(i)}, \quad e_\alpha^{(i)} = 1 \otimes \cdots \otimes \underset{i}{e_\alpha} \otimes \cdots \otimes 1,$$

where $e_\alpha^{(i)}$, $i = 1, \ldots, n$ is a basis of representation we choose for a quantisation and the first $e_\alpha^{(0)}$ corresponds to auxiliary space $\mathcal{H}_0$ given by the connection. The Schlesinger Hamiltonians then transform to

$$\hat{H}_i = \sum_{j \neq i} \frac{\mathrm{Tr}^{(0)}(\hat{A}^{(i)} \hat{A}^{(j)})}{u_i - u_j}, \tag{5.7.2}$$

where $\mathrm{Tr}^{(0)}$ is a trace in the auxiliary space $\mathcal{H}_0$. The quantum Schlesinger Hamiltonians $\hat{H}_i$ are the solutions for the classical Yang-Baxter equations and may be written as

$$\hat{H}_i = \sum_{j \neq i} \frac{r_{ij}}{u_i - u_j}, \tag{5.7.3}$$

where $r_{ij}$ is a solution of the classical Yang-Baxter equation

$$[r_{ij}, r_{ik}] + [r_{ij}, r_{jk}] + [r_{ik}, r_{jk}] = 0.$$

The corresponding set of Schrödinger equations are called Knizhnik–Zamolodchikov equations and take form

$$\nabla_i \Psi = \left( \frac{\partial}{\partial u_i} - \sum_{j \neq i} \frac{r_{ij}}{u_i - u_j} \right) \Psi = 0.$$

Moreover, the Knizhnik–Zamolodchikov operators commute, i.e.

$$[\nabla_i, \nabla_j] = 0 \quad \Longleftrightarrow \quad \frac{\partial}{\partial u_i} \hat{H}_j = \frac{\partial}{\partial u_j} \hat{H}_i, \quad [\hat{H}_i, \hat{H}_j] = 0.$$

Reproducing the same scheme for the Takiff co-algebras, we obtain the quantisation map that acts by replacing the co-algebra with the Lie algebra

$$\hat{\mathfrak{g}}^\star_{r_1} \times \hat{\mathfrak{g}}^\star_{r_2} \times \ldots \hat{\mathfrak{g}}^\star_{r_n} \times \hat{\mathfrak{g}}^\star_{r_\infty} \quad \longrightarrow \quad \hat{\mathfrak{g}}_{r_1} \otimes \ldots \hat{\mathfrak{g}}_{r_n} \otimes \hat{\mathfrak{g}}_{r_\infty}. \tag{5.7.4}$$

The quantum connection then takes the form

$$\hat{A}(\lambda) = \sum_i^n \left( \sum_{j=0}^{r_i} \frac{\hat{B}_j^{(i)} \left( t_1^{(i)}, t_2^{(i)} \ldots t_{r_i}^{(i)} \right)}{(\lambda - u_i)^{j+1}} \right),$$

where $\hat{B}^{(i)}$'s are given by

$$\hat{B}_j^{(i)}(t_1^{(i)},\dots t_{r_i}^{(i)}) = \sum_{k=j}^{r} \hat{A}_k^{(i)} \mathcal{M}_{j,k}^{(r_i)}(t_1^{(i)}, t_2^{(i)} \dots t_{r_i}^{(i)}), \quad \hat{A}_k^{(i)} = \sum_\alpha e_\alpha^{(0)} \otimes e_\alpha^{(i)} \otimes z_i^k, \quad e_\alpha^{(i)} = 1 \otimes \dots \otimes \underset{i}{e_\alpha} \otimes \dots \otimes 1.$$

The Hamiltonians which correspond to the position of poles are given as in the Fuchsian case, i.e.

$$\hat{H}_{u_i} = \frac{1}{2} \operatorname*{res}_{\lambda=u_i} \mathrm{Tr}_0 \hat{A}(\lambda)^2,$$

where $\mathrm{Tr}_0$ is the tarce in the 0-th space, so we now have to choose a quantum ordering, for example lexigraphical ordering. The irregular Hamiltonians have to be calculated according to the Theorem 2.0.6 at each irregular singularity changing Tr by $\mathrm{Tr}_0$. Again we will choose a quantum ordering. Thus, we obtain that the irregular Hamiltonians are given by

$$\mathcal{M}^{(r_i)} \begin{pmatrix} \hat{H}_1^{(i)} \\ \hat{H}_2^{(i)} \\ \dots \\ \hat{H}_{r_i}^{(i)} \end{pmatrix} = \begin{pmatrix} \hat{S}_1^{(u_i)} \\ \hat{S}_2^{(u_i)} \\ \dots \\ \hat{S}_{r_i}^{(u_i)} \end{pmatrix}, \quad \hat{S}_k^{(u_i)} = \frac{1}{2} \oint_{\Gamma_{u_i}} (\lambda - u_i)^k \mathrm{Tr}_0 \hat{A}^2 \mathrm{d}\lambda$$

at the point $u_i$ with the Poincare rank $r_i$. To prove Theorem 2.0.9 we need to show that the confluent KZ gives a quantum integrable system, namely that the differential operators defined in (2.0.18), (2.0.19)

$$\nabla_{u_j} := \frac{\partial}{\partial u_j} - \widehat{H}_{u_j}, \quad j = 1, \dots, n$$

$$\nabla_k^{(i)} := \frac{\partial}{\partial t_k^{(i)}} - \widehat{H}_k^{(i)}, \quad i = 1, \dots, n, \infty, \quad k = 1, \dots, r_i$$

commute. This is a simple consequence of the fact that in the quantisation process the derivatives remain commutative, i.e. for example $[\frac{\partial}{\partial u_j}, \frac{\partial}{\partial t_k^{(i)}}] = 0$, and that the quantum Hamiltonians are linear combinations of the quantum Gaudin spectral invariants $\hat{S}_k^{(u_i)}$, $k = 0, \dots, r_i$, which commute as proved in [71].

We have to mention that for the Fuchsian times the isomonodromic Hamiltonian depends on each phase space $\mathfrak{g}_{r_i}$ linearly – which means that it may be written as

$$\hat{H}_{u_i} \in \hat{\mathfrak{g}}_{r_1} \otimes \dots \hat{\mathfrak{g}}_{r_n} \otimes \hat{\mathfrak{g}}_{r_\infty} \subset U\left(\hat{\mathfrak{g}}_{r_1} \oplus \dots \oplus \hat{\mathfrak{g}}_{r_\infty}\right).$$

In the case of irregular times Hamiltonians becomes more complicated – there are quadratic terms which contains elements from the same space and in general we have that

$$\hat{H}_k^{(i)} \in U(\hat{\mathfrak{g}}_{r_1}) \otimes \ldots U(\hat{\mathfrak{g}}_{r_n}) \otimes U(\hat{\mathfrak{g}}_{r_\infty}).$$

The problem of the explicit form of the Hamiltonians introduced in this thesis has to deal with the $U(\hat{g}_{r_i})$ representation theory, which is rather complicated. In order to avoid this representational theoretic problems, we write down the quantum Hamiltonians for the irregular isomonodromic deformations using intermediate Darboux coordinates. For the classical examples of the Painlevé equations, we provide invariant subspaces for these Hamiltonians. These subspaces give finite dimensional representations for the Hamiltonians which are the quantum reduction of the irregular Hamiltonians introduced in this section.

### 5.7.2 Intermediate Quantum Hamiltonians for Painlevé equations.

In this subsection, we write quantum Hamiltonians for the Painlevé equations written in Darboux coordinates before the reduction with respect to the gauge group action. In the case of Painlevé VI the gauge group action is not taken into account. For other cases, we partly resolve the gauge group action by diagonalizing the leading term, but we do not complete the reduction, we ignore additional Cartan torus action, to obtain quantum operators which leave invariant the homogeneous polynomials of fixed degree. Since that in the Painlevé VI example the number of coordinates for $\mathfrak{sl}_2$ for 4 punctures is 6 while in other examples the number of intermidiate coordinates is 4 (2 moments + 2 positions). Since we are dealing with Darboux coordinates, the quantisation process becomes fairly straightforward. In this subsection, we show that for each of the non-ramified Painlevé differential equations, there is a choice of quantisation such that the quantum operator acts nicely on the space of homogeneous polynomials. More precisely, we show that the quantum Hamiltonians invariant subspaces are the homogeneous polynomials in several variables (3 for Painlevé VI and 2 for others) with fixed degree. In this section we keep $\hbar$ explicit as that makes it clearer how to extract semi-classical limits.

**Painlevé VI**

For the $\mathfrak{sl}_2$ Fuchsian system we have that the Hamiltonians in the intermediate coordinates take form

$$H_i = \sum_{j\neq i} \frac{h_{ij}}{u_i - u_j}, \quad h_{ij} = 2p_ip_jq_iq_j - p_i^2 q_iq_j - p_j^2 q_iq_j - \\ 2\theta_jp_iq_i - 2\theta_ip_jq_j + 2\theta_ip_iq_j + 2\theta_jp_jq_i + 2\theta_i\theta_j \tag{5.7.5}$$

The quantisation problem is not trivial since we have to choose the ordering for the mixed parts of Hamiltonian. There are three standard ways of the ordering, which are given by

$$: \widehat{p_i}\widehat{q_j} :=: \widehat{q_j}\widehat{p_i} := \widehat{q_j}\widehat{p_i} + \delta_{ij}\varepsilon^{(i)}, \quad \varepsilon^{(i)} = \begin{cases} 0, & \text{left} \\ i\hbar, & \text{right} \\ \frac{i\hbar}{2}, & \text{Weyl} \end{cases}$$

This leads to the following forms of Hamiltonians

$$\hat{H}_i = \sum_{j\neq i} \frac{\hat{h}_{ij}}{u_i - u_j}$$

where

$$\hat{h}_{ij} = 2\hat{q}_i\hat{q}_j\hat{p}_i\hat{p}_j - \hat{q}_i\hat{q}_j\hat{p}_i^2 - \hat{q}_i\hat{q}_j\hat{p}_j^2 - 2(\theta_j - \varepsilon^{(j)})\hat{q}_i\hat{p}_i - 2(\theta_i - \varepsilon^{(i)})\hat{q}_j\hat{p}_j + 2(\theta_i - \varepsilon^{(i)})\hat{q}_j\hat{p}_i + \\ +2(\theta_j - \varepsilon^{(j)})\hat{q}_i\hat{p}_j + 2(\theta_i - \varepsilon^{(i)})(\theta_j - \varepsilon^{(j)}).$$

Here we see that different ordering leads to the different shifts of the local monodromies $\theta_i \to \theta_i - \varepsilon^{(i)}$. Since that we may consider left ordering without loss of generality, first of all because different ordering shifts the constants and also this shift is of the order $\hbar$.

The most remarkable property is that Hamiltonians $\hat{H}_i$ leave invariant the space of homogeneous polynomials of $q_i$ with fixed degree in the following choice of the quantisation $\hat{p}_i = -i\hbar\frac{\partial}{\partial x_i}\cdot$, $\hat{q}_i = x_i\cdot$. So we may look for a solutions for the set of quantum Schrodinger equations

$$i\hbar\partial_{u_i}\Psi = \hat{H}_i\Psi \tag{5.7.6}$$

in the following form

$$\Psi^{(n)} = \sum_{|\alpha|=n} w_\alpha(u_1, .., u_i.., u_m) \prod_{i=1}^{m} x_i^{\alpha_k}, \quad |\alpha| = \sum_{i=1}^{m} \alpha_i$$

which will lead to the non-autonomous linear system of ODE for the $w_\alpha(\mathbf{u})$-s. The resulting equations in fact are KZ equations, since the equations for $w_\alpha$ inherit singularities of $\hat{H}_i$. Let's consider vector

$$W^{(n)} = \begin{pmatrix} w_{\alpha_1} \\ w_{\alpha_2} \\ .. \\ .. \\ w_{\alpha_N} \end{pmatrix}$$

where $\alpha_i$ are the distinct partitions of $n$ with height $m$ (with zero entries). Then $W^{(n)}$ satisfies the equations

$$i\hbar \frac{\partial}{\partial u_i} W^{(n)} - \sum_{j \neq i} \frac{M_n^{(i,j)}}{u_i - u_j} W^{(n)} = 0$$

where $M_n^{(i,j)}$ is action of $\hat{h}_{ij}$ on homogeneous polynomials of degree $n$. These equations are Knizhnik–Zamolodchikov–type equations.

In the case of the Painlevé VI equation we deal with 4-punctured sphere $0, 1, t, \infty$. The quantum Hamiltonian then writes as

$$\begin{aligned} \hat{H} =& \frac{1}{t} \left( 2\hat{q}_1\hat{q}_2\hat{p}_1\hat{p}_2 - \hat{q}_1\hat{q}_2\hat{p}_1^2 - \hat{q}_1\hat{q}_2\hat{p}_2^2 - 2\theta_2\hat{q}_1\hat{p}_1 - 2\theta_1\hat{q}_2\hat{p}_2 + 2\theta_1\hat{q}_2\hat{p}_1 + +2\theta_2\hat{q}_1\hat{p}_2 + 2\theta_1\theta_2 \right) + \\ & \frac{1}{t-1} \left( 2\hat{q}_1\hat{q}_3\hat{p}_1\hat{p}_3 - \hat{q}_1\hat{q}_3\hat{p}_1^2 - \hat{q}_1\hat{q}_3\hat{p}_3^2 - 2\theta_3\hat{q}_1\hat{p}_1 - 2\theta_1\hat{q}_3\hat{p}_3 + 2\theta_1\hat{q}_3\hat{p}_1 + +2\theta_3\hat{q}_1\hat{p}_3 + 2\theta_1\theta_3 \right) \end{aligned} \tag{5.7.7}$$

Let's consider simple case where $|\alpha| = 1$. Substitution of the following function

$$\Psi^{(1)} = w_1 x_1 + w_2 x_2 + w_3 x_3$$

into Schrodinger equation (5.7.6) gives the following system

$$\begin{aligned} i\hbar\frac{\mathrm{d}}{\mathrm{d}t}w_1 &= \frac{2i\hbar\theta_2(w_2-w_1)+2\theta_1\theta_2w_1}{t}+\frac{2i\hbar\theta_3(w_3-w_1)+2\theta_1\theta_3w_1}{t-1} \\ i\hbar\frac{\mathrm{d}}{\mathrm{d}t}w_2 &= \frac{2i\hbar\theta_1(w_1-w_2)+2\theta_1\theta_2w_2}{t}+\frac{2\theta_1\theta_3w_2}{t-1} \\ i\hbar\frac{\mathrm{d}}{\mathrm{d}t}w_3 &= \frac{2\theta_1\theta_2w_3}{t}+\frac{2i\hbar\theta_1(w_1-w_3)+2w_3\theta_1\theta_3}{t-1} \end{aligned} \tag{5.7.8}$$

which solution is given by the hypergeometric function in the follwing way

$$\begin{aligned} w_1 =& C_1t^{-\frac{2i}{\hbar}\theta_1\theta_2}(t-1)^{-\frac{2i}{\hbar}\theta_1\theta_3}+C_2t^{-\frac{2i}{\hbar}\theta_1\theta_2}(t-1)^{-\frac{2i}{\hbar}\theta_1\theta_3-2(\theta_1+\theta_3)}{}_2F_1(2\theta_2,-2\theta_3+1;2(\theta_1+\theta_2)+1;t) \\ & C_3t^{-\frac{2i}{\hbar}\theta_1\theta_2-2(\theta_1+\theta_2)}(t-1)^{-\frac{2i}{\hbar}\theta_1\theta_3-2(\theta_1+\theta_3)}{}_2F_1(-2\theta_1,-2(\theta_1+\theta_2+\theta_3)+1;-2(\theta_1+\theta_2)+1;t). \end{aligned}$$

**Painlevé V**

Hamiltonian in the intermediate coordinates is given by

$$tH = 2t\theta_\infty q_1p_1 - q_0q_1p_0^2 + 2q_0q_1p_0p_1 - q_0q_1p_1^2 + 2\theta_0q_1p_0 + 2\theta_1q_0p_1 + 2\theta_0\theta_1 \tag{5.7.9}$$

Using the same argument as in the previous case, we consider left ordering. Moreover, we see that if quantise in the following way

$$\hat{q}_i = x_i\cdot, \quad \hat{p}_i = -i\hbar\frac{\partial}{\partial x_i} \tag{5.7.10}$$

the space of homogeneous polynomials in $x_0$ and $x_1$ is invariant under the action of the Hamiltonian. Considering example of the degree 2

$$\Psi^{(2)} = w_1x_1^2 + w_2x_0^2 + w_3x_0x_1$$

we get the following system of ordinary differential equations for coefficients

$$\begin{aligned} it\hbar\frac{d}{dt}w_1 &= -4it\theta_\infty\hbar w_1 - 2\,i\theta_0\hbar w_3 \\ it\hbar\frac{d}{dt}w_2 &= -2i\theta_1\hbar w_3 \\ it\hbar\frac{d}{dt}w_3 &= 2\hbar^2(w_1+w_2) - 2i\theta_\infty t\hbar w_3 - 4\theta_1 i\hbar w_1 - 4\theta_0 i\hbar w_2\theta_0 - 2\hbar^2 w_3 \end{aligned} \tag{5.7.11}$$

**Painlevé IV**

Hamiltonian in the intermidiate coordinates takes form

$$H = q_t p_t^2 p_3 - 2\theta_t p_t p_3 - 2\left(p_t q_t - \theta_t\right)\left(t\theta_3 + \theta_2\right) + 2\theta_3 q_3 q_t. \tag{5.7.12}$$

The choice of the Lagrangian submanifold for quantisation procedure defines the properties of the quantum Hamiltonian. Here quantum Hamiltonian will not preserve homogeneous polynomials if we choose standard quantisation (5.7.10). However, choice of the Lagrangian submanifold is irrelevant when we deal with Darboux coordinates and corresponds to the integral transformation on the quantum level. If we choose the following quantisation

$$\hat{q}_3 = x\cdot, \quad \hat{p}_3 = \hbar\frac{\partial}{\partial x}, \quad \hat{q}_t = \hbar\frac{\partial}{\partial y}, \quad \hat{p}_t = y\cdot$$

quantum Hamiltonian will preserve degree of homogeneous polynomials. Moreover choice of the ordering shifts monodromy parameter $\theta_t$ by $\hbar$-small values. Hamiltonian writes as

$$\hat{H} = y^2\frac{\partial^2}{\partial x\partial y} - 2\theta_t y\frac{\partial}{\partial x} - 2\left(t\theta_3 + \theta_2\right)\left(y\frac{\partial}{\partial y} - \theta_t\right) + 2\theta_3 x\frac{\partial}{\partial y} \tag{5.7.13}$$

Writing down system for second order polynomial wave function

$$\Psi^{(2)} = w_1 x^2 + w_2 y^2 + w_3 xy,$$

we obtain system

$$\frac{i\hbar}{2}\frac{d}{dt}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} -(t\theta_3+\theta_2)\theta_t & 0 & -\theta_3 \\ & -(t\theta_3+\theta_2)\theta_t & 2\theta_t - 1 \\ 2\theta_t & -2\theta_3 & -(t\theta_3+\theta_2)\theta_t \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}, \tag{5.7.14}$$

which may be solved via exponents.

**Painlevé III**

Hamiltonian is

$$tH = p_2^2 q_2^2 + 4t\theta_2\theta_3 q_1 q_2 - 2\theta_1 p_2 q_2 + p_1 p_2.$$

quantisation is

$$\hat{q}_1 = x\cdot, \quad \hat{p}_1 = i\hbar\frac{\partial}{\partial x}, \quad \hat{q}_2 = i\hbar\frac{\partial}{\partial y}, \quad \hat{p}_2 = y\cdot$$

Quantum Hamiltonian (up to $\hbar$ shifts of $\theta_1$) takes form

$$\hat{H} = y^2\frac{\partial^2}{\partial y^2} - 2\theta_1 y\frac{\partial}{\partial y} + 4t\theta_2 x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}. \tag{5.7.15}$$

Writing down system for second order polynomial wave function

$$\Psi^{(2)} = w_1 x^2 + w_2 y^2 + w_3 xy,$$

we obtain system

$$i\hbar\frac{d}{dt}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 4t \\ 0 & 2-4\theta_1 & 1 \\ 2 & 8t & -2\theta_1 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}, \tag{5.7.16}$$

**Painlevé II**

Intermediate Darboux coordinates Hamiltonian is

$$\begin{aligned} H = (q_3^5 q_4^5 + 8q_3^4 q_4^4 + 18q_3^3 q_4^3 + 12q_3^2 q_4^2 + (4t+1)q_3 q_4)\theta_4^2 + \\ + (-2q_3^4 q_4^4 - 10q_3^3 q_4^3 - 10q_3^2 q_4^2 - 2q_3 q_4)\theta_3\theta_4 - \\ - (p_3 q_3^3 q_4^2 - p_4 q_3^2 q_4^3 - 4p_3 q_3^2 q_4 - 4p_4 q_3 q_4^2 + 4q_3 q_4\theta_2 + 2t\theta_2 - p_3 q_3 - p_4 q_4)\theta_4 + \\ + (q_3^3 q_4^3 + 2q_3^2 q_4^2)\theta_3^2 + (p_3 q_3^2 q_4 + p_4 q_3 q_4^2 + 2p_4 q_4)\theta_3 + p_3 p_4 \end{aligned} \tag{5.7.17}$$

Choice of the following quantisation

$$\hat{q}_3 = -i\hbar\frac{\partial}{\partial x}, \quad \hat{p}_3 = x\cdot, \quad \hat{q}_4 = y\cdot, \quad \hat{p}_4 = -i\hbar\frac{\partial}{\partial y}$$

leads to the invariance of degree of homogeneous polynomial with respect to action. Indeed, all the monomials in $q_3, p_3, q_4$ and $p_4$ are such that after quatisation Hamiltonian goes to the operator which consists of operators with the same number of derivatives and multiplications in each member. We do not provide explicit form of quantum Hamiltonian and the action on the eigenspaces since the calculation is straightforward but the answer is too long.

**Remark 5.7.1.** *In this section, we consider deformation quantisation of the intermediate Darboux coordinates. This means that the quantised Hamiltonians are elements of the Weyl algebra in two variables* $\mathbb{W}[x,y] = \mathbb{C}[x,\partial_x,y,\partial_y]/\langle[\partial_x,x]=1,[\partial_y,y]=1\rangle$. *However, we know that the Hamiltonian we quantise allows additional symmetry, which lifts to an additional vector field* $\hat{I}$ *which commutes with quantum Hamiltonian vector field. For example, in the case of the Painlevé III, the quantum Hamiltonian* (5.7.15) *commutes with*

$$\hat{I} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

*By restricting to the eigenfunctions of* $\hat{I}$ *with some chosen eigenvalue* $I_0$*, we produce quantum Hamiltonian reduction, which is simply given by the quotient of the algebra* $\mathbb{W}[x,y]/\langle\hat{I}-I_0\rangle$. *As a result we obtain the following quantum Hamiltonian*

$$\hat{H}_{\rm III} = q^2\frac{\partial}{\partial q} + \left(-q^2 - 2q\theta_1 + 4t\theta_2\right)\frac{\partial}{\partial q} + I_0 q, \quad q = \frac{y}{x},$$

*which is just the Dirac quantisation of the Hamiltonian for the Painlevé III equation* (5.6.22). *Such reduction may be performed for all examples, the resulting quantum Hamiltonians coincide with the quantum Hamiltonians introduced in [54, 72] up to change of variables and ordering.*

In order to extend the Reshetikhin theorem for the irregular singularity it is useful to work with the lifted coordinates. In the next subsection, we give a simple proof of this extended theorem for singularities of any type

### 5.7.3 Semi-classical solution of the confluent Knizhnik–Zamolodchikov equation

In this section we discuss the semi-classical solutions of the confluent Knizhnik–Zamolodchikov equations in terms of the isomonodromic tau function. In this subsection, we use only the lifted

Darboux coordinates and quantise them according to (5.7.1)

$$[\widehat{P}_{i_{ab}}, \widehat{Q}_{j_{cd}}] = i\hbar\, \delta_{ij}\delta_{cb}\delta_{ad}. \tag{5.7.18}$$

Such quantisation leads to the infinite dimensional representation of the isomonodromic Hamiltonians as differential operators on a Hilbert space of functions depending on some coordinates $x_{j_{ab}}$, $j = 1 \dots d, a, b = 1 \dots m$ and the isomonodromic times. In particular we put

$$\widehat{Q}_{j_{ab}} = x_{j_{ab}} \cdot, \quad \widehat{P}_{i_{cd}} = \hbar \frac{\partial}{\partial x_{i_{dc}}}.$$

To study the semi-classical solutions $\Psi_S$, we use the following standard quantum mechanical formula

$$\Psi \sim \Psi_S := \exp\left(\frac{i}{\hbar}\mathcal{S}\right),$$

where $\mathcal{S}$ is the classical action functional which explicitly depends on entries of classical variables $Q$ and the isomonodromic times. The dependence of $\mathcal{S}$ on $P$ is implicit, since

$$P_{i_{kl}} = \frac{\partial \mathcal{S}}{\partial Q_{i_{lk}}}.$$

In this section we prove Theorem 2.0.10, namely that $\Psi_{\mathcal{S}}$ evaluated along solutions of the classical system may be written as the isomonodromic $\tau$-function. This statement already appeared in [80] for the Knizhnik–Zamolodchikov equations with Fuchsian singularities. However, our approach works also for irregular systems.

*Proof of Theorem 2.0.10.* The semi-classical solution by definition is given by

$$\Psi_{\mathcal{S}} = \exp\left(\frac{i}{\hbar}\mathcal{S}\right),$$

where $\mathcal{S}$ is a classical action functional. In our case, we have a Hamiltonian system with Hamiltonians $H^{(i)}_{u_i}$ and $H^{(i)}_1, \dots, H^{(i)}_{r_i}$ for $i = 1, \dots, n$ and Darboux coordinates $P_1, P_2 \dots P_d, Q_1, Q_2 \dots Q_d$ the action functional satisfies the following relation

$$\mathrm{d}\mathcal{S} = \sum_{j=1}^{d} P_j \mathrm{d}Q_j - \sum_i \left( H_{u_i}\mathrm{d}u_i + \sum_{k=1}^{r_i} H^{(i)}_k \mathrm{d}t^{(i)}_k \right) = \sum_{j=1}^{d} P_j \mathrm{d}Q_j - \mathrm{d}\ln(\tau), \tag{5.7.19}$$

along the solutions of the system. It is easy to see that the logarithmic differential of the $\tau$ function is already in the definition of the action functional:

**Lemma 5.7.2.** *(Malgrange [65]) If the Hamiltonians are homogeneous polynomials of degree two in $P_1, \dots, P_d$, then along solutions one has*

$$\mathrm{d}\mathcal{S} = \sum_i \left( H_{u_i}^{(i)} \mathrm{d}u_i + \sum_{k=1}^{r_i} H_k^{(i)} \mathrm{d}t_k^{(i)} \right). \tag{5.7.20}$$

*Proof.* Evaluating the first term in (5.7.19) along the solutions of the isomonodromic deformation equations, we obtain

$$\sum_j \mathrm{Tr}(P_j \mathrm{d}Q_j) = \sum_j \mathrm{Tr}\left( P_j \sum_l \left( \frac{\mathrm{d}Q_j}{\mathrm{d}u_l} \mathrm{d}u_l + \sum_{k=1}^{r_l} \frac{\mathrm{d}Q_j}{\mathrm{d}t_k^{(l)}} \mathrm{d}t_k^{(l)} \right)\right) =$$
$$= \sum_j \mathrm{Tr}\left( P_j \sum_l \left( \frac{\partial H_{u_l}}{\partial P_j} \mathrm{d}u_l + \sum_{k=1}^{r_l} \frac{\partial H_k^{(l)}}{\partial P_j} \mathrm{d}t_k^{(l)} \right)\right).$$

Using the fact that the Hamiltonians are homogeneous of degree two in $P_1, \dots, P_d$, we obtain that

$$\sum_j \mathrm{Tr}\left( P_j \sum_l \left( \frac{\partial H_{u_l}}{\partial P_j} \mathrm{d}u_l + \sum_{k=1}^{r_l} \frac{\partial H_k^{(l)}}{\partial P_j} \mathrm{d}t_k^{(l)} \right)\right) = 2 \sum_i \left( H_{u_i}^{(i)} \mathrm{d}u_i + \sum_{k=1}^{r_i} H_k^{(i)} \mathrm{d}t_k^{(i)} \right).$$

which leads to the statement of the lemma. □

According to the previous lemma which works for any homogeneous polynomial Hamiltonians we get close to the proof of the theorem for the general isomonodromic Hamiltonians. In the case of the Fuchsian isomonodromic deformation are given by (4.3.15)

$$H_i = \sum_{j \neq i} \frac{\mathrm{Tr}(Q_i P_i Q_j P_j)}{u_i - u_j},$$

which are definitely homogeneous of degree 2 in the entries of matrices $P_1, P_2 \dots P_n$. This provides a simple proof that the semi-classical solution is a $\tau$-function in the Fuchsian case. The same holds for the irregular singularities - indeed, the irregular Hamiltonians are given by the quadratic spectral invariants, i.e.

$$H = \sum_{\alpha,\beta} \sum_{i,j} C_{i,j}^{\alpha,\beta} \mathrm{Tr}\left(A_i^{(\alpha)} A_k^{(\beta)}\right), \tag{5.7.21}$$

where $\alpha, \beta$ are the indices of the singular points, while $i$ and $j$ are the indices which correspond to the coefficients of local expansion near singularity and $C_{i,j}^{\alpha,\beta}$ are coefficients which can be explicitly computed by using the formulas from section 4. Thanks to Lemma 5.4.2, all the terms $\mathrm{Tr}\left(A_i^{(\alpha)} A_k^{(\beta)}\right)$ are homogeneous polynomials of degree 2 in the variables $P_i$ (as well as homogeneous polynomials of degree 2 in the $Q_i$). This fact allows us to apply Lemma 5.7.2 to conclude. ∎

Observe that this proof depends on the coordinates we use to quantise. In general, the property of semi-classical solution to be a power of an isomonodromic $\tau$-function breaks for the reduced systems. On the classical side this phenomenon is a straightforward statement that reduced Hamiltonians are not hoimogeneous in moments or coordinates. This can be seen on the Painlevé II example - in the fully reduced coordinates Hamiltonian writes as

$$H = \frac{p^2}{2} - \frac{1}{2}\left(q^2 + \frac{t}{2}\right) - \theta q$$

while the action along solution writes as

$$\mathrm{d}\mathcal{S} = p\mathrm{d}q - H\mathrm{d}t = \left[p\frac{\partial q}{\partial t} - H\right]\mathrm{d}t = \left[p^2 - H\right]\mathrm{d}t = \left[\frac{p^2}{2} + \frac{1}{2}\left(q^2 + \frac{t}{2}\right) + \theta q\right]\mathrm{d}t \neq H\mathrm{d}t$$

The classical action now differs from $\tau$-function by some function depends on time. This deviation from the classical action functional was investigated in the paper by Its and Prokhorov [52] for the classical Painlevé equations written in fully reduced coordinates. From the quantum point of view reduction is a restriction to the eigenspace of the Casimir operator which provides partially separation of variables in the quantum problem. By passing to the less number of coordinates the parts which were depending on the lifted coordinates vanish, so the structure of solution changes rapidly. However, despite the fact that the theorem doesn't work in the reduced case we still see the avatars of this statement since $\tau$-function still enters quasiclassical solution in some way, see paper [52] and formula (2.27) in [94].

## 5.8 Symplectic reduction for the Painlevé equations. Algebraic description

The Marsden-Weinstein-Meyer theorem gives a rich information about the phase space obtained by performing symplectic reduction with respect to the Hamiltonian action of a Lie group. In the previous sections, for all Painlevé equations, we have written down Darboux coordinates explicitly, which leads to the local description of the reduced phase space. In this section, we approach the problem of the Marsden-Weinstein-Meyer reduction from the algebraic perspective. Our aim is to describe the reduced phase space as an affine variety for all isomonodromic equations which are related to the Painlevé equations. Let us now briefly explain the approach we use to archive this aim.

We start considering symplectic manifolds $(M, \omega)$ given by the zero locus of the set of polynomials $P_1(x_1, x_2, \dots, x_n), P_2(x_1, x_2, \dots, x_n), \dots P_l(x_1, x_2, \dots, x_n)$ in the affine space $\mathbb{A}^n$, namely *algebraic symplectic varieties*. Assume that there is a Hamiltonian action of a Lie group $G$ on $(M, \omega)$, with algebraic moment map $\mu$. By algebraic moment map we mean that the components of the moment map are the elements of the coordinate ring of $M$, i.e. the moment map may be written as the set of the polynomials in $\mathbb{A}^n$ restricted to $M$. In detail, assume that $\mathfrak{g}^\star$ is spanned by $\Theta_1, \Theta_2, \dots \Theta_k$, then the moment map evaluated at the point $m \in M$ is

$$\mu(m) = \sum_{i=1}^{k} q_i(m) \Theta_i.$$

Then algebraicity of the moment map means that each function $q_i$ may be seen as the restriction of some polynomial $Q_i(x_1, x_2, \dots x_n) \in \mathbb{K}[x_1, x_2, \dots x_n]$ to the algebraic variety $M$. In the case when $G$ acts on $\mu^{-1}(0)$ freely and properly, we may use the Marsden-Weinstein-Meyer theorem and perform the symplectic reduction $M /\!\!/ G = \mu^{-1}(0)/G$. Since $M$ is an algebraic symplectic variety and the moment map is algebraic, the zero set of the moment map $\mu^{-1}(0)$ is given by the system of the following set of algebraic equations

$$\begin{cases} P_1(x_1, x_2, \dots x_n) = 0 \\ P_2(x_1, x_2, \dots x_n) = 0 \\ \dots \\ P_l(x_1, x_2, \dots x_n) = 0 \\ Q_1(x_1, x_2, \dots x_n) = 0 \\ \dots \\ Q_k(x_1, x_2, \dots x_n) = 0. \end{cases}$$

Then the coordinate ring of $\mu^{-1}(0)$ is given by $\mathbb{K}[x_1, x_2, \dots x_n]/\langle P \cup Q \rangle$, where $P$ is the ideal generated by $P_1, P_2, \dots P_l$ and $Q$ is the ideal generated by $Q_1, P_2, \dots Q_k$. Finally, to describe the reduced space, we have to perform a quotient with respect to the action of the Lie group $G$, which gives the following affine scheme

$$M_{\text{red}} = \text{Spec}\left[ (\mathbb{K}[x_1, \dots x_{2n}]/\langle P \cup Q \rangle)^G \right].$$

Roughly speaking, the reduced phase space is given by the spectrum of the $G$-invariant functions over the algebraic variety given by the union of the ideals $P$ and $Q$. For details on the symplectic reduction in the category of algebraic varieties we refer to [50, 78]. In some cases, affine schemes may be described explicitly by a set of algebraic equations in some polynomial ring. However, the obtained reduced space may happen to be non-compact or even to have singular points. In this section we provide such description for the phase space which corresponds to the Painlevé equations. In the previous sections, we provided a local description via Darboux coordinates, however, the reduced phase space is not topologically trivial, so the algebraic description gives some new information about phase space for the Painlevé equations.

In the case of the Painlevé equations, we always deal with four $\mathfrak{sl}_2$-matrices with some chosen ideals $P$ and $Q$. The ideal $P$ corresponds to the Casimirs of the corresponding Lie-Poisson bracket, while the moment map ideal $Q$ in the all cases gives that the sum residues of the connection equals to zero, which just repeats the Cauchy residue theorem. In this section we assume that the coordinates are chosen in such a way that the isomonodromic linear system has no pole at infinity for technical reasons.

### 5.8.1 Painlevé VI

As we mentioned in the beginning, the equation Painlevé VI which reads as

$$\begin{aligned}\frac{d^2y}{dt^2} &= \tfrac{1}{2}\left(\tfrac{1}{y}+\tfrac{1}{y-1}+\tfrac{1}{y-t}\right)\left(\tfrac{dy}{dt}\right)^2-\left(\tfrac{1}{t}+\tfrac{1}{t-1}+\tfrac{1}{y-t}\right)\tfrac{dy}{dt}\\ &+\tfrac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha+\beta\tfrac{t}{y^2}+\gamma\tfrac{t-1}{(y-1)^2}+\delta\tfrac{t(t-1)}{(y-t)^2}\right),\end{aligned}$$

is a reduction of the Schlesinger equations for the four copies of the coadjoint orbit of $\mathfrak{sl}_2$. This space may be represented by 4 matrices with constraints

$$(A_1,A_2,A_3,A_4)\in\mathcal{O}^\star\times\mathcal{O}^\star\times\mathcal{O}^\star\times\mathcal{O}^\star,\quad A_i=\begin{pmatrix}x_i & y_i\\ z_i & -x_i\end{pmatrix},\quad X_i:\,x_i^2+y_iz_i=\theta_i^2.$$

To avoid useless indices we use the following notation

$$\begin{aligned}&A_1=A,\quad A_2=B,\quad A_3=C,\quad A_4=D\\ &\theta_1=\alpha,\quad \theta_2=\beta,\quad \theta_3=\gamma,\quad \theta_4=\delta.\end{aligned}$$

The corresponding meromorphic $\mathfrak{sl}_2$ connection with 4 poles is given by

$$\nabla=d+\left(\frac{A}{z}+\frac{B}{z-1}+\frac{C}{z-t}+\frac{D}{z+1}\right)dz=d+A(z)dz$$

In $\mathfrak{sl}_2$ case such symplectic space may be seen as a symplectic leaf for the Lie-Poisson bracket, which writes as

$$\{x_i,y_j\}=\delta_{ij}y_j,\quad \{x_i,z_j\}=-\delta_{ij}z_i,\quad \{y_i,z_j\}=\delta_{ij}2x_i.$$

Moreover we assume that there are no other singularities (i.e. $\infty$ is a regular point) which means that there is a relation

$$A+C+B+D=0.$$

Taking into account the gauge group action, the phase space is described as

$$\mathcal{M}_{\text{red}}=\mathcal{O}_1^\star\times\mathcal{O}_2^\star\times\mathcal{O}_3^\star\times\mathcal{O}_4^\star//SL_2=\{A+C+B+D=0\}/SL_2, \tag{5.8.1}$$

where $SL_2$ acts by a simultaneous conjugation. Algebraically this space may be seen as

$$\mathcal{M}_{\rm red} \simeq \operatorname{Spec}(\mathbb{C}[x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, z_1 z_2, z_3, z_4]/\mathfrak{I})^{SL_2}$$

where ideal $\mathfrak{I}$ is given by the equations

$$\mathfrak{I}: \begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ y_1 + y_2 + y_3 + y_4 = 0 \\ z_1 + z_2 + z_3 + z_4 = 0 \\ x_i^2 + y_i z_i = \theta_i^2, \quad i = 1 \dots 4. \end{cases},$$

i.e. we describe the reduced phase space in terms of function over the algebraic variety given by $\mathfrak{I}$ which are invariant with respect to $SL_2$ action. Consider the following finctions

$$Y_K = \operatorname{Tr} \prod_{i \in K} A_i, \quad K = \{K_1, K_2, \dots K_m\}, \quad Y_{ij} = Y_{ji} = \operatorname{Tr} A_i A_j, \quad Y_{ijk} = \operatorname{Tr} A_i A_j A_k$$

here $K$ is a multi-index. Choose the following coordinates

$$Y_{12}, Y_{13}, Y_{14}, Y_{23}, Y_{24}, Y_{34}, Y_{123}, \tag{5.8.2}$$

so we have all the traces of the product of two matrices and trace of product of three matrices. Using Gröbner basis algorithm we obtain that (5.8.2) satisfies the following set of equations

$$Q_1 = 2\theta_1^2 + Y_{1,2} + Y_{1,3} + Y_{1,4} = 0$$

$$Q_2 = 2\theta_2^2 + Y_{1,2} + Y_{2,3} + Y_{2,4} = 0$$

$$Q_3 = 2\theta_3^2 + Y_{1,3} + Y_{2,3} + Y_{3,4} = 0$$

$$Q_4 = 2\theta_4^2 + Y_{1,4} + Y_{2,4} + Y_{3,4} = 0$$

$$\begin{aligned} Q_5 = -2Y_{1,2,3}^2 + Y_{1,2}Y_{1,3}Y_{1,4} + Y_{1,2}Y_{1,3}Y_{2,4} + Y_{1,2}Y_{1,3}Y_{3,4} + Y_{1,2}Y_{1,4}Y_{2,3} + \\ + Y_{1,2}Y_{1,4}Y_{3,4} + Y_{1,2}Y_{2,3}Y_{2,4} + Y_{1,2}Y_{2,3}Y_{3,4} + Y_{1,2}Y_{2,4}Y_{3,4} + Y_{1,3}Y_{1,4}Y_{2,3} + Y_{1,3}Y_{1,4}Y_{2,4} + \\ + Y_{1,3}Y_{2,3}Y_{2,4} + Y_{1,3}Y_{2,3}Y_{3,4} + Y_{1,3}Y_{2,4}Y_{3,4} + Y_{1,4}Y_{2,3}Y_{2,4} + Y_{1,4}Y_{2,3}Y_{3,4} + Y_{1,4}Y_{2,4}Y_{3,4} = 0 \end{aligned} \tag{5.8.3}$$

which defines an affine surface in $\mathbb{A}^7$. All the other invariant functions $Y_K$ can be written via linear combinations of (5.8.2). Such surface has a natural Poisson bracket given by

$$\{f,g\}=\frac{df\wedge dg\wedge dQ_1\wedge dQ_2\wedge dQ_3\wedge dQ_4\wedge dQ_5\wedge dQ_6}{dY_{12}\wedge dY_{13}\wedge dY_{14}\wedge dY_{23}\wedge dY_{24}\wedge dY_{34}\wedge dY_{123}}.$$

and the Schlesinger Hamiltonian becomes linear in such coordinates

$$H=\operatorname*{Res}_{z=t}\operatorname{Tr}\frac{A(z)^2}{2}=\frac{\operatorname{Tr}AC}{t}+\frac{\operatorname{Tr}BC}{t-1}+\frac{\operatorname{Tr}CD}{t+1}=\frac{Y_{1,3}}{t}+\frac{Y_{2,3}}{t-1}+\frac{Y_{3,4}}{t+1}.$$

Since first four equations in (5.8.3) we may rewrite obtained variety as a surface in $\mathbb{A}^3$. Instead of resolving system (5.8.3), let us choose the following set of coordinates

$$X=\frac{1}{2}\operatorname{Tr}(A+C)^2,\quad Y=\frac{1}{2}\operatorname{Tr}(C+D)^2,\quad Z=\operatorname{Tr}ABC.$$

Then surface (5.8.3) reads as

$$\begin{aligned}Z^2&=XY(X+Y)-\left(\alpha^2+\beta^2+\delta^2+\gamma^2\right)XY+\\&+(\gamma-\delta)(\gamma+\delta)(\alpha-\beta)(\alpha+\beta)X+(\beta-\delta)(\beta+\delta)(\alpha-\gamma)(\alpha+\gamma)Y+\\&+\left(\alpha^2-\beta^2+\delta^2-\gamma^2\right)(\alpha\delta-\beta\gamma)(\alpha\delta+\beta\gamma)\end{aligned}\quad (5.8.4)$$

The Hamiltonian takes form

$$H=\operatorname*{Res}_{z=t}\operatorname{Tr}\frac{A(z)^2}{2}=\frac{X}{t}+\frac{Y}{t+1}-\frac{X+Y}{t-1}-\frac{\alpha^2}{t}-\frac{\delta^2}{t+1}+\frac{4t^2+t-1}{t(t-1)(t+1)}\gamma^2.$$

In these coordinates Hamiltonian is a linear function, while the bracket which corresponds to the surface is given by the nonlinear Poisson bracket of the form

$$\{X,Y\}=F_Z=-2Z,\quad \{Y,Z\}=F_X=2XY+Y^2-\left(\alpha^2+\beta^2+\delta^2+\gamma^2\right)Y+(\gamma-\delta)(\gamma+\delta)(\alpha-\beta)(\alpha+\beta),$$

$$\{X,Z\}=-F_y=-2XY-X^2+\left(\alpha^2+\beta^2+\delta^2+\gamma^2\right)X-(\beta-\delta)(\beta+\delta)(\alpha-\gamma)(\alpha+\gamma).$$

Such representation for the Painlevé VI equation already appeared in the paper by N.Hitchin [49]. Using isomonodromic equations it is easy to check that the obtained Hamiltonian with introduced bracket gives the isomonodromic dynamics. The affine surface (5.8.4) is a double cover of the $\mathbb{A}^2$ branched along an elliptic curve. Indeed, the equation has the form

$$F_{VI}(X,Y,Z;\alpha,\beta,\gamma,\delta) = Z^2 - P_3(X,Y;\alpha,\beta,\gamma,\delta) = 0.$$

where $P_3$ is a polynomial of degree 3 in $X$ and $Y$. Affine part of such surface has no singularities if $P_3(X,Y;\alpha,\beta,\gamma,\delta) = 0$ is a smooth elliptic curve. For (5.8.4) discriminant of the elliptic curve is

$$\Delta_{VI} = \alpha^2\beta^2\delta^2\gamma^2 (\alpha+\beta-\gamma+\delta)^2 (\alpha+\beta-\gamma-\delta)^2 (\alpha-\beta-\gamma+\delta)^2 (\alpha-\beta-\gamma-\delta)^2$$
$$(\alpha-\beta+\gamma-\delta)^2 (\alpha-\beta+\gamma+\delta)^2 (\alpha+\beta+\gamma-\delta)^2 (\alpha+\beta+\gamma+\delta)^2 .$$

So surface (5.8.4) is a smooth affine surface for generic $\alpha,\beta,\gamma,\delta$. Discriminant is invariant under the action of the Weyl group for $D4$ root system, we may permute $\alpha,\beta,\gamma$ and $\delta$, as well as multiply each of the parameters by $-1$.

### 5.8.2 Painlevé V

Taking limit $t = \varepsilon t$ and performing the expansion in $\varepsilon$ due to the algorithm described in Section 5.2, Fuchsian connection transforms to the following one

$$\nabla_z = d + \left(\frac{A_1 t}{z^2} + \frac{A_0}{z} + \frac{C}{z-1} + \frac{D}{z+1}\right) dz.$$

Isomonodromic Hamiltonian takes form

$$H = \frac{1}{2t}\operatorname{Res}_{z=0}\operatorname{Tr}(zA(z)^2) = \operatorname{Tr}\left[-A_1 C + A_1 D + \frac{A_0^2}{2t}\right]$$

The Casimirs for the corresponding Poisson bracket are

$$\operatorname{Tr}A_1^2 = 2\alpha^2, \quad \operatorname{Tr}A_1 A_0 = 4\alpha\beta, \quad \operatorname{Tr}C^2 = 2\gamma^2, \quad \operatorname{Tr}D^2 = 2\delta^2.$$

The moment map condition writes as

$$A_0 + C + D = 0$$

**Theorem 5.8.1.** *The reduced space*

$$\mathcal{M}_{\mathrm{red}} = \hat{\mathcal{O}}_2^{\star} \times \mathcal{O}^{\star} \times \mathcal{O}^{\star}//SL_2$$

*is an affine cubic in* $\mathbb{A}^3$ *which can be described as a double cover of* $\mathbb{A}^2$ *ramified along the specific elliptic curve. In particular, choosing the following coordinates*

$$X = \frac{1}{2}\mathrm{Tr}A_0^2, \quad Y = \mathrm{Tr}A_1 D, \quad Z = \mathrm{Tr}A_1 A_0 D,$$

*the affine part of the reduced space is given by the equation*

$$Z^2 = XY^2 + \alpha^2 X^2 + 4\alpha\beta XY - 2\alpha^2\left(\gamma^2+\delta^2\right)X - 4\alpha\beta\left(\gamma-\delta\right)\left(\gamma+\delta\right)Y + \\ + \alpha^2\left(\gamma^4 - 2\gamma^2\delta^2 + 16\beta^2\delta^2 + \delta^4\right) \quad (5.8.5)$$

The discriminant of the elliptic curve in the rhs of (5.8.5) is

$$\Delta_V = (-16)\alpha^{12}\delta^2\gamma^2(\gamma-2\beta+\delta)^2(\gamma-2\beta-\delta)^2(\gamma+2\beta-\delta)^2(\gamma+2\beta+\delta)^2. \quad (5.8.6)$$

Hamiltonian rewrites as

$$H = 2Y + \frac{X}{t} + 2\alpha^2.$$

with Poisson bracket given by the Poincaré residue with respect to the cubic (5.8.5). Direct computations shows that the Hamiltons equations coincide with the isomondromic flow.

**Theorem 5.8.2.** *Consider the cubic for Painlevé VI* (5.8.4) *and do the following change of coordinates*

$$X = \tilde{X} + O(\varepsilon), \quad Y = \frac{\tilde{\alpha}^2}{\varepsilon^2} + \frac{1}{\varepsilon}\tilde{Y} + O(1), \quad Z = \frac{1}{\varepsilon}\tilde{Z} + O(1), \quad (5.8.7)$$

*and constants*

$$\alpha = \frac{\tilde{\alpha}}{\varepsilon} + 0 + O(\varepsilon), \quad \beta = -\frac{\tilde{\alpha}}{\varepsilon} + \tilde{\beta} + O(\varepsilon), \quad \gamma = \tilde{\gamma} + O(\varepsilon), \quad \delta = \tilde{\delta} + O(\varepsilon) \quad (5.8.8)$$

*then Painlevé V cubic* (5.8.5) *is a limit*

$$F_V(\tilde{X}, \tilde{Y}, \tilde{Z}; \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}) = \lim_{\varepsilon \to 0} \varepsilon^2 F_{VI}(X, Y, Z; \alpha, \beta, \gamma, \delta). \tag{5.8.9}$$

*Proof.* Can be checked explicitly □

### 5.8.3 Phase spaces for the Painlevé equations

In the same way, we introduce affine surfaces for all other Painlevé equations. The results are presented in the table 5.1. According to this table, for each Painlevé equation there exists a 4-parameter family of affine cubic surfaces in $\mathbb{A}^3$. The coalescence of poles in the isomonodromic problem should lead to the special limiting procedure, same as in the theorem 5.8.2, while the ramified systems correspond to the special fibers in the family.

In principle, the results obtained in this section are related to the co-adjoin orbits, but not specificly to the Painlevé equations. However there is a nice parallel with the confluence procedure for the Painlevé equations. Moreover, the isomonodromic flows (Hamiltonians for the Painlevé equations) become linear functions on the obtained algebraic space which leads to a new description of these equations. Moreover such point of view allows us to claim that we give an algebraic description for the affine part of the de-Rham moduli space of the corresponding connections. In particular, such interpretation may be useful in the study of the Riemann-Hilbert map as a map between algebraic varieties (or more precisely between the del Pezzo surfaces of different type).

| Painlevé | Orbits | Reduced cubic | Discriminant |
|---|---|---|---|
| VI | $\mathcal{O}\times\mathcal{O}\times\mathcal{O}\times\mathcal{O}$ | $Z^2=XY(X+Y)-$<br>$-\left(\alpha^2+\beta^2+\delta^2+\gamma^2\right)XY$<br>$+(\gamma-\delta)(\gamma+\delta)(\alpha-\beta)(\alpha+\beta)X$<br>$+(\beta-\delta)(\beta+\delta)(\alpha-\gamma)(\alpha+\gamma)Y+$<br>$+\left(\alpha^2-\beta^2+\delta^2-\gamma^2\right)(\alpha\delta-\beta\gamma)(\alpha\delta+\beta\gamma)$ | $\alpha^2\beta^2\delta^2\gamma^2$<br>$(\alpha+\beta-\gamma+\delta)^2(a+\beta-\gamma-\delta)^2$<br>$(a-\beta-\gamma+\delta)^2(a-\beta-\gamma-\delta)^2$<br>$(a-\beta+\gamma-\delta)^2(a-\beta+\gamma+\delta)^2$<br>$(a+\beta+\gamma-\delta)^2(a+\beta+\gamma+\delta)^2$ |
| V | $\hat{\mathcal{O}}_2\times\mathcal{O}\times\mathcal{O}$ | $Z^2=XY^2+\alpha^2X^2+4\alpha\beta XY-$<br>$-2\alpha^2\left(\gamma^2+\delta^2\right)X-4\alpha\beta(\gamma-\delta)(\gamma+\delta)Y$<br>$+\alpha^2\left(\gamma^4-2\gamma^2\delta^2+16\beta^2\delta^2+\delta^4\right)$ | $\alpha^{12}\delta^2\gamma^2$<br>$(\gamma-2\beta+\delta)^2(\gamma-2\beta-\delta)^2$<br>$(\gamma+2\beta-\delta)^2(\gamma+2\beta+\delta)^2$ |
| $IV$<br>$FN\,II:\ \alpha=0$ | $\hat{\mathcal{O}}_3\times\mathcal{O}$ | $4Z^2=\frac{X^3}{2}+\left(\alpha^2+\beta-2\gamma\right)$<br>$X^2+\alpha^2Y^2-\left(2\alpha^2+\beta\right)XY$<br>$+2\left(\gamma^2-2\alpha^2\delta^2+\beta\delta^2-\beta\gamma\right)X$<br>$+2\left(\beta\gamma-2\alpha^2\delta^2\right)Y+4\delta^2\left(\alpha^2\delta^2+\beta^2-\beta\gamma\right)$ | $\delta^2\alpha^6$<br>$(8\alpha^3\delta-4\alpha^2\gamma+\beta^2)^2$<br>$(8\alpha^3\delta+4\alpha^2\gamma-\beta^2)^2$ |
| $III(D6)$<br>$D7:\alpha=0$<br>$D8:\alpha=\gamma=0$ | $\hat{\mathcal{O}}_2\times\hat{\mathcal{O}}_2$ | $Z^2=-2X^2Y+16\beta\delta X+8\alpha^2\gamma^2Y$<br>$+16\beta^2\gamma^2+16\alpha^2\delta^2$ | Ramification along<br>rational curve<br>singular when $\alpha=0$ or $\gamma=0$ |
| $II$<br>$I:\alpha=0$ | $\hat{\mathcal{O}}_4$ | $Z^2=\frac{1}{8}X^3-\frac{1}{2}\gamma X^2-\alpha^2XY$<br>$+\beta\delta X+\frac{1}{2}\beta^2Y+\frac{1}{2}\gamma^2X+4\alpha^2\delta^2-2\beta\gamma\delta$ | Ramification along<br>rational curve<br>singular at "$\infty$" $\sim(0,1,0)\in\mathbb{P}^2$<br>$\alpha\neq0$- double point at $\infty$<br>$\alpha=0$- cusp at $\infty$ |

Table 5.1: Affine surfaces corresponding to the Painlevé equations isomonodromic problems

# CHAPTER 6

# CONCLUSION

The results of this thesis summarized in the introduction are related to the Hamiltonian description of the isomonodromic deformation equations. Here we want to underline the importance of the universality of our results - in most of the theorems related to the Poisson structures we work with an arbitrary Lie algebra which has a non-degenerate bilinear pairing. The description via lifted Darboux coordinates is given for $\mathfrak{sl}_n$ isomonodromic systems, however it should be possible to extend these results to other Lie algebras. On the other hand, such universality restricts us to special families of deformations arising from the confluence of Schlesinger flows, but not to the general isomonodromic deformation in the sense of Jimba-Miwa-Ueno. However, in the case of $\mathfrak{sl}_2$, our class of the isomonodromic deformation equations coincides with the one introduced by Jimbo, Miwa and Ueno and covers the all examples of the Painlevé equations and their generalizations. As a consequence, our approach allows to write down the Hamiltonians for the Painlevé equations by using explicit close formula in terms of the Takiff algebra pairing.

.

The first part of the thesis contains local results, while in the last section we do global analysis of the phase spaces for the isomonodromic systems by performing symplectic reduction without introducing the local Darboux coordinates. Such description is topologicaly non-trivial, and it still has to be compactified. One of the most important questions which we don't answer here is how to compactify such phase spaces, in such a way that the compactification has geometrical meaning. Our plan is to investigate this question in the future. Moreover, we wish to use the same approach for the higher rank Lie algebras (for example $\mathfrak{sl}_3$) as well as for more complicated configurations of the isomonodromic problems (which originates by a confluence from the Fuchsian systems with more than four poles).

We hope that this algebraic approach allows to re-interpret a famous result by Manin [66] about the elliptic form of the sixth Painlevé equation, firstly introduced by Painlevé himself in [76], in terms of the very specific geometry of the phase space in connection with the moduli spaces of elliptic curves. Our aim is also to investigate how the algebraic description of the phase space for the other Painlevé equations are related to the Manin form of these equations which are non-autonomous versions of the one particle Inozemtsev systems with trigonometric and rational potentials (see [56, 86]).

Another application of the algebraic description for the phase space is related to the study of the Riemann-Hilbert map and its connection to mirror symmetry. Our computations show that the phase spaces for Painlevé equations are 4-parameter families of possibly singular cubic surfaces in the affine space $\mathbb{A}^3$ which are essentially del Pezzo surfaces. These families of del Pezzo surfaces have a very nice geometric interpretation - the generic member of the family can be seen as a double cover of the family of (possibly degenerated) elliptic curves. Such description brings new insights on the space of initial conditions for the Painlevé VI equation, bi-rational Okamoto transformations and global properties of the solutions. Another feature of such approach is that the Hamiltonian for the system becomes a linear function of the coordinates on $\mathcal{A}$, while the non-linearity transfers to the Poisson structure. This interesting "linearisation" of the Hamiltonian was already discovered by Hitchin in [49] by parameterising the phase space by traces of products of matrices. While Hitchin's result was somewhat mysterious, we now understand that the symplectic structure he computes is nothing else that the Nambu bracket on our 4-parameter family of singular cubic surfaces. An essential thing here is that now we may consider a Riemann-Hilbert map as a correspondence between two families of del Pezzo surfaces – one given by the monodromy manifolds the other given by the space of initial conditions. A mirror pair for the del Pezzo family on the monodromy manifold side was constructed in [23] applying Gross-Hacking-Keel construction [22]. For the initial condition space side we have to introduce a suitable compactification to apply the mirror construction. To do this we are planing to consider different compactifications to the weighted projective spaces and try to find a mirror pair. The natural question is if there is some evidence of the Riemann-Hilbert correspondence on the mirror side. Our aim for the future research is to investigate this mirror avatar of the Riemann-Hilbert correspondence.

# LIST OF REFERENCES


[1] M. Adams, J. Harnad, J. Hurtubise, Isospectral Hamiltonian Flows in Finite and Infinite Dimensions II. Integration of Flows, *Comm. Math. Phys.,* **134** (1990) 555–585.

[2] M. Adams, J. Harnad, J. Hurtubise, Darboux Coordinates and Liouville-Arnold integration in Loop Algebras, *Comm. Math. Phys.* **155** (1993) 385-413.

[3] M. Adams, J. Harnad, E. Previato, Isospectral Hamiltonian flows in finite and infinite dimensions, *Comm. Math. Phys.,* **117** (1988) 3:451–500.

[4] M. Audin Lectures on gauge theory and integrable systems, *Gauge theory and symplectic geometry.* Springer, Dordrecht, 1997, 1-48.

[5] O.Babelon, D.Bernard, M. Talon, *Introduction to classical integrable systems*, Cambridge University Press (2003).

[6] M. Babich, Rational Symplectic Coordinates on the Space of Fuchs Equations $m \times m$-Case, *Lett. Math. Phys.,* **96** (2008) 63–77.

[7] M.V. Babich and S.E. Derkachov, On rational symplectic parametrisation of the coadjoint orbit of $\mathrm{GL}(N)$. Diagonalizable case, *St. Petersburg Math. J.* **22** (2011) 347–357.

[8] H.M. Babujian, A.V. Kitaev, Generalised Knizhnik-Zamolodchikov equations and isomonodromy quantisation of the equations integrable via the inverse scattering transform: Maxwell-Bloch system with pumping, *J. Math. Phys.,* **39** (1998) 5:2499–2506.

[9] A.A. Belavin, A.M. Polyakov, A.B. Zamolodchikov, Infinite conformal symmetry in two-dimensional quantum field theory, *Nuclear Phys. B* **241** (1984) 2:333–380.

[10] M. Bertola, M. Cafasso, V. Rubtsov, Noncommutative Painlevé equations and systems of Calogero type, *Comm. Math. Phys.,* **363** (2018) 2:503–530.

[11] M. Bertola, D. Korotkin Tau-functions and monodromy symplectomorphisms *arXiv preprint arXiv:1910.03370* (2020).

[12] P. Boalch, Simply-laced isomonodromy systems, *Publ. Math. Inst. Hautes Études Sci.* **116** (2012) 1–68.

[13] P. Boalch, G -bundles, isomonodromy, and quantum Weyl groups, *Int. Mat. Res. Not.* **22** (2002), 1129–1166.

[14] P. Boalch, Quasi-Hamiltonian geometry of meromorphic connections, *Duke Mathematical Journal* **139** (2007) 2:369-405.

[15] P. Boalch, Symplectic manifolds and isomonodromic deformations, *Advances in Mathematics* **163** (2001) 2:137-205.

[16] L. Chekhov, M. Mazzocco and V. Rubtsov, Painlevé Monodromy Manifolds, Decorated Character Varieties, and Cluster Algebras, *Int. Math. Res. Not.,* **24** (2016) 7639–7691.

[17] L. Chekhov, M. Mazzocco and V. Rubtsov, Algebras of quantum monodromy data and character varieties, *Geometry and Physics I,* Oxford University Press (2018).

[18] Yu. Chernyakov, Integrable systems obtained by puncture fusion from rational and elliptic gaudin systems, *Theoretical and mathematical physics* **141** (2004) 1:1361-1380.

[19] N. Chriss, V.Ginzburg, *Representation theory and complex geometry* Vol. **42**. Boston: Birkhäuser.

[20] G. Cotti, B. Dubrovin and D. Guzzetti, Isomonodromy deformations at an irregular r with coalescing eigenvalues, *Duke Math, J,* 8 (2019) no.6:967–1108.

[21] G. Cotti, B. Dubrovin and D. Guzzetti, Local moduli of semisimple Frobenius coalescent structures, *SIGMA Symmetry Integrability Geom. Methods Appl.,* **16** (2020) no.40, 105pp.

[22] Gross M., Hacking P., Keel S., Mirror symmetry for log Calabi-Yau surfaces I, *Publ. Math. Inst. Hautes Études Sci.,* **122** (2015), 65–168.

[23] M.Gross, P.Hacking, S.Keel, B.Siebert The mirror of the cubic surface, *arXiv:1910.08427* (2019)

[24] A. Chervov, D. Talalaev Hitchin Systems on Singular Curves II: Gluing Subschemes. *International Journal of Geometric Methods in Modern Physics* **4** (2007) 5:751–787.

[25] A. Chervov, G. Falqui, L. Rybnikov, Limits of Gaudin systems: classical and quantum cases, *SIGMA. Symmetry, Integrability and Geometry: Methods and Applications* (2009) **9**;5:029.

[26] C. de Concini, C. Procesi, Hyperplane arrangements and holonomy equations, *Selecta Math. (N.S.),* **1** (1995) 3:495–535.

[27] B. Dubrovin, Integrable systems and classification of 2–dimensional topological field theories, *Integrable Systems, Luminy 1991,* Progr. Math. **115** (1993).

[28] D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry.* Springer-Verlag (1995) ISBN 978-3-540-78122-6.

[29] B. Enriquez, S. Pakuliak, V.Rubtsov, Basic representations of quantum current algebras in higher genus. *Quantum groups*, 177–190, Contemp. Math., **433**, Israel Math. Conf. Proc., Amer. Math. Soc., Providence, RI, 2007.

[30] G. Felder, Y. Markov, V. Tarasov, A. Varchenko, *Differential equations compatible with KZ equations*, *Math. Phys. Anal. Geom.,* **3** (2000) 139–177.

[31] L. Faddeev, L. Takhtajan, *Hamiltonian methods in the theory of solitons.* Springer Science & Business Media (2007) ISBN: 978-3-540-69969-9.

[32] R. M. Fedorov, Algebraic and Hamiltonian approaches to isoStokes deformations, *Transform. Groups* **11** (2006) 2:137–160.

[33] B. Feigin, E. Frenkel, L. Rybnikov, Opers with irregular singularity and spectra of the shift of argument subalgebra, *Duke Math. J.* **155**, (2010) 337–363.

[34] B. Feigin, E. Frenkel, V. Toledano-Laredo, Gaudin models with irregular singularities, *Adv. in Math.,* **223** (2010) 873–948.

[35] H. Flaschka, AC. Newell Monodromy- and spectrum-preserving deformations I. *Communications in Mathematical Physics* (1980) **76**(1) 65-116.

[36] A. Fokas, A. Its, V. Novokshenov, A. Kapaev, *Painlevé transcendents: the Riemann-Hilbert approach* 128 American Mathematical Soc. (2006).

[37] I. Gaiur, M. Mazzocco, V. Rubtsov Isomonodromic deformations: Confluence, Reduction & Quantisation arXiv:2106.13760

[38] I. Gaiur, V. Rubtsov Dualities for rational multi-particle Painlevé systems: Spectral versus Ruijsenaars arXiv:1912.12588

[39] I. Gaiur, N. Kudryashov Weak nonlinear asymptotic solutions for the fourth order analogue of the second Painlevé equation *Regular and Chaotic Dynamics* (2017), 22(3), 266-271

[40] I. Gaiur, N. Kudryashov Asymptotic solutions of a fourth—order analogue for the Painlevé equations *Journal of Physics: Conference Series* (2017) Vol. 788, No. 1

[41] I. Gaiur, N. Kudryashov Isomonodromic problem for analogue of the Painlevé equations *Journal of Physics: Conference Series* (2017), Vol. 788, No. 1.

[42] J. Gibbons, T. Hermsen, A generalisation of the Calogero-Moser system, *Physica D Nonlinear Phenomena* (1984) **11**(3):337-48.

[43] J. Harnad, Quantum isomonodromic deformations and the Knizhnik-Zamolodchikov equations, *Symmetries and integrability of difference equations* **9** (1996) 155.

[44] J. Harnad, Dual Isomonodromic Deformations and Moment Maps to Loop Algebras, *Commun. Math. Phys.,* **166** (1994) 337–365.

[45] J. Harnad, M. Routhier, R–matrix construction of electromagnetic models for the Painlevé transcendents, *J. Math. Phys.,* **36** (1995) 9:4863–4881.

[46] K. Hiroe, D. Yamakawa, Moduli spaces of meromorphic connections and quiver varieties, *Adv. Math.* **266** (2014) 120–151.

[47] N. Hitchin, Frobenius manifolds *Gauge theory and symplectic geometry.* Springer, Dordrecht, (1997) 69-112.

[48] N. Hitchin, Flat connections and geometric quantisation, *Commun. Math. Phys.,* **131** (1990) 2:347–380.

[49] N.Hitchin, Geometrical aspects of Schlesinger's equation *Journal of Geometry and Physics* **23** (1997): 287-300.

[50] V. Hoskins, Geometric invariant theory and symplectic quotients, *available at V. Hoskins homepage*

[51] J. Hurtubise, On the geometry of isomonodromic deformations it J. Geom. Phys., **58** (2008) 1394–1406.

[52] A.R. Its, A. Prokhorov, On some Hamiltonian properties of the isomonodromic tau functions, *Reviews in Mathematical Physics* **30** (2018) 7:1840008.

[53] M. Jimbo , T. Miwa, Y. Mori, M. Sato, Density Matrix of an Impenetrable Bose Gas and the Fifth Painlevé Equation, *Physica D,* **1** (1980) 1:80–158.

[54] M. Jimbo, H. Nagoya, J. Sun, Remarks on the confuent KZ equation for $\mathfrak{sl}_2$ and quantum Painlevé equations, *J. Phys. A: Math. Theor.,* **41** (2008)

[55] M. Jimbo , T. Miwa, K. Ueno, Monodromy preserving deformation of linear ordinary differential equations with rational coefficients: I. General theory and $\tau$-function, *Physica D,* **2** (1981) 2:306–352.

[56] Levin A., Olshanetsky M., Painlevé—Calogero Correspondence, *In Calogero—Moser—Sutherland Models* Springer, New York, NY, 313-332 (2000).

[57] K.Kalinin, M.Babich Parametrisation of phase space of Painlevé V equation *DAYS on DIFFRACTION 2021,* in print

[58] K. Kajiwara, T. Masuda, M. Noumi, Y. Ohta, Y. Yamada Cubic pencils and Painlevé hamiltonians. *Funkcialaj Ekvacioj,* **48** (2005) 1:147-160.

[59] N. M. Katz, *Rigid Local Systems* Princeton University Press, (1996).

[60] H. Kawakami, Matrix Painlevé systems, *Journal of Mathematical Physics* **56**, (2015) 033503.

[61] H. Kawakami, A. Nakamura, H. Sakai, Degeneration scheme of 4-dimensional Painlevé-type equations, *arXiv:1209.3836* (2012).

[62] V. G. Knizhnik, A. B. Zamolodchikov, Current algebra and Wess-Zumino model in two dimensions, *Nuclear Phys. B* **247** (1984) 1:83–103.

[63] D. Korotkin and H. Samtleben, Quantization of coset space $\sigma$-models coupled to two-dimensional gravity, *Comm. Math. Phys.* **190** (1997) 2:411–457.

[64] I. Krichever, Isomonodromy equations on algebraic curves, canonical transformations and Whitham equations, *Mosc. Math. J.,* **2** (2002) 4:717–752.

[65] B. Malgrange, Déformations isomonodromiques, forme de Liouville, fonction $\tau$ *Ann. Inst. Fourier,* **54** (2004) 1371–1392.

[66] Manin Y., Sixth Painlevé Equation, Universal Elliptic Curve, and Mirror of $\mathbf{P}^2$, *Geometry of differential equations, Amer. Math. Soc. Transl. Ser. 2, 186, Adv. Math. Sci., 39, Amer. Math. Soc., Providence, RI* 131–151 (1998) .

[67] J. Marsden, A. Weinstein, Reduction of symplectic manifolds with symmetry, *Rep. Math. Phys.* 5 (1974), 121–130.

[68] K. Meyer, *Symmetries and integrals in mechanics, Dynamical systems* (M. Peixoto, ed.), Academic Press New York, (1973), pp. 259–273.

[69] J.J. Millson, V. Toledano Laredo, Casimir operators and monodromy representations of generalised braid groups, *Transform. Groups,* **10** (2005) 2:217–254.

[70] A.I. Molev Casimir elements and Sugawara operators for Takiff algebras, *Journal of Mathematical Physics* **62(1)** (2021) 011701.

[71] E. Mukhin, V. Tarasov, A. Varchenko, Bethe eigenvectors of higher transfer matrices, *J. Stat. Mech. Theory Exp.* **8** (2006) P08002.

[72] H. Nagoya, Hypergeometric solutions to Schrödinger equations for the quantum Painlevé equations, *Journal of mathematical physics* **52** (2011) 083509.

[73] W. Nahm, The construction of all self-dual multimonopoles by the ADHM method *International Centre for Theoretical Physics* **IC–82/16** (1982).

[74] S. Pakuliak, V. Rubtsov, A. Silantyev, Classical elliptic current algebras. I, *J. Gen. Lie Theory Appl.* **2** (2008) 2:65–78.

[75] S. Pakuliak, V. Rubtsov, A. Silantyev, Classical elliptic current algebras. II, *J. Gen. Lie Theory Appl.* **2** (2008) 2:79–93.

[76] P. Painlevé, Sur les équations différentielles du second ordre à points critiques fixes *CR Acad. Sci. Paris.* (1906) **143** 1111-1117.

[77] D. Panyushev and O. Yakimova, Takiff algebras with polynomial rings of symmetric invariants, *Transform. Groups* (2019).

[78] V.L. Popov, E.B. Vinberg, Invariant theory *In Algebraic geometry* **IV** (1994) pp. 123-278.

[79] G. Rembado, Simply-laced quantum connections generalising KZ, *Comm. Math. Phys.* **368** (2019) 1–54.

[80] N. Reshetikhin, The Knizhnik-Zamolodchikov system as a deformation of the isomonodromy problem, *Lett. Math. Phys.,* **26** (1992) 167–177.

[81] P. Saksida, Nahm's equations and generalizations of the Neumann system, *Proceedings of the London Mathematical Society* **78** (1999) 3:701-720.

[82] L. Schlesinger, Über eine Klasse von Differentialsystemen beliebiger Ordnung mit festen kritischen Punkten. *J. fur Math.,* **141** (1912) 96–145.

[83] Y. Sibuya, *Linear differential equations in the complex domain: problems of analytic continuation*, Vol. 82. American Mathematical Soc. (2008)

[84] D. Sinelshchikov, I. Gaiur, N. Kudryashov Lax representation and quadratic first integrals for a family of non-autonomous second-order differential equations *Journal of Mathematical Analysis and Applications* (2019).

[85] D. Talalaev, Quantum spectral curve method, *Geometry and quantisation, Trav. Math.* **19** (2011) 203–271.

[86] Takasaki K., Painlevé–Calogero correspondence revisited, *Journal of Mathematical Physics*, **42(3)**, 1443-1473 (2001).

[87] S. J. Takiff, Rings of invariant polynomials for a class of Lie algebras, *Trans. Amer. Math. Soc.* **160 (1971), 249–262.**

[88] B. Vicedo, C. Young, $(\mathfrak{gl}M, \mathfrak{gl}N)-$dualities in Gaudin models with irregular singularities, *SIGMA Symmetry Integrability Geom. Methods Appl.* **14** (2018) 40:28 pp.

[89] W.Wasow, *Asymptotic expansions for ordinary differential equations,* (2018) Courier Dover Publications.

[90] N.M.J. Woodhouse The symplectic and twistor geometry of the general isomonodromic deformation problem, *J. Geom. Phys.* **39** (2001) 2:97-128.

[91] N.M.J. Woodhouse Duality for the general isomonodromy problem, *J.Geom.Phys.* **57** (2007) 4:1147-1170.

[92] D. Yamakawa, Fundamental two-forms for isomonodromic deformations. *J. Integrable Syst.* **4** (2019) 1:1-35.

[93] D. Yamakawa, Tau functions and Hamiltonians of isomonodromic deformations, *Josai Mathematical Monographs* **10** (2017) 139 – 160

[94] A. Zabrodin, A. Zotov Quantum Painleve-Calogero Correspondence *Journal of mathematical physics* **53** (2012) 7:073507.