LOCAL GROUP THEORY, THE AMALGAM METHOD, AND FUSION SYSTEMS

by

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Abstract

In this thesis, we provide a framework in which certain configurations in saturated fusion systems can be characterized via the amalgam method. Along the way, we identify several rank 2 amalgams involving strongly p-embedded subgroups, and recognize some finite simple groups as associated completions. In addition, as an application, we determine all saturated fusion systems supported on a Sylow p-subgroup of $\mathrm{G}_2(p^n)$ and $\mathrm{PSU}_4(p^n)$ for all primes p and $n \in \mathbb{N}$.

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CONTENTS

CHAPTER 1

INTRODUCTION

For a finite group G and a prime p dividing the order of G, the p-fusion category of G provides a means to concisely express properties of the conjugacy of p-elements within a Sylow p-subgroup S of G. Fusion systems may then be viewed as an abstraction of fusion categories without the need to specify any enveloping finite group G, instead focusing only on the conjugacy properties of some fixed p-group S.

Fusion systems were first introduced by Puig in the 1990s, under the moniker "Frobenius categories," as a way to capture properties of the defect group of a p-block in modular representation theory. These Frobenius categories were then revived by Broto, Levi and Oliver in [BLO03], where they found purchase in algebraic topology as a mechanism to investigate p-completions of classifying spaces. There, they were renamed fusion systems, a terminology which has now become standard.

More recently, fusion systems have found use in finite group theory, specifically in revisiting the classification of finite simple groups, through a program initiated by

Aschbacher (see [Asc19]). Aschbacher's program aims to classify the finite simple groups of "component type" using "semisimple" methods from local group theory which have been translated to fusion systems, and specifically focusing on the case where $p = 2$. Indeed, several of the more difficult results in the proof of the classification of finite simple groups are easier and often have more gratifying statements in the context of fusion systems.

Alongside the program of Aschbacher, there is another "next generation" scheme to reprove large parts of the classification. This program, headed by Meierfrankenfeld, Stellmacher and Stroth and dubbed the "MSS program", aims to determine the finite simple groups of "local characteristic p" by using mostly "unipotent" methods (see [MSS03] for an overview). Pivotal to this approach is the use of *amalgams* to identify finite simple groups, a methodology which we utilize heavily in this thesis.

Within the MSS program, there is scope to investigate a larger class of "characteristic p" groups than in the original proof of the classification. Indeed, it may be possible here to determine the finite simple groups which are of *parabolic characteristic p* (but probably only for the prime 2), and this improvement would substantially ease the burden on the treatment of component type groups. Because of the Gorenstein–Walter Dichotomy Theorem, and a suitable analysis of some small cases, the net result of the union of these two programs should be a shortened proof of the classification of finite simple groups.

The results in this thesis lie somewhere in between these two programs: applying unipotent, or characteristic p, methods from group theory to saturated fusion systems. While some equivalent notion of parabolic characteristic p for fusion systems is not needed for this work, the results in this thesis would certainly fit more in this framework. Important to note is the *dichotomy theorem* for saturated fusion systems which says that every saturated fusion system is either of "characteristic p-type" or of "component type." Following the proof of this theorem, due to Aschbacher [AKO11, Theorem II.4.3], it is not hard to generalize to a dichotomy theorem partitioning fusion systems into "parabolic characteristic p" and "parabolic component type."

Within the realm of fusion systems, one of the more active areas of research is the hunt for *exotic* fusion systems: those which do not correspond to the p-fusion categories of finite groups. Notably, when $p = 2$ there is only one known family of exotic fusion systems: the Benson–Solomon systems constructed by Oliver and Levi [LO02]. As for odd primes, there are far more examples to draw from, and so we will not provide a comprehensive list here. In this work, we uncover some previously unknown exotic systems supported on a Sylow 3-subgroup of the sporadic finite simple group F_3 (see Section 3.3), and so this work may be viewed as another contribution to the following research direction suggested by Oliver [AKO11, III.7.4]:

"Try to better understand how exotic fusion systems arise at odd primes; or (more realistically) look for patterns which explain how certain large families of them arise."

The primary purpose of this thesis is to classify saturated fusion systems $\mathcal{F}$, supported on a p-group S, which are generated by automorphisms of two subgroups of S which satisfy certain properties. The subgroups in question are *maximally essential subgroups* of $\mathcal{F}$, and by the Alperin–Goldschmidt fusion theorem, in this setting the automizers of these essential subgroups completely determine $\mathcal{F}$. Then the characterization of $\mathcal{F}$ is achieved by identifying a rank two amalgam within the fusion system, via a result of Robinson [Rob07, Theorem 1], and utilizing the *amalgam method*. The amalgam method was first conceived by Goldschmidt [Gol80], building on earlier work of Sims. In our interpretation, we closely follow the version of the method developed and refined by Delgado and Stellmacher [DS85]. Fortunately, given our hypothesis motivated by fusion systems, we can often prove that the amalgam we obtain is a so called *weak BN-pair of rank* 2, and we can directly appeal to [DS85] where such configurations are already classified.

Within this work, we very often use a $\mathcal{K}$-group hypothesis when investigating automizers of essential subgroups and a local $\mathcal{CK}$-system hypothesis on the fusion system $\mathcal{F}$. Recall that a $\mathcal{K}$-group is a finite group in which every simple section is isomorphic to a known finite simple group. A local $\mathcal{CK}$-system is then a saturated fusion system in which the induced automorphism groups on all p-subgroups are $\mathcal{K}$-groups. At some stage in the analysis, unfortunately, we make explicit use of the classification of finite simple groups (CFSG), specifically when $\mathcal{F}$ is exotic. However, up to that point, we are still able to determine the isomorphism type of the p-group on which $\mathcal{F}$ is supported, as well the important local actions, within a local $\mathcal{CK}$-system hypothesis and only appeal to the classification to prove that the fusion system is exotic. Thus, we believe this result would still be suitable for use in any investigation of fusion systems in which induction via a minimal counterexample is utilized.

The majority of the work in this thesis is in proving the following theorem.

Main Theorem. *Let $\mathcal{F}$ be a local $\mathcal{CK}$-system on a p-group S such that $O_p(\mathcal{F}) = \{1\}$. Assume that $\mathcal{F}$ has two $\mathrm{Aut}_{\mathcal{F}}(S)$-invariant maximally essential subgroups $E_1, E_2 \trianglelefteq S$ with the property $\mathcal{F} = \langle N_{\mathcal{F}}(E_1), N_{\mathcal{F}}(E_2)\rangle$. Then $\mathcal{F}$ is one of the following:*

(i) *$\mathcal{F} = \mathcal{F}_S(G)$, where $F^*(G)$ is isomorphic to a rank 2 simple group of Lie type in characteristic p;*

(ii) *$\mathcal{F} = \mathcal{F}_S(G)$, where $G \cong \mathrm{M}_{12}, \mathrm{Aut}(\mathrm{M}_{12}), \mathrm{J}_2, \mathrm{Aut}(\mathrm{J}_2), \mathrm{G}_2(3)$ or $\mathrm{PSp}_6(3)$ and $p = 2$;*

(iii) *$\mathcal{F} = \mathcal{F}_S(G)$, where $G \cong \mathrm{Co}_2, \mathrm{Co}_3, \mathrm{McL}, \mathrm{Aut}(\mathrm{McL}), \mathrm{Suz}, \mathrm{Aut}(\mathrm{Suz})$ or Ly and $p = 3$;*

(iv) *$\mathcal{F} = \mathcal{F}_S(G)$, where $G \cong \mathrm{PSU}_5(2), \mathrm{Aut}(\mathrm{PSU}_5(2)), \Omega_8^+(2), \mathrm{O}_8^+(2), \Omega_{10}^-(2), \mathrm{Sp}_{10}(2), \mathrm{PSU}_6(2)$ or $\mathrm{PSU}_6(2).2$ and $p = 3$;*

(v) *$\mathcal{F}$ is simple fusion system on a Sylow 3-subgroup of F_3 and, assuming CFSG, $\mathcal{F}$ is an exotic fusion system uniquely determined up to isomorphism;*

(vi) *$\mathcal{F} = \mathcal{F}_S(G)$, where $F^*(G) \cong \mathrm{Ly}, \mathrm{HN}, \mathrm{Aut}(\mathrm{HN})$ or B and $p = 5$; or*

(vii) *$\mathcal{F}$ is a simple fusion system on a Sylow 7-subgroup of $\mathrm{G}_2(7)$ and, assuming CFSG, $\mathcal{F}$ is an exotic fusion system uniquely determined up to isomorphism.*

We include $\mathrm{G}_2(2)' \cong \mathrm{PSU}_3(3)$, $\mathrm{Sp}_4(2)' \cong \mathrm{Alt}(6)$ and the Tits groups ${}^2\mathrm{F}_4(2)'$ as groups of Lie type in characteristic 2.

In the above classification, where $\mathcal{F}$ is *realizable* by finite group, we provide only one example of a group which realizes the fusion system. In several instances, this example is not unique, even amongst finite simple groups. In particular, if $\mathcal{F}$ is realized by a simple group of Lie type in characteristic coprime to p, then there are lots of examples which realize the fusion system, see for instance [BMO12]. Note also that we manage to capture a large number of fusion systems at odd primes associated to sporadic simple groups. Indeed, as can be witnessed in the tables provided in [AH12], almost all of the p-fusion categories of the Sporadic simple groups at odd primes are either constrained, supported on an extraspecial group of exponent p and so are classified in [RV04], or satisfy the hypothesis of the Main Theorem.

It is surprising that in the conclusion of the Main Theorem there are so few exotic fusion systems. It has seemed that, at least for odd primes, exotic fusion systems were reasonably abundant. Perhaps an explanation for the apparent lack of exotic fusion systems is that the setup from the Main Theorem somehow reflects some of the geometry present in rank 2 groups of Lie type. Additionally, we remark that in the two exotic examples in the classification, the fusion systems are obtained by "pruning" a particular class of essential subgroups, as defined in [PS21]. Indeed, these essential subgroups, along with their automizers, seem to resemble Aschbacher blocks, the minimal counterexamples to the Local $C(G,T)$-theorem [BHS06]. Most of the exotic fusion systems the author is aware of either have a set of essentials resembling blocks, or are obtained by pruning a class of essentials resembling blocks out of the fusion category of some finite group. For instance, pearls in fusion systems, investigated in [Gra18] and [GP20], are the smallest examples of blocks in fusion systems.

Given the hypothesis of the Main Theorem, there are some fairly natural questions and extensions to consider. First, is it necessary to demand that the essential subgroups E_1 and E_2 are maximally essential in the fusion system $\mathcal{F}$? It appears that the truly difficult case here is where the outer automorphism group of the essential subgroup induced by the fusion system is p-solvable and has a Sylow p-subgroup of p-rank 1. Outside of these cases, given suitable characterization of quadratic 2F-modules for groups with strongly p-embedded subgroups, it seems likely the techniques employed in this thesis could be adapted in order to remove the maximality condition on the essential subgroups. Second, is the condition that the essential subgroups are $\mathrm{Aut}_{\mathcal{F}}(S)$-invariant truly necessary? This should be related to notion of "pushing up" in finite groups. Fortunately, there are a large number of results which may be applicable in this setting. The hope is then to maintain some control of the automorphisms present in the fusion system so that the methodology described in this thesis should still be applicable. A final question to consider is whether we need to restrict to only two classes of essential subgroups. In the analogous situation in finite group theory, groups of Lie type of rank n are "controlled" by their rank 2 residues. This indicates that perhaps there should be an equivalent "Lie theory" of saturated fusion systems. Work towards this has already been initiated in [Ono11], wherein chamber systems and parabolic systems for fusion systems are explored.

The work we undertake in the proof of the Main Theorem may be regarded as a generalization of some of the results in [AOV13], where only certain configurations at the prime 2 are considered. There, the authors exhibit a situation in which a pair of subgroups of the automizers of pairs of essential subgroup generate a subsystem, and then describe the possible actions present in the subsystem, utilizing Goldschmidt's pioneering results in the amalgam method. With this in mind, we provide the following corollary (proved as Corollary 5.5.1) along the same lines which, at least with regards to essential subgroups, may also be considered as the minimal situation in which a saturated fusion system satisfies $O_p(\mathcal{F}) = \{1\}$.

Corollary A. *Suppose that $\mathcal{F}$ is a saturated fusion system on a p-group S such that $O_p(\mathcal{F}) = \{1\}$. Assume that $\mathcal{F}$ has exactly two essential subgroups E_1 and E_2. Then $N_S(E_1) = N_S(E_2)$ and writing $\mathcal{F}_0 := \langle N_{\mathcal{F}}(E_1), N_{\mathcal{F}}(E_2)\rangle_{N_S(E_1)}$, $\mathcal{F}_0$ is a saturated normal subsystem of $\mathcal{F}$ and either*

(i) *$\mathcal{F} = \mathcal{F}_0$ is determined by the Main Theorem;*

(ii) *p is arbitrary, $\mathcal{F}_0$ is isomorphic to the p-fusion category of H, where $F^*(H) \cong \mathrm{PSL}_3(p^n)$, and $\mathcal{F}$ is isomorphic to the p-fusion category of G where G is the extension of H by a graph or graph-field automorphism;*

(iii) *$p = 2$, $\mathcal{F}_0$ is isomorphic to the 2-fusion category of H, where $F^*(H) \cong \mathrm{PSp}_4(2^n)$, and $\mathcal{F}$ is isomorphic to the 2-fusion category of G where G is the extension of H by a graph or graph-field automorphism; or*

(iv) *$p = 3$, $\mathcal{F}_0$ is isomorphic to the 3-fusion category of H, where $F^*(H) \cong \mathrm{G}_2(3^n)$, and $\mathcal{F}$ is isomorphic to the 3-fusion category of G where G is the extension of H by a graph or graph-field automorphism.*

As intimated earlier in this introduction, we utilize the amalgam method to classify the fusion systems in the statement of the Main Theorem. Here, we work in a purely group theoretic setting and so, as a consequence of the work in the thesis, we obtain some generic results concerning amalgams of finite groups which apply outside of fusion systems. We operate under the following hypothesis, and note that the relevant definitions are provided in Section 5.1:

Hypothesis B. $\mathcal{A} := (G_1, G_2, G_{12})$ is a characteristic p amalgam of rank 2 with faithful completion G satisfying the following:

(i) for $S \in \mathrm{Syl}_p(G_{12})$, $G_{12} = N_{G_1}(S) = N_{G_2}(S)$; and

(ii) writing $\overline{G_i} := G_i/O_p(G_i)$, $\overline{G_{12}}$ is a strongly p-embedded subgroup of $\overline{G_i}$.

It transpires that all the amalgams satisfying Hypothesis B are either weak BN-pairs of rank 2; or $p \leqslant 7$, $|S| \leqslant 2^9$ when $p = 2$, and $|S| \leqslant p^7$ when p is odd. Moreover, in the latter exceptional cases we can generally describe, at least up to isomorphism, the parabolic subgroups of the amalgam.

What is remarkable about these results is that amalgams produced have "critical distance" (defined in Notation 5.2.5) bounded above by 5 . In the cases where the amalgam is not a weak BN-pair of rank 2, the critical distance is bounded above by 2, and when this distance is equal to 2, the amalgam is *symplectic* and was already known about by work of Parker and Rowley [PR12]. We present an undetailed version of the theorem summarizing the amalgam theoretic results below.

Theorem C. *Suppose that $\mathcal{A} = \mathcal{A}(G_1, G_2, G_{12})$ satisfies Hypothesis B. Then one of the following occurs:*

(i) *$\mathcal{A}$ is a weak BN-pair of rank 2;*

(ii) *$p = 2$, $\mathcal{A}$ is a symplectic amalgam, $G_1/O_2(G_1) \cong \mathrm{Sym}(3)$, $G_2/O_2(G_2) \cong (3 \times 3) : 2$ and $|S| = 2^6$;*

(iii) *$p = 2$, $\Omega(Z(S)) \trianglelefteq G_2$, $\langle (\Omega(Z(S))^{G_1})^{G_2} \rangle \not\leq O_2(G_1)$, $O^{2'}(G_1)/O_2(G_1) \cong \mathrm{SU}_3(2)'$, $O^{2'}(G_2)/O_2(G_2) \cong \mathrm{Alt}(5)$ and $|S| = 2^9$;*

(iv) *$p = 3$, $\Omega(Z(S)) \trianglelefteq G_2$, $\langle (\Omega(Z(S))^{G_1}) \rangle \not\leq O_3(G_2)$, $O_3(G_1) = \langle (\Omega(Z(S))^{G_1}) \rangle$ is cubic $2F$-module for $O^{3'}(G_1/O_3(G_1))$ and $|S| \leqslant 3^7$; or*

(v) *$p = 5$ or 7, $\mathcal{A}$ is a symplectic amalgam and $|S| = p^6$.*

Much more information about the amalgams is provided where they arise in the proofs.

Naturally, an interesting question to ask is whether the results concerning these amalgams have any direct application to finite group theory, and in particular, in classifying certain finite simple groups by their p-local structure. In Section 5.5, we collect various results already present in the literature which, when augmented with some additional hypotheses, characterize some finite simple groups from the garnered amalgam data.

As a first substantial application of the Main Theorem, which we provide before the proof of the Main Theorem to ease exposition, we approach a slightly different research problem. Namely, we classify all saturated fusion systems supported on a p-group isomorphic to a Sylow p-subgroup of $\mathrm{G}_2(p^n)$ or $\mathrm{PSU}_4(p^n)$. This work has a different flavour to the methods used in the proof of the Main Theorem. There, the hypothesis enforced restrictions on the global structure of the fusion system without necessarily demanding any specific structure of the p-group on which the system is supported whereas in this application, we impose restrictions on the p-group itself. This work forms part of a program to classify all saturated fusion systems supported on Sylow p-subgroups of rank 2 groups of Lie type, complementing the results in [Cle07] and [HS19]. Moreover, we generalize results already obtained in [PS18], [BFM19] and [Mon20] where only the case where the field of definition is of order p is considered. Furthermore, we remove some of the other restrictions in those works, where only fusion systems $\mathcal{F}$ satisfying $O_p(\mathcal{F}) = \{1\}$ are considered, at little cost to the exposition. The work here draws heavily from results and ideas within those papers and most of the 'interesting' examples we uncover occur in this 'small' setting.

Although a number of the the results applied to classify these fusion systems (particularly those results occurring as corollaries of the Main Theorem) rely on a $\mathcal{K}$-group hypothesis on the local actions, within the restricted setting of an enforced structure on a the p-group S, we are almost always able to circumvent the need for such strong assumptions. Where appropriate, we describe the required modifications to make these results independent of any $\mathcal{K}$-group hypothesis. In this way, we are able to almost completely rid ourselves of any reliance on the classification of finite simple groups, and only make use of it to prove the exoticity of some fusion systems supported on a Sylow 7-subgroup of $\mathrm{G}_2(7)$, a check already completed in [PS18], and to recognize $\mathrm{PSL}_2(q^2)$ acting on a natural $\Omega_4^-(q)$-module to classify fusion systems on supported on a Sylow p-subgroup of $\mathrm{PSU}_4(q)$, where $q = p^n$ and p is odd. We do, however, make use of some of the results listed in [GLS98] concerning known facts about known finite simple groups. We present the main results below.

Theorem D. *Let $\mathcal{F}$ be a saturated fusion system over a Sylow p-subgroup of $\mathrm{G}_2(q)$ where $q = p^n$, and identify Q_1 and Q_2 with the unipotent radicals of two non-conjugate maximal parabolic subgroups of $\mathrm{G}_2(q)$. Then one of the following holds:*

(i) $\mathcal{F} = \mathcal{F}_S(S : \mathrm{Out}_{\mathcal{F}}(S))$;

(ii) $\mathcal{F} = \mathcal{F}_S(Q_1 : \mathrm{Out}_{\mathcal{F}}(Q_1))$ *where* $O^{p'}(\mathrm{Out}_{\mathcal{F}}(Q_1)) \cong \mathrm{SL}_2(q)$, *or* $\mathrm{Out}_{\mathcal{F}}(Q_1)$ *is isomorphic to a subgroup of* $(3 \times 3) : 2$ *and* $p = q = 2$, *or* $p = q \in \{5, 7\}$ *and the possibilities for* $O^{p'}(\mathrm{Out}_{\mathcal{F}}(Q_1))$ *are given in [PS18, Lemma 5.2];*

(iii) $\mathcal{F} = \mathcal{F}_S(Q_2 : \mathrm{Out}_{\mathcal{F}}(Q_2))$ *where* $O^{p'}(\mathrm{Out}_{\mathcal{F}}(Q_2)) \cong \mathrm{SL}_2(q)$;

(iv) $\mathcal{F} = \mathcal{F}_S(M)$ *where* $M \cong 2^3.\mathrm{PSL}_3(2)$ *is non-split and* $p = q = 2$;

(v) $\mathcal{F} = \mathcal{F}_S(M)$ *where* $M \cong 5^3.\mathrm{SL}_3(5)$ *is non-split and* $p = q = 5$;

(vi) $\mathcal{F} = \mathcal{F}_S(G)$ *where* $G \cong \mathrm{G}_2(3)$ *or* M_{12} *and* $p = q = 2$;

(vii) $\mathcal{F} = \mathcal{F}_S(G)$ *where* $G \cong \mathrm{Ly}$, HN, $\mathrm{HN}.2$ *or* B *and* $p = q = 5$;

(viii) $\mathcal{F} = \mathcal{F}_S(G)$ *where* $G \cong \mathrm{M}$ *and* $p = q = 7$*;*

(ix) $\mathcal{F}$ *is one of the exotic fusion systems listed in [PS18, Table 5.1] and* $p = q = 7$*; or*

(x) $\mathcal{F} = \mathcal{F}_S(G)$ *where* $F^*(G) = O^{p'}(G) \cong \mathrm{G}_2(p^n)$*.*

Theorem E. *Let* $\mathcal{F}$ *be a saturated fusion system over a Sylow* p*-subgroup of* $\mathrm{PSU}_4(q)$ *where* $q = p^n$*, and let* X *be the preimage in* S *of* $J(S/Z(S))$*. Then one of the following occurs:*

(i) $\mathcal{F} = \mathcal{F}_S(S : \mathrm{Out}_{\mathcal{F}}(S))$*;*

(ii) $\mathcal{F} = \mathcal{F}_S(X : \mathrm{Out}_{\mathcal{F}}(X))$ *where* $O^{p'}(\mathrm{Out}_{\mathcal{F}}(X)) \cong \mathrm{SL}_2(q)$*, or* $\mathrm{Out}_{\mathcal{F}}(X)$ *is determined in [BFM19] and* $q = p = 3$*;*

(iii) $\mathcal{F} = \mathcal{F}_S(J(S) : \mathrm{Out}_{\mathcal{F}}(J(S)))$ *where* $J(S)$ *is a natural* $\Omega_4^-(q)$*-module for* $O^{p'}(\mathrm{Out}_{\mathcal{F}}(J(S)) \cong \mathrm{PSL}_2(q^2)$*;*

(iv) $\mathcal{F} = \mathcal{F}_S(Q : \mathrm{Out}_{\mathcal{F}}(Q_x))$ *where* $x \in S' \setminus Z(S)$*,* $Q_x = C_S(x)$*,* $\mathrm{Out}_{\mathcal{F}}(Q_x) \cong \mathrm{Sym}(3)$ *and* $q = p = 2$*;*

(v) $\mathcal{F} = \mathcal{F}_S(M)$ *where* $M \cong 2^4 : (\mathrm{Sym}(3) \times \mathrm{Sym}(3))$ *and* $q = p = 2$*;*

(vi) $\mathcal{F} = \mathcal{F}_S(M)$ *where* $M \cong 2^3 : \mathrm{PSL}_3(2)$ *and* $q = p = 2$*;*

(vii) $\mathcal{F} = \mathcal{F}_S(G)$ *where* $G \cong \mathrm{PSL}_4(2)$ *and* $q = p = 2$*;*

(viii) $\mathcal{F} = \mathcal{F}_S(G)$ *where* $G = \mathrm{Co}_2$*,* McL*,* $\mathrm{McL}.2$*,* $\mathrm{PSU}_6(2)$ *or* $\mathrm{PSU}_6(2).2$ *and* $p = q = 3$*; or*

(ix) $\mathcal{F} = \mathcal{F}_S(G)$ *where* $F^*(G) = O^{p'}(G) \cong \mathrm{PSU}_4(q)$*.*

Additionally, with a small amount of extra effort, for S a Sylow p-subgroup of $\mathrm{PSU}_4(p^n)$ or $\mathrm{G}_2(p^n)$, we are able to give a good description of all possible radical, centric subgroups of a fusion system (or group) containing S as a Sylow p-subgroup. This has implications beyond the rest of the results in this thesis. For example, several results concerning weight conjectures for groups and fusion systems rely on detailed information of the radical, centric subgroups of a Sylow p-subgroup, see for instance [Kes+19] and [KMS20].

As in the Main Theorem, something interesting to note in Theorem D and Theorem E is the small number of exotic fusion systems unearthed. The only exotic fusion systems that arise were already identified in [PS18] and are related to the Monster sporadic simple group. This gives credence to [PS21, Conjecture 2] that, aside from a few exceptions in small rank and small prime cases, the structure of a Sylow p-subgroup of a group of Lie type in characteristic p is too rigid to support any exotic fusion systems. This is in complete contrast to the case where the fusion system is supported on a Sylow p-subgroup of a group of Lie type in characteristic coprime to p, where exotic fusion systems are ubiquitous (see [OR20]).

In terms of progressing towards the goal of determining all fusion systems on Sylow p-subgroups of rank 2 groups of Lie type, this still leaves $\mathrm{PSU}_5(p^n)$, ${}^3\mathrm{D}_4(p^n)$ and ${}^2\mathrm{F}_4(2^n)$, where necessarily $p = 2$ in the last case. As in this work, a suitable methodology for classifying fusion systems over the Sylow p-subgroups of these groups boils down to determining a complete set of essential subgroups and, after treating small values of n and p separately, applying the Main Theorem.

It feels prudent at this point to mention some important results which play some part in the proof of the results above, but which should be widely applicable in other works on saturated fusion systems and amalgams. The first of which involves *critical subgroups*, specified subgroups of p-groups first used by Feit and Thompson in the "Odd Order" paper. As far as the author is aware, critical subgroups have not been heavily utilized in fusion systems or in the amalgam method. In an earlier draft of this work, critical subgroups were used to obtain strong control of the actions of parabolic subgroups of in the amalgam method when $p \geqslant 5$. However, we later found methods to treat these cases alongside the cases where $p \in \{2, 3\}$ and so this approach was abandoned. We still believe that it should be recorded here for posterity.

Proposition F. *Let $\mathcal{A} = \mathcal{A}(G_1, G_2, G_{12})$ be a characteristic p amalgam. Then writing $\overline{G} := G_i/O_p(G_i)$, for some $i \in \{1, 2\}$ there is a $\overline{G}$-module V on which p'-elements of $\overline{G}$ act faithfully and a p-subgroup C of $\overline{G}$ such that $[V, C, C, C] = \{1\}$.*

A further result which may have application outside of this thesis is the following proposition.

Proposition G. *Let $\mathcal{A} = \mathcal{A}(G_1, G_2, G_{12})$ be a characteristic p amalgam satisfying Hypothesis B. Then, writing $Q_i := O_p(G_i)$, $Q_1 \cap Q_2 \ntrianglelefteq G_i$ for $i \in \{1, 2\}$.*

Again, peering into the world of finite groups, given the classification of weak BN-pairs of rank 2 in [DS85], one hopes to determine higher rank groups of Lie type in characteristic p using the rank 2 residues to identify their associated building. In this line of work, Timmesfeld [Tim88] associates a graph using local data, where two points, corresponding to rank 1 parabolic subgroups P_i and P_j, are joined if and only if $O_p(P_i) \cap O_p(P_j)$ is not normal in P_i or P_j. See [ST98] for how this method is used to gain control in the rank 3 setting. If one hopes to develop a theory of fusion systems akin to the notion of parabolic systems in groups, then it seems sensible that an “equivalent” result should be proved. The above proposition provides one direction of such a result.

We now describe the strategy to prove the main results of this thesis.

In Chapter 2, we set up the requisite group and module theoretic results needed to examine the local actions within a fusion system, and within the amalgam method. Most importantly, we characterize groups with strongly p-embedded subgroups, groups with associated FF-modules and 2F-modules, groups which contain elements which act quadratically, and exhibit situations in which these phenomena occur. The typical examples of automizers in our investigations are rank 1 groups of Lie type in characteristic p and, because of this, large parts of Chapter 2 are devoted to the properties of such groups and their “natural” modules.

In Chapter 3, we introduce fusion systems and, for the most part, reproduce definitions and properties associated to fusion systems which may be readily found in the literature. Importantly, here we describe the necessary tools to describe a complete set of essential subgroups for a saturated fusion system $\mathcal{F}$ and determine their automizers. Then, using the model theorem, we are able to able to investigate finite groups whose fusion categories are isomorphic to normalizer subsystems of the two distinguished essential subgroups. We close this chapter with a discussion and construction on the unearthed exotic fusion systems supported on a 3-group isomorphic to a Sylow 3-subgroup of F_3.

In Chapter 4, we classify saturated fusion systems $\mathcal{F}$ which are supported on S where S is isomorphic to a Sylow p-subgroup of $\mathrm{G}_2(p^n)$ or $\mathrm{PSU}_4(p^n)$, assuming the validity the of the Main Theorem which is proved in Chapter 5. The sections within this chapter deal with the cases where S is isomorphic to a Sylow p-subgroup of $\mathrm{G}_2(2^n)$, $\mathrm{G}_2(3^n)$, $\mathrm{G}_2(p^n)$ for $p \geqslant 5$, and $\mathrm{PSU}_4(p^n)$. For $\mathrm{G}_2(p^n)$, the separation in cases is brought about due to some degeneracies in the Chevalley commutator formulas when $p = 2$ or 3, resulting in some exceptional structural properties. While there are differences when $p = 2$ and p is odd for $\mathrm{PSU}_4(p^n)$, the differences are not so drastic to affect the methodology.

In each of the cases, it transpires that, barring some small exceptions, there are only two potential essential subgroups of $\mathcal{F}$: those which coincide with the unipotent radicals of maximal parabolic subgroups in $\mathrm{G}_2(p^n)$ and $\mathrm{PSU}_4(p^n)$. Upon deducing the potential automizers of these subgroups, we then distinguish between the case where there is at most one essential subgroup (where necessarily $O_p(\mathcal{F}) \neq \{1\}$), and where both subgroups are essential. In this latter case, we apply the Main Theorem which identifies a rank 2 amalgam in $\mathcal{F}$ and then, with the aid of the results in [DS85], completely determines the fusion system. Importantly within this work, since the only exotic fusion systems we engage with are determined in [PS18], we do not need to concern ourselves with checks on saturation and exoticity as in other works. As mentioned previously, there is some exceptional behaviour for small values of p and n where the fusion systems of some other finite simple groups appear. In these instances, we generally appeal to previous results in the literature or apply a package in MAGMA [PS21] to determine a list of radical, centric subgroups and a list of saturated fusion systems supported on S.

In Chapter 5, we first demonstrate how to identify a rank 2 amalgam given certain hypotheses on a fusion system and begin setting up the group theoretic framework needed for the amalgam method. We also provide some classification results for fusion systems based on known amalgam results where it is easy to do so. For several arguments, we investigate a minimal counterexample where minimality is imposed on the order of the models of the normalizers of essential subgroups. Then, in the amalgam method, the case division separates fairly naturally, and we follow the divisions used in [DS85]. Then the following sections and subsections deal with these partitioned cases.

For several of the amalgams we investigate, their completions are unique up to "local isomorphism" and, as it turns out, this is enough to determine the fusion system up isomorphism. However, in some cases, at least from a fusion system perspective, we do not go so far and instead aim only to bound the order of the p-group on which $\mathcal{F}$ is supported and apply a package in MAGMA [PS21] which identifies the fusion system. In fact, in two instances there are no finite groups which realize the amalgam appropriately and we uncover two exotic fusion systems, one of which was known about previously by work of Parker and Semeraro [PS18], and another which was previously undocumented. With that said, given the information we gather about the amalgams, it does not seem such a stretch to at least provide a characterization of these amalgams up to some weaker notion of isomorphism. Finally, we close this chapter by providing some useful corollaries to the Main Theorem and provide some identifications of finite simple groups which satisfy Hypothesis B.

The notation used throughout generally follows the standard conventions, but we mention some particular practices we adopt. With regards to notation concerning simple groups, we will generally follow the Atlas [Con+85], with some caveats regarding the classical groups. We include the prefix "P" to indicate a quotient by the center, and "S" indicates the subgroup of matrices with determinant 1 e.g. we use $\mathrm{PSL}_n(q)$ where the Atlas uses $\mathrm{L}_n(q)$. In addition, we reserve the notations $O_n^+(q)$ and $O_n^-(q)$ for the full orthogonal groups, while $\Omega_n^\varepsilon(q)$ denotes the commutator subgroup of $SO_n^\varepsilon(q)$ for $\varepsilon \in \{+,-\}$. For the sporadic groups, we follow the Atlas with the exception of Thompson's sporadic simple group, which we refer to as F_3 instead of the usual Th. We make this choice to emphasize the connection with "amalgams of type F_3" as defined in [DS85] and [Del88]. We denote by $\mathrm{Sym}(n)$ and $\mathrm{Alt}(n)$ the symmetric and alternating groups of degree n, and $\mathrm{Dih}(n)$ represents the dihedral group of order n so that n is necessarily even. The notation Q_{4n} is used for generalized quaternion groups of order $4n$. When $p = 2$, 2_+^{1+2n} is the extraspecial group obtained by taking the central product of r groups isomorphic to $\mathrm{Dih}(8)$ and $n-r$ groups isomorphic to Q_8 where $n-r$ is even, and 2_-^{1+2n} is the extraspecial group obtained by a taking the central product of r groups isomorphic to $\mathrm{Dih}(8)$ and $n-r$ groups isomorphic to Q_8 where $n-r$ is odd. For p an odd prime, we reserve the notation p_+^{1+2n} and p_-^{1+2n} for extraspecial p-groups of exponent p and p^2 respectively. We will use Atlas notation for the "shape" of p-groups, often to exhibit the structure of their chief factors in some enveloping group G e.g. q^{1+2} is a group of order q^3 for q some prime power, with some grouped collection of G-chief factors having orders q and q^2. Where unambiguous, we will often present cyclic groups uniquely by their order, and elementary abelian p-groups by their expression as p-powers e.g. $r \times s$ is the direct product of a cyclic group of order r and a cyclic group of order s, and p^n is an elementary group of order p^n. Finally, we mention that as the majority of the modules we study occur "internally", we will use multiplicative notation for modules throughout.

CHAPTER 2

GROUP THEORY, REPRESENTATION THEORY AND PRELIMINARIES

We reserve this chapter for any general results in group theory or representation theory which will be useful in proving later results concerning fusion systems and amalgams. Several are well known or elementary, and where possible, we aim to give explicit references or rudimentary proofs.

Of particular importance in this chapter is the notion of a group with a strongly p-embedded subgroup, and we provide some classification results regarding this class of groups. Since rank 1 groups of Lie type in characteristic p provide the standard examples of groups with strongly p-embedded subgroups, we devote a large part of this chapter for recording several facts about such groups and their associated actions. Finally, of key importance in this work, is the identification of these groups along with their modules and, because of this, FF-modules, 2F-modules, quadratic action and Hall–Higman type theorems are also a focus of this chapter.

As background texts, we use [Asc00], [Gor07], [Hup13] and [KS06].

2.1 Group Theoretic Methods

We first start with concepts and results which are ubiquitous across all of finite group theory. Set G to be a finite group throughout.

Lemma 2.1.1 (Dedekind Modular Law). *Suppose that $X, Y, Z \leq G$ and $X \leq Y$. Then $X(Y \cap Z) = Y \cap XZ$.*

Lemma 2.1.2 (Three Subgroup Lemma). *Let $X, Y, Z \leq G$. If $[X, Y, Z] = [Y, Z, X] = \{1\}$, then $[Z, X, Y] = \{1\}$. Moreover, if $N \trianglelefteq G$ and both $[X, Y, Z]$ and $[Y, Z, X]$ are contained in N, then $[Z, X, Y] \leq N$.*

Lemma 2.1.3 (Frattini Argument). *Let $A \trianglelefteq G$ and $T \in \mathrm{Syl}_p(A)$. Then $G = AN_G(T)$.*

Lemma 2.1.4 (Gaschutz's Theorem). *Let A be an abelian normal subgroup of G and $R \leq G$ such that $A \leq R$ and $(|A|, |G : R|) = 1$. Then A has a complement in R if and only if A has a complement in G.*

Definition 2.1.5. Let G act on a group A. A *G-chief series* for A is a normal series

$$\{1\} = A_0 \trianglelefteq A_1 \trianglelefteq \ldots \trianglelefteq A_n = A$$

such that A_i is normal in the internal semidirect product $A : G$ and the series cannot be further refined with respect to this condition i.e. there does not exists $A_i < N < A_{i+1}$ such that $N \trianglelefteq A : G$. The factors A_i/A_{i-1} are referred to as the G-chief factors and a factor is central if $[G, A_i] \leq A_{i-1}$ and non-central otherwise. We refer to a $\{1\}$-chief series as a *chief series* for A and the $\{1\}$-chief factors as the *chief factors* of A.

Remark. In a similar way to composition series, one can show that finite groups with a G-action always have a G-chief series and that the G-chief factors are unique up to isomorphism and reordering, independent of the particular G-chief series constructed. Thus we are justified in describing *the* chief factors of a group A.

Of particular importance in this work is coprime action. We will often use the results described below without explicit reference, and where we do reference, we will refer to the totality of the techniques as “coprime action.”

Definition 2.1.6. Suppose G acts on a group A. Say the action of G on A is *coprime* if $(|G|, |A|) = 1$ and one of $|A|$ or $|G|$ is solvable. Note that if the first condition holds, the second automatically does by the Feit–Thompson theorem.

Lemma 2.1.7 (Coprime Action)**.** *Suppose that a group G acts on a group A coprimely, and B is a G-invariant subgroup of A. Then the following hold:*

(i) $C_{A/B}(G) = C_A(G)B/B$;

(ii) *if G acts trivially on A/B and B, then G acts trivially on A;*

(iii) $[A, G] = [A, G, G]$;

(iv) $A = [A, G]C_A(G)$ *and if A is abelian* $A = [A, G] \times C_A(G)$;

(v) *if G acts trivially on $A/\Phi(A)$, then G acts trivially on A;*

(vi) *if $p \neq 2$, A is a p-group and G acts trivially on $\Omega(A)$, then G acts trivially on A; and*

(vii) *for $S \in \mathrm{Syl}_p(G)$, if $m_p(S) \geqslant 2$ then $A = \langle C_A(s) \mid s \in S \setminus \{1\}\rangle$.*

Proof. See, for instance, [KS06, Chapter 8]. □

In conclusion (v) in the statement above, one can say a little more. The following is a classical result of Burnside, but the version we use is [Gor07, (I.5.1.4)]. We also provide a related result further below.

Lemma 2.1.8 (Burnside)**.** *Let S be a finite p-group. Then $C_{\mathrm{Aut}(S)}(S/\Phi(S))$ is a normal p-subgroup of* $\mathrm{Aut}(S)$*.*

Lemma 2.1.9. *Let E be a finite p-group and $Q \leq A$ where $A \leq \mathrm{Aut}(E)$ and Q is a p-group. Suppose there exists a normal chain $\{1\} = E_0 \trianglelefteq E_1 \trianglelefteq E_2 \trianglelefteq \ldots \trianglelefteq E_m = E$ of subgroups such that for each $\alpha \in A$, $E_i\alpha = E_i$ for all $0 \leq i \leq m$. If for all $1 \leq i \leq m$, Q centralizes E_i/E_{i-1}, then $Q \leq O_p(A)$.*

Proof. See [Gor07, (I.5.3.2)]. □

The final result we describe here which still falls under the umbrella of "coprime action" is the A×B-lemma due to Thompson.

Lemma 2.1.10 (A×B-Lemma)**.** *Let AB be a finite group which acts on a p-group V. Suppose that B is a p-group, $A = O^p(A)$ and $[A, B] = \{1\} = [A, C_V(B)]$. Then $[A, V] = \{1\}$.*

Proof. See [Asc00, (24.2)]. □

We now introduce concepts and techniques more familiar in local group theory, and which are heavily used in the proof of the classification of finite simple groups.

Definition 2.1.11. A finite group G is a $\mathcal{K}$-group if every simple section of G is a known finite simple group.

Definition 2.1.12. Let G be a finite group and p a prime dividing $|G|$. Then G is of *characteristic* p if $C_G(O_p(G)) \leq O_p(G)$.

Lemma 2.1.13. *Let G be a finite group of characteristic p. If $H \trianglelefteq\trianglelefteq G$ or $O_p(G) \leq H$, then H is of characteristic p.*

Proof. This is elementary. □

Definition 2.1.14. Say a group K is quasisimple if K is perfect and $K/Z(K)$ is a simple group. A subgroup $K \leq H$ is a *component* of H if K is quasisimple and subnormal in H.

Lemma 2.1.15. *Let K be a component of G and $H \trianglelefteq\trianglelefteq G$. Then*

(i) *either K is a component of H, or H centralizes K;*

(ii) *every component of H is a component of G; and*

(iii) *for L a component of G not equal to K, $[L, K] = \{1\}$.*

Proof. See [Asc00, (31.3)-(31.5)]. □

Definition 2.1.16. We denote by $F(G)$ the *Fitting subgroup* of G, the largest normal nilpotent subgroup of G, and by $E(G)$ the *layer* of G, the subgroup of G generated by all of its components. Define $F^*(G)$, the *generalized Fitting subgroup* of G, to be the product of $F(G)$ and $E(G)$.

The following results may be found in [Asc00, (31.7)-(31.13)], for example.

Lemma 2.1.17. *Let G a finite group. Then*

(i) *$F(G)$, $E(G)$ and $F^*(G)$ are characteristic subgroups of G;*

(ii) *if G is solvable then $F(G) = F^*(G)$ and $C_G(F(G)) \leq F(G)$;*

(iii) *$F(G) = \prod_r O_r(G)$ where r ranges over the prime divisors of G;*

(iv) *$E(G)$ is the central product of the components of G;*

(v) *$F^*(G)$ is a central product of $E(G)$ and $F(G)$;*

(vi) *$C_G(F^*(G)) \leq F^*(G)$; and*

(vii) *G is of characteristic p if and only if $F^*(G) = O_p(G)$.*

We now move on to some more specialized results pertaining to the work in this thesis.

Definition 2.1.18. Let G be a finite group and $S \in \mathrm{Syl}_p(G)$. Then G is *p-minimal* if $S \ntrianglelefteq G$ and S is contained in a unique maximal subgroup of G.

Lemma 2.1.19 (McBride's Lemma)**.** *Let G be a finite group, $S \in \mathrm{Syl}_p(G)$ and $\mathcal{P}_G(S)$ denote the collection of p-minimal subgroups of G over S. Then $G = \langle \mathcal{P}_G(S) \rangle N_G(S)$. Moreover, $O^{p'}(G) = \langle \mathcal{P}_G(S) \rangle$.*

Proof. If $G \in \mathcal{P}_G(S)$ the the result holds trivially so assume that G is counterexample to the first statement with $|G|$ minimal. Since G is not p-minimal over S, there are maximal subgroups M_1, M_2 of G which contain S. But then, since G was a minimal counterexample, $M_i = \langle \mathcal{P}_{M_i}(S) \rangle N_{M_i}(S)$ for $i \in \{1, 2\}$. Since $\mathcal{P}_{M_i}(S) \subseteq \mathcal{P}_G(S)$, $N_{M_i}(S) \leq N_G(S)$ and $G = \langle M_1, M_2 \rangle$, the result holds.

Now, let $P \in \mathcal{P}_G(S)$ and $x \in N_G(S)$. Then for M the unique maximal subgroup of P containing S, M^x is the unique maximal subgroup of P^x containing $S^x = S$, and $S \ntrianglelefteq P^x$. It follows that $N_G(S)$ normalizes $\langle \mathcal{P}_G(S) \rangle$ and by the definition of $O^{p'}(G)$ and since $G = \langle \mathcal{P}_G(S) \rangle N_G(S)$, $O^{p'}(G) \leq \langle \mathcal{P}_G(S) \rangle$. Now, suppose that there is $P \in \mathcal{P}_G(S)$ with $P \not\leq O^{p'}(G)$. Then $O^{p'}(P) \leq P \cap O^{p'}(G) < P$ and so $O^{p'}(P)$ is contained in the unique maximal subgroup of P which contains S. Since S is not normal in P, $N_P(S)$ is also contained in the unique maximal subgroup of P containing S. But then, by the Frattini argument, $P = O^{p'}(G)N_G(S) < P$, a contradiction. Therefore, $\langle \mathcal{P}_G(S) \rangle \leq O^{p'}(G)$ and the lemma holds. □

Lemma 2.1.20. *Suppose that H is p-minimal over S and R is a normal p-subgroup of H. Then H/R is p-minimal.*

Proof. This is elementary. □

Definition 2.1.21. Let G be a finite group and $H < G$. Then H is *strongly p-embedded* in G if and only if $|H|_p > 1$ and $N_G(P) \leq H$ for each non-trivial p-subgroup Q with $Q \leq H$.

Lemma 2.1.22. *Suppose that G contains a strongly p-embedded subgroup X. Then the following hold:*

(i) *X contains a Sylow p-subgroup of G;*

(ii) *if $H \leq G$ with $H \not\leq X$ then provided $|H \cap X|_p > 1$, $H \cap X$ is strongly p-embedded in H;*

(iii) *$O^{p'}(G) \cap X$ is strongly p-embedded in $O^{p'}(G)$; and*

(iv) *if $G \neq XO_{p'}(G)$, then $XO_{p'}(G)/O_{p'}(G)$ is strongly p-embedded in $G/O_{p'}(G)$.*

Proof. See [PStr09, Lemmas 3.2, 3.3]. □

Lemma 2.1.23. *If G has a cyclic or generalized quaternion Sylow p-subgroup T and $O_p(G) = 1$, then $N_G(\Omega(T))$ is strongly p-embedded in G.*

Proof. For $X \leq T$ a non-trivial subgroup, X is also cyclic or generalized quaternion and so also has a unique subgroup of order p. Thus, $\Omega(X) = \Omega(T)$ and since $O_p(G) \neq 1$, we have that $N_G(X) \leq N_G(\Omega(X)) = N_G(\Omega(T)) < G$ so that $N_G(\Omega(T))$ is strongly p-embedded in G. $\square$

Quite remarkably, possessing a strongly p-embedded subgroup is a surprisingly limiting condition. In the following two propositions, we roughly determine the structure of groups with strongly p-embedded subgroups. For $p = 2$, we refer to work of Bender [Ben71], while if p is odd we make use of the classification of finite simple groups. In the application of these results, groups with strongly p-embedded subgroups will only ever appear in the local analysis of fusion systems. Particularly, these groups appear as automizers of certain p-subgroups and so would fit into the framework of any proofs utilizing a "minimal counterexample" hypothesis.

Proposition 2.1.24. *Suppose that $G = O^{p'}(G)$ has a strongly p-embedded subgroup. Let $S \in \mathrm{Syl}_p(G)$ and denote $\widetilde{G} := G/O_{p'}(G)$. If $m_p(S) = 1$ then one of the following holds:*

(i) *p is an odd prime, S is cyclic, G is perfect and $\widetilde{G}$ is a non-abelian finite simple group;*

(ii) *S is cyclic, $G = SO_{p'}(G)$ and G is p-solvable; or*

(iii) *$p = 2$, S is generalized quaternion and $G = O_{2'}(G)C_G(\Omega(S))$.*

Moreover, in cases (ii) and (iii), $\langle \Omega(S)^G \rangle = \Omega(S)[\Omega(S), O_{p'}(G)]$ is the unique normal subgroup of G which is divisible by p and minimal with respect to this condition.

Proof. Since $m_p(S) = 1$, S is either cyclic or generalized quaternion by [Gor07, I.5.4.10 (ii)]. If S is generalized quaternion, then $p = 2$ and (iii) follows from a result of Bender [Ben71]. Moreover, if S is cyclic and $p = 2$, then G has a normal 2-complement (see [Gor07, Theorem 7.4.3]) and (ii) holds. Hence, we may assume from now that S is cyclic and p is odd. Notice that $F(\widetilde{G}) = O_p(\widetilde{G})$ since $O_{p'}(\widetilde{G}) = \{1\}$. If $F^*(\widetilde{G}) = F(\widetilde{G}) = O_p(\widetilde{G})$, then $O_p(\widetilde{G})$ is self-centralizing and as $\widetilde{S}$ is abelian, we have that $O_p(\widetilde{G}) = \widetilde{S}$ and $SO_{p'}(G) \trianglelefteq G$. In particular, $G = O^{p'}(G) \leq SO_{p'}(G) \leq G$, G is p-solvable and (ii) holds.

Suppose now that $\widetilde{G}$ has a component $\widetilde{L}$. If $p \nmid |\widetilde{L}|$, then $L \leq O_{p'}(E(\widetilde{G})) \leq O_{p'}(\widetilde{G})$, a contradiction. Hence, p divides the order of any component of $\widetilde{G}$. Since $\widetilde{S}$ is cyclic, $\widetilde{L}$ has cyclic Sylow p-subgroups. By [Asc00, Lemma 33.14], $Z(\widetilde{L})$ is a p'-prime group, and so $Z(\widetilde{L}) \leq O_{p'}(E(\widetilde{G})) = \{1\}$ and $\widetilde{L}$ is simple. Notice also that since each component is simple, $E(\widetilde{G})$ is a direct product of components, and since p divides the order of any component, $E(\widetilde{G}) = \widetilde{L}$ is the unique component of $\widetilde{G}$, else $m_p(\widetilde{G}) = m_p(G) > 1$. Since $O_p(\widetilde{G}) \cap E(\widetilde{G}) = \{1\}$, we have that $F^*(\widetilde{G}) = O_p(\widetilde{G}) \times E(\widetilde{G})$ and since $m_p(\widetilde{G}) = 1$, $O_p(\widetilde{G}) = \{1\}$. Therefore, $F^*(\widetilde{G})$ is a non-abelian simple group.

It remains to prove that $\widetilde{S} \leq F^*(\widetilde{G})$ to show that (i) holds. Form the group $\widetilde{H} = F^*(\widetilde{G})\widetilde{S}$ and assume that $\widetilde{H} \neq F^*(\widetilde{G})$. Note that by the Frattini argument, $\widetilde{H} = F^*(\widetilde{G})N_{\widetilde{H}}(R)$ for all $R \in \mathrm{Syl}_r(\widetilde{F}^*(\widetilde{G}))$. Moreover, for $r \neq p$ a prime, $\mathrm{Syl}_r(F^*(\widetilde{G})) \subseteq \mathrm{Syl}_r(\widetilde{H})$. Then for $R \in \mathrm{Syl}_r(F^*(\widetilde{G}))$ with $r \neq p$, let $P \in \mathrm{Syl}_p(N_{\widetilde{H}}(R))$ and $T \in \mathrm{Syl}_p(\widetilde{H})$ containing P. Then $F^*(\widetilde{H}) \cap T < T$ and as T is cyclic and $\widetilde{H} = F^*(\widetilde{G})N_{\widetilde{H}}(R)$, we deduce that $P = T$ and $N_{\widetilde{H}}(R)$ contains a Sylow p-subgroup of $\widetilde{H}$. Hence, by conjugacy, $\widetilde{S}$ normalizes a Sylow r-subgroup of $\widetilde{H}$, for all primes r. But then $\widetilde{S}$ normalizes a Sylow r-subgroup of $N_{\widetilde{H}}(\widetilde{S})$

for all r, and so centralizes a Sylow r-subgroup of $N_{\widetilde{H}}(\widetilde{S})$ for all r. Applying [Gor07, Theorem 7.4.3], $\widetilde{H}$ has a normal p-complement, a contradiction since $\widetilde{H}$ contains a component of $\widetilde{G}$. Thus, $\widetilde{S} \leq F^*(\widetilde{G})$ and since $G = O^{p'}(G)$ it follows that $\widetilde{G}$ is a non-abelian simple group. Hence, $\widetilde{G'} = \widetilde{G}$ and so $S \leq G'$. Then $G = O^{p'}(G) \leq G' \leq G$, G is perfect and (i) holds.

Suppose case (ii) or (iii) occurs and let N be a normal subgroup of G whose order is divisible by p. Then, as $m_p(S) = 1$, $\Omega(S) \leq N$ and so $\Omega(S)[\Omega(S), O_{p'}(G)] = \Omega(S)[\Omega(S), G] = \langle \Omega(S)^G \rangle \leq N$, and the result follows. $\square$

Remark. Notice that if H is a non-abelian finite simple with cyclic Sylow p-subgroups, then for $S \in \mathrm{Syl}_p(H)$, $N_G(\Omega(S))$ is strongly p-embedded in H by Lemma 2.1.23. Thus, the description in case (i) is best possible up to a better understanding of $O_{p'}(G)$. It is also worth noting that every non-abelian finite simple group has a cyclic Sylow p-subgroup for some odd prime p.

Proposition 2.1.25. *Suppose that $G = O^{p'}(G)$ is a $\mathcal{K}$-group with a strongly p-embedded subgroup X. Let $S \in \mathrm{Syl}_p(G)$ and set $\widetilde{G} := G/O_{p'}(G)$. If $m_p(G) \geqslant 2$ then $\widetilde{G}$ is isomorphic to one of:*

(i) $\mathrm{PSL}_2(p^{a+1})$ *or* $\mathrm{PSU}_3(p^b)$ *for p arbitrary, $a \geqslant 1$ and $p^b > 2$;*

(ii) $\mathrm{Sz}(2^{2a+1})$ *for $p = 2$ and $a \geqslant 1$;*

(iii) $\mathrm{Alt}(2p)$ *for $p > 3$;*

(iv) $\mathrm{Ree}(3^{2a+1}), \mathrm{PSL}_3(4)$ *or* M_{11} *for $p = 3$ and $a \geqslant 0$;*

(v) $\mathrm{Sz}(32) : 5, {}^2\mathrm{F}_4(2)', \mathrm{McL}$ *or* Fi_{22} *for $p = 5$; or*

(vi) J_4 *for $p = 11$.*

Proof. If $G \neq XO_{p'}(G)$, then this follows from [PStr09, (2.5), (3.3)] which in turn uses [GLS98, Theorem 7.6.1]. So assume that $G = XO_{p'}(G)$. By coprime action,

$$O_{p'}(G) = \langle C_{O_{p'}(G)}(a) | 1 \neq a \in S \rangle$$

since $m_p(G) \geqslant 2$ and so $O_{p'}(G) \leq X$ and $G = X$, a contradiction. □

The final concept in this section is that of *critical subgroups*, which first arose in the proof of the Feit–Thompson theorem. Originally in this work, critical subgroups provided a means to control the automizer of some p-group Q whenever $p \geqslant 5$. In the context of the amalgam method, they force "cubic action" on some faithful section of Q and from there, one can apply Hall–Higman type results to deduce information about Q and its automizer. Where this methodology was previous employed, we now have methods to treat these cases uniformly across all primes and so critical subgroups now play a far lesser role in this work. However, we believe they still provide some interesting consequences in the amalgam method and we still include some of these consequences (see Corollary 5.2.21). We present the *critical subgroup theorem*, due to Thompson, below.

Theorem 2.1.26. *Let Q be a p-group. Then there exists $C \leq Q$ such that the following hold:*

(i) *C is characteristic in Q;*

(ii) *$\Phi(C) \leq Z(C)$ so that C has class at most 2;*

(iii) *$[C, Q] \leq Z(C)$;*

(iv) *$C_Q(C) \leq C$; and*

(v) *C is coprime automorphism faithful.*

Proof. This is [Gor07, (I.5.3.11)]. $\square$

We call such a subgroup $C \leq Q$ a *critical subgroup* of Q.

2.2 Properties of Rank 1 Groups Of Lie Type

As witnessed in Section 2.1, the generic examples of groups with a strongly p-embedded subgroup are rank 1 groups of Lie type in characteristic p. These are the groups which will appear most often in later work, and so we take this opportunity to list some of their important properties. While almost all of these results are well known, we aim to provide explicit references or proofs of these results.

Lemma 2.2.1. *Let $G \cong \mathrm{PSL}_2(p^n)$ or $\mathrm{SL}_2(p^n)$ and $S \in \mathrm{Syl}_p(G)$. Then the following hold:*

(i) *S is elementary abelian of order p^n;*

(ii) *$\mathrm{SL}_2(2) \cong \mathrm{Sym}(3)$, $\mathrm{PSL}_2(3) \cong \mathrm{Alt}(4)$ and $\mathrm{SL}_2(3)$ are all solvable;*

(iii) *if $p = 2$, then for $U \leq S$ with $|U| = 4$, there is $x \in G$ such that $G = \langle U, u^x\rangle$ for $1 \neq u \in U$;*

(iv) *if $p = 2$, all involutions in S are conjugate and so, for $1 \neq u \in S$ an involution, there is $x, y \in G$ such that $G = \langle u, u^x, u^y\rangle$;*

(v) *if p is odd, then for $1 \neq u \in S$, there is $x \in G$ such that $G = \langle u, u^x\rangle$ unless $p^n = 9$ in which case there is $x \in G$ such that $H := \langle u, u^x\rangle < G$ is maximal subgroup of G and $H/Z(H) \cong \mathrm{PSL}_2(5)$;*

(vi) *$N_G(S)$ is a solvable maximal subgroup of G and for K a Hall p'-subgroup of $N_G(S)$, $K/Z(G)$ is cyclic of order $(p^n-1)/(p^n-1,2)$ and acts fixed point freely on $S \setminus \{1\}$;*

(vii) *if $p^n \geqslant 4$, then G is perfect and if $\widetilde{G}$ is a perfect central extension of G by a group of p'-order, then $\widetilde{G} \cong \mathrm{PSL}_2(p^n)$ or $\mathrm{SL}_2(p^n)$; and*

(viii) *if x is a non-trivial automorphism of G which centralizes S, then $x \in \mathrm{Aut}_S(G)$.*

Proof. The proofs of (i)-(vi) are written out fairly explicitly in [Hup13, II.6–II.8]. Detailed information on automorphism groups and Schur multipliers is provided in [GLS98, Theorem 2.5.12] and [GLS98, Theorem 6.1.2]. □

Lemma 2.2.2. *Let $G \cong \mathrm{PSU}_3(p^n)$ or $\mathrm{SU}_3(p^n)$ and $S \in \mathrm{Syl}_p(G)$. Then the following hold:*

(i) *S is a special p-group of order p^{3n} with $|Z(S)| = p^n$;*

(ii) *$\mathrm{SU}_3(2)$ is solvable, a Sylow 2-subgroup of $\mathrm{SU}_3(2)$ is isomorphic to the quaternion group of order 8 and $\mathrm{SU}_3(2)' \cong 3_+^{1+2} : 2$ has index 4 in $\mathrm{SU}_3(2)$;*

(iii) *for $p^n > 2$, $N_G(S)$ is a solvable maximal subgroup of G and for K a Hall p'-subgroup of $N_G(S)$, $|K/Z(G)| = (p^{2n}-1)/(p^{2n}-1,3)$ and K acts irreducibly on $S/Z(S)$;*

(iv) *$N_G(Z(S)) = N_G(S)$ and for K a Hall p'-subgroup of $N_G(S)$, $|C_K(Z(S))| = p^n+1$ and $C_K(Z(S))$ acts fixed point freely on $S/Z(S)$;*

(v) *for any $x \in G \setminus N_G(S)$, $\langle Z(S), Z(S)^x\rangle \cong \mathrm{SL}_2(p^n)$ and $G = \langle Z(S), S^x\rangle$;*

(vi) *for* $\{1\} \neq U \leq Z(S)$*, unless* $p^n = 9$ *and* $|U| = 3$ *or* $p = 2$ *and* $|U| = 2$*, there is* $x, z \in G$ *such that* $G = \langle U, U^x, U^z \rangle$*;*

(vii) *for* $\{1\} \neq U \leq Z(S)$*, if* $p^n = 9$ *and* $|U| = 3$ *or* $p = 2 < p^n$ *and* $|U| = 2$*, then there is* $x, y, z \in G$ *such that* $G = \langle U, U^x, U^y, U^z \rangle$*;*

(viii) *for* $\{1\} \neq U \trianglelefteq S$ *with* $U \not\leq Z(S)$*, if* $p^n \neq 2$ *then there is* $x \in G$ *such that* $G = \langle U, U^x \rangle$*;*

(ix) *if* $p^n > 2$*, then* G *is perfect and if* $\widetilde{G}$ *is a perfect central extension of* G *by a group of* p'*-order, then* $\widetilde{G} \cong \mathrm{PSU}_3(p^n)$ *or* $\mathrm{SU}_3(p^n)$*; and*

(x) *if* x *is a non-trivial automorphism of* G *which centralizes* S*, then* $x \in \mathrm{Aut}_{Z(S)}(G)$*.*

Proof. The proofs of (i)-(v) may be found in [Hup13, II.10]. Again, information on automorphism groups and Schur multipliers may be found in [GLS98, Theorem 2.5.12, Theorem 6.1.2]. It remains to prove (vi)-(viii).

For (vi) and (vii) suppose that $U \leq Z(S)$, $p^n \neq 2$ and set $H := \langle Z(S), Z(S)^x \rangle \cong \mathrm{SL}_2(p^n)$ for $x \in G \setminus N_G(S)$. By Lemma 2.2.1 (iv), (v), H is generated by two or three conjugates of U, and by [Mit11], H is contained in a unique maximal subgroup $M \cong \mathrm{GU}_2(p^n) \cong (p^n+1).\mathrm{SL}_2(p^n)$. Since $G = \langle U^G \rangle$, there is z such that $U^z \not\leq M$. It then follows from the maximality of M in G that $G = \langle H, U^z \rangle$ and (vi) and (vii) are proved.

Suppose now that $U \not\leq Z(S)$, $U \trianglelefteq S$ and $p^n \neq 2$. Since $U \not\leq Z(S)$, $\{1\} \neq [U, S] \leq Z(S) \cap U$. Set $C := C_{N_G(S)}(Z(S))$ and observe that C is irreducible on $S/Z(S)$ by (iv). Then, since $[U, S] \leq Z(S)$, $[U, S] = [U, S]^C = [\langle U^C \rangle, \langle S^C \rangle]$. By the irreducibility of C on $S/Z(S)$, $(UZ(S)/Z(S))^C = S/Z(S)$ and so $[\langle U^C \rangle, \langle S^C \rangle] =$

$Z(S) = [U, S] \leq U$. Now, there is $x \in G \setminus N_G(S)$ such that $\langle Z(S), Z(S)^x \rangle \cong \mathrm{SL}_2(p^n)$ is contained in a unique maximal subgroup $M \cong \mathrm{GU}_2(p^n)$. Then, as $U > Z(S)$, $|U| > p^n$, $\langle Z(S), Z(S)^x \rangle < \langle U, U^x \rangle$ and (viii) follows. □

Lemma 2.2.3. *Let $G \cong \mathrm{Sz}(2^n)$ and $S \in \mathrm{Syl}_2(G)$. Then the following hold:*

(i) *n is odd and 3 does not divide the order of G;*

(ii) *$\mathrm{Sz}(2) \cong 5 : 4$ is a Frobenius group, $\Phi(\mathrm{Sz}(2)) \cong \mathrm{Dih}(10)$, $|\mathrm{Sz}(2)'| = 5$ and a Sylow 2-subgroup of $\mathrm{Sz}(2)$ is cyclic of order 4;*

(iii) *if $n > 1$ then $\Phi(S) = Z(S) = \Omega(S)$ and $S/\Phi(S) \cong \Phi(S)$ is elementary abelian of order 2^n;*

(iv) *$N_G(S)$ is a solvable maximal subgroup of G and for K a Hall $2'$-subgroup of $N_G(S)$,$|K| = 2^n - 1$ and K acts irreducibly on $S/\Phi(S)$ and $\Phi(S)$;*

(v) *there is $x \in G$ such that $G = \langle Z(S), Z(S)^x \rangle$;*

(vi) *all involutions in S are conjugate and if $n > 1$, for $1 \neq u \in Z(S)$, there is $x, y \in G$ such that $G = \langle u, u^x, u^y \rangle$;*

(vii) *for $U \trianglelefteq S$ with $U \not\leq Z(S)$, there is $x \in G$ such that $G = \langle U, U^x \rangle$;*

(viii) *if $n > 1$ then G is perfect and has trivial Schur multiplier; and*

(ix) *if x is a non-trivial automorphism of G which centralizes S, then $x \in \mathrm{Aut}_{Z(S)}(G)$.*

Proof. Most of the proofs of these facts may be found in [Suz62, Sections 13 - 16], except the proof of (viii) which may be gleaned from [GLS98, Theorem 6.1.2]. □

Lemma 2.2.4. *Let $G \cong \mathrm{Ree}(3^n)$ and $S \in \mathrm{Syl}_3(G)$. Then the following hold:*

(i) *n is odd;*

(ii) *the Sylow 2-subgroups of G are abelian;*

(iii) *if $n = 1$, then $G \cong \mathrm{PSL}_2(8) : 3$, $G' \cong \mathrm{PSL}_2(8)$, $S \cong 3_-^{1+2}$, $Z(S) = \Phi(S)$ has order 3, $\Omega(S) = S \cap G'$ has order 9 and $|S| = 27$;*

(iv) *if $n > 1$, then S has order 3^{3n}, $\Phi(S) = \Omega(S)$ has order 3^{2n}, $Z(S) = [S, \Phi(S)]$ has order 3^n and $S/\Phi(S) \cong \Phi(S)/Z(S) \cong Z(S)$ is elementary abelian of order 3^n;*

(v) *$N_G(S)$ is a solvable maximal subgroup of G and for K a Hall $3'$-subgroup of $N_G(S)$, $|K| = 3^n - 1$ and K acts irreducibly on $S/\Omega(S)$, $\Omega(S)/Z(S)$ and $Z(S)$;*

(vi) *for $\{1\} \neq U \trianglelefteq S$, if $n > 1$ then there is $x, y \in G$ such that $G = \langle U, U^x, U^y \rangle$;*

(vii) *if $n > 1$ then G is perfect and has trivial Schur multiplier, and $\mathrm{Ree}(3)'$ is perfect and has trivial Schur multiplier; and*

(viii) *if x is a non-trivial automorphism of G which centralizes S, then $x \in \mathrm{Aut}_{Z(S)}(G)$.*

Proof. The proofs of (i) to (v) follow from the main theorem of [War66] while (vii) and (viii) follow from [GLS98, Theorem 2.5.12, Theorem 6.1.2]. We make use of results in [War66] to prove (vi). Since the results when $n = 1$ are easily verified, we assume that $n > 1$ throughout.

Suppose that $U \not\leq Z(S)$ and $U \trianglelefteq S$. Then $U \cap Z(S) \neq \{1\}$ and $\{1\} \neq \Omega(U) \leq \Omega(S) \cap U$. Suppose first that there is $u \in U$ such that $u \in \Omega(U) \setminus Z(S)$. Then by (v), it follows that $C_{N_G(S)}(u) = \Omega(S)\langle i \rangle$, where $i \in K$ is an involution. Then

$u \in C_G(i)$ and by [War66], $C_G(i) \cong \langle i\rangle \times L$, where $L \cong \mathrm{PSL}_2(3^n)$, and $C_G(i)$ is a maximal subgroup of G (see also [Kle88, Theorem C]). Since $n > 1$ is odd, there is $x \in L$ such $L = \langle u, u^x\rangle$ by Lemma 2.2.1 (v). Further, $C_G(i) \cap Z(S) = \{1\}$ and since $U \cap Z(S) \neq \{1\}$ as $U \trianglelefteq S$, $L < \langle U, U^x\rangle$ and since $C_G(i)$ is maximal, it follows that $G = \langle U, U^x\rangle$.

Suppose now that $\Omega(U) \leq Z(S)$, $U \not\leq Z(S)$ and $U \trianglelefteq S$. Let $x \in G \setminus N_G(S)$ such that $U^x \neq N_G(S)$. Since $U \not\leq Z(S)$, it follows that $U \not\leq \Omega(S)$. If $G \neq \langle U, U^x\rangle$, then $\langle U, U^x\rangle$ is contained in a maximal subgroup of G. Since $|U| \geqslant 9$, $U \cap \Omega(S) \leq Z(S)$ and $U^x \not\leq N_G(S)$, comparing with the list of maximal subgroups in [Kle88, Theorem C], $\langle U, U^x\rangle$ lies in a subfield subgroup of G. But then, as K acts transitively on $Z(S)$, there is $y \in N_G(S)$ such that for some $u \in \Omega(U)$, u^y is not represented by elements of a subfield. Hence, $G = \langle U, U^x, U^y\rangle$.

Finally, suppose that $U \leq Z(S)$ with $|U| \geqslant 9$. Again, considering the maximal subgroup structure of G, since $|U| \geqslant 9$ and there is $x \in G$ such that $U^x \not\leq N_G(S)$, we may assume that $\langle U, U^x\rangle$ is contained in a subfield subgroup of G. Then, as K is irreducible on $Z(S)$, there is $y \in N_G(S)$ such that for some $u \in U$, u^y is not represented by elements of a subfield. Hence, $G = \langle U, U^x, U^y\rangle$. Suppose that $|U| = 3$ and let $x \in G$ such that $U^x \not\leq N_G(S)$ and $y \in G$ such that $U^y \leq S$ but U^y is not in a subfield subgroup. Then $\langle U, U^y\rangle$ is elementary abelian of order 9 and contained in some maximal subgroup. Comparing with the list of maximal subgroups in [Kle88, Theorem C] and using that the centralizer of an involution in K intersects $Z(S)$ trivially, $\langle U, U^y\rangle$ lies in a unique maximal subgroup, namely $N_G(S)$. It follows that $\langle U, U^x, U^y\rangle$ is not contained in any maximal subgroup so that $G = \langle U, U^x, U^y\rangle$. $\square$

Pivotal to the analysis of local actions in the amalgam method and within a fusion system is recognizing $\mathrm{SL}_2(p^n)$ acting on its modules in characteristic p. Below, we list the most important modules for this work.

Definition 2.2.5. Let $X \cong \mathrm{SL}_2(q)$, $q = p^n$, $k = \mathrm{GF}(q)$ and V a faithful 2-dimensional kX-module.

- $V|_{\mathrm{GF}(p)X}$ is a *natural* $\mathrm{SL}_2(q)$*-module* for X.
- A *natural* $\Omega_3(q)$*-module* for X is the 3-dimensional submodule of $V \otimes_k V$ regarded as a $\mathrm{GF}(p)X$-module by restriction, and is irreducible whenever p is an odd prime.
- If $n = 2a$ for some $a \in \mathbb{N}$, a *natural* $\Omega_4^-(q^{\frac{1}{2}})$*-module* for X is any non-trivial irreducible submodule of $(V \otimes_k V^\tau)|_{\mathrm{GF}(q^{\frac{1}{2}})X}$, where τ is an automorphism of $\mathrm{GF}(q)$ of order 2, regarded as a $\mathrm{GF}(p)X$-module by restriction.
- If $n = 3a$ for some $a \in \mathbb{N}$, a *triality module* for X is any non-trivial irreducible submodule of $(V \otimes V^\tau \otimes V^{\tau^2})|_{\mathrm{GF}(q^{\frac{1}{3}})X}$, where τ is an automorphism of k of order 3, regarded as a $\mathrm{GF}(p)X$-module by restriction.

Lemma 2.2.6. *Suppose $G \cong \mathrm{SL}_2(p^n)$, $S \in \mathrm{Syl}_p(G)$ and V is natural $\mathrm{SL}_2(p^n)$-module. Then the following hold:*

(i) $[V, S, S] = \{1\}$*;*

(ii) $|V| = p^{2n}$ *and* $|C_V(S)| = p^n$*;*

(iii) $C_V(s) = C_V(S) = [V, S] = [V, s] = [v, S]$ *for all* $v \in V \backslash C_V(S)$ *and* $1 \neq s \in S$*;*

(iv) $V = C_V(S) \times C_V(S^g)$ *for* $g \in G \setminus N_G(S)$*;*

(v) *every p'-element of G acts fixed point freely on V; and*

(vi) *$V/C_V(S)$ and $C_V(S)$ are irreducible $\mathrm{GF}(p)N_G(S)$-modules upon restriction.*

Proof. See [PR06, Lemma 4.6] $\square$

Lemma 2.2.7. *Suppose that $G \cong \mathrm{SL}_2(p)$ and V is a direct sum of natural two $\mathrm{SL}_2(p)$-modules. If $U \leq C_V(S)$ is $N_G(S)$-invariant and of order p, then $|\langle U^G \rangle| = p^2$.*

Proof. By [Gor07, (I.3.5.6)], the number of distinct irreducible submodules of V is $p + 1 = (p^2 - 1)/p - 1$. For each W an irreducible submodule, $C_W(S)$ is $N_G(S)$-invariant and of order p, and since $|C_V(S)| = p^2$, $C_V(S)$ has $p+1$ subgroups of order p and each subgroup of order p uniquely determines a submodule. Thus, U uniquely determines a submodule W of order p^2 for which $W = \langle U^G \rangle$. $\square$

Lemma 2.2.8. *Suppose that $G \cong \mathrm{SL}_2(p^n)$, p an odd prime, $S \in \mathrm{Syl}_p(G)$ and V is a natural $\Omega_3(p^n)$-module for G. Then the following hold:*

(i) *$C_G(V) = Z(G)$;*

(ii) *$[V, S, S, S] = \{1\}$;*

(iii) *$|V| = p^{3n}$ and $|V/[V, S]| = |C_V(S)| = p^n$;*

(iv) *$[V, S] = [V, s]$ and $[V, S, S] = [V, s, s] = C_V(s) = C_V(S)$ for all $1 \neq s \in S$;*

(v) *$[V.S]/C_V(S)$ is centralized by $N_G(S)$; and*

(vi) *$V/[V, S]$ and $C_V(S)$ are irreducible $\mathrm{GF}(p)N_G(S)$-modules upon restriction.*

Proof. See [PR06, Lemma 4.7]. $\square$

Lemma 2.2.9. *Let $G \cong (\mathrm{P})\mathrm{SL}_2(p^{2n})$, $S \in \mathrm{Syl}_p(G)$ and V a natural $\Omega_4^-(p^n)$-module for G. Then the following hold:*

(i) $C_G(V) = Z(G)$;

(ii) $[V, S, S, S] = \{1\}$;

(iii) $|V| = p^{4n}$ *and* $|V/[V, S]| = |C_V(S)| = p^n$;

(iv) $|C_V(s)| = |[V, s]| = p^{2n}$ *and* $[V, S] = C_V(s) \times [V, s]$ *for all* $1 \neq s \in S$; *and*

(v) $V/[V, S]$ *and* $C_V(S)$ *are irreducible* $\mathrm{GF}(p)N_G(S)$*-modules upon restriction.*

Moreover, for $\{1\} \neq F \leq S$*, one of the following occurs:*

(a) $[V, F] = [V, S]$ *and* $C_V(F) = C_V(S)$;

(b) $p = 2$, $[V, F] = C_V(F)$ *has order* p^{2n}, F *is quadratic on* V *and* $|F| \leqslant p^n$; *or*

(c) p *is odd,* $|[V, F]| = |C_V(F)| = p^{2n}$, $[V, S] = [V, F]C_V(F)$, $C_V(S) = C_{[V,F]}(F)$ *and* $|F| \leqslant p^n$.

Proof. See [PR06, Lemma 4.8] and [PR12, Lemma 3.15]. □

We require one miscellaneous result concerning the exceptional 1-cohomology of $\mathrm{PSL}_2(9)$ on an $\Omega_4^-(3)$-module.

Lemma 2.2.10. *Suppose that* $G \cong \mathrm{PSL}_2(p^2)$, $p \in \{2, 3\}$ *and* $S \in \mathrm{Syl}_p(G)$*. If* V *is a* 5*-dimensional* $\mathrm{GF}(p)G$*-module such that* $V/C_V(G)$ *is isomorphic to a natural* $\Omega_4^-(p)$*-module, then either* $V = [V, G] \times C_V(G)$*; or* $p = 3$ *and* $[V, S, S]$ *is* 2*-dimensional as a* $\mathrm{GF}(3)S$*-module.*

Proof. This follows from direct computation in $\mathrm{GL}_5(p)$. □

Lemma 2.2.11. *Suppose that $G \cong (\mathrm{P})\mathrm{SL}_2(p^{3n})$, $S \in \mathrm{Syl}_p(G)$ and V is a triality module for G. Then the following hold:*

(i) $[V, S, S, S, S] = \{1\}$;

(ii) $|V| = p^{8n}$, $|V/[V,S]| = |C_V(S)| = |[V,S,S,S]| = p^n$ *and* $|[V,S,S]| = p^{4n}$;

(iii) *if p is odd then $|V/C_V(s)| = p^{5n}$, while if $p = 2$ then $|V/C_V(s)| = p^{4n}$, for all $1 \neq s \in S$; and*

(iv) *$V/[V,S]$ and $C_V(S)$ are irreducible $\mathrm{GF}(p)N_G(S)$-modules upon restriction.*

Proof. See [PR06, Lemma 4.10]. □

We are also interested in the natural modules for $\mathrm{SU}_3(p^n)$ and $\mathrm{Sz}(2^n)$.

Definition 2.2.12. The natural modules for $\mathrm{SU}_3(p^n)$ and $\mathrm{Sz}(2^n)$ are the unique irreducible $\mathrm{GF}(p)$-modules of smallest dimension. Equivalently, they may be viewed as the restrictions of a "natural" $\mathrm{SL}_3(p^{2n})$-module and $\mathrm{Sp}_4(2^n)$-module respectively.

Lemma 2.2.13. *Suppose $G \cong \mathrm{SU}_3(p^n)$, $S \in \mathrm{Syl}_p(G)$ and V is a natural module. Then the following hold:*

(i) $C_V(S) = [V, Z(S)] = [V, S, S]$ *is of order* p^{2n};

(ii) $C_V(Z(S)) = [V, S]$ *is of order* p^{4n}; *and*

(iii) *$V/[V,S]$, $[V,S]/C_V(S)$ and $C_V(S)$ are irreducible $\mathrm{GF}(p)N_G(S)$-modules upon restriction.*

Proof. See [PR06, Lemma 4.13]. □

Lemma 2.2.14. *Suppose $G \cong \mathrm{Sz}(2^n)$, $S \in \mathrm{Syl}_2(G)$ and V is the natural module. Then the following hold:*

(i) $[V, S]$ *has order* 2^{3n}*;*

(ii) $[V, \Omega(S)] = C_V(\Omega(S)) = [V, S, S]$ *has order* 2^{2n}*;*

(iii) $C_V(S) = [V, S, \Omega(S)] = [V, \Omega(S), S] = [V, S, S, S]$ *has order* 2^n*; and*

(iv) $V/[V, S]$, $[V, S]/C_V(\Omega(S))$, $C_V(\Omega(S))/C_V(S)$ *and* $C_V(S)$ *are all irreducible* $\mathrm{GF}(p)N_G(S)$*-modules upon restriction.*

Proof. This is an elementary calculation in $\mathrm{Sp}_4(2^n)$. □

2.3 Module Results, Minimal Polynomials and FF-Actions

Given the descriptions of rank 1 Lie type groups and their modules in Section 2.2, we now require ways to identify them. Furthermore, we would like to have ways to completely determine a group G with a strongly p-embedded subgroup, and its actions, given reasonably general hypotheses. In this section, we provide some methods which aid in these goals. Importantly, this is where we introduce FF-modules, quadratic action and Hall–Higman type arguments. We also take this opportunity to list some generic module results which will be used throughout this work.

Lemma 2.3.1 (Maschke's Theorem)**.** *Let G be a finite group and k a field whose characteristic does not divide the order of G. If V is a kG-module, then $V = V_1 \times \cdots \times V_n$, where each V_i is a simple kG-module for $i \in \{1, \ldots, n\}$.*

Proof. See [Asc00, (12.9)]. $\square$

Lemma 2.3.2. *Let G be a group and V be a faithful $\mathrm{GF}(p)G$-module. Let $T \in \mathrm{Syl}_p(O^p(G))$ and assume that $V = \langle C_V(T)^G \rangle$. Then $V = [V, O^p(G)]C_V(O^p(G))$.*

Proof. See [Che01, Lemma 1.1]. $\square$

We require, at least when p is an odd prime, a way to distinguish between $\mathrm{SL}_2(p^n)$ and $\mathrm{PSL}_2(p^n)$ from a strongly p-embedded hypothesis. Additionally, as can be seen from the Main Theorem, none of the configurations we are interested in have Ree groups as their automizers, so we will also have to dispel of this case later on. Generally, we achieve this using quadratic action.

Definition 2.3.3. Let G be a finite group and V a $\mathrm{GF}(p)G$-module. If $A \leq G$ satisfies $[V, A, A] = \{1\} \neq [V, A]$, then A acts *quadratically* on V and if $[V, A, A, A] = \{1\}$ and A is not quadratic or trivial on V, then A acts *cubically.*

Lemma 2.3.4. *Suppose that V is an irreducible $\mathrm{GF}(p)$-module for $G \cong \mathrm{Ree}(3^n)$ or $G \cong \mathrm{PSL}_2(p^n) \not\cong \mathrm{SL}_2(p^n)$. If there is a non-trivial subgroup A of G with $[V, A, A] = \{1\}$, then $[V, A] = [V, G] = \{1\}$.*

Proof. Since the Sylow 2-subgroups of $\mathrm{PSL}_2(p^n)$ are either abelian or dihedral and the Sylow 2-subgroups of $\mathrm{Ree}(3^n)$ are abelian, this follows from [Gor07, (I.3.8.4)]. $\square$

For $p \geqslant 5$, the pairs (G, V) where G is a group acting faithfully on a module V such that G is generated by elements which act quadratically on V were classified by Thompson. Thompson's results were extended to the prime 3 by work of Ho. It seems imperative to emphasize that the these works predate the classification of finite simple groups. For convenience, the version we use here is by Chermak and utilizes the classification of finite simple groups, although as we stressed earlier, these groups will only ever appear as local subgroups in any arguments.

Lemma 2.3.5. *Suppose G is a $\mathcal{K}$-group which has a strongly p-embedded subgroup for p an odd prime and V be a faithful, irreducible* $\mathrm{GF}(p)$*-module for G. Suppose there is a p-subgroup $A \leq G$ such that $[V, A, A] = \{1\}$ and $G = \langle A^G \rangle$. Then one of the following occurs:*

(i) $G \cong \mathrm{SL}_2(p^n)$ *where p is any odd prime;*

(ii) $G \cong \mathrm{(P)SU}_3(p^n)$ *where p is any odd prime;*

(iii) $G \cong 2 \cdot \mathrm{Alt}(5) \cong \mathrm{SL}_2(5)$ *when $p = 3$; or*

(iv) $G \cong 2^{1+4}_{-}.\mathrm{Alt}(5)$ *when $p = 3$.*

Proof. This follows from [Che02], [Che04], Lemma 2.3.4 and a comparison with the groups listed in Proposition 2.1.24, Proposition 2.1.25. □

More than just a quadratic module, the natural module for $\mathrm{SL}_2(p^n)$ provides the minimal example of an *FF-module*. FF-modules are named due to how they arise as counterexamples to *Thompson factorization* (see [Asc00, 32.11]), which aims to factorize a group into two p-local subgroups. One of these p-local subgroups is the normalizer of the Thompson subgroup of a fixed Sylow p-subgroup. Independent of

FF-modules, the Thompson subgroup is incredibly useful in studying the structure of a p-group and will play an important role in the analysis of subgroups of Sylow p-subgroups of $\mathrm{G}_2(p^n)$ and $\mathrm{PSU}_4(p^n)$ later.

Definition 2.3.6. Let S be a finite p-group. Set $\mathcal{A}(S)$ to be the set of all elementary abelian subgroups of S of maximal rank. Then the *Thompson subgroup* of S is defined as $J(S) := \langle A \mid A \in \mathcal{A}(S)\rangle$.

Proposition 2.3.7. *Let S be a finite p-group. Then the following hold:*

(i) *$J(S)$ is a non-trivial characteristic subgroup of S;*

(ii) *for $A \in \mathcal{A}(S)$, $A = \Omega(C_S(A))$;*

(iii) *$\Omega(C_S(J(S))) = \Omega(Z(J(S))) = \bigcap_{A\in\mathcal{A}(S)} A$; and*

(iv) *if $J(S) \leq T \leq S$, then $J(S) = J(T)$.*

Proof. See [KS06, 9.2.8]. □

Definition 2.3.8. Let G be a finite group and V a $\mathrm{GF}(p)$-module. If there exists $A \leq G$ such that

(i) $A/C_A(V)$ is an elementary abelian p-group;

(ii) $[V, A] \neq \{1\}$; and

(iii) $|V/C_V(A)| \leqslant |A/C_A(V)|$

then V is a *failure to factorize module* (abbrev. FF-module) for G and A is an *offender* on V.

The following proposition describes a fairly natural situation in which one can identify an FF-module from a group failing to satisfy Thompson factorization. This result is well known and the proof is standard (see [KS06, 9.2]).

Proposition 2.3.9. *Let G be a finite group with $S \in \mathrm{Syl}_p(G)$ and $F^*(G) = O_p(G)$. Set $V := \langle \Omega(Z(S))^G \rangle$. Then $O_p(G) = O_p(C_G(V))$ and $O_p(G/C_G(V)) = \{1\}$. Furthermore, if $\Omega(Z(S)) < V$ and $J(S) \not\leq C_S(V)$ then V is an FF-module for $G/C_G(V)$.*

As a counterpoint to the determination of groups with a strongly p-embedded subgroup, whenever a group with a strongly p-embedded subgroup has an associated FF-module, we can almost completely determine the group and its action without the need for a $\mathcal{K}$-group hypothesis. Indeed, the following lemma relies only on a specific case in the Local $C(G,T)$-theorem [BHS06].

Lemma 2.3.10. *Suppose $G = O^{p'}(G)$ has a strongly p-embedded subgroup and a faithful FF-module V. Then $G \cong \mathrm{SL}_2(p^n)$ and $V/C_V(O^p(G))$ is the natural module.*

Proof. See [Hen10, Theorem 5.6]. □

Given a way to characterize a natural $\mathrm{SL}_2(p^n)$-module, it is a natural to ask whether we can characterize some of the other modules, particularly those irreducible modules described in Section 2.2.

Lemma 2.3.11. *Let $G \cong \mathrm{SL}_2(p^n)$ and $S \in \mathrm{Syl}_p(G)$. Suppose that V is a module for G over $\mathrm{GF}(p)$ such that $[V, S, S] = \{1\}$ and such that $[V, O^p(G)] \neq \{1\}$. Then $[V/C_V(O^p(G)), O^p(G)]$ is a direct sum of natural modules for G.*

Proof. See [Che04, Lemma 2.2]. □

Lemma 2.3.12. *Let $G \cong \mathrm{SL}_2(p^n)$, $S \in \mathrm{Syl}_p(G)$ and V an irreducible $\mathrm{GF}(p)G$-module. If $|V| \leqslant p^{3n}$ then both $C_V(S)$ and $V/[V,S]$ are irreducible as $N_G(S)$-modules, $|C_V(S)| = |V/[V,S]|$ and either*

(i) *V is natural $\mathrm{SL}_2(p^n)$-module for $G \cong \mathrm{SL}_2(p^n)$, $|V| = p^{2n}$ and $|C_V(S)| = p^n$;*

(ii) *V is natural $\Omega_4^-(p^{n/2})$, n is even, $|V| = p^{2n}$ and $|C_V(S)| = p^{n/2}$;*

(iii) *V is natural $\Omega_3(p^n)$, p is odd, $|V| = p^{3n}$ and $|C_V(S)| = p^n$; or*

(iv) *V is a triality module, $n = 3r$ for some $r \in \mathbb{N}$, $|V| = p^{8n/3}$ and $|C_V(S)| = p^{n/3}$.*

Proof. This is [CD91, Lemma 2.6]. □

We may relax the restrictions in the definition of an FF-module to allow for a greater class of module setups. An an example, the natural modules for $\mathrm{SU}_3(p^n)$ and $\mathrm{Sz}(2^n)$ are not FF-modules but satisfy the ratio $|V/C_V(A)| \leqslant |A/C_A(V)|^2$ for V the module and A an elementary abelian p-group. Such modules are referred to as *2F-modules.*

Definition 2.3.13. Let G be a finite group and V a $\mathrm{GF}(p)$-module. If there exists $A \leq G$ such that

(i) $A/C_A(V)$ is an elementary abelian p-group;

(ii) $[V, A] \neq \{1\}$; and

(iii) $|V/C_V(A)| \leqslant |A/C_A(V)|^2$

then V is *2F-module* for G.

If G is an almost quasisimple group with a 2F module V, then both G and V are known by work of Guralnick, Lawther and Malle [GM02], [GM04], [GLM07]. Importantly for applications in this work, even when G is not almost quasisimple, we have good idea of the structure of groups which have a strongly p-embedded subgroup and a 2F-module which admits a quadratically acting element.

First we introduce two groups that have associated $\mathrm{GF}(p)$-modules which exhibit 2F-action and arise heavily in the local actions in later chapters. In addition, we provide some "characterizations" of these groups, and some structural properties of the groups and the associated 2F-module we are interested in.

Lemma 2.3.14. *There is a unique group G of shape $(3 \times 3) : 2$ which has a faithful quadratic $2F$-module V, namely the generalized dihedral group of order 18. Moreover, for $S \in \mathrm{Syl}_2(G)$ and V an associated faithful quadratic $2F$-module, the following hold:*

(i) $|V| = 2^4$ *and G is unique up to conjugacy in* $\mathrm{GL}_4(2)$*;*

(ii) $\{G, \mathrm{Dih}(18)\} = \{H \mid |H| = 18, O_2(H) = \{1\}$ *and* $H = O^{2'}(H)\}$*;*

(iii) *there are exactly four overgroups of S in G which are isomorphic to* $\mathrm{Sym}(3)$*, any two of which generate G; and*

(iv) $C_{\mathrm{GL}_4(2)}(G) = \{1\}$ *and* $|\mathrm{Out}_{\mathrm{GL}_4(2)}(G)| = 4$.

Proof. This follows directly from calculations in MAGMA, working explicitly with matrices in $\mathrm{GL}_4(2)$ and comparing with the Small Groups Library. □

Indeed, in the above lemma G is also isomorphic to $\mathrm{PSU}_3(2)'$ and is listed in the Small Groups Library as $SmallGroup(18, 4)$.

Lemma 2.3.15. *There is a unique group G of shape $(Q_8 \times Q_8) : 3$ which has a faithful quadratic $2F$-module V. Moreover, for $S \in \mathrm{Syl}_3(G)$ and V an associated faithful quadratic $2F$-module, the following hold:*

(i) $|V| = 3^4$ *and G is determined uniquely up to conjugacy in* $\mathrm{GL}_4(3)$*;*

(ii) *G is the unique group of order $2^4.3$ or $2^6.3$ such that $O_3(G) = \{1\}$, $Z(G) \neq \{1\}$, $G = O^{3'}(G)$ and, if the order is $2^6.3$, there exists at least two distinct normal subgroups of G of order* 8*;*

(iii) *there are exactly five overgroups of S in G which are isomorphic to* $\mathrm{SL}_2(3)$*, any two of which generate G;*

(iv) $N_{O_2(G)}(S) = Z(G) \cong 2 \times 2$*;*

(v) $\mathrm{Aut}(G) = \mathrm{Aut}_{\mathrm{GL}_4(3)}(G)$, $C_{\mathrm{GL}_4(3)}(G) = Z(G)$ *and* $|\mathrm{Out}(G)| = 2^2.3$*; and*

(vi) *if $U < V$ is $N_G(S)$-invariant and $|U| = 3$, then* $|\langle U^G \rangle| = 9$.

Proof. This follows directly from calculations in MAGMA, working explicitly with matrices in $\mathrm{GL}_4(3)$ and comparing with the Small Groups Library. □

The above group is listed in the Small Groups Library as $\mathit{SmallGroup}(192, 1022)$.

We now give an important characterization of certain "small" groups which have an associated non-trivial quadratic 2F-module. The proof of this result will be broken up over a series of lemmas.

Lemma 2.3.16. *Assume that $G = O^{p'}(G)$ is a $\mathcal{K}$-group that has a strongly p-embedded subgroup, $S \in \mathrm{Syl}_p(G)$, V is a faithful $\mathrm{GF}(p)$-module with $C_V(O^p(G)) = \{1\}$ and $V = \langle C_V(S)^G \rangle$. Furthermore, assume that $m_p(S) \geqslant 2$*

and $O_{p'}(G) \leq Z(G)$*. If there is a* p*-element* $1 \neq x \in S$ *such that* $[V, x, x] = \{1\}$ *and* $|V/C_V(x)| = p^2$ *then either:*

(i) p *is odd,* $G = L \cong (\mathrm{P})\mathrm{SU}_3(p)$ *and* V *is the natural module;*

(ii) p *is arbitrary,* $G \cong \mathrm{SL}_2(p^2)$ *and* V *is the natural module; or*

(iii) $p = 2$, $G = L \cong \mathrm{PSL}_2(4)$ *and* V *is a natural* $\Omega_4^-(2)$*-module.*

Proof. Applying the characterization in Proposition 2.1.25 and using Lemma 2.3.5 when p is odd, we deduce that G is a quasisimple group and $G/Z(G)$ is isomorphic to a simple rank 1 group of Lie type. It follows now from Lemma 2.3.4 that $G \cong \mathrm{SL}_2(p^{n+1}), (\mathrm{P})\mathrm{SU}_3(p^n)$ or $\mathrm{Sz}(2^{2n+1})$ for $n \geqslant 1$, and by [DS85, (5.10)] we may assume that $x \in \Omega(Z(S))$. Then, applying Lemma 2.2.1 (iv), (v), Lemma 2.2.2 (vi), (vii) and Lemma 2.2.3 (vi), we have that G is generated by three, four or three conjugates of x respectively and as $|V/C_V(x)| = p^2$, we infer that that $|V| \leqslant p^6, p^8$ and 2^6 respectively. Since the minimal degree of a $\mathrm{GF}(p)$-representation is $2(n+1)$, $6n$ or $4(2n+1)$ respectively, we deduce that $G \cong (\mathrm{P})\mathrm{SU}_3(p)$ and $p \geqslant 3$; or $G \cong \mathrm{SL}_2(p^2)$. In the former case, since $(\mathrm{P})\mathrm{SU}_3(p)$ is generated by three conjugates of x by Lemma 2.2.2 (vi), it follows that $|V| \leqslant p^6$ so that V is a natural module and (i) holds. In the latter case, since $\mathrm{SL}_2(p^2)$ is generated by at most three conjugates, $|V| \leqslant p^6$ and comparing with Lemma 2.3.12, there is a unique irreducible constituent within V, and as V is admits quadratic action, this constituent is a natural $\mathrm{SL}_2(p^2)$-module, or a natural $\Omega_4^-(2)$-module when $p = 2$. Then using that $C_V(O^p(G)) = \{1\}$, Lemma 2.3.2 implies that $V = [V, O^p(G)]$ is irreducible, yielding outcomes (ii) and (iii). □

Lemma 2.3.17. *Assume that* $G = O^{p'}(G)$ *is a* $\mathcal{K}$*-group,* $S \in \mathrm{Syl}_p(G)$*,* V *is a faithful* $\mathrm{GF}(p)$*-module with* $C_V(O^p(G)) = \{1\}$ *and* $V = \langle C_V(S)^G \rangle$*. Furthermore,* *assume that* $m_p(S) = 1$, $N_G(S) = N_G(\Omega(Z(S)))$ *is strongly* p*-embedded in* G*, and* G *is not* p*-solvable. If there is a* p*-element* $1 \neq x \in S$ *such that* $[V, x, x] = \{1\}$ *and* $|V/C_V(x)| = p^2$ *then either:*

(i) $p = 3$, $G = L \cong 2 \cdot \mathrm{Alt}(5)$ *or* $2^{1+4}_{-}.\mathrm{Alt}(5)$ *and* V *is the unique irreducible quadratic* $2F$*-module of dimension* 4*; or*

(ii) p *is arbitrary,* $G = L \cong \mathrm{SL}_2(p)$ *and* V *is the direct sum of two natural* $\mathrm{SL}_2(p)$*-modules.*

Proof. Suppose first that $p = 2$. Applying Proposition 2.1.24, we deduce that S is generalized quaternion and $G = O_{2'}(G)C_G(\Omega(S))$. But now, $C_G(\Omega(S)) = N_G(\Omega(Z(S))) = N_G(S)$ is solvable so that G itself is solvable, a contradiction to the initial hypothesis. Hence, p is odd. Applying Lemma 2.3.5 and using that G is not p-solvable, we deduce that for $L := \langle x^G \rangle$, $L/C_L(U) \cong \mathrm{SL}_2(p)$ for $p \geqslant 5$, $2 \cdot \mathrm{Alt}(5)$ or $2^{1+4}_{-}.\mathrm{Alt}(5)$ for U some non-trivial irreducible constituent of $V|_L$. Indeed, applying Proposition 2.1.24, $G = L$ and $C_G(U)$ is a p'-group. Now, by coprime action $V = C_V(C_G(U)) \times [V, C_G(U)]$ and $U \leq C_V(C_G(U))$. Applying Lemma 2.3.10, if $2 \cdot \mathrm{Alt}(5)$ or $2^{1+4}_{-}.\mathrm{Alt}(5)$ when $p = 3$, we have that $|U/C_U(s)| = 3^2$ so that $[V, C_G(U)] \leq C_V(s)$ so that $[V, C_G(U)] \leq C_V((G)) = \{1\}$ and as V is a faithful module, $C_G(U) = \{1\}$. Indeed, by Lemma 2.3.2 and using that $C_V(G) = \{1\}$, $V = U$ is an irreducible module and outcome (i) holds.

Hence, we may assume that $G/C_G(U) \cong \mathrm{SL}_2(p)$ and $p \geqslant 5$. Then $C_V(C_G(U))$ is a quadratic module for $G/C_G(U)$ and Lemma 2.3.11 and using that $C_V(G) = \{1\}$, $C_V(C_G(U))$ is a direct sum of at most two natural $\mathrm{SL}_2(p)$-modules. Suppose first that $C_V(C_G(U))$ is a natural $\mathrm{SL}_2(p)$-modules so that $U = C_V(C_G(U))$ and $|U/C_U(s)| = p$. Then $|[V, C_G(U)]/C_{[V, C_G(U)]}(s)| = p$ and applying

Lemma 2.3.10, we deduce that $G/C_G([V, C_G(U)]) \cong \mathrm{SL}_2(p)$ and $[V, C_G(U)]$ is a natural $\mathrm{SL}_2(p)$-module. Since $[V, C_G(U)]$ is acted upon non-trivially by $C_G(U)$ and $C_G(U)$ is a $p'-$group, we conclude that $C_G([V, C_G(U)])C_G(U)/C_G(U) = Z(G/C_G(U))$, $C_G([V, C_G(U)])C_G(U)/C_G([V, C_G(U)]) = Z(G/C_G([V, C_G(U)]))$ and $G/C_G([V, C_G(U)]) \cap C_G(U)$ is a central extension of $\mathrm{PSL}_2(p)$ by a fours group. Since the 2-part of the Schur multiplier of $\mathrm{PSL}_2(p)$ has order 2, G is perfect and $G = O^{p'}(G)$, this is a contradiction. Suppose now that $C_V(C_G(U))$ is a direct sum of two natural $\mathrm{SL}_2(p)$-modules. Then $|C_V(C_G(U))/C_{C_V(C_G(U))}(s)| = p^2$ and we deduce that $[V, C_G(U)] \leq C_V(s)$ so that $[V, C_G(U)] \leq C_V((G)) = \{1\}$ and as V is a faithful module, $C_G(U) = \{1\}$ and outcome (ii) holds. □

Lemma 2.3.18. *Assume that $G = O^{p'}(G)$, $S \in \mathrm{Syl}_p(G)$, V is a faithful $\mathrm{GF}(p)$-module with $C_V(O^p(G)) = \{1\}$ and $V = \langle C_V(S)^G \rangle$. Furthermore, assume that $m_p(S) = 1$, $N_G(S) = N_G(\Omega(Z(S)))$ is strongly p-embedded in G, and G is p-solvable. If there is a p-element $1 \neq x \in S$ such that $[V, x, x] = \{1\}$ and $|V/C_V(x)| = p^2$ then, setting $L := \langle x^G \rangle$, one of the following holds:*

(i) *$p = 2$, $L \cong \mathrm{SU}_3(2)'$, G is isomorphic to a subgroup of $\mathrm{SU}_3(2)$ which contains $\mathrm{SU}_3(2)'$ and V is a natural $\mathrm{SU}_3(2)$-module viewed as an irreducible $\mathrm{GF}(2)G$-module by restriction;*

(ii) *$p = 2$, $L \cong \mathrm{Dih}(10)$, $G \cong \mathrm{Dih}(10)$ or $\mathrm{Sz}(2)$ and V is a natural $\mathrm{Sz}(2)$-module viewed as an irreducible $\mathrm{GF}(2)G$-module by restriction;*

(iii) *$p = 3$, $G = L \cong (Q_8 \times Q_8) : 3$ and $V = V_1 \times V_2$ where V_i is a natural $\mathrm{SL}_2(3)$-module for $G/C_G(V_i) \cong \mathrm{SL}_2(3)$;*

(iv) *$p = 2$, $G = L \cong (3 \times 3) : 2$ and $V = V_1 \times V_2$ where V_i is a natural $\mathrm{SL}_2(2)$-module for $G/C_G(V_i) \cong \mathrm{Sym}(3)$; or*

(v) $p = 2$, $L \cong (3 \times 3) : 2$, $G \cong (3 \times 3) : 4$, V *is irreducible as a* $\mathrm{GF}(2)G$*-module and* $V|_L = V_1 \times V_2$ *where* V_i *is a natural* $\mathrm{SL}_2(2)$*-module for* $L/C_L(V_i) \cong \mathrm{Sym}(3)$.

Proof. Let $L := \langle x^G \rangle$ so that $L = [\Omega(S), O_{p'}(G)]\Omega(S)$ by Proposition 2.1.24. Since $N_G(S) = N_G(\Omega(S))$, we deduce that $G = LS$ so that $O^p(L) = O^p(G) = [\Omega(S), O_{p'}(G)]$ and $C_V(O^p(L)) = \{1\}$. Moreover, any element of S centralizes $\Omega(Z(S)) \in \mathrm{Syl}_p(L)$ but does not centralize L, for otherwise, since S contains a unique subgroup of order p, $[\Omega(Z(S)), L] = \{1\}$ and $\Omega(Z(S)) \trianglelefteq G$. Thus, $S/\Omega(S)$ embeds into $\mathrm{Out}(L)$. Finally, using Lemma 2.3.2, $V = [V, O^p(L)]$ and so both L and V are determined in [Che01, Lemma 4.3]. We examine each of the cases individually, using MAGMA for the explicit calculation in $\mathrm{Out}(L)$.

First, if $L \cong \mathrm{SL}_2(p)$ then it follows from Lemma 2.2.1 (viii) that $\mathrm{Out}_S(L) = \{1\}$, $L = G$ and V is a direct sum of two natural modules. If $L \cong \mathrm{Dih}(10)$ then $\mathrm{Aut}(L) \cong \mathrm{Sz}(2)$ and it follows that $G = \mathrm{Dih}(10)$ or $\mathrm{Sz}(2)$, and V is the restriction of a natural $\mathrm{Sz}(2)$-module to G.

Suppose that $L \cong \mathrm{SU}_3(2)'$. Then a Sylow 2-subgroup of $\mathrm{Aut}(L)$ is isomorphic to a semidihedral group of order 16 and since $m_p(S) = 1$, $|S| \leqslant 8$ and S is either cyclic or quaternion. Moreover, $54 \leqslant |G| \leqslant 216$ and $|G| = 54$ if and only if $G = L \cong \mathrm{SU}_3(2)'$. Suppose that $|G| = 216$ and S is cyclic. Utilizing the small group library in MAGMA, we identify a unique group H such that $\langle \Omega(S)^H \rangle \cong \mathrm{SU}_3(2)'$. But in such a group, $N_H(T) < N_H(\Omega(T))$ for $T \in \mathrm{Syl}_2(H)$, a contradiction to our hypothesis. Employing similar methods when $|G| = 108$, or when $|G| = 216$ and S is quaternion, gives that G is isomorphic to any index 2 subgroup of $\mathrm{SU}_3(2)$ resp. $G \cong \mathrm{SU}_3(2)$. In all cases, V is the restriction of a natural $\mathrm{SU}_3(2)$-module to G.

Suppose that $L \cong (Q_8 \times Q_8) : 3$. Since G acts faithfully on V, of order 3^4, G embeds into $\mathrm{GL}_4(3)$ and since the embedding of L is uniquely determined up to conjugacy in $\mathrm{GL}_4(3)$, it follows that G embeds into its normalizer in $\mathrm{GL}_4(3)$. For H the image of L in $\mathrm{GL}_4(3)$, we have that a Sylow 3-subgroup of $N_{\mathrm{GL}_4(3)}(H)$ is elementary abelian of order 9. Since $m_p(S) = 1$, we have that $G = L$ in this case and V is as described in [Che01, Lemma 4.3].

Finally, suppose that $L \cong (3 \times 3) : 2$. Since G acts faithfully on V, of order 2^4, G embeds into $\mathrm{GL}_4(2)$ and since the embedding of L is uniquely determined up to conjugacy in $\mathrm{GL}_4(2)$, it follows that G embeds into the normalizer of its image. For H the image of L in $\mathrm{GL}_4(2)$, we have that a Sylow 2-subgroup of $N_{\mathrm{GL}_4(2)}(H)$ is a dihedral group of order 8 and there is a unique proper overgroup of H in $N_{\mathrm{GL}_4(2)}(H)$ with a cyclic Sylow 2-subgroup. Moreover, this group is irreducible in $\mathrm{GL}_4(2)$, is defined uniquely up to conjugacy in $\mathrm{GL}_4(2)$ and is isomorphic to any index 2 subgroup of $\mathrm{PSU}_3(2)$. We denote this group $(3 \times 3) : 4$ and it follows that either $G = L \cong (3 \times 3) : 2$ or $G \cong (3 \times 3) : 4$. then V is as given in [Che01, Lemma 4.3]. $\square$

The following proposition is the summation of the previous three lemmas. This situation occurs frequently throughout the later sections of this work.

Proposition 2.3.19. *Assume that $G = O^{p'}(G)$ is a $\mathcal{K}$-group that has a strongly p-embedded subgroup, $S \in \mathrm{Syl}_p(G)$, V is a faithful* GF(p)*-module with $C_V(O^p(G)) = \{1\}$ and $V = \langle C_V(S)^G \rangle$. Furthermore, assume that $N_G(S) = N_G(\Omega(Z(S)))$ and if $m_p(S) \geqslant 2$, assume that $O_{p'}(G) \leq Z(G)$. Suppose that there is a p-element $1 \neq x \in S$ such that $[V, x, x] = \{1\}$ and $|V/C_V(x)| = p^2$. Setting*

$L := \langle x^G \rangle$ one of the following holds:

(i) *p is odd, $G = L \cong \mathrm{(P)SU}_3(p)$ and V is the natural module;*

(ii) *p is arbitrary, $G \cong \mathrm{SL}_2(p^2)$ and V is the natural module;*

(iii) *$p = 2$, $G = L \cong \mathrm{PSL}_2(4)$ and V is a natural $\Omega_4^-(2)$-module;*

(iv) *$p = 3$, $G = L \cong 2 \cdot \mathrm{Alt}(5)$ or $2_-^{1+4}.\mathrm{Alt}(5)$ and V is the unique irreducible quadratic $2F$-module of dimension 4;*

(v) *p is arbitrary, $G = L \cong \mathrm{SL}_2(p)$ and V is the direct sum of two natural $\mathrm{SL}_2(p)$-modules;*

(vi) *$p = 2$, $L \cong \mathrm{SU}_3(2)'$, G is isomorphic to a subgroup of $\mathrm{SU}_3(2)$ which contains $\mathrm{SU}_3(2)'$ and V is a natural $\mathrm{SU}_3(2)$-module viewed as an irreducible $\mathrm{GF}(2)G$-module by restriction;*

(vii) *$p = 2$, $L \cong \mathrm{Dih}(10)$, $G \cong \mathrm{Dih}(10)$ or $\mathrm{Sz}(2)$ and V is a natural $\mathrm{Sz}(2)$-module viewed as an irreducible $\mathrm{GF}(2)G$-module by restriction;*

(viii) *$p = 3$, $G = L \cong (Q_8 \times Q_8) : 3$ and $V = V_1 \times V_2$ where V_i is a natural $\mathrm{SL}_2(3)$-module for $G/C_G(V_i) \cong \mathrm{SL}_2(3)$;*

(ix) *$p = 2$, $G = L \cong (3 \times 3) : 2$ and $V = V_1 \times V_2$ where V_i is a natural $\mathrm{SL}_2(2)$-module for $G/C_G(V_i) \cong \mathrm{Sym}(3)$; or*

(x) *$p = 2$, $L \cong (3 \times 3) : 2$, $G \cong (3 \times 3) : 4$, V is irreducible as a $\mathrm{GF}(2)G$-module and $V|_L = V_1 \times V_2$ where V_i is a natural $\mathrm{SL}_2(2)$-module for $L/C_L(V_i) \cong \mathrm{Sym}(3)$.*

While most of the groups and modules above have been described earlier in this section, we list some properties of the groups and modules occurring in (i) and (ix) above.

Lemma 2.3.20. *Suppose that $G \cong 2 \cdot \mathrm{Alt}(5)$ or $2_-^{1+4}.\mathrm{Alt}(5)$, $S \in \mathrm{Syl}_3(G)$ and V is the associated faithful quadratic $2F$-module. Then $C_V(S) = [V, S]$ has order 3^2 and $V/[V, S]$ and $[V, S]$ are irreducible as $\mathrm{GF}(3)N_G(S)$-modules.*

Proof. This follows directly from calculations in MAGMA, working explicitly with the matrices in $\mathrm{Sp}_4(3)$. □

Lemma 2.3.21. *Suppose that $G \cong (3\times3) : 4$, $S \in \mathrm{Syl}_2(G)$ and V is the associated faithful quadratic $2F$-module. Then the following hold:*

(i) *$[V, S]$ has order 2^3;*

(ii) *$[V, \Omega(S)] = C_V(\Omega(S)) = [V, S, S]$ has order 2^2; and*

(iii) *$C_V(S) = [V, S, \Omega(S)] = [V, \Omega(S), S] = [V, S, S, S]$ has order 2.*

Proof. This follows directly from calculations in MAGMA, working explicitly with the matrices in $\mathrm{GL}_4(2)$. □

Lemma 2.3.22. *Suppose that (G, V) satisfies the hypothesis of Proposition 2.3.19. In addition, assume that V is generated as a $\mathrm{GF}(p)G$-module by an $N_G(S)$-invariant subspace of order p. Then $G \cong \mathrm{PSL}_2(4), \mathrm{Dih}(10), \mathrm{Sz}(2)$, $(3\times3) : 2$ or $(3 \times 3) : 4$ and V is as described in Proposition 2.3.19.*

Proof. We apply Proposition 2.3.19 to get the list of candidates for G and V. By Lemma 2.2.13 (iii), Lemma 2.2.6 (vi) and Lemma 2.3.20, if (G, V) satisfy (i), (ii), (iv) or (vi), then there are no $N_G(S)$-invariant subspaces of order p. By Lemma 2.2.7 and Lemma 2.3.15 (vi), if (G, V) satisfy (v) or (viii) then V is not generated by a subspace of order p. This leaves outcomes (iii), (vii), (ix) and (x), as required. □

We now generalize even further than quadratic or cubic action by investigating the *minimal polynomial* of p-elements in a representation, noticing that in quadratic and cubically acting elements, the minimal polynomial is of degree 2 and 3 respectively. We cannot hope to make such strong statements as in the earlier cases, but for larger primes and solvable groups, we have decent control due to the Hall–Higman theorem.

Theorem 2.3.23 (Hall–Higman Theorem)**.** *Suppose that G is p-solvable group with $O_p(G) = \{1\}$ and V a faithful $\mathrm{GF}(p)$-module for G. If $x \in G$ has order p^n and $[V, x; r] = \{1\}$ then one of the following holds:*

(i) $r = p^n$*;*

(ii) *p is a Fermat prime, the Sylow 2-subgroups of G are non-abelian and $r \geqslant p^n - p^{n-1}$; or*

(iii) *$p = 2$, the Sylow q-subgroups of G are non-abelian for some Mersenne prime $q = 2^m - 1 < 2^n$ and $r \geqslant 2^n - 2^{n-m}$.*

Proof. See [HH56, Theorem B]. □

Whenever $p \geqslant 5$, applying the Hall–Higman theorem to the situation where the group G has a strongly p-embedded subgroup and some associated cubic module, we can characterize G completely. As intimated in Section 2.1, a nice way to impose cubic action, particularly in the amalgam method, is through the use of critical subgroups.

Corollary 2.3.24. *Suppose that $G = O^{p'}(G)$ is a $\mathcal{K}$-group which has a strongly p-embedded subgroup, $S \in \mathrm{Syl}_p(G)$ and V is a faithful $\mathrm{GF}(p)$-module. Suppose that $p \geqslant 5$ and there is $s \in S$ of order p^n such that $[V, s, s, s] = \{1\}$. Then $G \cong (\mathrm{P})\mathrm{SL}_2(p^n)$ or $(\mathrm{P})\mathrm{SU}_3(p^n)$ for any prime $p \geqslant 5$, or $p = 5$, $G \cong 3 \cdot \mathrm{Alt}(6)$ or $3 \cdot \mathrm{Alt}(7)$ and for W some irreducible constituent of V, $|W| \geqslant 5^6$.*

Proof. Suppose first that $m_p(S) = 1$. Then, by [Gor07, I.5.4.10 (ii)], S is cyclic and so we may as well assume that $[V, \Omega(S), \Omega(S), \Omega(S)] = \{1\}$. Suppose first that G is p-solvable. Since $p^n - p^{n-1} = p^{n-1}(p-1) \geqslant 4$, the Hall–Higman theorem implies that $O_p(G) \neq \{1\}$, a contradiction since G has a strongly p-embedded subgroup.

Suppose now that $m_p(S) = 1$ and G is not p-solvable. Since $G = O^{p'}(G)$, by Proposition 2.1.24 we have that $G/O_{p'}(G)$ is a simple group with a cyclic Sylow p-subgroup. Form $X := \Omega(S)O_{p'}(G)$. Then X is a p-solvable group and V is a faithful module for X by restriction. Since $p \geqslant 5$, $p^n - p^{n-1} = p^{n-1}(p-1) \geqslant 4$ and by the Hall–Higman theorem $O_p(X) \neq \{1\}$. In particular, $\Omega(S) \trianglelefteq X$ and $[O_{p'}(G), \Omega(S)] \leq O_{p'}(G) \cap \Omega(S) = \{1\}$. But then, since $G/O_{p'}(G)$ is simple, $[O_{p'}(G), G] = [O_{p'}(G), \langle \Omega(S)^G \rangle] = \{1\}$ and $O_{p'}(G) \leq Z(G)$. Hence, G is a quasisimple group with a cyclic Sylow p-subgroup such that the degree of the minimal polynomial of some p-element is 3. Such groups and their associated modules are determined in [Zal99].

Suppose that $m_p(S) \geqslant 2$ so that $G/O_{p'}(G)$ is determined by Proposition 2.1.25, and let $X = O_{p'}(G)\Omega(Z(S))$. Unless $G/O_{5'}(G) \cong \mathrm{Sz}(32) : 5$, we have that for any

$1 \neq s \in \Omega(S)$, $G = \langle s^G \rangle$. In this case, forming $X := \langle s \rangle O_{p'}(G)$, we have that X acts faithfully on V with s acting cubically, and by the Hall–Higman theorem, $\langle s \rangle \trianglelefteq X$. But then $[s, O_{p'}(G)] \leq \langle s \rangle \cap O_{p'}(G) = \{1\}$. Thus, $[G, O_{p'}(G)] = [\langle s^G \rangle, O_{p'}(G)] = [s, O_{p'}(G)]^G = \{1\}$ and $O_{p'}(G) \leq Z(G)$. Since $G = O^{p'}(G)$ is perfect, G is a perfect central extension of $G/O_{p'}(G)$. If $G/O_{p'}(G)$ is isomorphic to a rank 1 simple group of Lie type in characteristic p, then the result follows from Lemma 2.2.1 (vii) and Lemma 2.2.2 (ix). If $G/O_{p'}(G) \cong \mathrm{Alt}(2p)$ then, as $p \geqslant 5$, G has no faithful modules which witness cubic action by [KZ04]. Hence, by Proposition 2.1.25, we are left with a finite number of perfect p'-central extensions of simple groups. We verify that none of these groups have a faithful module which witness cubic action using MAGMA, although there exists results in the literature which substantiate this claim.

So assume that $G/O_{5'}(G) \cong \mathrm{Sz}(32):5$. Then, for $s \in \Omega(S)$, we have that for $L := \langle s^G \rangle$, $L/O_{5'}(L) \cong \mathrm{Sz}(32)$ and following the reasoning above, we have that $O_{5'}(L) \leq Z(L)$. Since the Schur multiplier of $\mathrm{Sz}(32)$ is trivial and $\mathrm{Sz}(32)$ is perfect, we have that $O^{5'}(L) \cong \mathrm{Sz}(32)$. But $O^{5'}(L)$ acts faithfully on V, with $s \in S \cap L$ acting cubically, and since $\mathrm{Sz}(32)$ has no cubic modules, we have a contradiction. Hence, the result. $\square$

CHAPTER 3

FUSION SYSTEMS

In this chapter, we begin by setting up concepts, terminologies and elementary results related to fusion systems, with an emphasis on saturated fusion systems. All of these results are available in the literature, and we follow the standard conventions there. Then, we provide results which aid in determining automizers of essential subgroups of fusion systems. These results are crucial in the determination of fusion systems in the Main Theorem, as well as Theorem D and Theorem E. While these results are probably well known among those working on fusion systems, some of them do not appear to be formally recorded anywhere and so we take the opportunity here to write them down, along with proofs. Finally in this section, we unearth some exotic fusion systems supported on a Sylow 3-subgroup of the sporadic simple group F_3. One of these exotic systems appears as a configuration when applying the amalgam method later in this work and so, we take time to construct this system here, as well as proving some results about it, to ease presentation in later chapters.

3.1 An Introduction to Fusion Systems

In this section, we set up notation and terminology, and list some properties of fusion systems. The standard references for the study of fusion systems are [AKO11] and [Cra11] and most of what follows may be gleaned from these texts.

Definition 3.1.1. Let G be a finite group with $S \in \mathrm{Syl}_p(G)$. The *fusion category* of G over S, written $\mathcal{F}_S(G)$, is the category with object set $\mathrm{Ob}(\mathcal{F}_S(G)) := \{Q : Q \leq S\}$ and for $P, Q \leq S$, $\mathrm{Mor}_{\mathcal{F}_S(G)}(P,Q) := \mathrm{Hom}_G(P,Q)$, where $\mathrm{Hom}_G(P,Q)$ denotes maps induced by conjugation by elements of G. That is, all morphisms in the category are induced by conjugation by elements of G.

Definition 3.1.2. Let S be a p-group. A fusion system $\mathcal{F}$ over S is a category with object set $\mathrm{Ob}(\mathcal{F}) := \{Q : Q \leq S\}$ and whose morphism set satisfies the following properties for $P, Q \leq S$:

- $\mathrm{Hom}_S(P,Q) \subseteq \mathrm{Mor}_{\mathcal{F}}(P,Q) \subseteq \mathrm{Inj}(P,Q)$; and
- each $\phi \in \mathrm{Mor}_{\mathcal{F}}(P,Q)$ is the composite of an $\mathcal{F}$-isomorphism followed by an inclusion,

where $\mathrm{Inj}(P,Q)$ denotes injective homomorphisms between P and Q. To motivate the group analogy, we write $\mathrm{Hom}_{\mathcal{F}}(P,Q) := \mathrm{Mor}_{\mathcal{F}}(P,Q)$ and $\mathrm{Aut}_{\mathcal{F}}(P) := \mathrm{Hom}_{\mathcal{F}}(P,P)$.

Two subgroups of S are said to be *$\mathcal{F}$-conjugate* if they are isomorphic as objects in $\mathcal{F}$. We write $Q^{\mathcal{F}}$ for the set of all $\mathcal{F}$-conjugates of Q. We say a fusion system is *realizable* if there exists a finite group G with $S \in \mathrm{Syl}_p(G)$ and $\mathcal{F} = \mathcal{F}_S(G)$. Otherwise, the fusion system is said to be *exotic*.

Definition 3.1.3. Let $\mathcal{F}$ be a fusion system on a p-group S. Then $\mathcal{H}$ is a *subsystem* of $\mathcal{F}$, written $\mathcal{H} \leq \mathcal{F}$, on a p-group T if $T \leq S$, $\mathcal{H} \subseteq \mathcal{F}$ as sets and $\mathcal{H}$ is itself a fusion system. Then, for $\mathcal{F}_1, \mathcal{F}_2$ subsystems of $\mathcal{F}$, write $\langle \mathcal{F}_1, \mathcal{F}_2 \rangle$ for the smallest subsystem of $\mathcal{F}$ containing $\mathcal{F}_1$ and $\mathcal{F}_2$.

Following are the most important concepts concerning p-subgroups of a fusion system $\mathcal{F}$, at least for the purposes of this thesis.

Definition 3.1.4. Let $\mathcal{F}$ be a fusion system over a p-group S and let $Q \leq S$. Say that Q is

- *fully $\mathcal{F}$-normalized* if $|N_S(Q)| \geq |N_S(P)|$ for all $P \in Q^{\mathcal{F}}$;
- *fully $\mathcal{F}$-centralized* if $|C_S(Q)| \geq |C_S(P)|$ for all $P \in Q^{\mathcal{F}}$;
- *fully $\mathcal{F}$-automized* if $\mathrm{Aut}_S(Q) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(Q))$;
- *receptive* in $\mathcal{F}$ if for each $P \leq S$ and each $\phi \in \mathrm{Iso}_{\mathcal{F}}(P, Q)$, setting

  $$N_\phi = \{g \in N_S(P) : {}^{\phi}c_g \in \mathrm{Aut}_S(Q)\},$$

  there is $\overline{\phi} \in \mathrm{Hom}_{\mathcal{F}}(N_\phi, S)$ such that $\overline{\phi}|_P = \phi$;
- *S-centric* if $C_S(Q) = Z(Q)$ and *$\mathcal{F}$-centric* if P is S-centric for all $P \in Q^{\mathcal{F}}$;
- *S-radical* if $O_p(\mathrm{Out}(Q)) \cap \mathrm{Out}_S(Q) = \{1\}$;
- *$\mathcal{F}$-radical* if $O_p(\mathrm{Out}_{\mathcal{F}}(Q)) = \{1\}$; or
- *$\mathcal{F}$-essential* if Q is $\mathcal{F}$-centric, fully $\mathcal{F}$-normalized and $\mathrm{Out}_{\mathcal{F}}(Q)$ contains a strongly p-embedded subgroup.

If it is clear which fusion system we are working in, we will refer to subgroups as being fully normalized (centralized, centric etc.) without the $\mathcal{F}$ prefix.

For a fusion system $\mathcal{F}$, we set $\mathcal{E}(\mathcal{F})$ to be the set of essential subgroups of $\mathcal{F}$ and note that essential subgroups of S are fully $\mathcal{F}$-normalized, $\mathcal{F}$-centric, $\mathcal{F}$-radical subgroups by definition. We also remark that any $\mathcal{F}$-radical subgroup is also S-radical.

We mostly care about *saturated* fusion systems as they most closely parallel groups and have the most interesting applications.

Definition 3.1.5. Let $\mathcal{F}$ be a fusion system over a p-group S. Then $\mathcal{F}$ is *saturated* if the following conditions hold:

(i) Every fully $\mathcal{F}$-normalized subgroup is also fully $\mathcal{F}$-centralized and fully $\mathcal{F}$-automized.

(ii) Every fully $\mathcal{F}$-centralized subgroup is receptive in $\mathcal{F}$.

By a theorem of Puig [Pui76], the fusion category of a finite group $\mathcal{F}_S(G)$ is a saturated fusion system.

From this point on, we implicitly assume that the fusion systems we study are *saturated*, although some of the results we describe apply in wider contexts and can even be used to determine whether or not a fusion system is saturated.

Definition 3.1.6. A *local $\mathcal{CK}$-system* is a saturated fusion system $\mathcal{F}$ on a p-group S such that $\mathrm{Aut}_{\mathcal{F}}(P)$ is a $\mathcal{K}$-group for all $P \leq S$.

Local $\mathcal{CK}$-systems provides a means to apply the results from Chapter 2 which relied on a $\mathcal{K}$-group hypothesis. This allows for minimal counterexample arguments in fusion systems and provides a link between fusion systems and the classification of finite simple groups. That is, if G is a finite group which is a counterexample to the classification with $|G|$ minimal subject to these constraints, then $\mathcal{F}_S(G)$ is a local $\mathcal{CK}$-system for $S \in \mathrm{Syl}_p(G)$.

We now present arguably the most important tool in classifying saturated fusion systems. Because of this, we need only investigate the local action on a relatively small number of p-subgroups to obtain a global characterization of a saturated fusion system.

Theorem 3.1.7 (Alperin – Goldschmidt Fusion Theorem)**.** *Let $\mathcal{F}$ be a saturated fusion system over a p-group S. Then*

$$\mathcal{F} = \langle \mathrm{Aut}_{\mathcal{F}}(Q) \mid Q \textit{ is essential or } Q = S \rangle.$$

Proof. See [AKO11, Theorem I.3.5]. □

Along these lines, another important notion is for a p-subgroup to be *normal* in a saturated fusion system.

Definition 3.1.8. Let $\mathcal{F}$ be a fusion systems over a p-group S and $Q \leq S$. Say that Q is *normal* in $\mathcal{F}$ if $Q \trianglelefteq S$ and for all $P, R \leq S$ and $\phi \in \mathrm{Hom}_{\mathcal{F}}(P, R)$, ϕ extends to a morphism $\overline{\phi} \in \mathrm{Hom}_{\mathcal{F}}(PQ, RQ)$ such that $\overline{\phi}(Q) = Q$.

It may be checked that the product of normal subgroups is itself normal. Thus, we may talk about the largest normal subgroup of $\mathcal{F}$ which we denote $O_p(\mathcal{F})$ (and occasionally refer to as the p-core of $\mathcal{F}$). Further, it follows immediately from the saturation axioms that any subgroup normal in S is fully normalized and fully centralized.

Definition 3.1.9. Let $\mathcal{F}$ be a fusion system over a p-group S and let Q be a subgroup. The *normalizer fusion subsystem* of Q, denoted $N_{\mathcal{F}}(Q)$, is the largest subsystem of $\mathcal{F}$, supported over $N_S(Q)$, in which Q is normal.

It is clear from the definition that if $\mathcal{F}$ is the fusion category of a group G i.e. $\mathcal{F} = \mathcal{F}_S(G)$, then $N_{\mathcal{F}}(Q) = \mathcal{F}_{N_S(Q)}(N_G(Q))$. The following result is originally attributed to Puig [Pui06].

Theorem 3.1.10. *Let $\mathcal{F}$ be a saturated fusion system over a p-group S. If $Q \leq S$ is fully $\mathcal{F}$-normalized then $N_{\mathcal{F}}(Q)$ is saturated.*

Proof. See [AKO11, Theorem I.5.5]. □

Definition 3.1.11. Let $\mathcal{F}$ be a fusion system over a p-group S and $P \leq Q \leq S$. Say that P is $\mathcal{F}$-*characteristic* in Q if $\mathrm{Aut}_{\mathcal{F}}(Q) \leq N_{\mathrm{Aut}(Q)}(P)$.

Plainly, if $Q \trianglelefteq \mathcal{F}$ and P is $\mathcal{F}$-characteristic in Q, then $P \trianglelefteq \mathcal{F}$.

A slightly weaker notion of normality in fusion systems in *strong closure*.

Definition 3.1.12. Let $\mathcal{F}$ be a fusion system over a p-group S. Then Q is strongly closed in $\mathcal{F}$ if $x\alpha \leq Q$ for all $\alpha \in \mathrm{Hom}_{\mathcal{F}}(x, S)$ whenever $x \in Q$.

We now present a link between normal subgroups of a saturated fusion system $\mathcal{F}$ and its essential subgroups.

Proposition 3.1.13. *Let $\mathcal{F}$ be a saturated fusion system over a p-group S. Then Q is normal in $\mathcal{F}$ if and only if Q is contained in each essential subgroup, Q is $\mathrm{Aut}_{\mathcal{F}}(E)$-invariant for any essential subgroup E of $\mathcal{F}$ and Q is $\mathrm{Aut}_{\mathcal{F}}(S)$-invariant.*

Proof. See [AKO11, Proposition I.4.5]. □

As for finite groups, we desire a more global sense of normality in fusion systems, not just restricted to p-subgroups. That is, we are interested in subsystems of a fusion system $\mathcal{F}$ which are *normal*.

Definition 3.1.14. Let $\mathcal{F}$ be a saturated fusion system over a p-group S. A fusion system $\mathcal{E}$ is *weakly normal* in $\mathcal{F}$ if the following conditions hold:

(i) $\mathcal{E}$ is a saturated subsystem of $\mathcal{F}$ over $T \leq S$;

(ii) T is strongly $\mathcal{F}$-closed in S;

(iii) ${}^{\alpha}\mathcal{E} = \mathcal{E}$ for all $\alpha \in \mathrm{Aut}_{\mathcal{F}}(T)$; and

(iv) for each $P \leq T$ and each $\phi \in \mathrm{Hom}_{\mathcal{F}}(P, T)$ there are $\alpha \in \mathrm{Aut}_{\mathcal{F}}(T)$ and $\phi_0 \in \mathrm{Hom}_{\mathcal{E}}(P, T)$ such that $\phi = \alpha \circ \phi_0$.

A fusion system $\mathcal{E}$ is *normal* in $\mathcal{F}$, denoted $\mathcal{E} \trianglelefteq \mathcal{F}$, if $\mathcal{E}$ is weakly normal in $\mathcal{F}$ and each $\alpha \in \mathrm{Aut}_{\mathcal{E}}(T)$ extends to some $\overline{\alpha} \in \mathrm{Aut}_{\mathcal{F}}(TC_S(T))$ which fixes every coset of $Z(T)$ in $C_S(T)$.

Conditions (iii) and (iv) are referred to as the invariance condition and Frattini condition respectively. As one would hope, for a p-subgroup Q, if $Q \trianglelefteq \mathcal{F}$, then $\mathcal{F}_Q(Q) \trianglelefteq \mathcal{F}$. As is the case with groups, we refer to a saturated fusion system as *simple* if it contains no proper non-trivial normal subsystems.

We shall describe some important subsystems associated to a saturated fusion which have a natural analogues in finite group theory. More details on the construction of such subsystems may be found in Section I.7 of [AKO11].

Definition 3.1.15. Let $\mathcal{F}$ be a saturated fusion system on a p-group S. Say a subsystem $\mathcal{E}$ has *index prime to* p in $\mathcal{F}$ if $\mathcal{E}$ is a fusion system on S and $\mathrm{Aut}_{\mathcal{E}}(P) \geq O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P))$ for all $P \leq S$.

Moreover, by [AKO11, Theorem I.7.7], there is a unique minimal saturated fusion system of index prime to p in $\mathcal{F}$ denoted by $O^{p'}(\mathcal{F})$ and $O^{p'}(\mathcal{F})$ is a normal subsystem of $\mathcal{F}$.

Definition 3.1.16. Let $\mathcal{F}$ be a saturated fusion system on a p-group S. Then the *hyperfocal subgroup* $\mathfrak{hyp}(\mathcal{F})$ of $\mathcal{F}$ is defined as

$$\mathfrak{hyp}(\mathcal{F}) := \langle g^{-1}\alpha(g) \mid g \in P \leq S, \alpha \in O^p(\mathrm{Aut}_{\mathcal{F}}(P))\rangle.$$

A subsystem $\mathcal{E}$ has *p-power index* in $\mathcal{F}$ if $\mathcal{E}$ is a fusion system on $T \geq \mathfrak{hyp}(\mathcal{F})$ and $\mathrm{Aut}_{\mathcal{E}}(P) \geq O^p(\mathrm{Aut}_{\mathcal{F}}(P)$ for all $P \leq S$.

Moreover, by [AKO11, Theorem I.7.4], there is a unique minimal fusion subsystem of p-power index in $\mathcal{F}$ denoted by $O^p(\mathcal{F})$, over $\mathfrak{hyp}(\mathcal{F})$, and $O^p(\mathcal{F})$ is a normal subsystem of $\mathcal{F}$.

Definition 3.1.17. A saturated fusion system is *reduced* if $O_p(\mathcal{F}) = \{1\}$ and $\mathcal{F} = O^p(\mathcal{F}) = O^{p'}(\mathcal{F})$.

Naturally, an important consideration in fusion systems is the notion of isomorphism. After defining what isomorphism means in the context of fusion systems, it follows readily that the "sensible" properties hold, which we state below.

Definition 3.1.18. Let $\mathcal{F}$ be a fusion system on a p-group S and $\mathcal{E}$ a fusion system on a p-group T. A *morphism* $\phi : \mathcal{F} \to \mathcal{E}$ is a tuple $(\phi_S, \phi_{P,Q} \mid P, Q \leq S)$ such that $\phi_S : S \to T$ is a group homomorphism and $\phi_{P,Q} : \mathrm{Hom}_{\mathcal{F}}(P, Q) \to \mathrm{Hom}_{\mathcal{E}}(P\phi, Q\phi)$

is such that $\alpha\phi_S = \phi_S(\alpha\phi_{P,Q})$ for all $\alpha \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$.

Say that ϕ is *injective* if $\phi_S \colon S \to T$ is injective, and ϕ is *surjective* if ϕ_S is surjective and, for all $P, Q \leq S$, $\phi_{P_0,Q_0} \colon \mathrm{Hom}_{\mathcal{F}}(P_0, Q_0) \to \mathrm{Hom}_{\mathcal{E}}(P\phi, Q\phi)$ is surjective, where P_0, Q_0 denote the preimages in S of $P\phi, Q\phi$. Then, ϕ is an *isomorphism* of fusion systems if $\phi : \mathcal{F} \to \mathcal{E}$ is an injective, surjective morphism.

Lemma 3.1.19. *Let $G \cong H$ be finite groups with $S \in \mathrm{Syl}_p(G)$ and $T \in \mathrm{Syl}_p(H)$. Then $\mathcal{F}_S(G) \cong \mathcal{F}_T(H)$.*

Lemma 3.1.20. *Let $\mathcal{F} = \mathcal{F}_S(G)$ be a saturated fusion system and set $\overline{G} = G/O_{p'}(G)$. Then $\mathcal{F}_S(G) \cong \mathcal{F}_{\overline{S}}(\overline{G})$.*

In order to investigate the local actions in a saturated fusion system, and in particular in its normalizer subsystems, it will often be convenient to work in a purely group theoretic context. The *model theorem* guarantees that we may do this for a certain class of p-subgroups of a saturated fusion system $\mathcal{F}$.

Theorem 3.1.21 (Model Theorem)**.** *Let $\mathcal{F}$ be a saturated fusion system over a p-group S. Fix $Q \leq S$ which is $\mathcal{F}$-centric and normal in $\mathcal{F}$. Then the following hold:*

(i) *There are models for $\mathcal{F}$.*

(ii) *If G_1 and G_2 are two models for $\mathcal{F}$, then there is an isomorphism $\phi : G_1 \to G_2$ such that $\phi|_S = \mathrm{Id}_S$.*

(iii) *For any finite group G containing S as a Sylow p-subgroup such that $Q \leq G$, $C_G(Q) \leq Q$ and $Aut_G(Q) = Aut_{\mathcal{F}}(Q)$, there is $\beta \in \mathrm{Aut}(S)$ such that $\beta|_Q = \mathrm{Id}_Q$ and $\mathcal{F}_S(G) = {}^{\beta}\mathcal{F}$. Thus, there is a model for $\mathcal{F}$ which is isomorphic to G.*

Proof. See [AKO11, Theorem I.4.9]. □

Fusion systems satisfying the hypothesis of the above theorem are referred to as *constrained* fusion systems. It is clear that if E is an essential subgroup of $\mathcal{F}$, E is a centric normal subgroup of $N_{\mathcal{F}}(E)$, $N_{\mathcal{F}}(E)$ is constrained and there is a model G for $N_{\mathcal{F}}(E)$ with $O_p(G) = E$.

We record two further results regarding the saturation of fusion systems. The first describes a situation in which a certain class of essentials are excised out. This has been referred to as "pruning" in the literature.

Lemma 3.1.22. *Suppose that $\mathcal{F}$ is a saturated fusion system on S and P is an $\mathcal{F}$-essential subgroup of S. Let $\mathcal{C}$ be a set of $\mathcal{F}$-class representatives of $\mathcal{F}$-essential subgroups with $P \in \mathcal{C}$. Assume that if $Q < P$ then Q is not S-centric. Letting $H_{\mathcal{F}}(P)$ be the subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$ which is generated by $\mathcal{F}$-automorphisms of P which extend to $\mathcal{F}$-isomorphisms between strictly larger subgroups of S, if $H_{\mathcal{F}}(P) \leq K \leq \mathrm{Aut}_{\mathcal{F}}(P)$, then $\mathcal{G} = \langle \mathrm{Aut}_{\mathcal{F}}(S), K, \mathrm{Aut}_{\mathcal{F}}(E) \mid E \in \mathcal{C} \setminus \{P\} \rangle$ is saturated.*

Proof. See [PS21, Lemma 6.4]. □

We now provide the results promising the opposite situation, where one can append suitably small essential subgroups to a saturated fusion system, while maintaining saturation.

Theorem 3.1.23. *Let $\mathcal{F}_0$ be a saturated fusion system on a finite p-group S. Let $V \leq S$ be a fully $\mathcal{F}_0$-normalized subgroup, set $H = \mathrm{Out}_{\mathcal{F}_0}(V)$ and let $\widetilde{\Delta} \leq \mathrm{Out}(V)$ be such that H is a strongly p-embedded subgroup of $\widetilde{\Delta}$. For Δ the full preimage of $\widetilde{\Delta}$ in $\mathrm{Aut}(V)$, write $\mathcal{F} = \langle \mathcal{F}_0, \Delta \rangle$. Assume further that*

(i) *V is $\mathcal{F}_0$-centric and minimal under inclusion amongst all $\mathcal{F}$-centric subgroups; and*

(ii) *no proper subgroup of V is $\mathcal{F}_0$-essential.*

Then $\mathcal{F}$ is saturated.

Proof. See [Sem14, Theorem C]. □

3.2 Controlling Automizers of Essential Subgroups

With the aim of applying the Alperin–Goldschmidt fusion theorem, we present the following lemmas which provide the main tools for determining whether a p-group is an essential subgroup of saturated fusion system $\mathcal{F}$.

Lemma 3.2.1. *Let S be a p-group, $E \leq S$ and $A \leq \mathrm{Aut}(E)$. Set $\{1\} = E_0 \trianglelefteq E_1 \trianglelefteq E_2 \trianglelefteq \ldots \trianglelefteq E_m = E$ such that, for all $0 \leq i \leq m$, $E_i\alpha = E_i$ for each $\alpha \in A$. Let $Q \leq \mathrm{Aut}_S(E)$ with the property $[Q, E_i] \leq E_{i-1}$ for all $1 \leq i \leq m$.*

(i) *If $A = \mathrm{Aut}(E)$ and E is S-radical, then $Q \leq \mathrm{Inn}(E)$.*

(ii) *If $\mathcal{F}$ is a saturated fusion system on S, E is $\mathcal{F}$-radical and $\mathrm{Aut}_{\mathcal{F}}(E) \leq A$, then $Q \leq \mathrm{Inn}(E)$.*

Proof. We apply Lemma 2.1.9 to E, Q and A to deduce that in both (i) and (ii), $Q \leq O_p(A) \cap \mathrm{Aut}_S(E)$. In (i), since E is S-radical, it follows directly from the definition that $Q \leq \mathrm{Inn}(E)$. In (ii), we have that $O_p(A) \leq \mathrm{Aut}_S(E)$ and $O_p(A)$ is normalized by $\mathrm{Aut}_{\mathcal{F}}(E)$. Thus, $Q \leq O_p(A) \leq O_p(\mathrm{Aut}_{\mathcal{F}}(E)) = \mathrm{Inn}(E)$ since E is $\mathcal{F}$-radical, and the result holds. $\square$

Lemma 3.2.2. *Suppose that $\mathcal{F}$ is a saturated fusion system and E is an essential subgroup. Assume that $\mathrm{Aut}_{\mathcal{F}}(E)$ is a $\mathcal{K}$-group. Then $|E/\Phi(E)| \geqslant |\mathrm{Out}_S(E)|^2$.*

Proof. This is [PS21, Proposition 4.8 (4)]. $\square$

Now that we have a way to determine whether a subgroup is essential, in order to make use of the Alperin–Goldschmidt fusion theorem, we must also determine the induced automorphism group by $\mathcal{F}$. The first result along these lines determines the potential automizer $\mathrm{Aut}_{\mathcal{F}}(E)$ of an essential subgroup E whenever some non-central chief factor of E is an FF-module. It is important to note that this theorem does not rely on a $\mathcal{K}$-group hypothesis, and it is essentially the fusion theoretic equivalent of Lemma 2.3.10.

Theorem 3.2.3. *Suppose that E is an essential subgroup of a saturated fusion system $\mathcal{F}$ over a p-group S, and assume that there is an $\mathrm{Aut}_{\mathcal{F}}(E)$-invariant subgroup $V \leq \Omega(Z(E))$ such that V is an FF-module for $G := \mathrm{Out}_{\mathcal{F}}(E)$. Then, writing $L := O^{p'}(G)$, we have that $L/C_L(V) \cong \mathrm{SL}_2(p^n)$, $C_L(V)$ is a p'-group and $V/C_V(O^p(L))$ is a natural $\mathrm{SL}_2(q)$-module.*

Proof. This is [Hen10, Theorem 1.2]. $\square$

Armed with the analysis of groups with strongly p-embedded subgroups from Chapter 2, we now investigate the limitations of $\mathrm{Out}_{\mathcal{F}}(E)$ for E an essential subgroup of $\mathcal{F}$. In our analysis, the most important case of study is that where E is *maximally essential*.

Definition 3.2.4. Suppose that $\mathcal{F}$ is a saturated fusion system on a p-group S. Then $E \leq S$ is *maximally essential* in $\mathcal{F}$ if E is essential and, if $F \leq S$ essential in $\mathcal{F}$ and $E \leq F$, then $E = F$.

Coupled with saturation arguments and the Alperin–Goldschmidt theorem, this definition further limits the possibilities for $\mathrm{Out}_{\mathcal{F}}(E)$.

Lemma 3.2.5. *Let $\mathcal{F}$ be a saturated fusion systems on a p-group S with E a maximally essential subgroup of $\mathcal{F}$. Then $N_{\mathrm{Out}_{\mathcal{F}(E)}}(\mathrm{Out}_S(E))$ is strongly p-embedded in $\mathrm{Out}_{\mathcal{F}}(E)$.*

Proof. Let $T \leq N_S(E)$ with $E < T$. Now, since E is receptive, for all $\alpha \in N_{\mathrm{Aut}_{\mathcal{F}(E)}}(\mathrm{Aut}_T(E))$, α lifts to a morphism $\widehat{\alpha} \in \mathrm{Hom}_{\mathcal{F}}(N_\alpha, S)$ with $N_\alpha > E$. Since E is maximally essential, applying the Alperin–Goldschmidt theorem, $\widehat{\alpha}$ is the restriction of a morphism $\overline{\alpha} \in \mathrm{Aut}_{\mathcal{F}}(S)$. But then, α normalizes $\mathrm{Aut}_S(E)$ and so $N_{\mathrm{Aut}_{\mathcal{F}(E)}}(\mathrm{Aut}_T(E)) \leq N_{\mathrm{Aut}_{\mathcal{F}(E)}}(\mathrm{Aut}_S(E))$.This induces the inclusion $N_{\mathrm{Out}_{\mathcal{F}(E)}}(\mathrm{Out}_T(E)) \leq N_{\mathrm{Out}_{\mathcal{F}(E)}}(\mathrm{Out}_S(E)$. Since this holds for all $T \leq N_S(E)$ with $E < T$, we infer that $N_{\mathrm{Out}_{\mathcal{F}(E)}}(\mathrm{Out}_S(E))$ is strongly p-embedded in $\mathrm{Out}_{\mathcal{F}}(E)$, as required. □

As in the earlier analysis of groups with strongly p-subgroups, we divide into two cases, where $m_p(\mathrm{Out}_S(E)) = 1$ or $m_p(\mathrm{Out}_S(E)) \geqslant 2$.

Proposition 3.2.6. *Let $\mathcal{F}$ be a saturated fusion systems on a p-group S with E a maximally essential subgroup of $\mathcal{F}$, and set $G = \mathrm{Out}_{\mathcal{F}}(E)$. If $m_p(G) = 1$ then either*

(i) $\mathrm{Out}_S(E)$ *is cyclic or generalized quaternion and*

$$\begin{aligned} O^{p'}(G) &= \mathrm{Out}_S(E)[O_{p'}(O^{p'}(G)), \Omega(\mathrm{Out}_S(E))] \\ &= \mathrm{Out}_S(E)\langle \Omega(\mathrm{Out}_S(E))^{O^{p'}(G)} \rangle \end{aligned}$$

is p*-solvable; or*

(ii) $O^{p'}(G)/O_{p'}(O^{p'}(G))$ *is a non-abelian simple group,* p *is odd and* $\mathrm{Out}_S(E)$ *is cyclic.*

Proof. Since G has a strongly p-embedded subgroup, so does $O^{p'}(G)$ and we apply Proposition 2.1.24 and (ii) follows immediately. In the other cases of Proposition 2.1.24, since $\Omega(\mathrm{Out}_S(E))[O_{p'}(O^{p'}(G)), \Omega(\mathrm{Out}_S(E))] \trianglelefteq O^{p'}(G)$, by the Frattini argument,

$$\begin{aligned} O^{p'}(G) &= N_{O^{p'}(G)}(\Omega(\mathrm{Out}_S(E)))[O_{p'}(O^{p'}(G)), \Omega(\mathrm{Out}_S(E))] \\ &= N_{O^{p'}(G)}(\Omega(\mathrm{Out}_S(E)))\langle \Omega(\mathrm{Out}_S(E))^{O^{p'}(G)} \rangle. \end{aligned}$$

Since E is maximally essential, applying Lemma 3.2.5, $N_{O^{p'}(G)}(\Omega(\mathrm{Out}_S(E))) \leq N_G(\Omega(\mathrm{Out}_S(E)))$ so that $N_{O^{p'}(G)}(\Omega(\mathrm{Out}_S(E))) = N_G(\mathrm{Out}_S(E))$. But then $\mathrm{Out}_S(E)[O_{p'}(O^{p'}(G)), \Omega(\mathrm{Out}_S(E))] \trianglelefteq O^{p'}(G)$ and by the definition of $O^{p'}(G)$, we have that $O^{p'}(G) = \mathrm{Out}_S(E)[O_{p'}(O^{p'}(G)), \Omega(\mathrm{Out}_S(E))]$. $\square$

Proposition 3.2.7. *Let* $\mathcal{F}$ *be a local* $\mathcal{CK}$*-system on a* p*-group* S *and let* E *be an essential subgroup of* $\mathcal{F}$*. Suppose further that* E *is maximal by inclusion with respect to this property. Set* $G = \mathrm{Out}_{\mathcal{F}}(E)$*. If* $m_p(G) \geqslant 2$ *then* $O^{p'}(G)$ *is isomorphic to a central extension by a group of* p'*-order of one of the following groups:*

(i) $\mathrm{PSL}_2(p^{a+1})$ *or* $\mathrm{PSU}_3(p^b)$ *for* p *arbitrary,* $a \geqslant 1$ *and* $p^b > 2$*;*

(ii) $\mathrm{Sz}(2^{2a+1})$ *for* $p = 2$ *and* $a \geqslant 1$*;*

(iii) $\mathrm{Ree}(3^{2a+1}), \mathrm{PSL}_3(4)$ *or* M_{11} *for* $p = 3$ *and* $a \geqslant 0$*;*

(iv) $\mathrm{Sz}(32) : 5, {}^2\mathrm{F}_4(2)'$ *or* McL *for* $p = 5$*; or*

(v) J_4 *for* $p = 11$.

Furthermore, either $O^{p'}(G)$ *is a perfect central extension, or* $O^{p'}(G) \cong \mathrm{Ree}(3)$ *resp.* $\mathrm{Sz}(32) : 5$ *and* $p = 3$ *resp.* $p = 5$.

Proof. Set $\widetilde{G} = G/O_{p'}(G)$ and $K = O^{p'}(G)$. By Lemma 3.2.5, $N_G(\mathrm{Out}_S(E))$ is strongly p-embedded in G. In particular, we deduce that $N_K(\mathrm{Out}_S(E))$ is strongly p-embedded in K. Let $A \leq \mathrm{Out}_S(E)$ be elementary abelian of order p^2. By coprime action, $O_{p'}(K) = \langle C_{O_{p'}(K)}(a) \mid a \in A^\# \rangle$. Since $N_K(\mathrm{Out}_S(E))$ is strongly p-embedded in K, we have that $O_{p'}(K) \leq N_K(\mathrm{Out}_S(E))$ so that $[O_{p'}(K), \mathrm{Out}_S(E)] = \{1\}$. Then

$$[O_{p'}(K), K] = [O_{p'}(K), \langle \mathrm{Out}_S(E)^K \rangle] = [O_{p'}(K), \mathrm{Out}_S(E)]^K = \{1\}$$

and $O_{p'}(K) \leq Z(K)$.

Now, $\widetilde{K} \cong K/O_{p'}(K)$ is determined as in Proposition 2.1.25. Moreover, $\widetilde{N_K(\mathrm{Out}_S(E))} = N_{\widetilde{K}}(\widetilde{\mathrm{Out}_S(E)}))$ is strongly p-embedded in $\widetilde{K}$ and applying [GLS98, Theorem 7.6.2], $\widetilde{K} \ncong \mathrm{Alt}(2p)$ or Fi_{22}. Unless $\widetilde{K} \cong \mathrm{Ree}(3)$ or $\mathrm{Sz}(32) : 5$, using that $\widetilde{K}$ is simple and $K = O^{p'}(K)$, K is perfect central extension of $\widetilde{K}$ by a group of p'-order. If $\widetilde{K} \cong \mathrm{Ree}(3)$ or $\mathrm{Sz}(32) : 5$ then $O^{p'}(O^p(K))$ is a perfect central extension of $\mathrm{Ree}(3)' \cong \mathrm{PSL}_2(8)$ resp. $\mathrm{Sz}(32)$ by a p'-group so that $O^{p'}(O^p(K)) \cong \mathrm{PSL}_2(8)$ resp. $\mathrm{Sz}(32)$. Since $O_{p'}(K) \leq N_K(\mathrm{Out}_S(E))$ and $K = O^{p'}(O^p(K))O_{p'}(K)\mathrm{Out}_S(E)$, we conclude that $O_{p'}(K) = \{1\}$ and $K = O^{p'}(K) = O^{p'}(O^p(K))\mathrm{Out}_S(E) \cong \mathrm{Ree}(3)$ resp. $\mathrm{Sz}(32):5$. $\square$

As intimated in the introduction, a valid question to consider is whether the requirement that E be maximally essential in the Main Theorem is truly necessary. Observe that this condition implies that $N_{\mathrm{Out}_{\mathcal{F}}(E)}(\mathrm{Out}_S(E))$ is strongly p-embedded in $\mathrm{Out}_{\mathcal{F}}(E)$. We begin this discussion with a somewhat trivial example.

Example 3.2.8. *Let V be a 4-dimensional vector space over $\mathrm{GF}(2)$ and let $\mathrm{Dih}(10)$ act irreducibly on it. In its embedding in $\mathrm{GL}_4(2)$, $\mathrm{Dih}(10)$ is centralized by a 3-element and so we may form a subgroup of $\mathrm{GL}_4(2)$ of shape $\mathrm{Dih}(10) \times 3$. This group is normalized by an element t of order 4 such that $\langle \mathrm{Dih}(10), t\rangle \cong \mathrm{Sz}(2)$, $t^2 \in \mathrm{Dih}(10)$ and t inverts the 3-element which centralizes $\mathrm{Dih}(10)$. Thus, we may construct a group H of shape $\mathrm{Dih}(10).\mathrm{Sym}(3)$ in $\mathrm{GL}_4(2)$. Form the semidirect product $G := V \rtimes H$ and consider the 2-fusion category of G over some Sylow 2-subgroup S. Since H has cyclic Sylow 2-subgroups and $O_2(H) = \{1\}$, we have that V is essential in the 2-fusion category of G. Moreover, for s the unique involution in $H \cap S$, we have that $E := V\langle s\rangle$ has order 2^5 and $N_G(E)/E \cong \mathrm{Sym}(3)$. Therefore, E is also an essential subgroup which properly contains another essential subgroup V.*

It is easy to construct other examples in which smaller essentials are contained in some larger essential, even when imposing the condition the the essential subgroups are $\mathrm{Aut}_{\mathcal{F}}(S)$-invariant. But it is reasonable to ask whether such examples actually occur in an amalgam setting motivated by the hypothesis of the Main Theorem.

To this end, let E be an $\mathrm{Aut}_{\mathcal{F}}(S)$-invariant essential subgroup of a saturated fusion system $\mathcal{F}$ on a p-group S, let G be a model for $N_{\mathcal{F}}(E)$ and suppose that $\Omega(Z(S)) \ntrianglelefteq G$. In the midst of the amalgam method, to determine $\mathrm{Out}_{\mathcal{F}}(E)$ and its actions, we work "from the bottom up" by determining $\mathrm{Out}_{\mathcal{F}}(E)$-chief factors of E, starting with those in $\langle \Omega(Z(S))^G \rangle$ and taking progressively larger subgroups of E, so working "up." Taking the above example as inspiration, one might imagine a situation in which $\mathrm{Out}_{\mathcal{F}}(E)$ induces a $\mathrm{Sym}(3)$-action on almost all $\mathrm{Out}_{\mathcal{F}}(E)$-chief factors in E. Without examining an ever increasing sequence of subgroups and chief factors, it may be hard to eventually uncover a chief factor which witnesses non-trivial action by a 5-element (although this would probably only happen for amalgams with large "critical distance", see Notation 5.2.5, and even then it seems unlikely). It seems some additional tricks and techniques (or perhaps an even more granular case division) are required to treat these types of examples.

3.3 Exotic Fusion Systems on a Sylow 3-subgroup of F_3

In this section, we describe some exotic fusion systems supported on a Sylow 3-subgroup of F_3. One of these systems appear in the conclusion of the Main Theorem, and we focus effort on constructing this system and proving its exoticity here so as to not impede the exposition later. Throughout, we require some Lie theoretic terminology and refer to [Car89] or [GLS98] for the appropriate definitions.

For some structural results concerning S and its internal actions, we appeal to the Atlas [Con+85]. We begin by noting the following 3-local maximal subgroups of

F_3:

$$M_1 \cong 3^{2+3+2+2} : \mathrm{GL}_2(3)$$

$$M_2 \cong 3^{1+2+1+2+1+2} : \mathrm{GL}_2(3)$$

$$M_3 \cong 3^5 : \mathrm{SL}_2(9).2$$

remarking that $|S| = 3^{10}$. We set $E_i = O_3(M_i)$ and compute (e.g. using MAGMA) that $E_1 = C_S(Z_2(S)) = J(S)$ and $E_2 = C_S(Z_3(S)/Z(S))$ are characteristic subgroups of S, and so are $\mathrm{Aut}_{\mathcal{F}}(S)$-invariant in any fusion system $\mathcal{F}$ on S. Indeed, the above list exhausts all essential subgroups of the 3-fusion category of F_3.

Proposition 3.3.1. *Let $\mathcal{F} = \mathcal{F}_S(F_3)$. Then $\mathcal{F}^{frc} = \{E_1, E_2, E_3^S, S\}$. In particular, $\mathcal{E}(\mathcal{F}) = \{E_1, E_2, E_3^S\}$.*

Proof. This follows from [Wil88]. □

Lemma 3.3.2. *Every $\mathcal{G}$-conjugate of E_3 is contained in E_1 and not contained in E_2.*

Proof. Since $\{E_3^{\mathcal{F}}\} = \{E_3^S\}$ and both E_1 and E_2 are normal in S, it suffices to show that $E_3 \leq E_1$ and $E_3 \not\leq E_2$. To this end, we note that $[Z_2(S), E_3] = \{1\}$. One can see this in the 3-fusion category of F_3 for otherwise, since E_3 is elementary abelian, we would have that $Z_2(S) \not\leq E_3$ and $[Z_2(S), E_3] \leq Z(S)$, a contradiction since $\mathrm{Out}_{F_3}(E_3) \cong \mathrm{SL}_2(9).2$ has no non-trivial modules exhibiting this behaviour. If $E_3 \leq E_2$, then since $E_2 \trianglelefteq S$, we would have that $E_1 = \langle E_3^S \rangle \leq E_2$, a clear contradiction. Thus, $E_3 \not\leq E_2$. □

Throughout the remainder of this section, we set $\mathcal{G}$ to be the 3-fusion category of F_3 so that $\mathcal{E}(\mathcal{G}) = \{E_1, E_2, E_3^S\}$. Set $\mathcal{H} = \langle \mathrm{Aut}_{\mathcal{G}}(E_1), \mathrm{Aut}_{\mathcal{G}}(E_2)\rangle$ and $\mathcal{D} = \langle \mathrm{Aut}_{\mathcal{G}}(E_1), \mathrm{Aut}_{\mathcal{G}}(E_3)\rangle$.

We now prove that the fusion system $\mathcal{H}$ is exotic. There is no known way to do this without invoking the classification of finite simple groups. This is also the case for the fusion systems supported on a Sylow 7-subgroup of $\mathrm{G}_2(7)$ mentioned in the Main Theorem and Theorem D.

Proposition 3.3.3. *$\mathcal{H}$ is a saturated simple exotic fusion system with $\mathcal{H}^{frc} = \{E_1, E_2, S\}$.*

Proof. That $\mathcal{H}$ is saturated follows immediately from Lemma 3.1.22. Since $\mathcal{H}$ is a subsystem of $\mathcal{G}$, the deduction of $\mathcal{H}^{frc}$ is straightforward. Assume that $\mathcal{N} \trianglelefteq \mathcal{H}$ and $\mathcal{N}$ is supported on T. Then T is a strongly closed subgroup of $\mathcal{H}$ and we calculate using MAGMA that $S = T$ and $\mathcal{N}$ has index prime to 3 in $\mathcal{H}$ by [AKO11, Lemma I.7.6]. Since $\mathrm{Aut}_{\mathcal{H}}(S)$ is generated by lifted morphisms from $O^{3'}(\mathrm{Aut}_{\mathcal{H}}(E_1))$ and $O^{3'}(\mathrm{Aut}_{\mathcal{H}}(E_2))$, applying [AKO11, Lemma I.7.6], we have that $\mathcal{H} = \mathcal{N}$ is simple.

Suppose that $\mathcal{H} = \mathcal{F}_S(G)$ for some finite group G with $S \in \mathrm{Syl}_3(G)$. We may as well assume that $O_3(G) = O_{3'}(G) = \{1\}$ so that $F^*(G) = E(G)$ is a direct product of non-abelian simple groups, all divisible by 3. Furthermore, since $|\Omega(Z(S))| = 3$, we deduce that $F^*(G)$ is simple and G is an almost simple group. Since $\Omega(Z(S)) \leq F^*(G)$, the action of $\mathrm{Aut}_{\mathcal{G}}(E_1)$ and $\mathrm{Aut}_{\mathcal{G}}(E_2)$ implies that $S \leq \langle \Omega(Z(S))^G\rangle \leq F^*(G)$. In particular, we reduce to searching for simple groups with a Sylow 3-subgroup of order 3^{10} and 3-rank 5. Since E_3 is not normal in S, S does not have a unique elementary abelian subgroup of maximal rank.

If $F^*(G) \cong \mathrm{Alt}(n)$ for some n then $m_3(\mathrm{Alt}(n)) = \lfloor \frac{n}{3} \rfloor$ by [GLS98, Proposition

5.2.10] and so $n < 18$. But a Sylow 3-subgroup of $\mathrm{Alt}(18)$ has order 3^8 and so $F^*(G) \not\cong \mathrm{Alt}(n)$ for any n. If $F^*(G)$ is isomorphic to a group of Lie type in characteristic 3, then comparing with [GLS98, Table 3.3.1], we see that the groups with a Sylow 3-subgroup which has 3-rank 5 are $\mathrm{PSL}_2(3^5)$, $\Omega_7(3)$, ${}^3\mathrm{D}_4(3)$ and $\mathrm{PSU}_5(3)$, and only $\mathrm{PSU}_5(3)$ has a Sylow 3-subgroup of order 3^{10} of these examples. If G is a $3'$-extension of $\mathrm{PSU}_5(3)$, the unipotent radicals of parabolic subgroups of $\mathrm{PSU}_5(3)$ are essential subgroups and since neither has index 3 in a Sylow 3-subgroup, we have shown that $F^*(G)$ is not a group of Lie type of characteristic 3.

Assume now that $F^*(G)$ is a group of Lie type in characteristic $r \neq 3$. By [GLS98, Theorem 4.10.3], S has a unique elementary abelian subgroup of 3-rank 5 unless $F^*(G) \cong \mathrm{G}_2(r^a), {}^2\mathrm{F}_4(r^a), {}^3\mathrm{D}_4(r^a), \mathrm{PSU}_3(r^a)$ or $\mathrm{PSL}_3(r^a)$. Moreover, by [GLS98, Theorem 4.10.2], there is a normal abelian subgroup S_T of S such that S/S_T is isomorphic to a subgroup of the Weyl group of $F^*(G)$. But $|S_T| \leqslant 3^5$ so that $|S/S_T| \geqslant 3^5$. All of the candidate groups above have Weyl group with 3-part strictly less than 3^5 and so $F^*(G)$ is not isomorphic to a group of Lie type in characteristic r.

Finally, checking the orders of the Sporadic groups, we have that F_3 is the unique Sporadic simple group with a Sylow 3-subgroup of order 3^{10}. Since F_3 has trivial outer automorphism group and the 3-fusion category of F_3 has 3 classes of essential subgroups, $F^*(G) \not\cong \mathrm{F}_3$ and $\mathcal{H}$ is exotic. □

Taking G_i to be the model for $N_{\mathcal{G}}(E_i)$, in the above situation the induced amalgam is *parabolic isomorphic* to an F_3-type amalgam. This general idea forms the fundamental concept of this thesis and we refer to Section 5.1 for its initial treatment.

In the following, we calculate normal closures of certain 3-subgroups of S by particular groups of automorphisms induced by $\mathcal{D}$. All of these actions come from $\mathcal{G}$ and the calculations may be performed using MAGMA and the necessary maximal subgroups of F_3.

Lemma 3.3.4. *E_1 is the unique proper non-trivial strongly closed subgroup of $\mathcal{D}$.*

Proof. Since every essential subgroup of $\mathcal{D}$ is contained in E_1, and since E_1 is characteristic in S, we deduce that E_1 is strongly closed in $\mathcal{D}$. Assume that T is any proper non-trivial strongly closed subgroup of $\mathcal{D}$. Then $T \trianglelefteq S$ and so $Z(S) \leq T$ and $Z_2(S) = \langle Z(S)^{\mathrm{Aut}_{\mathcal{D}}(E_1)}\rangle \leq T$. Suppose first that $T \cap \Phi(E_1) = Z_2(S)$. Since $\Phi(E_1) \trianglelefteq S$ we have that $[\Phi(E_1), T] = Z_2(S)$ so that $T \leq E_1$. But then $[E_1, T] \leq \Phi(E_1) \cap T = Z_2(S) = Z(E_1)$ and $T \leq Z_2(E_1) = \Phi(E_1)$ so that $T = Z_2(S)$. However, then $T < \langle T^{\mathrm{Aut}_{\mathcal{D}}(E_3)}\rangle$, a contradiction.

Thus, $T \cap \Phi(E_1) > Z_2(S)$ and since $\mathrm{Out}_{\mathcal{D}}(E_1)$ acts irreducibly on $\Phi(E_1)/Z_2(S)$, we must have that $\Phi(E_1) \leq T$. But now $E_3 = \langle (\Phi(E_1) \cap E_3)^{\mathrm{Aut}_{\mathcal{D}}(E_3)}\rangle \leq \langle (T \cap E_3)^{\mathrm{Aut}_{\mathcal{D}}(E_3)}\rangle \leq T$. Finally, since $E_1 = \langle E_3^S\rangle \leq T$, we deduce that $T = E_1$, as desired. □

Proposition 3.3.5. *$\mathcal{D}$ is a saturated simple exotic fusion system, and $\mathcal{D}^{frc} = \{E_1, E_3^{\mathcal{D}}, S\}$.*

Proof. In the statement of Theorem 3.1.23, letting $\mathcal{F}_0 = N_{\mathcal{G}}(E_1)$, $V = E_3$ and $\Delta = \mathrm{Aut}_{\mathcal{G}}(E_3)$ we have that $\mathcal{D}$ is saturated. Again, the deduction of $\mathcal{D}^{frc}$ is clear from the inclusion $\mathcal{D} \leq \mathcal{G}$. Let K be a Sylow 2-subgroup of $N_{O^{3'}(\mathrm{Aut}_{\mathcal{D}}(E_3))}(\mathrm{Aut}_S(E_3))$ which is cyclic of order 8. Then, by saturation, the morphisms in K lift to morphisms of larger subgroups of S and as E_1 is $\mathrm{Aut}_{\mathcal{D}}(S)$-invariant, and applying the Alperin–Goldschmidt theorem, we deduce that the morphisms in K lift to morphisms in $\mathrm{Aut}_{\mathcal{D}}(E_1)$. Hence, $\mathrm{Out}_{\mathcal{D}}(E_1)$ contains a cyclic group of order 8. Since $\mathrm{Out}_{\mathcal{D}}(E_1) \cong \mathrm{GL}_2(3)$, applying [AKO11, Lemma I.7.6] we must have that $O^{3'}(\mathcal{D}) = \mathcal{D}$.

If $\mathcal{D}$ is not simple with $\mathcal{N} \trianglelefteq \mathcal{D}$ then by Lemma 3.3.4 we have that $\mathcal{N}$ is supported on E_1. Then by [AKO11, Proposition I.6.4], $\mathrm{Aut}_{\mathcal{N}}(E_1) \trianglelefteq \mathrm{Aut}_{\mathcal{D}}(E_1)$ so that $\mathrm{Out}_{\mathcal{N}}(E_1)$ is isomorphic to a normal $3'$-subgroup of $\mathrm{Out}_{\mathcal{D}}(E_1) \cong \mathrm{GL}_2(3)$. In particular, E_3 is not essential in $\mathcal{N}$ for otherwise we could again lift a cyclic subgroup of order 8 to $\mathrm{Aut}_{\mathcal{N}}(E_1)$, using saturation. Then, performing the explicit calculations in MAGMA, we deduce that $\mathcal{E}(\mathcal{N}) = \emptyset$ and $E_1 = O_3(\mathcal{N})$, and so $E_1 \trianglelefteq \mathcal{D}$, a contradiction by Proposition 3.1.13.

Suppose that there is a finite group G with $\mathcal{F} = \mathcal{F}_S(G)$. Since $O_3(\mathcal{F}) = \{1\}$, we may as well assume that $O_{3'}(G) = O_3(G) = \{1\}$. Furthermore, since $\mathcal{D}$ is a simple fusion system, we infer that $S \leq F^*(G)$ for otherwise $\mathcal{F}_{S\cap F^*(G)}(F^*(G))$ is a proper normal subsystem of G. As in Proposition 3.3.3, using that $|\Omega(Z(S))| = 3$, we deduce that $F^*(G)$ is simple group containing S as a Sylow 3-subgroup. The remainder of the proof is the same as in Proposition 3.3.3, and we conclude that $\mathcal{D}$ is exotic. □

Using MAGMA [PS21] we see that there are three fusion systems supported on S with $O_3(\mathcal{F}) = \{1\}$, namely $\mathcal{D}, \mathcal{G}$ and $\mathcal{H}$. It would be desirable prove this result without using MAGMA.

CHAPTER 4

FUSION SYSTEMS ON A SYLOW p-SUBGROUP OF $\mathrm{G}_2(p^n)$ OR $\mathrm{PSU}_4(p^n)$

In this chapter we classify all saturated fusion systems supported on p-groups isomorphic to a Sylow p-subgroup of $\mathrm{G}_2(p^n)$ or $\mathrm{PSU}_4(p^n)$. We strive to achieve this without the need for a $\mathcal{K}$-group hypothesis. Indeed, barring an identification of $\mathrm{PSL}_2(q^2)$ acting on a natural $\Omega_4^-(q)$-module, the only real point of contact we have with the classification of finite simple groups is in proving that the exotic fusion systems supported on a Sylow 7-subgroup of $\mathrm{G}_2(7)$ are exotic.

Additionally, we do not assume that $O_p(\mathcal{F}) = 1$ for the fusion system $\mathcal{F}$ under consideration as in other works and so we obtain some generalizations of results already in the literature (see [PS18], [Mon20] and [BFM19]), although we often lean on these works for convenience. Often, at least for small values of q, we make use of MAGMA to ease some of the exposition although, with some minor alterations, we remark that the techniques we employ could also be used in these small cases.

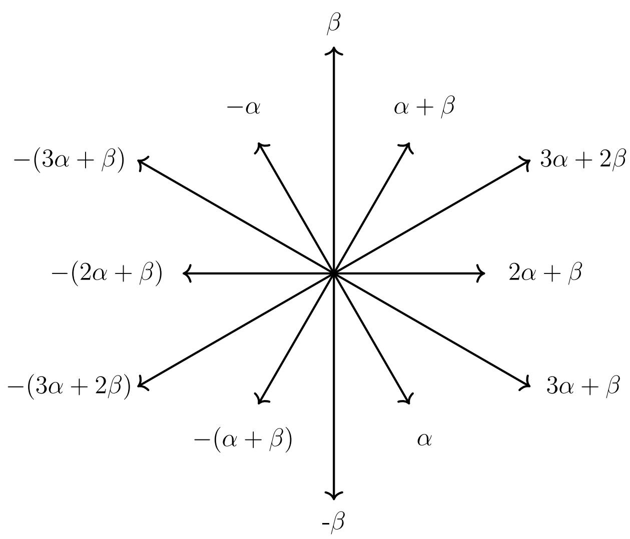
Finally, in all the situations considered, we also provide a list of all S-centric,

S-radical subgroups of Sylow p-subgroups of $\mathrm{G}_2(p^n)$ or $\mathrm{PSU}_4(p^n)$, which may be of independent interest.

4.1 Sylow p-subgroups of $\mathrm{G}_2(p^n)$ and $\mathrm{PSU}_4(p^n)$

In this section we construct Sylow p-subgroups of $\mathrm{G}_2(p^n)$ and $\mathrm{PSU}_4(p^n)$ and describe some of their basic properties. We refer to [Car89] for constructions and properties of $\mathrm{G}_2(q)$ and $\mathrm{PSU}_4(q)$, as well as generic properties and terminology regarding the simple groups of Lie type.

We present the root system of type G_2 below. We follow the choices of roots as in [Ree61, p. 443] and depict a slightly altered root system than what is given in that paper [Ree61, Figure 1].



In this way, we can arrange that our six positive roots are

$$\Phi^+ = \{\alpha, \beta, \alpha+\beta, 2\alpha+\beta, 3\alpha+\beta, 3\alpha+2\beta\}.$$

For $\varepsilon \in \Phi^+$ we set $X_\varepsilon := \langle x_\varepsilon(t) \mid t \in \mathbb{K}\rangle$, where $\mathbb{K}$ is a field of order $q = p^n$. Thus, we have that

$$S = \langle X_\alpha, X_\beta, X_{3\alpha+\beta}, X_{\alpha+\beta}, X_{2\alpha+\beta}, X_{3\alpha+2\beta}\rangle \in \mathrm{Syl}_p(\mathrm{G}_2(q))$$

is of order q^6.

Using results from [Ree61, (3.10)], we have the following Chevalley commutator formulas for the root subgroups:

$$\begin{aligned}
[x_\alpha(t), x_\beta(u)] &= x_{\alpha+\beta}(-tu)x_{2\alpha+\beta}(-t^2u)x_{3\alpha+\beta}(t^3u)x_{3\alpha+2\beta}(-2t^3u^2)\\
[x_\alpha(t), x_{\alpha+\beta}(u)] &= x_{2\alpha+\beta}(-2tu)x_{3\alpha+\beta}(3t^2u)x_{3\alpha+2\beta}(3tu^2)\\
[x_\alpha(t), x_{2\alpha+\beta}(u)] &= x_{3\alpha+\beta}(3tu)\\
[x_\beta(t), x_{3\alpha+\beta}(u)] &= x_{3\alpha+2\beta}(tu)\\
[x_{\alpha+\beta}(t), x_{2\alpha+\beta}(u)] &= x_{3\alpha+2\beta}(3tu).
\end{aligned}$$

We remark that the coefficients in the commutator formulas showcase obvious degeneracies when $p = 2$ or 3. This is one of the reasons we treat these cases separately.

Lemma 4.1.1. *Suppose that S is isomorphic to a Sylow p-subgroup of $\mathrm{G}_2(p^n)$. Then the following holds:*

(i) *if $p = 2$, then S has exponent 8;*

(ii) *if $p \in \{3, 5\}$, then S has exponent p^2; and*

(iii) *if $p \geqslant 7$, then S has exponent p.*

Proof. Set $q = p^n$. Since $\mathrm{G}_2(q)$ has a 7 dimensional representation over $\mathrm{GF}(q)$ when p is odd, and $\mathrm{G}_2(q)$ has a 6 dimensional representation over $\mathrm{GF}(q)$ when $p = 2$, we can find an upper bound for the exponent of S by calculating the exponent of a Sylow p-subgroup of $\mathrm{GL}_r(q)$, where $r = 7$ when p is odd and $r = 6$ if $p = 2$. But a Sylow p-subgroup of $\mathrm{GL}_r(p^n)$ has exponent p^a with a minimal such that $p^a > r - 1$. Thus, S has exponent p when $p \geqslant 7$ and the exponent of S is bounded above by p^2 or 8 when $p \in \{3, 5\}$ or $p = 2$ respectively. One can compute directly that a Sylow p-subgroup of $\mathrm{G}_2(p)$ has exponent 8, 9 or 25 when $p =$ 2,3 or 5 respectively, and so the result follows. $\square$

We now proceed with the construction of a Sylow p-subgroup S of $\mathrm{PSU}_4(p^n)$. Let $\Phi^+ = \{a, b, c, a+b, a+c, b+c, a+b+c\}$ be a choice of positive roots for the root system A_3. In particular, under the symmetry of A_3, we may partition the positive roots into equivalence classes $\{a, c\}$, $\{b\}$, $\{a+b, b+c\}$ and $\{a+b+c\}$. Following [GLS98, Theorem 2.4.1] and setting $\widehat{\mathbb{K}}$ to be a finite field of order q^2, and $\mathbb{K}$ the subfield of order q, we may choose a set of fundamental roots $\{\alpha, \beta\}$ for ${}^2\mathrm{A}_3$ as

$$x_\alpha(t) = x_a(t)x_c(t^q),$$
$$x_\beta(u) = x_b(u),$$

where $t, u \in \widehat{\mathbb{K}}$ and $u = u^q \in \mathbb{K}$. We then retrieve a full set of positive roots and root subgroups for $\mathrm{PSU}_4(q)$

$$x_\alpha(t) = x_a(t)x_c(t^q),$$
$$x_\beta(u) = x_b(u),$$
$$x_{\alpha+\beta}(t) = x_{a+b}(t)x_{b+c}(t^q),$$

$$x_{2\alpha+\beta}(u) = x_{a+b+c}(u)$$

where $t, u \in \widehat{\mathbb{K}}$ and $u = u^q \in \mathbb{K}$. Hence, we infer that

$$|X_\alpha| = q^2,\ |X_\beta| = q,\ |X_{\alpha+\beta}| = q^2,\ |X_{2\alpha+\beta}| = q$$

and $S = \langle X_\alpha, X_\beta, X_{\alpha+\beta}, X_{2\alpha+\beta}\rangle$ is of order q^6.

We reproduce the Chevalley commutator formulas for $\mathrm{PSU}_4(q)$ and as, before, set $\mathbb{K}$ to be a field of order q. For more details, see [GLS98, Theorem 2.4.5].

$$[x_\alpha(t), x_\beta(u)] = x_{\alpha+\beta}(\varepsilon tu)x_{2\alpha+\beta}(\varepsilon' N(t)u)$$
$$[x_\alpha(t), x_{\alpha+\beta}(u)] = x_{2\alpha+\beta}(\varepsilon'' Tr(tu))$$

where $t, u \in \widehat{\mathbb{K}}$ and $u = u^q$, and Tr and N denote the field trace and norm from $\widehat{\mathbb{K}}$ down to $\mathbb{K}$. Moreover, $\varepsilon, \varepsilon', \varepsilon'' \in \{1, -1\}$ depend only on the roots in the commutators they are involved in. It then follows that

$$S' = X_{\alpha+\beta}X_{2\alpha+\beta},\ Z(S) = X_{2\alpha+\beta}.$$

For the purposes of this thesis, the exact values of $\varepsilon, \varepsilon'$ and ε'' are not important and all we require is that commutators with single elements generate entire $\mathrm{GF}(q)$ spaces of root subgroups e.g. $[x_\alpha(t), S'] = Z(S)$ and $|[x_\alpha(t), X_\beta X_{\alpha+\beta}X_{2\alpha+\beta}]| = q^2$ for all $t \neq 0$.

In the analysis of $S \in \mathrm{Syl}_p(\mathrm{PSU}_4(p^n))$, it will often be more useful to work with local subgroups of $\mathrm{PSU}_4(p^n)$, recognizing the internal modules within these local subgroups and obtaining information about S from its embedding in these groups.

In this way, we work with the elements as matrices explicitly, recognizing the isomorphism ${}^2\mathrm{A}_3(q^2) \cong \mathrm{PSU}_4(q) \leq \mathrm{PSL}_4(q^2)$ ([Car89, Theorem 14.5.1]). However, for some arguments, we still reference the commutator formulas.

Lemma 4.1.2. *Suppose that S is isomorphic to a Sylow p-subgroup of $\mathrm{PSU}_4(p^n)$. Then the following holds:*

(i) *if $p = 2$, then S has exponent 4;*

(ii) *if $p = 3$, then S has exponent 9; and*

(iii) *if $p \geqslant 5$, then S has exponent p.*

Proof. This proof is much the same as Lemma 4.1.1. Set $q = p^n$. Since $\mathrm{PSU}_4(q)$ is a subgroup of $\mathrm{PSL}_4(q^2)$, we can find an upper bound for the exponent of S by calculating the exponent of a Sylow p-subgroup of $\mathrm{GL}_4(q^2)$, which is p^a with a minimal such that $p^a > 3$. Thus, S has exponent p when $p \geqslant 5$ and the exponent of S is bounded above by 4 or 9 when $p = 2$ or $p = 3$ respectively. One can compute directly that a Sylow p-subgroup of $\mathrm{PSU}_4(p)$ has exponent p^2 when $p \in \{2, 3\}$, and so the result follows. $\square$

For identification arguments later in this chapter, we record the outcomes from the Main Theorem where S is isomorphic to either a Sylow p-subgroup of $\mathrm{G}_2(p^n)$ or $\mathrm{PSU}_4(p^n)$. Although the proof of the Main Theorem is the contents of Chapter 5, we assume its validity throughout this chapter.

Corollary 4.1.3. *Suppose the hypothesis of the Main Theorem and assume that S is isomorphic to a Sylow p-subgroup of $\mathrm{G}_2(p^n)$ for some $n \in \mathbb{N}$. Then either*

(i) *$\mathcal{F} = \mathcal{F}_S(G)$, where $F^*(G) = O^{p'}(G) \cong \mathrm{G}_2(p^n)$;*

(ii) $p = 2$ *and* $\mathcal{F} = \mathcal{F}_S(G)$ *where* $G \cong \mathrm{M}_{12}$ *or* $\mathrm{G}_2(3)$*; or*

(iii) $p = 7$, $\mathcal{F}$ *is a uniquely determined simple fusion system on a Sylow 7-subgroup of* $\mathrm{G}_2(7)$ *and, assuming the classification of finite simple groups,* $\mathcal{F}$ *is exotic.*

Corollary 4.1.4. *Suppose the hypothesis of the Main Theorem and assume that* S *is isomorphic to a Sylow* p*-subgroup of* $\mathrm{PSU}_4(p^n)$ *for some* $n \in \mathbb{N}$*. Then* $\mathcal{F} = \mathcal{F}_S(G)$*, where* $F^*(G) = O^{p'}(G) \cong \mathrm{PSU}_4(p^n)$*; or* $p = 3$ *and* $G \cong \mathrm{PSU}_6(2), \mathrm{PSU}_6(2).2$*,* McL*,* $\mathrm{Aut}(\mathrm{McL})$ *or* Co_2*.*

It is worth mentioning that aside from the above two corollaries, the methods utilized in this chapter are independent of Chapter 5 and the only concept which is relevant to the work in this chapter which has not been considered is that of a weak BN-pair of rank 2 (see Definition 5.1.7).

4.2 Fusion Systems on a Sylow 2-subgroup of $\mathrm{G}_2(2^n)$

In this section, we let $q = 2^n$, $\mathbb{K} = \mathrm{GF}(q)$ and S be isomorphic to a Sylow 2-subgroup of $\mathrm{G}_2(q)$. Assume throughout that $\mathcal{F}$ is a saturated fusion system on S.

We deal with the $q = 2$ case separately in order to streamline some of the arguments later in this section. Fortunately, since $|S| = 2^6$ is small, we can directly determine the list of S-centric, S-radical subgroups and their automizers. We employ MAGMA to do this, although remark that lemmas and propositions in the remainder of this section all apply when $q = 2$ and their proofs could adapted with minor alternations.

Proposition 4.2.1. *Let S be isomorphic to a Sylow 2-subgroup of* $\mathrm{G}_2(2)$*. The S-centric, S-radical subgroups of S are $S, C_S(Z_3(S)/Z(S)), C_S(Z_2(S))$ and the maximal elementary abelian subgroups of S of order 2^3.*

Proposition 4.2.2. *Let $\mathcal{F}$ be a saturated fusion system over a Sylow 2-subgroup of* $\mathrm{G}_2(2)$*. Set $Q_1 := C_S(Z_3(S)/Z(S))$ and $Q_2 = C_S(Z_2(S))$. Then one of the following holds:*

(i) $\mathcal{F} = \mathcal{F}_S(S)$;

(ii) $\mathcal{F} = \mathcal{F}_S(Q_1 : \mathrm{Out}_{\mathcal{F}}(Q_1))$ *where* $\mathrm{Out}_{\mathcal{F}}(Q_1)$ *is isomorphic to a subgroup of* $(3 \times 3) : 2$;

(iii) $\mathcal{F} = \mathcal{F}_S(Q_2 : \mathrm{Out}_{\mathcal{F}}(Q_2))$ *where* $\mathrm{Out}_{\mathcal{F}}(Q_2) \cong \mathrm{Sym}(3)$;

(iv) $\mathcal{F} = \mathcal{F}_S(M)$ *where* $M \cong 2^3.\mathrm{PSL}_3(2)$;

(v) $\mathcal{F} = \mathcal{F}_S(G)$ *where* $G \cong \mathrm{G}_2(2)$;

(vi) $\mathcal{F} = \mathcal{F}_S(G)$ *where* $G \cong \mathrm{G}_2(3)$; *or*

(vii) $\mathcal{F} = \mathcal{F}_S(G)$ *where* $G \cong \mathrm{M}_{12}$.

Remark. In case (iv) of the above theorem, one can take M to be a maximal subgroup of $\mathrm{G}_2(3)$.

We continue the analysis when $p = 2$ and suppose throughout the remainder of this section that $q > 2$. We may reduce the commutator formulas from Section 4.1 to the following:

$$
\begin{aligned}
[x_\alpha(t), x_\beta(u)] &= x_{\alpha+\beta}(tu)x_{2\alpha+\beta}(t^2u)x_{3\alpha+\beta}(t^3u) \\
[x_\alpha(t), x_{\alpha+\beta}(u)] &= x_{3\alpha+\beta}(t^2u)x_{3\alpha+2\beta}(tu^2) \\
[x_\alpha(t), x_{2\alpha+\beta}(u)] &= x_{3\alpha+\beta}(tu) \\
[x_\beta(t), x_{3\alpha+\beta}(u)] &= x_{3\alpha+2\beta}(tu) \\
[x_{\alpha+\beta}(t), x_{2\alpha+\beta}(u)] &= x_{3\alpha+2\beta}(tu).
\end{aligned}
$$

It follows that

$$Z_3(S) = \langle X_{\alpha+\beta}, X_{2\alpha+\beta}, X_{3\alpha+\beta}, X_{3\alpha+2\beta}\rangle$$

$$Z_2(S) = \langle X_{3\alpha+\beta}, X_{3\alpha+2\beta}\rangle$$

$$Z(S) = \langle X_{3\alpha+2\beta}\rangle$$

are characteristic subgroups of S of orders q^4, q^2 and q respectively.

We define

$$Q_1 := C_S(Z_3(S)/Z_1(S)) = \langle X_\beta, X_{\alpha+\beta}, X_{2\alpha+\beta}, X_{3\alpha+\beta}, X_{3\alpha+2\beta}\rangle$$

$$Q_2 := C_S(Z_2(S)) = \langle X_\alpha, X_{\alpha+\beta}, X_{2\alpha+\beta}, X_{3\alpha+\beta}, X_{3\alpha+2\beta}\rangle$$

both of order q^5 and characteristic in S. Moreover, we can identify Q_1 and Q_2 with unipotent radicals of two maximal parabolic subgroups in $\mathrm{G}_2(q)$. Therefore, $\Phi(Q_1) = Z(Q_1) = Z(S)$ and $\Phi(Q_2) = Z_2(S) = Z(Q_2)$.

The following lemma gives detailed information on involutions in S, their normalizers and the maximal elementary abelian subgroups of S.

Lemma 4.2.3. *Every involution in S is conjugate in S to one of the following: $x_\alpha(t_1)$, $x_\beta(t_2)x_{2\alpha+\beta}(t_2')$, $x_{2\alpha+\beta}(t_3)$, $x_{\alpha+\beta}(t_4)$, $x_{3\alpha+\beta}(t_5)$ or $x_{3\alpha+2\beta}(t_6)$, for $t_i \in \mathbb{K}^\times$ and $t_2' \in \mathbb{K}$. Moreover, each has centralizer of order q^3, q^4, q^4, q^4, q^5 or q^6 respectively. As a consequence, every maximal elementary abelian subgroup is conjugate in S to one of*

$$T := X_\alpha X_{3\alpha+\beta} X_{3\alpha+2\beta},$$

$$U := X_\beta X_{2\alpha+\beta} X_{3\alpha+2\beta},$$

$$V := X_\beta X_{\alpha+\beta} X_{3\alpha+2\beta},$$

$$W := X_{2\alpha+\beta} X_{3\alpha+\beta} X_{3\alpha+2\beta}, \mathit{or}$$

$$X := X_{\alpha+\beta} X_{3\alpha+\beta} X_{3\alpha+2\beta}.$$

All are of order q^3 and have normalizers in S equal to Q_2, Q_1, Q_1, S and S respectively.

Proof. See [Tho69, (3.6)-(3.10)]. □

Throughout this section, we retain the notation from the lemma and remark that $WX = Z_3(S)$, $T \le Q_2 \setminus Q_1$ and $U, V \le Q_1 \setminus Q_2$.

We can now begin to determine to the possible essential subgroups of $\mathcal{F}$. The primary technique used is Lemma 3.2.1 which, more generally, aids in proving that a candidate subgroup E is not an $\mathcal{F}$-radical subgroup of S. Moreover, if we can prove that a chain of characteristic subgroups of E is centralized by some p-group not contained in E, then E will be not be S-radical. For large parts of this section, we can operate in this more general setting, assuming only that E is S-centric and S-radical.

Proposition 4.2.4. *Let E be an S-centric and S-radical subgroup of S and suppose $Z_3(S) \leq E$. Then $E \in \{Q_1, Q_2, S\}$.*

Proof. Since $Z_3(S) \leq E$, $W, X \leq E$ and so $\mathcal{A}(E) \subseteq \mathcal{A}(S)$. Suppose first that $Q_i < E$ for some $i \in \{1, 2\}$. Then, W, X are the unique normal elementary abelian subgroups of maximal rank in E and so $Z_3(S) = WX$ is characteristic in E. Hence, $Z_2(S) = Z(Z_3(S))$ is also a characteristic subgroup. If $Q_1 \not\leq E$ and $Q_2 \not\leq E$, then $\mathcal{A}(E) = \{W, X\}$, $J(E) = Z_3(S)$ and again, $Z_3(S)$ and $Z_2(S)$ are characteristic subgroups of E. Thus, we have shown in either case that $Z_2(S)$ and $Z_3(S)$ are characteristic subgroups of E.

Now, if $Q_2 \not\leq E$, Q_2 centralizes the chain $\{1\} \trianglelefteq Z_2(S) \trianglelefteq Z_3(S) \trianglelefteq E$ and E is not S-radical by Lemma 3.2.1, a contradiction. So $Q_2 < E$. But then, it follows from the commutator formulas that $Z(E) = Z(S)$. Hence, Q_1 centralizes the chain $\{1\} \trianglelefteq Z(S) \trianglelefteq Z_2(S) \trianglelefteq Z_3(S) \trianglelefteq E$, and since E is S-radical, we conclude that $E = S$, as required. □

Lemma 4.2.5. *Let E be an S-centric, S-radical subgroup of S and suppose that $Z_3(S) \not\leq E$. Then $Z(S) < Z(E)$ and if $Z(S) < Z(E) \cap Z_2(S)$, then $Z_2(S) < E$ and $E < Q_2$. In particular $Z(E) \not\leq Z_2(S)$.*

Proof. Suppose first that $Z(S) = Z(E)$. Since $WX = Z_3(S) \not\leq E$, there exists $Y \in \{W, X\}$ with $Y \not\leq E$. Notice that $Z_2(S)$ centralizes the chain $\{1\} \trianglelefteq Z(E) \trianglelefteq E$ so that, as E is S-radical, $Z_2(S) \leq E$ and $Z_2(S) \leq Z_2(E)$. Suppose that $\Omega(Z_2(E)) \leq Q_1$. Then, as $Y \trianglelefteq S$, Y centralizes the chain $\{1\} \trianglelefteq Z(E) \trianglelefteq \Omega(Z_2(E)) \trianglelefteq E$, a contradiction since $Y \not\leq E$. Therefore, by Section 4.2, there exists an involution $e \in Z_2(E)$ which is conjugate in S to $x_\alpha(t)$, for some $t \in \mathbb{K}^\times$. Since $[E, e] \leq Z(E) = Z(S)$ it follows from the commutator formulas that elements of E are conjugate to elements of Q_2, and since $Q_2 \trianglelefteq S$ we deduce that $E \leq Q_2$. But then $Z(S) < Z_2(S) \leq Z(E)$, a contradiction. Hence, $Z(S) < Z(E)$.

Suppose now that $Z(S) < Z(E) \cap Z_2(S)$ and let $e \in (Z(E) \cap Z_2(S)) \setminus Z(S)$. Then $C_S(e) = Q_2$ by Section 4.2 and $E \leq C_S(e) = Q_2$. Because E is S-centric, $Z_2(S) \leq E$ from which it follows that $Z_2(S) \leq Z(E)$. Assume that $Z(E) = Z_2(S)$. Then, Q_2 centralizes the chain $\{1\} \trianglelefteq Z(E) \trianglelefteq E$ and since E is S-radical, $Q_2 \leq E$. But then $Z_3(S) \leq E$, a contradiction. Hence, if $Z(S) < Z(E) \cap Z_2(S)$ we deduce that $Z(E) > Z_2(S)$ and $E < Q_2$. □

Proposition 4.2.6. *Let E be an S-centric, S-radical subgroup of S and suppose that $Z_3(S) \not\leq E$. Then E is maximal elementary abelian, so is conjugate in S to W, X, T, U or V.*

Proof. By Lemma 4.2.5, we may assume that $Z(E) \not\leq Z_2(S)$. Suppose first that $\Omega(Z(E)) \leq Z_2(S)$. By Lemma 4.2.5, either $\Omega(Z(E)) = Z(S)$; or that $Z_2(S) < Z(E)$ and $E < Q_2$. Suppose the latter and, since $Z_3(S) \not\leq E$, choose $Y \in \{W, X\}$ with $Y \not\leq E$. Since $\Omega(Z(E)) \leq Z_2(S) < Z(E)$, E is centric and $Z_2(S)$ has exponent 2, we have that $\Omega(Z(E)) = Z_2(S)$ and Y centralizes the chain, $\{1\} \trianglelefteq \Omega(Z(E)) \trianglelefteq E$, a contradiction since E is S-radical and $Y \not\leq E$. Hence, we assume that $\Omega(Z(E)) = Z(S) = Z(E) \cap Z_2(S)$ and $E \not\leq Q_2$.

Since $Z_2(S)$ centralizes the chain $\{1\} \trianglelefteq \Omega(Z(E)) \trianglelefteq E$, $Z_2(S) \leq E$ and $Z(E) \leq Q_2$. Furthermore, $[Z_3(S), E] \leq Z_2(S) \leq E$ and so $Z_3(S) \leq N_S(E) \leq N_S(Z(E))$. In particular, $[Z_3(S), Z(E)] \leq Z(E) \cap [Z_3(S), Q_2] = Z(E) \cap Z_2(S) = \Omega(Z(E)) = Z(S)$ and so $Z(E) \leq C_S(Z_3(S)/Z(S)) = Q_1$. Therefore, $Z(E) \leq Z_3(S)$. Let $e \in E$ be an involution and suppose that $e \not\leq Q_1$. Then, by Section 4.2, e is conjugate in S to $x_\alpha(t)$ for some $t \in \mathbb{K}^\times$ by Section 4.2. Then $Z(E) \leq C_S(e) \leq T^s$ for some

$s \in S$ and since $Z(E) \leq Z_3(S) \trianglelefteq S$, it follows that $Z(E) \leq X_{3\alpha+\beta}X_{3\alpha+2\beta} = Z_2(S)$. But then $Z(E)$ has exponent 2 and $Z(E) = \Omega(Z(E)) = Z(S)$, a contradiction. Therefore, $\Omega(E) \leq E \cap Q_1$. In particular, $Z_2(S) \leq \Omega(E)$ so that $[E, Z_3(S)] \leq \Omega(E)$ and $Z_3(S)$ centralizes the chain $\{1\} \trianglelefteq \Omega(Z(E)) \trianglelefteq \Omega(E) \trianglelefteq E$, a contradiction since E is S-radical.

Hence, there exists an involution $e \in Z(E) \setminus Z_2(S)$ such that e is conjugate in S to $x_\alpha(t_1)$, $x_\beta(t_2)x_{2\alpha+\beta}(t_2')$, $x_{2\alpha+\beta}(t_3)$ or $x_{\alpha+\beta}(t_4)$ for $t_i \in \mathbb{K}^\times$ and $t_2' \in \mathbb{K}$ by Section 4.2. Suppose first that e is conjugate to $x_\alpha(t)$, some $t \in \mathbb{K}^\times$. Then $E \leq C_S(e) = T^s$ for some $s \in S$ and since E is S-centric, $E = T^s$.

Suppose now that e is conjugate to $x_{2\alpha+\beta}(t)$, $t \in \mathbb{K}^\times$. Then $E \leq C_S(e) = WU^s \leq Q_1$ for some $s \in S$ and $Z(C_S(e)) = (U \cap W)^s \leq Z(E)$. If $Z(C_S(e)) = Z(E)$, then $C_S(e)$ centralizes the series $\{1\} \trianglelefteq Z(E) \trianglelefteq E$ and $E = C_S(e)$. But now, X centralizes the series $\{1\} \trianglelefteq E' \trianglelefteq E$ and since E is S-radical and $X \not\leq E$, we have a contradiction. Thus, $Z(C_S(e)) < \Omega(Z(E))$ and $C_S(\Omega(Z(E)))$ is an elementary abelian subgroup of order q^3. Since E is S-centric, it follows that $|E| = q^3$ and $E = W$ or U^s for some $s \in S$, as required. If e is conjugate to $x_{\alpha+\beta}(t)$, we obtain $E \leq C_S(e) = XV^s$ for some $s \in S$ by Section 4.2. Arguing as before, we obtain that E is conjugate to either V or X in S.

Finally, we suppose that e is conjugate in S to some $x_\beta(t)x_{2\alpha+\beta}(t')$, for $t \in \mathbb{K}^\times$ and $t' \in \mathbb{K}$. Then, using the commutator formulas, one can calculate that $|C_S(e)| = q^4$, $E \leq C_S(e) \leq Q_1$ and $Z(S)X_\beta^s = Z(C_S(e)) \leq \Omega(Z(E))$ for some $s \in S$. If $\Omega(Z(E)) = Z(C_S(e))$ then $C_S(e)$ centralizes the series $\{1\} \trianglelefteq Z(E) \trianglelefteq E$ and since E is S-radical, $E = C_S(e)$. But then, $E' = Z(S)$ and Q_1 centralizes the series $\{1\} \trianglelefteq E' \trianglelefteq E$, a contradiction since E is S-radical and $Q_1 \not\leq E$. Hence, $Z(C_S(e)) < \Omega(Z(E))$, $|\Omega(Z(E))| > q^2$ and since $\Omega(Z(E))Z_3(S) \leq Q_1$, there is some $\widetilde{e} \in (\Omega(Z(E)) \cap Z_3(S)) \setminus Z(S)$. Indeed, $\widetilde{e}$ is not contained in $Z_2(S)$, for otherwise $E \leq Q_1 \cap Q_2 = Z_3(S)$, a contradiction since $e \not\leq Z_3(S)$. Therefore, $\widetilde{e}$ is conjugate in S to some $x_{2\alpha+\beta}(t)$ or $x_{\alpha+\beta}(t)$ and by the above, E is elementary abelian. Moreover, since there is $e \in E$ conjugate to some $x_\beta(t)x_{2\alpha+\beta}(t')$, we have that E is conjugate to U or V. □

We have shown that the S-centric, S-radical subgroups of S are S, Q_1, Q_2 or maximal elementary abelian subgroups of S. At this point, we restrict our attention to a saturated fusion system $\mathcal{F}$ on S and its essential subgroups. We make use of Lemma 3.2.2, and as stated, this appears to rely on a $\mathcal{K}$-group hypothesis on $\mathrm{Aut}_{\mathcal{F}}(E)$, where E is a candidate essential subgroup. Following the proof in [PS21, Proposition 4.8], the $\mathcal{K}$-group condition is only used to provide a list of candidates for groups with a strongly 2-embedded subgroup along with their Sylow 2-subgroups. Fortunately, when $p = 2$ a result of Bender [Ben71] classifies all such groups and so, we can determine the essential subgroups of $\mathcal{F}$ without the need to employ a $\mathcal{K}$-group hypothesis.

In addition, the proof of Proposition 3.2.7 relies on a $\mathcal{K}$-group hypothesis for the same reason as Lemma 3.2.2 and so when $p = 2$, utilizing Bender's result with the acknowledgment that $q > 2$, $O^{2'}(\mathrm{Out}_{\mathcal{F}}(E))$ is isomorphic to a central extension of a rank 1 group of Lie type in characteristic 2, independent of any $\mathcal{K}$-group hypothesis on $\mathrm{Aut}_{\mathcal{F}}(E)$. A final consideration is that we intend to use Corollary 4.1.3 which relies on the Main Theorem which again uses a $\mathcal{K}$-group hypothesis. Following the proof of that theorem, the determination of $\mathcal{F}$ from a rank 2 amalgam relies only on the work in [DS85] which is, again, independent of any $\mathcal{K}$-group hypothesis. Hence, when $p = 2$, we can apply all the necessary results to determine $\mathcal{F}$ without the need to enforce a $\mathcal{K}$-group hypothesis on $\mathrm{Aut}_{\mathcal{F}}(E)$.

Theorem 4.2.7. *Let $\mathcal{F}$ be a saturated fusion system over a Sylow 2-subgroup of* $\mathrm{G}_2(2^n)$ *for $n > 1$. Then one of the following holds:*

(i) $\mathcal{F} = \mathcal{F}_S(S : \mathrm{Out}_{\mathcal{F}}(S))$;

(ii) $\mathcal{F} = \mathcal{F}_S(Q_i : \mathrm{Out}_{\mathcal{F}}(Q_i))$ *where* $O^{2'}(\mathrm{Out}_{\mathcal{F}}(Q_i)) \cong \mathrm{SL}_2(2^n)$*; or*

(iii) $\mathcal{F} = \mathcal{F}_S(G)$*, where* $F^*(G) = O^{2'}(G) \cong \mathrm{G}_2(2^n)$.

Proof. Let $E \in \mathcal{E}(\mathcal{F})$ and suppose that E is elementary abelian. Then, in all cases, we deduce that $q^3 = |E| < q^4 \leqslant |\mathrm{Out}_S(E)|^2$, a contradiction by Lemma 3.2.2. Therefore, $\mathcal{E}(\mathcal{F}) \subseteq \{Q_1, Q_2\}$. If neither Q_1 nor Q_2 are essential then outcome (i) holds, and if $\mathcal{E}(\mathcal{F}) = \{Q_i\}$ for some $i \in \{1, 2\}$ then since Q_i is $\mathrm{Aut}_{\mathcal{F}}(S)$-invariant and maximally essential, outcome (ii) holds upon comparing with the list in Proposition 3.2.7. Thus, $\mathcal{E}(\mathcal{F}) = \{Q_1, Q_2\}$. Since Q_i is $\mathrm{Aut}_{\mathcal{F}}(S)$-invariant for $i \in \{1, 2\}$, if $O_2(\mathcal{F}) = \{1\}$ we apply Corollary 4.1.3 and the result follows.

Suppose that $Q := O_2(\mathcal{F}) \neq \{1\}$. By Proposition 3.1.13, $Q \leq Q_1 \cap Q_2 = Z_3(S)$ and so, $\Phi(Q) \leq Z(S)$. Now, $Z_2(S)$ is normalized by $\mathrm{Aut}_{\mathcal{F}}(Q_2)$ and $\mathrm{Out}_S(Q_2)$ centralizes $Z(S)$ which has index q in $Z_2(S)$, which is itself of order q^2. Moreover, since S does not centralize $Z_2(S)$, $\mathrm{Out}_S(Q_2)$ acts non-trivially on $Z_2(S)$ and, by Theorem 3.2.3, $Z_2(S)$ is an FF-module for $O^{2'}(\mathrm{Out}_{\mathcal{F}}(Q_2)) \cong \mathrm{SL}_2(2^n)$ and $Z_2(S)$ is irreducible. Since $\Phi(Q) \leq Z(S) \leq Z_2(S)$, we conclude that $\Phi(Q) = \{1\}$, Q is elementary abelian and $Z_2(S) \leq Q$.

If $Q = Z_2(S)$, then $Z_2(S)$ is $\mathrm{Aut}_{\mathcal{F}}(Q_1)$-invariant and so is $Z_3(S) = C_{Q_1}(Z_2(S))$. But then S centralizes the chain $\{1\} \trianglelefteq Z(S) \trianglelefteq Z_2(S) \trianglelefteq Z_3(S) \trianglelefteq Q_1$, a contradiction since Q_1 is $\mathcal{F}$-radical. Hence, $Z_2(S) < Q < Z_3(S)$ and there is an involution $x \in Q$ which is conjugate in S to $x_{2\alpha+\beta}(t)$ or $x_{\alpha+\beta}(t)$ for some

$t \in \mathbb{K}^\times$. But then $C_S(Q) \leq Q_1 \cap Q_2$ and so $C_S(Q)$ is $\mathrm{Aut}_{\mathcal{F}}(Q_i)$-invariant for $i \in \{1,2\}$. It follows from Proposition 3.1.13 that $C_S(Q) \trianglelefteq \mathcal{F}$ so that $Q = C_S(Q)$ is self-centralizing in S, $Q \in \{W, X\}$ and $\mathcal{F}$ is satisfies the hypothesis of Theorem 3.1.21.

By Theorem 3.1.21, there is a finite group G such that $F^*(G) = Q$ and $\mathcal{F} = \mathcal{F}_S(G)$. Moreover, $O^{p'}(\mathrm{Out}_G(Q_i)) \cong \mathrm{SL}_2(q)$ and $\mathrm{Out}_{\mathcal{F}}(Q_i)$ acts faithfully on Q_i/Q for $i \in \{1,2\}$. Set $\overline{G} := G/O_2(\mathcal{F})$ and notice that $\overline{Q_1}$ and $\overline{Q_2}$ are self-centralizing in $\overline{G}$. Moreover, $\overline{G} = \langle N_{\overline{G}}(\overline{Q_1}), N_{\overline{G}}(\overline{Q_2})\rangle$, and $\overline{Q_i}$ is $\mathrm{Aut}_{\overline{G}}(\overline{S})$-invariant for $i \in \{1,2\}$. It follows that $\overline{G}$ has a weak BN-pair of rank 2 in the sense of Definition 5.1.7. Moreover, since Q_2 centralizes $Z_2(S)$ which has index q in Q and Q_2/Q is elementary abelian of order q^2, we infer that Q is an FF-module for $\overline{G}$. Then, comparing with the completions in [DS85] and applying [CD91, Theorem A], Q is a “natural module” for $O^{p'}(\overline{G}) \cong \mathrm{PSL}_3(q)$. Notice that if S splits over Q, then S is isomorphic to a Sylow 2-subgroup of $\mathrm{PSL}_4(q)$. Then by [GLS98, Theorem 3.3.3], the 2-rank of S is $4n$, a contradiction to Section 4.2. Therefore, S is non-split and it follows by [Bel78, Table I], that $q = 2$, a contradiction to the original hypothesis. □

Combined with the classification provided in Proposition 4.2.2, this completely determines all saturated fusion systems on a Sylow 2-subgroup of $\mathrm{G}_2(2^n)$ for any n.

4.3 Fusion Systems on a Sylow 3-subgroup of $\mathrm{G}_2(3^n)$

Throughout this section, we suppose that $p = 3$, $q = 3^n$, $\mathbb{K}$ is a finite field of order q and S is isomorphic to a Sylow 3-subgroup of $\mathrm{G}_2(q)$. We may reduce the commutator formulas from Section 4.1 to the following:

$$
\begin{aligned}
[x_\alpha(t), x_\beta(u)] &= x_{\alpha+\beta}(-tu)x_{2\alpha+\beta}(-t^2u)x_{3\alpha+\beta}(t^3u)x_{3\alpha+2\beta}(t^3u^2)\\
[x_\alpha(t), x_{\alpha+\beta}(u)] &= x_{2\alpha+\beta}(tu)\\
[x_\beta(t), x_{3\alpha+\beta}(u)] &= x_{3\alpha+2\beta}(tu).
\end{aligned}
$$

Additionally, $Z(S) = \langle X_{2\alpha+\beta}, X_{3\alpha+2\beta}\rangle$ is a characteristic subgroup of S of order q^2.

We let

$$Q_1 = \langle X_\beta, X_{3\alpha+\beta}, X_{\alpha+\beta}, X_{2\alpha+\beta}, X_{3\alpha+2\beta}\rangle$$

$$Q_2 = \langle X_\alpha, X_{\alpha+\beta}, X_{3\alpha+\beta}, X_{3\alpha+2\beta}, X_{2\alpha+\beta}\rangle$$

and by removing one root subgroup at a time from Q_i, starting from the left, we get a chain of subgroups $Q_1 \cap Q_2 \to Z(Q_i) \to Z(S) \to \Phi(Q_i) \to \{1\}$ e.g.

$$Z(Q_1) = \langle X_{\alpha+\beta}, X_{2\alpha+\beta}, X_{3\alpha+2\beta}\rangle.$$

Before determining the essential subgroups of a saturated fusion system $\mathcal{F}$ on S, we state and prove some important properties of S, Q_1 and Q_2 which may be of interest in their own right.

Lemma 4.3.1. *The subgroup* $X := \langle X_\beta, X_{3\alpha+\beta}, X_{3\alpha+2\beta}\rangle \leq Q_1$ *is a subgroup of shape* q^{1+2} *and is isomorphic to a Sylow* 3*-subgroup of* $\mathrm{SL}_3(q)$*.*

Proof. Since the groups X_β and $X_{3\alpha+\beta}$ commute modulo $X_{3\alpha+2\beta}$, it follows that every element may be written as $x_{3\alpha+\beta}(t_1)x_\beta(t_2)x_{3\alpha+2\beta}(t_3)$. Then, using the commutator formulas, we calculate that the map $\theta : X \to \mathrm{SL}_3(q)$ such that

$$(x_{3\alpha+\beta}(t_1)x_\beta(t_2)x_{3\alpha+2\beta}(t_3))\theta = \begin{pmatrix} 1 & 0 & 0 \\ t_1 & 1 & 0 \\ t_3 & t_2 & 1 \end{pmatrix}$$

is an injective homomorphism, from which it follows that X is isomorphic to a Sylow 3-subgroup of $\mathrm{SL}_3(q)$. □

Remark. By symmetry, the subgroup $\langle X_\alpha, X_{\alpha+\beta}, X_{2\alpha+\beta}\rangle \leq Q_2$ is also isomorphic to a Sylow 3-subgroup of $\mathrm{SL}_3(q)$.

As $Q_1 = Z(Q_1)X$, we observe that Q_1 and Q_2 are isomorphic groups of shape $q^2 \times q^{1+2}$, where q^{1+2} denotes a special group of order q^3. We may identify Q_1, Q_2 with the radical subgroups of maximal parabolic subgroups of $\mathrm{G}_2(q)$ of shape $(q^2 \times q^{1+2}) : \mathrm{GL}_2(q)$.

Lemma 4.3.2. *Let* $i \in \{1, 2\}$*. Then* $S/Z(Q_i)$ *is isomorphic to a Sylow* 3*-subgroup of* $\mathrm{SL}_3(q)$*.*

Proof. Since $X_\alpha Z(Q_1)$, $X_\beta Z(Q_1)$ commute modulo $X_{3\alpha+\beta}Z(Q_1)/Z(Q_1)$ we may write any element of $S/Z(Q_1)$ as $x_\beta(t_1)x_\alpha(t_2)x_{3\alpha+\beta}(t_3)Z(Q_1)$. Then the map θ_1 :

$S/Z(Q_1) \to \mathrm{SL}_3(q)$ such that

$$(x_\beta(t_1)x_\alpha(t_2)x_{3\alpha+\beta}(t_3)Z(Q_1))\theta = \begin{pmatrix} 1 & 0 & 0 \\ t_1 & 1 & 0 \\ t_3 & t_2^3 & 1 \end{pmatrix}$$

is an injective homomorphism, from which it follows that $S/Z(Q_1)$ is isomorphic to a Sylow 3-subgroup of $\mathrm{SL}_3(q)$.

Similarly, since $X_\alpha Z(Q_2)/Z(Q_2)$, $X_\beta Z(Q_2)/Z(Q_2)$ commute modulo $X_{\alpha+\beta}Z(Q_2)/Z(Q_2)$ we may write any element of $S/Z(Q_2)$ as $x_\alpha(t_1)x_\beta(t_2)x_{\alpha+\beta}(t_3)Z(Q_2)$. Then the map $\theta_2 : S/Z(Q_2) \to \mathrm{SL}_3(q)$ such that

$$(x_\alpha(t_1)x_\beta(t_2)x_{\alpha+\beta}(t_3)Z(Q_2))\theta_2 = \begin{pmatrix} 1 & 0 & 0 \\ t_1 & 1 & 0 \\ t_3 & t_2 & 1 \end{pmatrix}$$

is an injective homomorphism, from which it follows that $S/Z(Q_2)$ is isomorphic to a Sylow 3-subgroup of $\mathrm{SL}_3(q)$. □

We summarize some further structural results concerning S, Q_1 and Q_2. Some are easily calculated using the commutator formulas, while others are lifted from [PR06, Definition 2.1] and [PR06, Lemma 6.5].

Lemma 4.3.3. *For $i \in \{1,2\}$, we have the following:*

(i) $Q_1 \cap Q_2 = Z(Q_1)Z(Q_2) \in \mathcal{A}(S)$ *has order* q^4*;*

(ii) S *has nilpotency class* 3*;*

(iii) $C_S(Z(Q_i)) = Q_i$, $|Z(Q_i)| = q^3$, $Z(Q_1) \cap Z(Q_2) = \Phi(Q_1) \times \Phi(Q_2) = Z(S)$ *is* *of order q^2 and $\Phi(Q_i)$ is of order q;*

(iv) $[Q_i, Z(Q_{3-i})] = \Phi(Q_i)$;

(v) *for $x \in S \setminus Q_i$ we have that $[x, Q_i]Z(Q_i) = Q_1 \cap Q_2$ and $[x, Z(Q_i)]\Phi(Q_i) = Z(S)$;*

(vi) *Q_i is of exponent 3, S is of exponent 9, $\Omega(S) = S$ and $\mho(S) = Z(S)$;*

(vii) *if $z \in S$ is of order 3 then $z \in Q_1 \cup Q_2$; and*

(viii) *if $x \in Q_1 \setminus Q_2$ and $y \in Q_2 \setminus Q_1$ then $[y, x, x] \neq 1 \neq [x, y, y]$.*

Lemma 4.3.4. *Suppose $R \leq S$ has exponent 3. Then $R \leq Q_1$ or $R \leq Q_2$.*

Proof. As R has exponent 3, $R \subset Q_1 \cup Q_2$ by Lemma 4.3.3 (vii). If $R \not\leq Q_1$ and $R \not\leq Q_2$, then there exists $r \in R \setminus Q_1$ and $s \in R \setminus Q_2$. But then $rs \notin Q_1 \cup Q_2$, which is impossible. □

Lemma 4.3.5. *Let S be isomorphic to a Sylow 3-subgroup of $\mathrm{G}_2(3^n)$. Then $Q_1 \cap Q_2$ is characteristic in S, $N_{\mathrm{Aut}(S)}(Q_1) = N_{\mathrm{Aut}(S)}(Q_2)$ has index at most 2 in $\mathrm{Aut}(S)$ and for $\alpha \in \mathrm{Aut}(S)$ with non-trivial image in $\mathrm{Aut}(S)/N_{\mathrm{Aut}(S)}(Q_i)$, $Q_i\alpha = Q_{3-i}$ for $i \in \{1, 2\}$.*

Proof. By Lemma 4.3.4, Q_1 and Q_2 are the only subgroups of S of order q^5 and exponent 3. Therefore $\mathrm{Aut}(S)$ permutes $\{Q_1, Q_2\}$. As Q_1 and Q_2 are exchanged in $\mathrm{Aut}(S)$, $N_{\mathrm{Aut}(S)}(Q_1)$ has index at most 2 in $\mathrm{Aut}(S)$ and $N_{\mathrm{Aut}(S)}(Q_1) = N_{\mathrm{Aut}(S)}(Q_2)$. Furthermore, it follows that $Q_1 \cap Q_2$ is a characteristic subgroup of S. □

Proposition 4.3.6. *Let S be isomorphic to a Sylow 3-subgroup of $\mathrm{G}_2(3^n)$. Then $\mathrm{Aut}(S) = CH$ where C is a normal 3-subgroup and $H = N_{\mathrm{Aut}(\mathrm{G}_2(q))}(S)$.*

Proof. We have that $|N_{\mathrm{Aut}(\mathrm{G}_2(q))}(S)| = q^6.(q-1)^2.2n$ where $q = 3^n$, and so $|\mathrm{Aut}(S)|_{3'} \geqslant (q-1)^2.2n$. Note that $N_{\mathrm{Aut}(S)}(Q_1) = N_{\mathrm{Aut}(S)}(Q_2)$ normalizes $Z(Q_1)$ and $Z(Q_2)$ and so acts on both $S/Z(Q_1)$ and $S/Z(Q_2)$. Let $\alpha \in N_{\mathrm{Aut}(S)}(Q_1)$. If α acts trivially on $S/Z(Q_1)$ and $S/Z(Q_2)$, then α acts trivially on $S/Z(S)$ and since $Z(S) \leq \Phi(S)$, α acts trivially on $S/\Phi(S)$. By Lemma 2.1.8, all such automorphism form a normal 3-subgroup of $\mathrm{Aut}(S)$. Now, every other automorphism acts non-trivially on $S/Z(Q_i)$ for some $i \in \{1,2\}$ and so embeds in $\mathrm{Aut}(S/Z(Q_i))$. Without loss of generality, let $i = 1$. By Lemma 4.3.2, $S/Z(Q_1)$ is isomorphic to a Sylow 3-subgroup of $\mathrm{SL}_3(q)$, and by [PR06, Proposition 5.3], $\mathrm{Aut}(S/Z(Q_1)) = A.\mathit{\Gamma}\mathrm{L}_2(q)$ where A is a normal 3-subgroup of $\mathrm{Aut}(S/Z(Q_1))$ which centralizes $S/Q_1 \cap Q_2$. In particular, setting $C = C_{\mathrm{Aut}(S)}(S/Q_1 \cap Q_2)$, C is a normal 3-subgroup of $\mathrm{Aut}(S)$ and $\mathrm{Aut}(S)/C$ has an index 2 subgroup which normalizes Q_1 and is isomorphic to a subgroup of $\mathit{\Gamma}\mathrm{L}_2(q)$. Specifically, $N_{\mathrm{Aut}(S/Z(Q_1))}(Q_1/Z(Q_1)) = N_{\mathrm{Aut}(S/Z(Q_1))}(T)$ where $T \in \mathrm{Syl}_3(\mathrm{Aut}(S/Z(Q_1)))$. Therefore, $|\mathrm{Aut}(S)|_{3'} \leqslant (q-1)^2.2n$ and it follows that $|\mathrm{Aut}(S)|_{3'} = |N_{\mathrm{Aut}(\mathrm{G}_2(q))}(S)|_{3'}$ and $\mathrm{Aut}(S) = CH$ where $C = C_{\mathrm{Aut}(S)}(S/Q_1 \cap Q_2)$ and $H = N_{\mathrm{Aut}(\mathrm{G}_2(q))}(S)$. $\square$

Lemma 4.3.7. *Let $x \in Q_i \setminus Z(Q_i)$. Then $|C_{Q_i}(x)| = q^4$ and $\mathcal{A}(Q_i) = \{C_{Q_i}(x) \mid x \in Q_i \setminus Z(Q_i)\}$.*

Proof. By symmetry, we may as well suppose that $i = 1$. Then Lemma 4.3.1 implies that $Q_1 = Z(Q_1)X$. Moreover, for $x \in Q_1 \setminus Z(Q_1)$, $C_{Q_1}(x) = Z(Q_1)C_X(x)$ and an easy calculation in X shows that $C_X(x)$ has order q^2. Hence $C_{Q_1}(x)$ is elementary abelian of order q^4. Since the maximal elementary abelian subgroups of X have order q^2, the result follows. $\square$

We now determine the set of essential subgroups of a saturated fusion system $\mathcal{F}$ on S over a series of lemmas and propositions. As in the case where $p = 2$, it is enough to assume that a candidate essential is S-radical and S-centric and so we perform the analysis in this more general setting.

Lemma 4.3.8. *Let E be an S-centric, S-radical subgroup of S and suppose that $Q_1 \cap Q_2 < E$. Then $Q_1 \leq E$ or $Q_2 \leq E$ or $E = S$.*

Proof. Suppose that E is an S-centric, S-radical subgroup with $Q_1 \cap Q_2 < E$, $Q_1 \not\leq E$ and $Q_2 \not\leq E$. Note that $E \trianglelefteq S$ as $S' \leq Q_1 \cap Q_2 < E$. Since all elements of S of order 3 are contained in $Q_1 \cup Q_2$ we deduce that $\Omega(E) = (Q_1 \cap E)(Q_2 \cap E)$. Let $\alpha \in \mathrm{Aut}(E)$ and notice that $\Omega(E)$ is characteristic in E, so is normalized by α. Suppose also that $(Q_1 \cap E)\alpha \neq (Q_1 \cap E)$. We follow the same argument as Proposition 4.3.6 to see that $(Q_1 \cap E)\alpha = (Q_2 \cap E)$ and $(Q_2 \cap E)\alpha = (Q_1 \cap E)$ so that α fixes $(Q_1 \cap Q_2 \cap E)$. Therefore, in all cases, at least one of $(Q_1 \cap E)$, $(Q_2 \cap E)$ or $(Q_1 \cap Q_2 \cap E) = Q_1 \cap Q_2$ is characteristic in E.

Suppose $Q_1 \cap Q_2$ is characteristic in E. If $E \leq Q_i$ for some $i \in \{1, 2\}$, then as E is S-centric, $Z(Q_i) \leq Z(E)$. If $Z(Q_i) = Z(E)$ then Q_i centralizes the chain $\{1\} \trianglelefteq Z(E) \trianglelefteq E$, a contradiction since $Q_i \not\leq E$ and E is S-radical. Hence, there is $e \in Z(E) \setminus Z(Q_1)$ and since $Q_1 \cap Q_2$ is a maximal elementary abelian subgroup of S which centralizes $Z(E)$, by Lemma 4.3.7, we conclude that $E \leq C_S(Z(E)) = Q_1 \cap Q_2$, a contradiction. Therefore, $E \not\leq Q_i$ for $i \in \{1, 2\}$. We have that $[E, S] \leq [S, S] = S' \leq Q_1 \cap Q_2$ and since $E \not\leq Q_i$, we have that $[Q_1 \cap Q_2, E] = [Z(Q_1), E][Z(Q_2), E] = Z(S) = [Q_1 \cap Q_2, S]$. But $[Q_1 \cap Q_2, E]$ is a commutator of two characteristic subgroups of E, so is characteristic in E. Thus, S centralizes the characteristic chain $\{1\} \trianglelefteq [Q_1 \cap Q_2, E] \trianglelefteq Q_1 \cap Q_2 \trianglelefteq E$, and since E is S-radical, we conclude that $E = S$.

Suppose now that $Q_1 \cap E$ is characteristic in E and $Q_1 \cap Q_2 \leq E$ is not characteristic. Then $Q_1 \cap Q_2 \le Q_1 \cap E$ and $Z(Q_1 \cap E)$ centralizes $Q_1 \cap Q_2$. Since $Q_1 \cap Q_2$ is maximal elementary abelian, $Z(S) \le Z(Q_1 \cap E) \le Q_1 \cap Q_2$. If there is $x \in Z(Q_1 \cap E) \setminus Z(Q_1)$ then by Lemma 4.3.7, $C_{Q_1}(x) = Q_1 \cap Q_2$. But then $Q_1 \cap E$ obviously centralizes x so that $Q_1 \cap E = Q_1 \cap Q_2$ is characteristic in E, a contradiction. Therefore, we deduce that $Z(Q_1 \cap E) = Z(Q_1)$. But now $[Q_1, E] \le Q_1 \cap E$, $[Q_1, Q_1 \cap E] \le Q_1' \le Z(Q_1 \cap E)$ and $[Q_1, Z(Q_1 \cap E)] = \{1\}$ so that Q_1 centralizes the chain $\{1\} \trianglelefteq Z(Q_1 \cap E) \trianglelefteq Q_1 \cap E \trianglelefteq E$ and since E is S-radical, $Q_1 = Q_1 \cap E$ is a characteristic subgroup of E. The argument when $Q_2 \cap E$ is characteristic in E is similar. □

Proposition 4.3.9. *Let E be an S-centric, S-radical subgroup of S such that $Q_1 \cap Q_2 < E < S$. Then $E = Q_i$.*

Proof. By Lemma 4.3.8, we may assume that $Q_1 \le E$ or $Q_2 \le E$. Without loss of generality, suppose that $Q_1 < E$ but $Q_2 \not\le E$. By the proof of Lemma 4.3.8, Q_1 is characteristic in E. By the Dedekind modular law, $E = E \cap S = E \cap Q_1Q_2 = Q_1(E \cap Q_2)$ so that there exists $x \in (E \cap Q_2) \setminus Q_1$. As a consequence, using the commutator formulas, we deduce that $E'Z(Q_1) = Q_1 \cap Q_2$ is a characteristic subgroup of E and $Z(E) = Z(S)$. But then Q_2 centralizes the chain $\{1\} \trianglelefteq Z(E) \trianglelefteq Q_1 \cap Q_2 \trianglelefteq E$, a contradiction since $Q_2 \not\le E$ and E is S-radical. Therefore, $E = Q_1$, as required. □

Proposition 4.3.10. *Let $E \le S$ be an S-centric, S-radical subgroup of S such that $Q_1 \cap Q_2 \not\le E$. Then for some $i \in \{1, 2\}$, $E \in \mathcal{A}(Q_i)$ is of order q^4 and $N_S(E) = Q_i$.*

Proof. Suppose that $Q_1 \cap Q_2 \not\le E$. If $Z(E) \le Q_1 \cap Q_2$, since $[E, Q_1 \cap Q_2] \le [S, Q_1 \cap Q_2] = Z(S) \le Z(E)$, $Q_1 \cap Q_2$ centralizes the chain $\{1\} \trianglelefteq Z(E) \trianglelefteq E$, a contradiction since E is S-radical. Thus, $Z(E) \not\leq Q_1 \cap Q_2$. Since $Q_1 \cap Q_2 \not\leq E$, and $Q_1 \cap Q_2 = Z(Q_1)Z(Q_2)$, we may assume without loss of generality that $Z(Q_1) \not\leq E$. If $\Omega(Z(E)) \leq Q_1$ then, since $[E, Z(Q_1)] \leq [S, Z(Q_1)] = Z(S) \leq \Omega(Z(E))$, $Z(Q_1)$ centralizes the chain $\{1\} \trianglelefteq \Omega(Z(E)) \trianglelefteq E$, a contradiction.

Hence, $\Omega(Z(E)) \not\leq Q_1$ and so, $\Omega(Z(E)) \leq Q_2$ by Lemma 4.3.4. Since E centralizes $\Omega(Z(E))$, it follows from the commutator formulas that $E \leq Q_2$ and since E is S-centric, we conclude $Z(Q_2) \leq \Omega(Z(E))$. Moreover, since $Z(E) \not\leq Q_1 \cap Q_2$, there exists $e \in Z(E) \setminus Z(Q_2)$ and therefore $E \leq C_S(e) \in \mathcal{A}(Q_2)$ by Lemma 4.3.1. Since E is S-centric, $E = C_S(e)$ is elementary abelian of order q^4 and calculating using the commutator formulas, it follows that $N_S(E) = Q_2$. A similar argument when $Z(Q_2) \not\leq E$ completes the proof. □

Having identified the S-centric, S-radical subgroups we now turn our attention to a fixed saturated fusion system $\mathcal{F}$ on S and its essential subgroups. In the following, to restrict the list of centric, radical subgroups, we make use of Lemma 2.3.10, again stressing that this result does not rely on $\mathcal{K}$-group hypothesis. Moreover, we use some results in [PS18] and even though the hypothesis there includes $O_3(\mathcal{F}) = \{1\}$, the results we use are independent of this. Thus, we can still operate in a completely general setting.

Lemma 4.3.11. *Let E be an essential subgroup of a saturated fusion system $\mathcal{F}$ on S. Then $Q_1 \cap Q_2 \leq E$.*

Proof. By Proposition 4.3.10, without loss of generality, we assume that E is a maximal elementary abelian subgroup of $N_S(E) = Q_2$, $E \cap Q_1 = Z(Q_2)$ and $E(Q_1 \cap Q_2) = Q_2$. Since $Z(Q_2)$ is an index q subgroup of E centralized by Q_2, it follows by Lemma 2.3.10 that $O^{3'}(\mathrm{Out}_{\mathcal{F}}(E)) \cong \mathrm{SL}_2(q)$ and $E/C_E(O^{3'}(\mathrm{Out}_{\mathcal{F}}(E)))$ is a natural $\mathrm{SL}_2(q)$-module. Set $Z_E := C_E(O^{3'}(\mathrm{Out}_{\mathcal{F}}(E))) \leq Z(Q_2)$ and let $1 \neq t_E \in Z(O^{3'}(\mathrm{Out}_{\mathcal{F}}(E)))$. By Proposition 4.3.9 and Proposition 4.3.10, every essential subgroup is contained in either Q_1 or Q_2. In particular, Q_2 is the only possible essential subgroup E is contained in. Since t_E normalizes $\mathrm{Out}_S(E)$, using that E is receptive, and applying the Alperin–Goldschmidt theorem, we conclude that t_E lifts to some automorphism of S or Q_2, and since $Q_2 = N_S(E)$, the lift of t_E normalizes Q_2 in both cases.

Suppose that t_E lifts to some automorphism of S and call this morphism t_E^*. Since t_E^* normalizes Q_2, by Lemma 4.3.5 t_E^* normalizes Q_1. Moreover, t_E^* centralizes $Z(Q_1)/Z(S) = Z(Q_1)/(Z(Q_1) \cap E) \cong Q_2/E$. Since t_E^* normalizes $\Phi(Q_2)$, either t_E^* inverts $\Phi(Q_2)$ or centralizes $\Phi(Q_2)$. If t_E^* centralizes $\Phi(Q_2)$, then $[Q_1 \cap Q_2, Q_2, t_E^*] = \{1\}$. But t_E^* centralizes $(Q_1 \cap Q_2)/Z(Q_2) = (Q_1 \cap Q_2)/(Q_1 \cap Q_2 \cap E) \cong Q_2/E$ so that $[t_E^*, Q_1 \cap Q_2, Q_2] = \{1\}$. Then, the three subgroup lemma yields $[t_E^*, Q_2, Q_1 \cap Q_2] = \{1\}$ so that $[t_E^*, Q_2] \leq E \cap Q_1 \cap Q_2 = Z(Q_2)$, a contradiction since $Z_E \leq Z(Q_2)$. Thus, t_E^* inverts $\Phi(Q_2)$ and since $Z_E \leq Q_2$ has order q^2, it follows that t_E^* centralizes $Z(Q_2)/\Phi(Q_2)$ and $(Q_1 \cap Q_2)/\Phi(Q_2) = C_{Q_2/\Phi(Q_2)}(t_E^*)$. Again, t_E^* either inverts S/Q_2 or centralizes S/Q_2. Suppose the latter. Then $t_E^* Q_2$ is normalized by S so that $[Q_2/\Phi(Q_2), t_E^*]$ is normalized by S. But $Z(S/\Phi(Q_2)) \leq (Q_1 \cap Q_2)/\Phi(Q_2) = C_{Q_2/\Phi(Q_2)}(t_E^*)$ from which it follows that $[Q_2/\Phi(Q_2), t_E^*] = \{1\}$, a clear contradiction. Thus, t_E^* inverts S/Q_2. Now, $[t_E^*, Q_1 \cap Q_2, Q_1] = [\Phi(Q_2), Q_1] = \{1\}$ and $[Q_1, (Q_1 \cap Q_2), t_E^*] = [\Phi(Q_1), t_E^*] = \{1\}$, since $\Phi(Q_1) \cap \Phi(Q_2) = \{1\}$. Therefore, by the three subgroup lemma, $[t_E^*, Q_1, Q_1 \cap Q_2] = \{1\}$ and t_E^* centralizes $Q_1/Q_1 \cap Q_2$, a contradiction since t_E^* inverts $S/Q_2 \cong Q_1/(Q_1 \cap Q_2)$.

Suppose that t_E does not lift to a morphism of S. In particular, we may assume that Q_2 is essential. Note that S acts non-trivially on $Z(Q_2)/\Phi(Q_2)$ and centralizes $Z(S)/\Phi(Q_2)$. By Lemma 2.3.10, setting $L_2 := O^{3'}(\mathrm{Out}_{\mathcal{F}}(Q_2))$, we have that $V := Z(Q_2)/\Phi(Q_2)$ is a natural $\mathrm{SL}_2(q)$-module for $L_2/C_{L_2}(V) \cong \mathrm{SL}_2(q)$ and $C_{L_2}(V)$ is a $3'$-group. Then, independently of a $\mathcal{K}$-group hypothesis, provided $q > 3$, Proposition 3.2.7 implies that L_2 is a central extension of $\mathrm{SL}_2(q)$ by a group of p'-order, and so $L_2 \cong \mathrm{SL}_2(q)$. If $q = 3$, then [PS18, Lemma 7.8] implies that $L_2 \cong \mathrm{SL}_2(3)$ and V is a natural $\mathrm{SL}_2(3)$-module. Since S acts non-trivially and quadratically on $Q_2/Z(Q_2)$, $Q_2/Z(Q_2)$ is also a natural $\mathrm{SL}_2(q)$-module for L_2. But then, L_2 is transitive on subgroups of $Q_2/Z(Q_2)$ of order q and there is $\alpha \in L_2$ such that $E\alpha = Q_1 \cap Q_2$, a contradiction since E is fully normalized. This completes the proof. $\square$

As with the case when $p = 2$, we can circumvent the need for a $\mathcal{K}$-group hypothesis. As in the above, we only make use of Lemma 2.3.10 to identify the automizer of an essential subgroup, and this is enough to show that for E an essential subgroup under consideration, $O^{3'}(\mathrm{Out}_{\mathcal{F}}(E)) \cong \mathrm{SL}_2(3^r)$ for some r. Moreover, as intimated when $p = 2$, under such circumstances the proof of Corollary 4.1.3 boils down to recognizing a weak BN-pair of rank 2 whose completion is completely determined using [DS85] which does not rely on any inductive hypothesis. In our application, we identify a specified subsystem of $\mathcal{F}$ within the fusion category of $\mathrm{G}_2(q)$ using this methodology, and then identify $\mathcal{F}$ using the relationship between $\mathrm{Aut}(S)$ and $\mathrm{Aut}(\mathrm{G}_2(q))$ demonstrated in Proposition 4.3.6.

Theorem 4.3.12. *Let $\mathcal{F}$ be a saturated fusion system over a Sylow 3-subgroup of $\mathrm{G}_2(3^n)$. Then one of the following occurs:*

(i) $\mathcal{F} = \mathcal{F}_S(S : \mathrm{Out}_{\mathcal{F}}(S))$;

(ii) $\mathcal{F} = \mathcal{F}_S(Q_i : \mathrm{Out}_{\mathcal{F}}(Q_i))$ *where* $O^{3'}(\mathrm{Out}_{\mathcal{F}}(Q_i)) \cong \mathrm{SL}_2(3^n)$*; or*

(iii) $\mathcal{F} = \mathcal{F}_S(G)$ *where* $F^*(G) = O^{3'}(G) \cong \mathrm{G}_2(3^n)$.

Proof. By Proposition 4.3.9 and Lemma 4.3.11, $\mathcal{E}(\mathcal{F}) \subseteq \{Q_1, Q_2, Q_1 \cap Q_2\}$. Suppose that $Q_1 \cap Q_2$ is essential. Since $S/Q_1 \cap Q_2$ is elementary abelian and of order q^2 and $Z(S)$ is of index q^2 in $Q_1 \cap Q_2$ and centralized by S, it follows by Theorem 3.2.3 that $Q_1 \cap Q_2$ is a natural $\mathrm{SL}_2(q^2)$-module for $L_{12} := O^{3'}(\mathrm{Out}_{\mathcal{F}}(Q_1 \cap Q_2)) \cong \mathrm{SL}_2(q^2)$. But then $|N_{L_{12}}(\mathrm{Out}_S(Q_1 \cap Q_2))| = q^2 - 1$ and since $Q_1 \cap Q_2$ is receptive, each morphism $\phi \in N_{L_{12}}(\mathrm{Out}_S(Q_1 \cap Q_2))$ lifts to some morphism in $\mathrm{Aut}_{\mathcal{F}}(S)$. Since $N_{\mathrm{Aut}_{\mathcal{F}}(S)}(Q_1)$ has index at most 2 in $\mathrm{Aut}_{\mathcal{F}}(S)$, it follows that upon restriction there is a group of index at most 2 in $N_{L_{12}}(\mathrm{Out}_S(Q_1 \cap Q_2))$ normalizing $\mathrm{Out}_{Q_1}(Q_1 \cap Q_2)$, a contradiction unless $q = 3$. If $q = 3$, then $Q_1 \cap Q_2$ is not essential in $\mathcal{F}$ by [PS18, Lemma 7.4].

We have reduced to the case where the set of essentials is contained in $\{Q_1, Q_2\}$. If neither Q_1 nor Q_2 is essential then outcome (i) holds. If Q_i is essential then following an argument in Lemma 4.3.11, we deduce that $O^{3'}(\mathrm{Out}_{\mathcal{F}}(Q_i)) \cong \mathrm{SL}_2(q)$ and both $Q_i/Z(Q_i)$ and $Z(Q_i)/\Phi(Q_i)$ are natural $\mathrm{SL}_2(q)$-modules. In particular, if only one of Q_1, Q_2 is essential then by Lemma 4.3.5 $\mathrm{Aut}_{\mathcal{F}}(S) = N_{\mathrm{Aut}_{\mathcal{F}}(S)}(Q_i)$ and outcome (ii) holds.

Assume that both Q_1 and Q_2 are essential and suppose $Q := O_3(\mathcal{F}) \neq \{1\}$. By Proposition 3.1.13, $Q \leq Q_1 \cap Q_2$. Then $Q \cap Z(S) \neq \{1\}$ and the irreducibility of $Z(Q_i)/\Phi(Q_i)$ under the action of $O^{3'}(\mathrm{Out}_{\mathcal{F}}(Q_i))$ implies that $Z(Q_1)Z(Q_1) \leq Q_1 \cap Q_2 \leq Q \leq Q_1 \cap Q_2$ and $Q = Q_1 \cap Q_2$. Then, the irreducibility of $O^{3'}(\mathrm{Out}_{\mathcal{F}}(Q_i))$ on $Q_i/Z(Q_i)$ gives a contradiction. Therefore, $O_3(\mathcal{F}) = \{1\}$.

Set $\mathcal{F}_0 = \langle N_{\mathcal{F}}(Q_1), N_{\mathcal{F}}(Q_2) \rangle$ so that $\mathrm{Aut}_{\mathcal{F}_0}(S)$ has index at most 2 in $\mathrm{Aut}_{\mathcal{F}}(S)$. It follows by [AKO11, Lemma I.7.6(b)] that $\mathcal{F}_0$ is a saturated subsystem of $\mathcal{F}$ and so $\mathcal{F}_0$ has index 2 in $\mathcal{F}$. In particular, by [AKO11, Theorem I.7.7(c)], $\mathcal{F}_0$ is a normal subsystem of $\mathcal{F}$ and $O^{3'}(\mathcal{F}) \leq O^{3'}(\mathcal{F}_0)$. Now, $\mathcal{F}_0$ satisfies the hypothesis of Corollary 4.1.3 and comparing with the list there, it follows that $O^{3'}(\mathcal{F}_0)$ is isomorphic to the 3-fusion system of $\mathrm{G}_2(3^n)$ and since $O^{3'}(\mathcal{F}_0)$ is simple, we deduce that $O^{3'}(\mathcal{F}_0) = O^{3'}(\mathcal{F})$. By Proposition 4.3.6, we have that $\mathrm{Aut}(S) = CH$, where C is a 3-group and $H = N_{\mathrm{Aut}(\mathrm{G}_2(3^n))}(S)$, and so choices of $\mathrm{Aut}_{\mathcal{F}}(S)$ correspond exactly to $G \leq \mathrm{Aut}(\mathrm{G}_2(q))$ such that $F^*(G) = O^{3'}(G) \cong \mathrm{G}_2(q)$, as required. $\square$

4.4 Fusion Systems on a Sylow p-subgroup of $\mathrm{G}_2(p^n)$ for $p \geqslant 5$

Suppose now that $p \geqslant 5$, $q = p^n$ and S is isomorphic to a Sylow p-subgroup of $\mathrm{G}_2(q)$. Again, we set $\mathbb{K}$ to be a finite field of order q and recall the Chevalley commutator formulas from Section 4.1:

$$\begin{aligned}
[x_\alpha(t), x_\beta(u)] &= x_{\alpha+\beta}(-tu)x_{2\alpha+\beta}(-t^2u)x_{3\alpha+\beta}(t^3u)x_{3\alpha+2\beta}(-2t^3u^2)\\
[x_\alpha(t), x_{\alpha+\beta}(u)] &= x_{2\alpha+\beta}(-2tu)x_{3\alpha+\beta}(3t^2u)x_{3\alpha+2\beta}(3tu^2)\\
[x_\alpha(t), x_{2\alpha+\beta}(u)] &= x_{3\alpha+\beta}(3tu)\\
[x_\beta(t), x_{3\alpha+\beta}(u)] &= x_{3\alpha+2\beta}(tu)\\
[x_{\alpha+\beta}(t), x_{2\alpha+\beta}(u)] &= x_{3\alpha+2\beta}(3tu).
\end{aligned}$$

It then follows that

$$Z_4(S) = S' = \langle X_{\alpha+\beta}, X_{2\alpha+\beta}, X_{3\alpha+\beta}, X_{3\alpha+2\beta}\rangle,$$

$$Z_3(S) = S'' = \langle X_{2\alpha+\beta}, X_{3\alpha+\beta}, X_{3\alpha+2\beta}\rangle,$$

$$Z_2(S) = S''' = \langle X_{3\alpha+\beta}, X_{3\alpha+2\beta}\rangle, \textit{and}$$

$$Z(S) = S'''' = S^{(2)} = \langle X_{3\alpha+2\beta}\rangle$$

are characteristic subgroups of S of orders q^4, q^3, q^2 and q respectively. In particular, the lower and upper central series for S coincide.

We define

$$Q_1 := C_S(Z_3(S)/Z_1(S)) = \langle X_\beta, X_{\alpha+\beta}, X_{2\alpha+\beta}, X_{3\alpha+\beta}, X_{3\alpha+2\beta}\rangle$$

$$Q_2 := C_S(Z_2(S)) = \langle X_\alpha, X_{\alpha+\beta}, X_{2\alpha+\beta}, X_{3\alpha+\beta}, X_{3\alpha+2\beta}\rangle$$

both of order q^5 and characteristic in S. Observe that we may identify Q_1 and Q_2 with the unipotent radical subgroups of maximal parabolic subgroups in $\mathrm{G}_2(q)$. Additionally, $\Phi(Q_1) = Z(Q_1) = Z(S)$ and $\Phi(Q_2) = Z_3(S)$.

We first record some useful structural properties of S, Q_1 and Q_2. There is much more to be said here but we only present the results required to prove Theorem D.

Lemma 4.4.1. *Q_1 is isomorphic to $X_1 * X_2$ where $Z(S) = Z(X_1) = Z(X_2)$ and $X_i \cong T \in \mathrm{Syl}_p(\mathrm{SL}_3(p^n))$ for $i \in \{1, 2\}$.*

Proof. Let $X_1 = X_\beta X_{3\alpha+\beta} X_{3\alpha+2\beta} \leq Q_1$. Since the groups X_β and $X_{3\alpha+\beta}$

commute modulo $X_{3\alpha+2\beta}$, it follows that every element may be written as $x_{3\alpha+\beta}(t_1)x_\beta(t_2)x_{3\alpha+2\beta}(t_3)$ for $t_i \in \mathbb{K}$. Then, using the commutator formulas, we calculate that the map $\theta_1 : X_1 \to \mathrm{SL}_3(q)$ such that

$$(x_{3\alpha+\beta}(t_1)x_\beta(t_2)x_{3\alpha+2\beta}(t_3))\theta_1 = \begin{pmatrix} 1 & 0 & 0 \\ t_1 & 1 & 0 \\ t_3 & t_2 & 1 \end{pmatrix}$$

is an injective homomorphism, from which it follows that X_1 is isomorphic to a Sylow 3-subgroup of $\mathrm{SL}_3(q)$. Similarly, letting $X_2 = X_{2\alpha+\beta}X_{\alpha+\beta}X_{3\alpha+2\beta} \le Q_1$. Then every element of X_2 may be written as $x_{2\alpha+\beta}(t_1)x_{\alpha+\beta}(t_2)x_{3\alpha+2\beta}(t_3)$ for $t_i \in \mathbb{K}$. Then, using the commutator formulas, we calculate that the map $\theta_2 : X_2 \to \mathrm{SL}_3(q)$ such that

$$(x_{2\alpha+\beta}(t_1)x_{\alpha+\beta}(t_2)x_{3\alpha+2\beta}(t_3))\theta_2 = \begin{pmatrix} 1 & 0 & 0 \\ t_1 & 1 & 0 \\ t_3 & 3t_2 & 1 \end{pmatrix}$$

is an injective homomorphism, from which it follows that X_2 is isomorphic to a Sylow 3-subgroup of $\mathrm{SL}_3(q)$. Thus, Q_1 is a central product (over $Z(S) = X_{3\alpha+2\beta}$) of two groups isomorphic to a Sylow p-subgroup of $\mathrm{SL}_3(q)$. □

In the literature, Q_1 is referred to as an *ultraspecial* group. The properties of such groups are well known. See, for example, [Bei77].

Lemma 4.4.2. *Let $x \in Z_3(S) \setminus Z_2(S)$. Then x is S-conjugate to $x_{2\alpha+\beta}(u)$ for some $u \in \mathbb{K}^\times$.*

Proof. Let $x \in Z_3(S) \setminus Z_2(S)$ so that $x = x_{2\alpha+\beta}(t_1)x_{3\alpha+\beta}(t_2)x_{3\alpha+2\beta}(t_3)$ for some $t_1, t_2, t_3 \in \mathbb{K}$ with $t_1 \neq 0$. Then the element $x_\beta(t_3t_2^{-1})x_\alpha(3^{-1}t_2t_1^{-1})$ conjugates x

to $x_{2\alpha+\beta}(t_1)$ if $t_2 \neq 0$ and the element $x_{\alpha+\beta}(3^{-1}t_3t_1^{-1})$ conjugates x to $x_{2\alpha+\beta}(t_1)$ if $t_2 = 0$. $\square$

As in the cases where $p = 2$ or 3, the main tool we use to determine whether a subgroup of S is essential is Lemma 3.2.1 and so for a large number of arguments in this section, we need only assume that any essential candidate is S-radical and S-centric.

Lemma 4.4.3. *Suppose that E is an S-centric, S-radical subgroup of S with $Q_1 \leq E$ or $Q_2 \leq E$. Then $E \in \{Q_1, Q_2, S\}$.*

Proof. Suppose that $Q_1 < E$. Then there is $e = x_\alpha(t_1) \in E$ with $t_1 \neq 0$, applying the commutator formulas, it follows that $Z(E) = Z(S)$, $Z_2(E) = Z_2(S)$, $Z_3(E) = Z_3(S)$ and $E' = S'$. But then Q_2 centralizes the chain $\{1\} \trianglelefteq Z_2(E) \trianglelefteq Z_3(E) \trianglelefteq E' \trianglelefteq E$, and since E is S-radical, $E = S$. In a similar manner, if $Q_2 < E$ then there is $e = x_\beta(t_1) \in E$ with $t_1 \neq 0$. Again, from the commutator formulas, $Z(E) = Z(S)$ and $E' = S'$. Now, Q_1 centralizes the chain $\{1\} \trianglelefteq Z(E) \trianglelefteq E' \trianglelefteq E$ and since E is S-radical, $E = S$. $\square$

Lemma 4.4.4. *Suppose that $E \leq S$ is an S-centric, S-radical subgroup of S with $Z_3(S) = S'' \leq E$. Then $E = Z_3(S)$ or $Z(E) \leq Z_2(S)$. Moreover, if E is essential, then $E \neq Z_3(S)$.*

Proof. Since $Z_3(S) \leq E$ is self-centralizing, we have that $Z(E) \leq Z_3(S)$. By Lemma 4.4.2, if $Z(E) \not\leq Z_2(S)$ then there is $e \in Z(E) \setminus Z_2(S)$ with e conjugate in S to some $x_{2\alpha+\beta}(u)$. Thus, $Z_3(S) \leq E \leq C_S(e) = Z_3(S)(X_\beta)^s$ for some $s \in S$. Suppose that $E > Z_3(S)$. Since E is self centralizing $Z(C_S(e)) = Z(S)(X_{2\alpha+\beta})^s \leq Z(E)$ and so $Z(E) = Z(C_S(e))$. Therefore, $C_S(e)$ centralizes the series $\{1\} \trianglelefteq$

$Z(E) \trianglelefteq E$ so that $E = C_S(e) \leq Q_1$. But now, Q_1 centralizes the series $\{1\} \trianglelefteq E' = Z(S) = Q_1' \trianglelefteq E$, a contradiction.

Suppose that $E = Z_3(S)$ is an essential subgroup of $\mathcal{F}$. Then Q_2/E is elementary abelian of order q^2 and centralizes $Z_2(S)$ which has index q in $Z_3(S)$. Then Lemma 2.3.10 provides a contradiction. $\square$

Lemma 4.4.5. *Suppose that E is an S-centric, S-radical subgroup of S with $Z_3(S) = S'' \leq E$ and $Z(E) = Z(S)$. Then $E \in \{Q_1, S\}$.*

Proof. Since $Z(E) = Z(S)$, we infer that $E \not\leq Q_2$. Moreover, if $E \leq Q_1$, then $[E, Q_1] \leq Q_1' = Z(S) = Z(E)$ and Q_1 centralizes the chain $\{1\} \trianglelefteq Z(E) \trianglelefteq E$. Since E is S-radical, it follows that $E = Q_1$ in this case. Hence, we may assume throughout that $E \not\leq Q_1$ or Q_2 and so there is $e := x_\alpha(t_1)x_\beta(t_2)x_{\alpha+\beta}(t_3) \in E$ with $t_1 \neq 0 \neq t_2$. Then, $[e, Z_2(S)] = Z(S) \leq E'$ and $[e, X_{2\alpha+\beta}]Z(S) = Z_2(S) \leq E'$. Therefore, $C_E(E') \leq E \cap Q_2$.

Suppose first that $[Z_3(S), E'] = \{1\}$. Since $Z_3(S)$ is self-centralizing, we have that $Z_2(S) \leq E' \leq Z_3(S)$. If $E' \neq Z_2(S)$, then $Z_3(S) = C_E(E')$ is a characteristic subgroup of E. Then $E \cap Q_1 = C_E(Z_3(S)/Z(S)) = C_E(Z_3(S)/Z(E))$ is also characteristic in E. Then, since S' normalizes E, S' centralizes the chain $\{1\} \trianglelefteq Z(E) \trianglelefteq E \cap Q_1 \trianglelefteq E$ and since E is radical, $S' \leq E$ by Lemma 3.2.1. But then $E \trianglelefteq S$ and Q_1 centralizes the chain $\{1\} \trianglelefteq Z(E) \trianglelefteq E \cap Q_1 \trianglelefteq E$ and so $Q_1 \leq E$. Then by Lemma 4.4.3, $E = Q_1$ or $E = S$ and since $Z_2(S) \leq E'$ and $[E', Z_3(S)] = \{1\}$, we have a contradiction in either case. Therefore, $E' = Z_2(S)$ and $E \cap Q_2 = C_E(E')$ is characteristic in E.

If $E \cap S' > Z_3(S)$, as $E' = Z_2(S)$, it follows from the commutator formulas that $E \cap Q_2 = E \cap S'$. But then S' centralizes the chain $\{1\} \trianglelefteq Z(E) \trianglelefteq E \cap S' \trianglelefteq E$

and since E is S-radical, $S' \leq E$, $E \trianglelefteq S$ and S' is characteristic in E. Now, Q_1 centralizes the chain $\{1\} \trianglelefteq Z(E) \trianglelefteq S' \trianglelefteq E$ so that $Q_1 \leq E$and, by Lemma 4.4.3, $E = S$ or $E = Q_1$. Since $E' = Z_2(S)$, we have a contradiction in either case. Thus, $E \cap S' = Z_3(S)$. If $E \cap Q_2 = Z_3(S)$, then S' centralizes the chain $\{1\} \trianglelefteq Z(E) \trianglelefteq Z_3(S) \trianglelefteq E$ and since E is S-radical, $S' \leq E$. Since $E \cap S' = Z_3(S)$, this is an obvious contradiction. Thus, $Z_3(S) = E \cap S' < E \cap Q_2$. Since $E \not\leq Q_2$, there is $e := x_\alpha(t_1)x_\beta(t_2)x_{\alpha+\beta}(t_3) \in E$ with $t_2 \neq 0$ and $\widetilde{e} := x_\alpha(\widetilde{t_1})x_{\alpha+\beta}(\widetilde{t_2}) \in E \cap Q_2$ with $\widetilde{t_1} \neq 0$. But then, $[e, \widetilde{e}] \not\leq Z_2(S) = E'$, a contradiction.

Suppose now that $[Z_3(S), E'] \neq \{1\}$. Since $Z_2(S) \leq E'$, it follows that there is $x := x_{\alpha+\beta}(t_1)x_{2\alpha+\beta}(t_2) \in E'$ with $t_1 \neq 0$. In particular, $S' \cap E \leq C_E(E'/Z(E)) \leq Q_1 \cap E$ and so S' centralizes the chain $\{1\} \trianglelefteq Z(E) \trianglelefteq C_E(E'/Z(E)) \trianglelefteq E$ and since E is S-radical, $S' \leq E$. Therefore, $S' \leq C_E(E'/Z(E))$, $E \trianglelefteq S$ and Q_1 centralizes the chain $\{1\} \trianglelefteq Z(E) \trianglelefteq C_E(E'/Z(E)) \trianglelefteq E$. Since E is S-radical, $Q_1 \leq E$ and since $[Z_3(S), E'] \neq \{1\}$, it follows from Lemma 4.4.3 that $E = S$. □

Lemma 4.4.6. *Suppose that E is an S-centric, S-radical subgroup of S with $Z_3(S) = S'' < E$ and $Z(E) \neq Z(S)$. Then $E = Q_2$; or $E \leq Q_2$ has order q^4, $\Phi(E) < Z_2(S) = Z(E)$, $|\Phi(E)| = q$ and $N_S(E) = Q_2$. Moreover, if E is essential then $E = Q_2$.*

Proof. By Lemma 4.4.4, $Z(S) < Z(E) \leq Z_2(S)$. Then $E \leq Q_2$ and $Z(E) = Z_2(S)$ is characteristic in E. If $S' = E$ then Q_1 centralizes the chain $\{1\} \trianglelefteq E' \trianglelefteq E$, a contradiction since E is assumed to be S-radical; and if $S' < E$, by the commutator formulas, it follows that $Z_2(E) = Z_3(S) = Q_2'$ is characteristic in E and so Q_2 centralizes the chain $\{1\} \trianglelefteq Z(E) \trianglelefteq Z_2(E) \trianglelefteq E$ and as E is S-radical, $E = Q_2$ in this case. Hence, $S' \not\leq E$. Moreover, if $E \leq S'$ then S' centralizes the series $\{1\} \trianglelefteq Z(E) \trianglelefteq E$ so $E \not\leq S'$. Suppose there exists $x \in (S' \cap E) \setminus Z_3(S)$ and let $e \in E \setminus S'$.

Since $Z_3(S) \leq S' \cap E$, we may take $x = x_{\alpha+\beta}(t_1)$. Then $Z(S) = [x, Z_3(S)] \leq E'$ and $Z_2(S) = Z(S)[e, Z_3(S)] \leq E'$. Thus, $Z_2(S) < Z_2(S)[e, x] \leq E' \leq Z_3(S)$, $C_E(E') = Z_3(S)$ is characteristic in E and S' centralizes the chain $\{1\} \trianglelefteq Z(E) \trianglelefteq C_E(E') \trianglelefteq E$, a contradiction since E is S-radical. Hence, $S' \cap E = Z_3(S)$ and since $S'E \leq Q_2$, $|E| \leqslant q^4$. Moreover, comparing with commutator formulas, it follows that $N_S(E) = Q_2$.

Now, analyzing Q_2 within $\mathrm{G}_2(q)$, we see that $Q_2/Z_3(S)$ is a natural $\mathrm{SL}_2(q)$ module for $O^{p'}(\mathrm{Out}_{\mathrm{G}_2(q)}(Q_2)) \cong \mathrm{SL}_2(q)$. In particular, E is contained in some subgroup X of order q^4 such that X is conjugate in $O^{p'}(\mathrm{Out}_{\mathrm{G}_2(q)}(Q_2))$ to S'. Since $S^{(2)} = Z(S)$, and $Z_2(S)$ is also a natural $\mathrm{SL}_2(q)$ module for $O^{p'}(\mathrm{Out}_{\mathrm{G}_2(q)}(Q_2)) \cong \mathrm{SL}_2(q)$, it follows that $\Phi(X)$ is a group of order q contained in $Z_2(S) = Z(E)$. In particular, if $E < X$, then X centralizes the chain $\{1\} \trianglelefteq Z(E) \trianglelefteq E$, a contradiction since E is S-radical. Therefore, $E = X$ is of order q^4 and satisfies the required properties.

Assume now that E is essential. By the results in [PS18, Lemma 4.4], we may assume that $q > p$ else the result holds. Note that Q_2 centralizes $Z_2(S)$ and since $Q_2 = N_S(E)$, $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E))$ centralizes $Z_2(S) = Z(E)$. Moreover, since $\Phi(E) \leq Z_2(S)$, $|Q_2/E| = q$,$|E/Z_3(S)| = q$ and $[Q_2, Z_3(S)] = Z_2(S)$, it follows by a similar argument to Lemma 2.3.10 that $E/Z(E)$ is a natural $\mathrm{SL}_2(q)$-module for $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E)) \cong \mathrm{SL}_2(q)$.

Suppose first that Q_2 is essential in $\mathcal{F}$. Moreover, by Lemma 4.4.3, Q_2 is maximally essential. Since $\Phi(Q_2) = Z_3(S)$ and $[S, S'] \leq Z_3(S)$, by Lemma 2.3.10 we ave that $Q_2/\Phi(Q_2)$ is a natural $\mathrm{SL}_2(q)$-module for $O^{p'}(\mathrm{Out}_{\mathcal{F}}(Q_2)) \cong \mathrm{SL}_2(q)$. But then, $O^{p'}(\mathrm{Out}_{\mathcal{F}}(Q_2))$ acts transitively on subgroups of Q_2 of order q^4 containing $\Phi(Q_2) = Z_3(S)$ so that E is conjugate in $\mathcal{F}$ to S'. Since E was assumed to be fully $\mathcal{F}$-normalized, this is a contradiction.

Hence, we may assume that Q_2 is not essential. Note that as any essential containing E contains S'', we may as well assume that E is not properly contained in any essential subgroup and so E is maximally essential. Let t_E be a non-trivial element in $Z(O^{p'}(\mathrm{Out}_{\mathcal{F}}(E)))$. Using that t_E normalizes $\mathrm{Out}_S(E)$, E is receptive and applying the Alperin–Goldschmidt theorem, t_E lifts to some morphism in $\mathrm{Aut}_{\mathcal{F}}(S)$ and so normalizes $Z_3(S)$ and S'. Moreover, since $E/Z(E)$ is natural $\mathrm{SL}_2(q)$-module, t_E inverts $Z_3(S)/Z(E)$, centralizes $Z(E)$ and centralizes $Q_2E/E \cong S'/Z_3(S)$. But now, $[t_E, S', Z_3(S)] = \{1\}$ since $Z_3(S)$ is abelian, and $[S', Z_3(S), t_E] = \{1\}$. By the three subgroup lemma, $[t_E, Z_3(S), S'] = \{1\}$ and so $[t_E, Z_3(S)] \leq Z(S') = Z_2(S) = Z(E)$, a contradiction. □

Lemma 4.4.7. *Suppose that E is an S-centric, S-radical subgroup of S with $Z_3(S) \not\leq E$ but $Z_2(S) \leq E$. Then $E \cap Z_3(S) = Z_2(S)$.*

Proof. Since $Z_2(S) \leq E$, we deduce that $Z(E) \leq Q_2$. Suppose that $E \cap Z_3(S) > Z_2(S)$. Since $Z(E)$ centralizes $E \cap Z_3(S)$ and $Z_3(S)$ is self-centralizing in S, it follows that $Z(E) \leq Z_3(S)$. If $Z(E) \cap Z_2(S) > Z(S)$, then $E \leq Q_2$ and $Z_2(S) \leq Z(E) \leq Z_3(S)$. Moreover, if $Z_2(S) < Z(E)$ then, again using that $Z_3(S)$ is self-centralizing, it follows that $E \leq Z_3(S)$ and since E is S-centric, $E = Z_3(S)$, a contradiction. Hence, if $Z_2(S) \cap Z_2(S) > Z(S)$ then $Z(E) = Z_2(S)$. But now, $Z_3(S)$ centralizes the chain $\{1\} \trianglelefteq Z(E) \trianglelefteq E$, a contradiction since E is S-radical and $Z_3(S) \not\leq E$. Therefore, if $E \cap Z_3(S) > Z_2(S)$, then $Z(E) \cap Z_2(S) = Z(S)$.

Suppose that $Z(E) \cap Z_3(S) > Z(S)$ and let $e \in (Z_3(S) \cap Z(E)) \setminus Z(S)$. By Lemma 4.4.2, e is conjugate in S to some element $x_{2\alpha+\beta}(t)$ with $t \neq 0$. Moreover, it follows from the commutator formulas that the centralizer of such an element is contained in Q_1 and intersects S' only in $Z_3(S)$. Since Q_1, S' and $Z_3(S)$ are normal in S, the centralizer of e is contained in Q_1 and intersects S' only in $Z_3(S)$.

But E centralizes $e \leq Z(E)$ and so if $E \leq S'$, then $E \leq Z_3(S)$ and since E is S-centric, we have a contradiction. Therefore, $E \leq Q_1$ and there is $x \in E \setminus S'$. Since $Z_2(S) \leq E$, $Z(S) = [x, Z_2(S)] \leq E' \leq Q_1' = Z(S)$ and so $Z(S) = E'$. But then Q_1 normalizers the chain $\{1\} \trianglelefteq E' \trianglelefteq E$, and since E is S-radical, we conclude that $Z_3(S) \leq Q_1 \leq E$, a contradiction.

Hence, we have shown that if $E \cap Z_3(S) > Z_2(S)$, then $Z(E) = Z(S)$. In particular, $E \not\leq Q_2$ since $Z_2(S) \not\leq Z(E)$ and $E \not\leq Q_1$, for otherwise Q_1 centralizes the chain $\{1\} \trianglelefteq Z(E) \trianglelefteq E$, a contradiction for then $Z_3(S) \leq Q_1 \leq E$ since E is S-radical. Now, $Z_2(S) \leq Z_2(E)$ and since $E \cap Z_3(S) > Z_2(S)$, it follows from the commutator formulas that $Z_2(E) \leq E \cap Q_1$. But then $[Z_3(S), Z_2(E)] \leq Z(S) = Z(E)$, $[Z_3(S), E] \leq Z_2(S) \leq Z_2(E)$ and $Z_3(S)$ centralizes the chain $\{1\} \trianglelefteq Z(E) \trianglelefteq Z_2(E) \trianglelefteq E$, a contradiction since E is S-radical. $\square$

Lemma 4.4.8. *Suppose that ES is an S-centric, S-radical subgroup of S with $Z_3(S) \not\leq E$ but $Z_2(S) \leq E$. Then either $E \leq S'$ is elementary abelian of order q^3, $N_S(E) = Q_1$ and E is not an essential subgroup of any saturated fusion system $\mathcal{F}$ on S; or $E \cap S' = Z_2(S)$.*

Proof. By Lemma 4.4.7, we may assume that $E \cap Z_3(S) = Z_2(S)$. Suppose that $E \cap S' > Z_2(S)$. It then follows from the commutator formulas that $Z(E) \leq S'$. If $Z(E) \cap Z_2(S) > Z(S)$, then $E \leq Q_2$. But then $Z_2(S) \leq Z(E)$ and since $Z_3(S) \not\leq E$ and E is S-radical, we conclude that $Z_2(S) < Z(E)$ for otherwise, $Z_3(S)$ centralizes the chain $\{1\} \trianglelefteq Z(E) \trianglelefteq E$. But then, there is $e \in (Z(E) \cap S') \setminus Z_3(S)$ and it follows from the commutator formulas that $E \leq S'$ and since $E \cap Z_3(S) = Z_2(S)$, $|E| \leqslant q^3$. We may set $e := x_{\alpha+\beta}(t_1)x_{2\alpha+\beta}(t_2)x$, where $x \in Z_2(S)$ and $t_1 \in \mathbb{K}^\times$. Then for $y := x_\alpha(-t_2 2^{-1} t^{-1})$, $e^y = x_{\alpha+\beta}(t_1)x'$ for some $x' \in Z_2(S)$. Then $C_S(e^y Z_2(S)) = X_{\alpha+\beta} Z_2(S)$ and it follows that $E \leq C_S(e)$ is conjugate to a subgroup of $X_{\alpha+\beta}Z_2(S)$. Moreover, since E is S-centric and $X_{\alpha+\beta}Z_2(S)$ is elementary abelian, E is conjugate to $X_{\alpha+\beta}Z_2(S)$ and a calculation using the commutator formulas gives that $N_S(E) = Q_1$.

Suppose that E is essential. Since $Z_3(S)E/E$ is elementary abelian of order q and $Z_3(S)$ centralizes $Z_2(S)$ which has index q in E, by Lemma 2.3.10 we deduce that $E/C_E(O^{p'}(\mathrm{Out}_{\mathcal{F}}(E)))$ is a natural $\mathrm{SL}_2(q)$-module for $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E)) \cong \mathrm{SL}_2(q)$ and $\mathrm{Out}_{Z_3(S)}(E) \in \mathrm{Syl}_p(\mathrm{Out}_{\mathcal{F}}(E))$. But $Q_1 \leq N_S(E)$ and we have a contradiction.

Hence, if $E \cap S' > Z_2(S)$, then $Z(E) \cap Z_2(S) = Z(S)$. If $Z(E) \neq Z(S)$, then there is $e \in (Z(E) \cap S') \setminus Z(S)$ and it follows from the commutator formulas that the centralizer of such an element is contained in Q_1. Therefore, $E \leq Q_1$ and $E' \leq Q_1' = Z(S)$. Moreover, if there is $x \in E \setminus S'$, then $Z(S) = [x, Z_2(S)] \leq E' = Z(S)$ and so, Q_1 centralizes the chain $\{1\} \trianglelefteq E' \trianglelefteq E$, a contradiction since $Q_1 \not\leq E$ and E is radical. Therefore, $E \leq S'$, which yields another contradiction for then $Z_2(S) \leq Z(E)$.

Finally, we suppose that $E \cap S' > Z_2(S)$, $E \cap Z_3(S) = Z_2(S)$ and $Z(E) = Z(S)$. In particular, $E \not\leq Q_2$ and since $Z_2(S) \leq E$, for $x \in E \setminus Q_2$, $Z(S) = [x, Z_2(S)] \leq E'$. Now, for $e \in (E \cap S') \setminus Z_3(S)$, $[e, Z_2(E)] = Z(E)$ and it follows from the commutator formulas that $Z_2(S) \leq Z_2(E) \leq Q_1$. In particular, $Z_3(S)$ centralizes the chain $\{1\} \trianglelefteq Z(E) \trianglelefteq Z_2(E) \trianglelefteq E$, a contradiction since E is S-radical and $Z_3(S) \not\leq E$. $\square$

Lemma 4.4.9. *Suppose that E is an S-centric, S-radical subgroup of S with $E \cap S' = Z_2(S)$. Then either*

(i) *$E \leq Q_2$ is elementary abelian of order q^3, $E \not\leq S'$ and $N_S(E) = EZ_3(S)$ has order q^4; or*

(ii) *$E \cong q^{1+2}$, $Z_2(S) = E \cap Q_1 = E \cap Q_2$, $Z(S) = Z(E) = \Phi(E)$ and $N_S(E) = EZ_3(S)$ has order q^4.*

Moreover, in both cases E is not essential in any saturated fusion system $\mathcal{F}$ on S.

Proof. Suppose that $E \leq Q_2$. Then $Z_2(S) \leq Z(E)$ and $|E| \leqslant q^3$. If $Z(E) = Z_2(S)$, then $Z_3(S)$ centralizes the chain $\{1\} \trianglelefteq Z(E) \trianglelefteq E$, a contradiction since E is S-radical. Therefore, there is $e \in Z(E) \backslash S'$ and write $e = x_\alpha(t_1)x_{\alpha+\beta}(t_2)x_{3\alpha+\beta}(t_3)x$ for some $x \in Z_2(S)$ and $t_1 \in \mathbb{K}^\times$. Then for $y := x_\beta(t_1^{-1}t_2)x_{\alpha+\beta}(2^{-1}t_1^{1}(t_3 - t_1t_2))$, we have that $e^y = x_\alpha(t_1)x'$ for some $x' \in Z_2(S)$. Then $C_S(e^y Z_2(S)) = X_\alpha Z_2(S)$ and by conjugation, $E \leq C_S(e)$ is conjugate to a subgroup of $X_\alpha Z_2(S)$. Moreover, since E is S-centric and $X_\alpha Z_2(S)$ is elementary abelian, we conclude that E is conjugate to $X_\alpha Z_2(S)$ and a calculation using the commutator formulas gives that $N_S(E) = EZ_3(S)$, as required.

Suppose now that E is essential in a saturated fusion system $\mathcal{F}$ on S. Then $Z_3(S)E/E$ is elementary abelian of order q and $Z_3(S)$ centralizes $Z_2(S)$ which has index q in E. By Lemma 2.3.10, $E/C_E(O^{p'}(\mathrm{Out}_{\mathcal{F}}(E)))$ is a natural $\mathrm{SL}_2(q)$-module for $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E)) \cong \mathrm{SL}_2(q)$ and $\mathrm{Out}_{Z_3(S)}(E) \in \mathrm{Syl}_p(\mathrm{Out}_{\mathcal{F}}(E))$. Since $E \not\leq Q_1$, we may assume by Lemma 4.4.5 and Lemma 4.4.6 that the only possible essential E is properly contained in is Q_2.

If Q_2 is essential then using that S centralizes $S'/Z_3(S) = S'/\Phi(Q_2)$ and $S'/Z_3(S)$ has index q in $Q_2/Z_3(S)$, it follows by Theorem 3.2.3 that $Q_2/Z_3(S)$ is a natural $\mathrm{SL}_2(q)$-module for $O^{p'}(\mathrm{Out}_{\mathcal{F}}(Q_2)) \cong \mathrm{SL}_2(q)$. But then, $O^{p'}(\mathrm{Out}_{\mathcal{F}}(Q_2))$ is transitive on subgroup of order q in $Q_2/\Phi(Q_2)$ and so $E\phi \leq S'$ for some $\phi \in O^{p'}(\mathrm{Out}_{\mathcal{F}}(Q_2))$. Therefore, $[E\phi, Q_1] \leq Z(S) \leq Z_2(S) \leq E\phi$ and $Q_1 \leq N_S(E\phi)$. Since $|N_S(E)| = q^4$, E is not fully normalized, a contradiction.

Hence, we may assume that Q_2 is not essential in $\mathcal{F}$ and for a non-trivial element $t_E \in Z(O^{p'}(\mathrm{Out}_{\mathcal{F}}(E)))$, using that E is receptive, t_E lifts to some $t_E^* \in \mathrm{Aut}_{\mathcal{F}}(S)$. Moreover, by coprime action, $E = [E, t_E^*] \times C_E(t_E^*)$ and either $Z(S) = C_E(t_E^*)$ or $Z(S) \cap C_E(t_E^*) = \{1\}$. Since $Z_2(S) = C_E(Z_3(S))$, it follows in the latter case that t_E^* centralizes $Z_2(S)/Z(S)$ and since $Z_3(S)E/E \cong Z_3(S)/Z_2(S)$, coprime action yields $[Z_3(S), t_E^*] = Z(S)$. Then, $[Z_3(S), S, t_E^*] = Z(S)$, $[t_E^*, Z_3(S), S] = \{1\}$ and the three subgroup lemma yields, $[S, t_E^*, Z_3(S)] \leq Z(S)$ and t_E^* centralizes $S/Q_1 \cong Q_2/S' = ES'/S' \cong E/Z_2(S)$, a contradiction. Thus, t_E^* centralizes $Z(S)$ and inverts $Z_2(S)/Z(S)$. Moreover, t_E^* centralizes $Z_3(S)/Z_2(S)$ and inverts $E/Z_2(S) = E/E \cap S' \cong Q_2/S'$. Now, since $[S', Z_3(S)] \leq Z(S)$ is centralized by t_E^* and $[Z_3(S), t_E^*] \leq Z_2(S)$ is centralized by S', it follows from the three subgroup lemma that $[t_E^*, S', Z_3(S)] = \{1\}$ and since $Z_3(S)$ is self-centralizing, $[t_E^*, S'] \leq Z_3(S)$. Indeed, coprime action implies that $[t_E^*, S'] \leq Z_2(S)$. But then $[t_E^*, S', Q_2] = \{1\}$, $[S', Q_2, t_E^*] \leq Z_2(S)$ and another application of the three subgroup lemma gives $[t_E^*, Q_2, S'] \leq Z_2(S)$. But t_E^* inverts Q_2/S' and a contradiction follows from the commutator formulas.

Assume now that $E \not\leq Q_2$ and since $Z_2(S) \leq E$, for $x \in E \setminus Q_2$, we have that $Z(S) = [x, Z_2(S)] \leq E' \leq E \cap S' = Z_2(S)$. If $Z(S) < E'$, then $C_E(E') = E \cap Q_2$ is characteristic in E. Moreover, $Z_2(S) < C_E(E')$ for otherwise $Z_3(S)$ centralizes the chain $\{1\} \trianglelefteq Z_2(S) \trianglelefteq E$, a contradiction since $Z_3(S) \not\leq E$ and E is S-radical. Furthermore, $Z(E) \cap Q_2 \leq S' \cap E = Z_2(S)$, otherwise $E \leq Q_2$. But then $Z(E) = Z(S)$ and since there is $e \in E \cap Q_2 \setminus Z_2(S)$, $Z_2(S) \leq Z_2(E) \leq E \cap Q_1$ and so $Z_2(S) = Z_2(E) \cap E \cap Q_2$ is characteristic in E and $Z_3(S)$ centralizes the chain $\{1\} \trianglelefteq Z_2(S) \trianglelefteq E$, a contradiction.

Finally, we suppose that $E \cap S' = Z_2(S)$, $E \not\leq Q_2$ and $Z(S) = E'$. If $E \cap Q_2 > Z_2(S)$

then, as $E \not\leq Q_2$, there is $e \in E \backslash Q_2$, with $[e, E \cap Q_2] \not\leq Z(S) = E'$. Hence, $E \cap Q_2 = Z_2(S)$ and $|E| \leqslant q^3$. Notice that if $E \leq Q_1$, then $[E, Q_1] \leq Q_1' = Z(S) = E'$ and Q_1 centralizes the chain $\{1\} \trianglelefteq E' \trianglelefteq E$, a contradiction since $Z_3(S) \not\leq E$ and E is S-radical. Hence, there is $e \in E \setminus (Q_1 \cup Q_2)$ and since $[e, E \cap Q_1] \leq E' = Z(S)$, it follows from the commutator formulas that $E \cap Q_1 = Z_2(S)$. Note that $EQ_1/Q_1 \cong E/Z_2(S)$ is elementary abelian and so, $\Phi(E) \leq Z_2(S)$. If $Z(S) < \Phi(E)$, then $Z_2(S) = C_E(\Phi(E))$ is characteristic in E, a contradiction for then $Z_3(S)$ centralizes then $\{1\} \trianglelefteq Z_2(S) \trianglelefteq E$. Therefore, $\Phi(E) = Z(E) = Z(S)$, $|E| = q^3$ and the commutator formulas imply that $N_S(E) = Z_3(S)E$, as required.

Suppose that E is essential on some saturated fusion system $\mathcal{F}$ supported on S. Since $E \not\leq Q_1, Q_2$, it follows by Lemma 4.4.5 and Lemma 4.4.6 that E is maximally essential. Moreover, $Z_3(S)E/E$ is elementary abelian of order q and $Z_3(S)$ centralizes $Z_2(S)$ which has index q in E. Then by Lemma 2.3.10, $E/Z(E)$ is a natural $\mathrm{SL}_2(q)$-module, $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E)) \cong \mathrm{SL}_2(q)$ and $\mathrm{Out}_{Z_3(S)}(E) \in \mathrm{Syl}_p(\mathrm{Out}_{\mathcal{F}}(E))$.

Let $\lambda \in N_{O^{p'}(\mathrm{Out}_{\mathcal{F}}(E))}(\mathrm{Out}_S(E))$ be an element of order $q-1$, isomorphic to a generator of a torus in $\mathrm{SL}_2(q)$. We can choose λ to act as the scalars μ^{-1} on $E/Z_2(S)$ and as μ on $Z_2(S)/Z(S)$, for $\mu \in \mathbb{K}^\times$. Since E is essential, it is receptive, so we may extend λ to some $\widehat{\lambda}$, and by the Alperin – Goldschmidt Theorem and since E is maximally essential, we may take $\widehat{\lambda} \in \mathrm{Aut}_{\mathcal{F}}(S)$ so that $\widehat{\lambda}$ acts on S', Q_1 and Q_2. Since $E/Z_2(S) \cong ES'/S'$, it follows that $\widehat{\lambda}$ acts as μ^{-1} on ES'/S'. Let $x_\alpha(t_1), x_\beta(t_2)$ be transversals in S/S' such that $x_\alpha(t_1)x_\beta(t_2)S' \in ES'/S'$. We have that

$$x_\alpha(t)\widehat{\lambda} = (x_\alpha(t)x_\beta(u)\widehat{\lambda})(x_\beta(-u)\widehat{\lambda}) = (x_\alpha(\mu^{-1}t)x_\beta(\mu^{-1}u)(x_\beta(-u)\widehat{\lambda})$$

and comparing coefficients, we have that $\widehat{\lambda}$ acts as μ^{-1} on both Q_1/S' and Q_2/S'. Then, by the commutator formula

$$[x_\alpha(t), x_{\alpha+\beta}(u)] = x_{2\alpha+\beta}(-2tu)x_{3\alpha+\beta}(3t^2u)x_{3\alpha+2\beta}(3tu^2)$$

and using that $\widehat{\lambda}$ acts as μ^2 on $N_S(E)/E \cong Z_3(S)/Z_2(S)$, we deduce that $\widehat{\lambda}$ acts as μ^3 on $S'/Z_3(S)$. Using the commutator relation $[x_{\alpha+\beta}(t), x_{2\alpha+\beta}(u)] = x_{3\alpha+2\beta}(3tu)$ we get that $\widehat{\lambda}$ acts as μ^5 on $Z(S)$. But since $Z(S) = C_E(O^{p'}(\mathrm{Out}_{\mathcal{F}}(E)))$ and since λ was of order $q-1$, it follows that $q = 6$, a contradiction. □

Given Lemma 4.4.5, Lemma 4.4.6 and Lemma 4.4.9, we finally assume that $Z_2(S) \not\leq E$. This is a particular interesting case as there is some exceptional behaviour when $q = p = 7$ related to the 7-fusion system of the Monster sporadic simple group. Indeed, this exceptional behaviour produces a distinct class of essentials and with it, a large number of exotic fusion systems. This phenomena was already known about by the work in [PS18].

Lemma 4.4.10. *Suppose that E is an S-centric, S-radical subgroup of S with $Z_2(S) \not\leq E$. Then either*

(i) *$E \leq Q_1$ is elementary abelian of order q^3, $E \not\leq S'$ and $N_S(E) = Q_1$; or*

(ii) *$p \geqslant 7$, E is elementary abelian of order q^2, $E \cap Q_1 = E \cap Q_2 = Z(S)$ and $N_S(E) = Z_2(S)E$.*

Proof. We may suppose $Z(E) \not\leq Q_2$ for otherwise $Z_2(S)$ centralizes the chain $\{1\} \trianglelefteq Z(E) \trianglelefteq E$, a contradiction since $Z_2(S) \not\leq E$ and E is S-radical. In particular, it follows by the commutator formulas that $E \cap Q_2 \leq S'$ and $E \cap Z_2(S) = Z(S)$.

Suppose that $E \cap Q_1 \neq Z(S)$. Then a calculation using the commutator formulas reveals that $Z(E) \leq Q_1$. Then, $Z(E) \not\leq S'$ for otherwise $Z_2(S)$ centralizes the chain $\{1\} \trianglelefteq Z(E) \trianglelefteq E$, and another calculation yields $E \leq Q_1$. Recall from Lemma 4.4.1 that $Q_1 \cong q^{1+2} * q^{1+2}$. Then, $m_p(Q_1) = 3n$ and for any element of order $x \in Q_1 \setminus Z(S)$ of order p, we have that $|C_{Q_1}(x)| = q^4$, $|Z(C_S(e))| = q^2$ and $C_S(e)' = Z(S)$. Since $Z(E) \not\leq Q_2$, there is $e \in Z(E)$ such that $E \leq C_S(e)$ where $C_S(e)$ has order at most q^4. Then, as E is S-centric, $Z(C_S(e)) \leq Z(E)$. Now, if $Z(E) = Z(C_S(e))$, then $C_S(e)$ centralizes the chain $\{1\} \trianglelefteq Z(E) \trianglelefteq E$, and since E is S-radical, $E = C_S(e)$. But then Q_1 centralizes the chain $\{1\} \trianglelefteq E' \trianglelefteq E$, a contradiction since $Z_2(S) \not\leq E$.

So assume that $Z(C_S(e)) < Z(E)$. It follows that there is $e' \in (Z(E) \cap S') \setminus Z(S)$ so that $E \leq C_S(e')$ and again $Z(C_S(e')) \leq Z(E)$. Thus, $Z(C_S(e'))Z(C_S(e))$ is elementary abelian of order q^3 and contained in $Z(E)$. But $m_p(Q_1) = 3n$ and so $E = Z(E) = Z(C_S(e'))Z(C_S(e))$ is elementary abelian of order q^3. It follows directly from the commutator formulas that $N_S(E) = Q_1$.

Thus, we have shown that $Z(S) = E \cap Q_1 = E \cap Q_2$ and $|E| \leqslant q^2$. If $p \geqslant 7$, then as S has exponent p and E is centric, we can explicitly construct elementary abelian subgroups of order q completing $Z(S)$ in E so that $E = \Omega(Z(E))$ is of order q^2. If $p = 5$, then S has exponent 25 and it follows that $\mho(E) = E \cap S' = Z(S)$ and $Z_2(S)$ centralizes the chain $\{1\} \trianglelefteq \mho(E) \trianglelefteq E$, a contradiction since E is S-radical. $\square$

Lemma 4.4.11. *Suppose that $E \leq S$ is an essential subgroup of $\mathcal{F}$ and $Z_2(S) \not\leq E$. Then $q = p = 7$ and $E = \langle Z(S), x\rangle$ for some $x \in S \setminus (Q_1 \cup Q_2)$.*

Proof. By Lemma 4.4.10, we may assume that E is elementary abelian of order q^3 and contained in Q_1; or E is elementary abelian of order q^2 and intersects Q_1 only in $Z(S)$. In the former case, $Z_2(S)E/E$ is elementary abelian of order q and $Z_2(S)$ centralizes $E \cap S'$ which has index q in E. Then by Lemma 2.3.10, it follows that $E/C_E(O^{p'}(\mathrm{Out}_{\mathcal{F}}(E)))$ is a natural $\mathrm{SL}_2(q)$-module for $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E)) \cong \mathrm{SL}_2(q)$. But $N_S(E) = Q_1$ and $|Q_1/E| = q^2$, a contradiction.

Thus, E is elementary abelian of order q^2 and $E \cap Q_1 = E \cap Q_2 = Z(S)$. Since $Z_2(S)$ centralizes $Z(S)$ which has index q in E, by Lemma 2.3.10, E is a natural $\mathrm{SL}_2(q)$-module for $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E)) \cong \mathrm{SL}_2(q)$ and $\mathrm{Out}_{Z_2(S)}(E) = \mathrm{Out}_S(E)$. By Lemma 4.4.5, Lemma 4.4.6 and Lemma 4.4.9 and since $E \not\leq Q_1, Q_2$, we assume that E is maximally essential.

Let $\lambda \in N_{O^{p'}(\mathrm{Out}_{\mathcal{F}}(E))}(\mathrm{Out}_S(E))$ be an element of order $q-1$, isomorphic to a generator of a torus in $\mathrm{SL}_2(q)$. Since E is a natural $\mathrm{SL}_2(q)$-module, for some $\mu \in K^\times$ of order $q-1$, we can choose λ to acts as μ on $Z(S)$ and μ^{-1} on $E/Z(S)$. Since E is receptive, and by the Alperin–Goldschmidt Theorem, λ extends to $\widehat{\lambda} \in \mathrm{Aut}_{\mathcal{F}}(S)$. Since Q_1, Q_2, S' are characteristic in S, λ acts on Q_1/S', Q_2/S' and $ES'/S' \cong E/Z(S)$. Let $x_\alpha(t)$ be a transversal of Q_2/S'. Then $x_\alpha(t)\widehat{\lambda} = (x_\alpha(t)x_\beta(u)x_\beta(-u))\widehat{\lambda}$ for all $u \in K^\times$. But, for some u, $x_\alpha(t)x_\beta(u)$ is a transversal of ES'/S' and $x_\beta(-u)$ is a transversal of Q_1/S' and $\widehat{\lambda}$ acts on ES'/S' as μ^{-1}.

Thus,

$$x_\alpha(t)\widehat{\lambda} = (x_\alpha(t)x_\beta(u)\widehat{\lambda})(x_\beta(-u)\widehat{\lambda}) = (x_\alpha(\mu^{-1}t)x_\beta(\mu^{-1}u)(x_\beta(-u)\widehat{\lambda})$$

and by comparing coefficients, $\widehat{\lambda}$ acts as μ^{-1} on both Q_1/S' and Q_2/S'. Using the commutator formulas on various elements on S, one has that $\widehat{\lambda}$ acts as μ^{-2}, μ^{-3}, μ^{-4} and μ^{-5} on $S'/Z_3(S)$, $Z_3(S)/Z_2(S)$, $Z_2(S)$ and $Z(S)$ respectively. But since $\widehat{\lambda}$ acts on $Z(S)$ as λ does, $\mu^{-5} = \mu$ and $\mu^6 = 1$. Since μ was of order $q-1$, we conclude that $q = p = 7$. In this case, S has exponent 7 and there is $x \in E \setminus (Q_1 \cup Q_2)$ of order 7 such that $E = \langle Z(S), x\rangle$, as required. $\square$

Before determining all possible saturated fusion systems on S, we sum up the results concerning S-centric, S-radical subgroups of S.

Proposition 4.4.12. *Suppose that E is an S-centric, S-radical subgroup of S. Then one of the following holds:*

(i) $E \in \{Q_1, Q_2, S\}$;

(ii) $E \leq Q_2$ *has order* q^4, $\Phi(E) < Z_2(S) = Z(E)$, $|\Phi(E)| = q$ *and* $N_S(E) = Q_2$;

(iii) $E \leq S'$ *is elementary abelian of order* q^3 *with* $E \trianglelefteq S$ *if* $E = Z_3(S)$; *and* $N_S(E) = Q_1$ *otherwise;*

(iv) $E \leq Q_2$ *is elementary abelian of order* q^3, $E \not\leq S'$ *and* $N_S(E) = EZ_3(S)$ *has order* q^4;

(v) $E \cong q^{1+2}$, $Z_2(S) = E \cap Q_1 = E \cap Q_2$, $Z(S) = Z(E) = \Phi(E)$;

(vi) $E \leq Q_1$ *is elementary abelian of order* q^3, $E \cap Z_2(S) = Z(S)$ *and* $N_S(E) = Q_1$; *or*

(vii) E *is elementary abelian of order* q^2, $Z(S) = E \cap Q_1 = E \cap Q_2 = Z(S)$ *and* $N_S(E) = EZ_2(S)$ *has order* q^3.

We now analyze the automizers of the potential essential subgroups of a saturated fusion system $\mathcal{F}$ over S. That is, Q_1, Q_2 and if $q = p = 7$, some conjugacy class of elementary abelian subgroups of order 7^2. For the latter class of essentials, we refer to [PS18] to determine the fusion system, where a large number of exotic fusion systems are uncovered. We analyze the automizer of Q_2 via Lemma 2.3.10, noting that this result is independent of a $\mathcal{K}$-group hypothesis. Analyzing the automizer of Q_1 is more complicated and, with the help of some supporting results, we conclude that $O^{p'}(\mathrm{Out}_{\mathcal{F}}(Q_1))$ is isomorphic to a subgroup of $\mathrm{Sp}_4(q)$. Since the maximal subgroups of $\mathrm{Sp}_4(q)$ are known by [Mit14], we compute the candidates for $O^{p'}(\mathrm{Out}_{\mathcal{F}}(Q_1))$ independent of any $\mathcal{K}$-group hypothesis. We omit the details here, and instead appeal to Proposition 3.2.7 and a result in [PS18].

Finally, we wish to apply Corollary 4.1.3 to determine $\mathcal{F}$. Except in the case where $q = p \in \{5, 7\}$, we have that Q_1, Q_2 are the only possible essentials and $O^{p'}(\mathrm{Out}_{\mathcal{F}}(Q_i)) \cong \mathrm{SL}_2(q)$ for $i \in \{1, 2\}$. In particular, the application of Corollary 4.1.3 via the Main Theorem relies only on the classification of weak BN-pairs of rank 2 provided in [DS85] and again, is independent of any $\mathcal{K}$-group hypothesis. We remark that there is currently no known way of determining whether a fusion system is exotic without appealing to the classification of finite simple groups, and instead appeal to [PS18, Theorem 6.2] for a proof of the exoticity of the fusion systems listed in (vii).

Theorem 4.4.13. *Let $\mathcal{F}$ be a saturated fusion system over a Sylow p-subgroup of $\mathrm{G}_2(p^n)$ with $p \geqslant 5$. Then one of the following holds*

(i) $\mathcal{F} = \mathcal{F}_S(S : \mathrm{Out}_{\mathcal{F}}(S))$;

(ii) $\mathcal{F} = \mathcal{F}_S(Q_1 : \mathrm{Out}_{\mathcal{F}}(Q_1))$ *where* $O^{p'}(\mathrm{Out}_{\mathcal{F}}(Q_1)) \cong \mathrm{SL}_2(q)$ *or* $q = p \in \{5, 7\}$ *and the possibilities for* $O^{p'}(\mathrm{Out}_{\mathcal{F}}(Q_1))$ *are given in [PS18, Lemma 5.2];*

(iii) $\mathcal{F} = \mathcal{F}_S(Q_2 : \mathrm{Out}_{\mathcal{F}}(Q_2))$ *where* $O^{p'}(\mathrm{Out}_{\mathcal{F}}(Q_2)) \cong \mathrm{SL}_2(q)$;

(iv) $\mathcal{F} = \mathcal{F}_S(M)$ *where* $M \cong 5^3.\mathrm{SL}_3(5)$, $p = 5$ *and* $n = 1$;

(v) $\mathcal{F} = \mathcal{F}_S(G)$ *where* $G \cong \mathrm{Ly}$, HN, $\mathrm{HN}.2$ *or* B, $p = 5$ *and* $n = 1$*;*

(vi) $\mathcal{F} = \mathcal{F}_S(G)$ *where* $G \cong \mathrm{M}$, $p = 7$ *and* $n = 1$*;*

(vii) $\mathcal{F}$ *is one of the exotic fusion systems listed in [PS18, Table 5.1],* $p = 7$ *and* $n = 1$*; or*

(viii) $\mathcal{F} = \mathcal{F}_S(G)$ *where* $F^*(G) = O^{p'}(G) \cong \mathrm{G}_2(p^n)$*.*

Proof. Suppose first that there is an essential $E \not\in \{Q_1, Q_2\}$. By Lemma 4.4.11, $p = q = 7$ and the action of $O^{7'}(\mathrm{Out}_{\mathcal{F}}(E))$ is irreducible on E. In particular, since $O_7(\mathcal{F})$ is normal in S and contained in each essential subgroup by Proposition 3.1.13, $O_7(\mathcal{F}) = \{1\}$. Then the hypothesis of [PS18, Theorem 5.1] are satisfied and $\mathcal{F}$ is one of the fusion systems described in [PS18, Table 5.1].

Hence, we may assume that $\mathcal{E}(\mathcal{F}) \subseteq \{Q_1, Q_2\}$. Suppose that Q_2 is essential and notice that $Z_3(S) = \Phi(Q_2)$. Since $[S, S'] \le Z_3(S)$ and S' has index q in Q_2, it follows in a similar manner to Lemma 2.3.10 that $Q_2/\Phi(Q_2)$ is a natural $\mathrm{SL}_2(q)$-module for $O^{p'}(\mathrm{Out}_{\mathcal{F}}(Q_2)) \cong \mathrm{SL}_2(q)$. Moreover, since S does not centralize $Z_2(S) = Z(Q_2)$ but acts quadratically on $Z(Q_2)$, it follows $Z(Q_2)$ is also a natural $\mathrm{SL}_2(q)$-module for $O^{p'}(\mathrm{Out}_{\mathcal{F}}(Q_2))$ and since S centralizes $Z_3(S)/Z_2(S)$, $O^{p'}(\mathrm{Out}_{\mathcal{F}}(Q_2))$ centralizes $Z_3(S)/Z_2(S)$. In particular, if Q_1 is not essential then (iii) is satisfied.

Suppose that Q_1 is essential. Notice that $O^{p'}(\mathrm{Out}_{\mathrm{G}_2(q)}(Q_1)) \cong \mathrm{SL}_2(q)$ acts irreducibly on $Q_1/\Phi(Q_1)$ and it follows that $\langle \mathrm{Out}_S(Q_1)^{\mathrm{Out}(Q_1)} \rangle$ acts irreducibly on $Q_1/\Phi(Q_1)$ and centralizes $\Phi(Q_1)$. Then by [PR12, Lemma 2.73], $\langle \mathrm{Out}_S(Q_1)^{\mathrm{Out}(Q_1)} \rangle$ is isomorphic to an irreducible subgroup of $\mathrm{Sp}_4(q)$ and so $O^{p'}(\mathrm{Out}_{\mathcal{F}}(Q_1))$ is isomorphic to a subgroup of $\mathrm{Sp}_4(q)$ with a strongly p-embedded subgroup.

Applying Proposition 3.2.7, it follows that $O^{p'}(\mathrm{Out}_{\mathcal{F}}(Q_1))$ is isomorphic to a central extension of $\mathrm{PSL}_2(q)$; or $q = p \in \{5, 7\}$ and the possibilities are determined in [PS18, Lemma 5.2].

If both Q_1 and Q_2 are essential, then since $O_p(\mathcal{F}) \leq Q_1 \cap Q_2$ by Proposition 3.1.13 and $O^{p'}(\mathrm{Out}_{\mathcal{F}}(Q_2))$ is irreducible on $Z_2(S)$ and $Q_2/Z_3(S)$, we have that $Z_2(S) \leq O_p(\mathcal{F}) \leq Z_3(S)$ or $O_p(\mathcal{F}) = \{1\}$. If $O_p(\mathcal{F}) = \{1\}$, then $\mathcal{F}$ is determined by Corollary 4.1.3, and the result holds. So suppose that $Z_2(S) \leq O_p(\mathcal{F}) \leq Z_3(S)$. If $Z_2(S) = O_p(\mathcal{F})$, then $C_{Q_1}(Z_2(S)) = S'$ is $\mathrm{Aut}_{\mathcal{F}}(Q_1)$-invariant and since Q_2 centralizes $Z_2(S)$, Q_1/S' and $Z_3(S)/Z_2(S)$, it follows from Lemma 2.3.10 that $S'/Z_2(S)$ is a natural module for $O^{p'}(\mathrm{Out}_{\mathcal{F}}(Q_1)) \cong \mathrm{SL}_2(q)$, and both $Z_2(S)$ and Q_1/S' are centralized by $O^{p'}(\mathrm{Out}_{\mathcal{F}}(Q_1))$. Letting $1 \neq t \in Z(O^{p'}(\mathrm{Out}_{\mathcal{F}}(Q_1)))$, by coprime action we have that for $V := Q_1/Z(S)$, $V = [V, t] \times C_V(t)$ and $[V, t]$ is normalized by S. Since $Z_2(S)$ is centralized by t, we deduce that $[V, t] \cap Z(S/Z(S)) = \{1\}$ so that $[V, t] = \{1\}$ and t centralizes V, a contradiction. Therefore, $Z_2(S) < O_p(\mathcal{F}) \leq Z_3(S)$ so that $Z_3(S) = C_S(O_p(\mathcal{F})) \leq Q_1 \cap Q_2$. Then by Proposition 3.1.13, $C_S(O_p(\mathcal{F})) \trianglelefteq \mathcal{F}$ and since $Z_3(S)$ is elementary abelian, $O_p(\mathcal{F}) = Z_3(S)$.

Setting $L_1 := O^{p'}(\mathrm{Out}_{\mathcal{F}}(Q_1))$, we have that $L_1/C_{L_1}(Q_1/Z_3(S) \cong \mathrm{SL}_2(q)$ and $L_1/C_{L_1}(Z_3(S)/Z(S)) \cong \mathrm{SL}_2(q)$, and either $C_{L_1}(Q_1/Z_3(S) = C_{L_1}(Z_3(S)/Z(S))$ and $L_1 \cong \mathrm{SL}_2(q)$; or L_1 is isomorphic to a central extension of $\mathrm{PSL}_2(q)$ by an elementary group of order 4. Since $p \geqslant 5$, $\mathrm{PSL}_2(q)$ is perfect and has the p'-part of its Schur multiplier of order 2 by Lemma 2.2.1 (vii), and as $L_1 = O^{p'}(L_1)$, we have a contradiction in the latter case. Therefore, $L_1 \cong \mathrm{SL}_2(q) \cong O^{p'}(\mathrm{Out}_{\mathcal{F}}(Q_2))$.

Now, $Z_3(S)$ is a normal, S-centric subgroup of $\mathcal{F}$. By Theorem 3.1.21, there is a finite group G such that $F^*(G) = Z_3(S)$ and $\mathcal{F} = \mathcal{F}_S(G)$. Moreover,

$O^{p'}(\mathrm{Out}_G(Q_i)) \cong \mathrm{SL}_2(q)$ and $\mathrm{Out}_{\mathcal{F}}(Q_i)$ acts faithfully on $Q_i/Z_3(S)$ for $i \in \{1,2\}$. Set $\overline{G} := G/Z_3(S)$ and notice that $\overline{Q_1}$ and $\overline{Q_2}$ are self-centralizing in $\overline{G}$. Moreover, $\overline{G} = \langle N_{\overline{G}}(\overline{Q_1}), N_{\overline{G}}(\overline{Q_2})\rangle$, and $\overline{Q_i}$ is $\mathrm{Aut}_{\overline{G}}(\overline{S})$-invariant for $i \in \{1,2\}$. It follows that $\overline{G}$ has a weak BN-pair of rank 2 in the sense of Definition 5.1.7. Moreover, since Q_2 centralizes $Z_2(S)$ which has index q in $Z_3(S)$ and $Q_2/Z_3(S)$ is elementary abelian of order q^2, we deduce that $Z_3(S)$ is an FF-module for $\overline{G}$ by Proposition 2.3.9. Then, comparing with the completions in [DS85] and applying [CD91, Theorem A], we conclude that $O^{p'}(\overline{G}) \cong \mathrm{SL}_3(q)$ and $Z_3(S)$ is a natural module for $O^{p'}(\overline{G})$. As in the case when $p = 2$, we observe that if S splits over $Z_3(S)$, then S is isomorphic to a Sylow p-subgroup of $\mathrm{SL}_4(q)$, which has p-rank $4n$ by [GLS98, Theorem 3.3.3], whereas S has p-rank $3n$. Therefore, S is non-split and by [Bel78, Table I], it follows that $q = p = 5$. One can check that there is a unique fusion system up to isomorphism on S with $O_5(\mathcal{F}) = Z_3(S)$. $\square$

Remark. In case (iv) of the above theorem, one can take M to be a maximal subgroup of Ly.

4.5 Fusion Systems on a Sylow p-subgroup of $\mathrm{PSU}_4(p^n)$

We set S to be a Sylow p-subgroup of $\mathrm{PSU}_4(q)$ where $q = p^n$ and $\mathcal{F}$ to be a saturated fusion system supported on S. Again, let $\mathbb{K}$ be the finite field of order q and recall the commutator formulas from Section 4.1.

Proposition 4.5.1. *Suppose that S is isomorphic to a Sylow p-subgroup of $\mathrm{PSU}_4(p^n)$. Then $J(S) = X_\beta X_{\alpha+\beta} X_{2\alpha+\beta}$ is the unique elementary abelian subgroup of S of order p^{4n}.*

Proof. Let $X := X_\beta X_{\alpha+\beta} X_{2\alpha+\beta}$, $q = p^n$, $G := \mathrm{PSU}_4(q)$ and $S \in \mathrm{Syl}_p(G)$ with $X \le S$. Then $O^{p'}(\mathrm{Aut}_G(Q_2)) \cong \mathrm{PSL}_2(q^2)$ by [BHR13]. Suppose there is $A \in \mathcal{A}(S)$ with $A \neq X$ and note that $C_S(X) = X$ so that $A \cap X \le C_S(AX) \le X$. Then by Lemma 2.2.9, $|C_X(AX)| \in \{q, q^2\}$ so that $|A \cap X| \leqslant q^2$. Then, since $|S/X| = q^2$, $q^2 \geqslant |AX/X| = |A/A \cap X| \geqslant q^4/q^2 = q^2$ so that $S = AX$, $|A| = q^4$ and $|A \cap X| = q^2$. But $A \cap X \le Z(AX) = Z(S)$ and as $|Z(S)| = q$, we have a contradiction. Hence, $\mathcal{A}(S) = \{X\}$ and the result holds. $\square$

Lemma 4.5.2. *There exists a unique subgroup $X := X_\alpha X_{\alpha+\beta} X_{2\alpha+\beta} \le S$ of order q^5 such that $X' = Z(S)$, $|X| > q^4$, $S' = X \cap J(S)$ and X is maximal by inclusion with respect to these properties. In particular, X is characteristic in S.*

Proof. By the definition of X, $|X| = q^5 > q^4$ and $X \cap J(S) = S'$. Moreover, it follows from the commutator relations that $X' = Z(S)$. Thus, X satisfies the required properties. Suppose there is $Y \not\le X$ such that Y also satisfies the required properties. Since $Y \not\le X$ and $Y \cap J(S) = S'$, there is $y := x_\alpha(t_1)x_\beta(t_2) \in Y$ with $t_1 \neq 0 \neq t_2$. By the requirements, $[Y, y] \le Y' = Z(S)$ and since $[y, x_\alpha(t)] \not\le Z(S)]$ is follows that $Y \cap X = S'$. However, $|Y| > q^4$ so that $|XY| = |X||Y|/|X \cap Y| > q^6 = |S|$, a clear contradiction. $\square$

Remark. We may uniquely define X as the preimage in S of $J(S/Z(S))$. Moreover, X is an ultraspecial special group with $Z(X) = X' = Z(S)$ of order q, but we will not require this fact.

We set $Q_1 := X$ and $Q_2 := J(S)$ with the intention of proving $\mathcal{E}(\mathcal{F}) \subseteq \{Q_1, Q_2\}$. As it turns out, this is true except when $q = p = 2$ where S is coincidentally isomorphic to a Sylow 2-subgroup of $\mathrm{PSL}_4(2)$. In this case, since $|S| = 2^6$, we can directly compute that S-radical, S-centric subgroups of S and classify all saturated fusion systems on S with the aid of MAGMA.

Proposition 4.5.3. *Let S be isomorphic to a Sylow 2-subgroup of* $\mathrm{PSU}_4(2)$*. The S-centric, S-radical subgroups of S are as S, Q_1, Q_2, $C_S(x)$ for any $x \in S' \setminus Z(S)$ so that $|C_S(x)| = 2^5$; and $A \in \mathcal{A}(Q_1)$ with $A \not\leq Q_2$ so that $|A| = 2^3$.*

Proposition 4.5.4. *Let $\mathcal{F}$ be a saturated fusion system over a Sylow 2-subgroup of* $\mathrm{PSU}_4(2)$*. Then one of the following holds:*

(i) $\mathcal{F} = \mathcal{F}_S(S : \mathrm{Out}_{\mathcal{F}}(S))$;

(ii) $\mathcal{F} = \mathcal{F}_S(Q_2 : \mathrm{Out}_{\mathcal{F}}(Q_2))$ *where* $\mathrm{Out}_{\mathcal{F}}(Q_2) \cong \mathrm{PSL}_2(4)$;

(iii) $\mathcal{F} = \mathcal{F}_S(Q_1 : \mathrm{Out}_{\mathcal{F}}(Q_1))$ *where* $\mathrm{Out}_{\mathcal{F}}(Q_1)$ *is isomorphic to a subgroup of* $\mathrm{Sym}(3) \times 3$;

(iv) $\mathcal{F} = \mathcal{F}_S(Q_x : \mathrm{Out}_{\mathcal{F}}(Q_x))$ *where* $Q_x = C_S(x)$ *for any* $x \in S' \setminus Z(S)$*, and* $\mathrm{Out}_{\mathcal{F}}(Q_x) \cong \mathrm{Sym}(3)$;

(v) $\mathcal{F} = \mathcal{F}_S(M)$ *where* $M \cong 2^4 : (\mathrm{Sym}(3) \times \mathrm{Sym}(3))$;

(vi) $\mathcal{F} = \mathcal{F}_S(M)$ *where* $M \cong 2^3 : \mathrm{PSL}_3(2)$;

(vii) $\mathcal{F} = \mathcal{F}_S(G)$ *where* $G \cong \mathrm{PSU}_4(2)$*; or*

(viii) $\mathcal{F} = \mathcal{F}_S(G)$ *where* $G \cong \mathrm{PSL}_4(2)$.

Henceforth, we suppose that $q > 2$. Consider Q_1, Q_2 and their normalizers as subgroups of $\mathrm{PSU}_4(q)$. By [PR06, Definition 2.1], as $\mathrm{GF}(p)$-modules, Q_2 is a natural $\Omega_4^-(q)$-module for $O^{p'}(\mathrm{Aut}_{\mathrm{PSU}_4(q)}(Q_2)) \cong \mathrm{PSL}_2(q^2)$ while $Q_1/Z(Q_1)$ is the direct sum of two natural $\mathrm{SL}_2(q)$-modules for $O^{p'}(\mathrm{Out}_{\mathrm{PSU}_4(q)}(Q_1)) \cong \mathrm{SL}_2(q)$. With this information, we can properly analyze the centralizers of elements in S.

Lemma 4.5.5. *Let $F \leq S$ be such that $F \not\leq Q_2$. Then of the following occurs:*

(i) $[Q_2, F] = [Q_2, S] = S'$ *and* $C_{Q_2}(F) = C_{Q_2}(S) = Z(S)$;

(ii) $p = 2$, $[Q_2, F] = C_{Q_2}(F)$ *has order* q^2 *and* $|FQ_2/Q_2| \leqslant q$; *or*

(iii) p *is odd,* $|[Q_2, F]| = |C_{Q_2}(F)| = q^2$, $S' = [Q_2, F]C_{Q_2}(F)$, $Z(S) = C_{[Q_2,F]}(F)$ *and* $|FQ_2/Q_2| \leqslant q$.

Proof. This is a restatement of Lemma 2.2.9. □

Lemma 4.5.6. *Let $x \in S' \setminus Z(S)$. Then $Q_2 \leq C_S(x)$, $|C_S(x)| = q^5$, $Z(C_S(x)) = C_{Q_2}(C_S(x))$ has order q^2 and $C_S(x)' = [Q_2, C_S(x)]$ has order q^2.*

Proof. Let $x \in S' \setminus Z(S)$. Then since $x \in Q_2$, and Q_2 is elementary abelian, $Q_2 \leq C_S(x)$ so that $Q_2 = J(S) = J(C_S(x))$ is characteristic in $C_S(x)$. Moreover, since $x \in Q_1 \setminus Z(Q_1)$, we have that $|C_{Q_1}(x)| = q^4$. Then $C_{Q_1}(x)Q_2 \leq C_S(x)$ and so $|C_S(x)| \geqslant q^5$. Suppose $|C_S(x)| > q^5$. Then $q^6 < |C_S(x)||Q_1|/|C_{Q_1}(x)| = |C_S(x)Q_1| \leqslant |S| = q^6$, a contradiction.

Since Q_2 is self-centralizing and $Q_2 \leq C_S(x)$, we have that $Z(C_S(x)) = C_{Q_2}(C_S(x))$ may be determined from the information provided in Lemma 4.5.5. Indeed, since $x \in Z(C_S(x)) \setminus Z(S)$, we have that $|[Q_2, C_S(x)]| = |Z(C_S(x))| = q^2$. Finally, it is clear from the commutator formulas that $C_S(x)' = [Q_2, C_S(x)]$, as required. □

Lemma 4.5.7. *Let $x \in Q_2 \setminus S'$. Then $C_S(x) = Q_2$.*

Proof. Let $x \in Q_2 \setminus S'$. Since Q_2 is abelian, $Q_2 \leq C_S(x)$ and $|C_S(x)| \geqslant q^4$. We have that $S' \leq C_{Q_1}(x)$ so that $C_{Q_1/Z(S)}(x)$ is of order at least q^2. But $Q_1/Z(S)$ is a direct sum of natural $\mathrm{SL}_2(q)$-modules so that $|C_{Q_1/Z(S)}(x)| = q^2$ from which it follows that $S' = C_{Q_1}(x)$. Then $q^6 = |S| \geqslant |C_S(x)Q_1| = |C_S(x)||Q_1|/|S'| \geqslant q^6$ so that $S = C_S(x)Q_1$, $|C_S(x)| = q^4$ and $C_S(x) = Q_2$. $\square$

Lemma 4.5.8. *Let $x \in S \setminus Q_2$ be of order p. Then $C_S(x) \leq Q_1$, $|C_S(x)| = q^4$, $|C_S(x) \cap Q_2| = q^2$, $m_p(C_S(x)) \leqslant 3n$, $C_S(x)' = Z(S)$ and $|Z(C_S(x))| = q^2$.*

Proof. Upon demonstrating that $C_S(x) \leq Q_1$, the results follow from the structure of Q_1. Since $C_S(x)$ is centralized by $x \notin Q_2$, it follows that $C_S(x) \cap Q_2 \leq S'$ and $C_S(x)S'$ has order q^5 and intersects Q_2 in S'. Hence, if $(C_S(x)S')' = Z(S)$, then $C_S(x)S' = Q_1$ by Lemma 4.5.2. It is clear from Lemma 4.5.5 that $[S', C_S(x)] = Z(S)$ and so it remains to show that $C_S(x)' \leq Z(S)$. Indeed, since S splits over Q_2, $C_S(x)$ splits over S' and since $C_S(x)S'/S'$ is elementary abelian, we need only show that $[C_S(x) \cap S', C_S(x)] = Z(S)$. But this follows from Lemma 4.5.5, and the result is proved. $\square$

With this information, we can determine the S-centric, S-radical subgroups of S, which we do over the following two propositions.

Proposition 4.5.9. *Suppose that E is an S-centric, S-radical subgroup of S and $S' \not\leq E$. Then E is elementary abelian of order q^3, $E \leq Q_1$ and either*

(i) *$p = 2$, $E \trianglelefteq S$ and $|E \cap S'| = q^2$;*

(ii) *p is odd, $N_S(E) = Q_1$ and $|E \cap S'| = q^2$; or*

(iii) *p is arbitrary, $N_S(E) = Q_1$ and $E \cap S' = Z(S)$.*

Moreover, in all cases, E is not essential in any saturated fusion system $\mathcal{F}$ over S.

Proof. Suppose that $S' \not\leq E$. Since $[E, S'] \leq [S, S'] \leq Z(S) \leq \Omega(Z(E))$, we must have that $[S', \Omega(Z(E))] \neq \{1\}$ for otherwise S' centralizes the chain $\{1\} \trianglelefteq \Omega(Z(E)) \trianglelefteq E$, a contradiction by Lemma 3.2.1 since S-radical. Since S' centralizes Q_2, there is $x \in \Omega(Z(E))$ with $x \in S \setminus Q_2$ and $E \leq C_S(x)$. In particular, $Z(C_S(x)) \leq Z(E)$, $|Z(E)Q_2/Q_2| \geqslant q$ and $E \leq Q_1$ by Lemma 4.5.8.

Suppose first that $E \cap S' > Z(S)$. Then for $e \in (E \cap S') \setminus Z(S)$, $Z(E) \leq C_S(e)$. In particular, $|Z(E)Q_2/Q_2| = q$. Moreover, $C_{S'}(\Omega(Z(E))) = Z(C_S(e))$ has order q^2 and centralizes the chain $\{1\} \trianglelefteq \Omega(Z(E)) \trianglelefteq E$ so that $C_{S'}(\Omega(Z(E))) = E \cap S'$ has order q^2. Suppose that $|EQ_2/Q_2| > q$. Then by Lemma 4.5.5, we have $Z(S) = [E, E \cap S'] \leq E'$ and either $E' = Z(S)$ and Q_1 centralizes the chain $\{1\} \trianglelefteq E' \trianglelefteq E$, a contradiction since E is S-radical and $S' \not\leq E$; or $Z(S) < E' \leq E \cap S'$, $C_E(E') = E \cap C_S(e) = Z(E)(E \cap S')$ has order q^3 and $[E, C_E(E')] = [E, S \cap E'] = Z(S)$ is characteristic in E and again, Q_1 centralizes a characteristic chain. Thus, $|EQ_2/Q_2| = q$ and $E = Z(E)(E \cap S')$ is elementary abelian of order q^3. Since $E \leq Q_1$ and $Q_1' = Z(S) \leq E$, we deduce that $E \trianglelefteq Q_1$. Moreover, when $p = 2$, it follows from Lemma 4.5.5 that $[C_S(e), E] \leq C_S(e)' = (S' \cap E)$ and so $E \trianglelefteq S = Q_1 C_S(e)$.

Suppose now that $E \cap S' = Z(S)$. Since $E \leq Q_1$, it follows that $E \cap Q_2 = Z(S)$ and $|E| \leqslant q^3$. If $\Omega(Z(E)) \leq Q_2$, then $\Omega(Z(E)) = Z(S)$ and so Q_1 centralizes the chain $\{1\} \trianglelefteq \Omega(Z(E)) \trianglelefteq E$, a contradiction since E is S-radical. Hence, there is $e \in \Omega(Z(E)) \setminus Q_2$ and so, $E \leq C_S(e)$. Since E is S-centric, we must have that $Z(C_S(e)) \leq \Omega(Z(E))$. If $\Omega(Z(E)) = Z(C_S(e))$, then as $C_S(e)' = Z(S)$, $C_S(e)$ centralizes the chain $\{1\} \trianglelefteq \Omega(Z(E)) \trianglelefteq E$, and since E is S-radical, $E = C_S(e)$. But then Q_1 centralizes the chain $\{1\} \trianglelefteq E' \trianglelefteq E$, a contradiction. So there is $e' \in \Omega(Z(E)) \setminus (Q_2 C_S(e))$ with $Z(C_S(e')) \cap Z(C_S(e)) = Z(S)$ and $Z(C_S(e')) \leq$

$\Omega(Z(E))$. In particular, $Z(C_S(e'))Z(C_S(e))$ is an elementary abelian subgroup of E of order q^3, and since E itself has order at most q^3, we conclude that $E = Z(C_S(e))Z(C_S(e'))$. Then for any $y \in Q_2 \setminus S'$, $[E, y] \not\leq Z(S)$ and so $N_{Q_2}(E) = S'$. Since $E \leq Q_1$ and $Q_1' = Z(S) \leq E$, we have that $N_S(E) = Q_1$.

Suppose that for any of the E considered, E is essential is some saturated fusion system $\mathcal{F}$ supported on S. Suppose first that we are in case (i) or (ii). Then S' centralizes $E \cap S'$ and since $|S'/E \cap S'| = |E/E \cap S'| = q$, it follows from Lemma 2.3.10 that $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E)) \cong \mathrm{SL}_2(q)$ and $\mathrm{Out}_{S'}(E) \in \mathrm{Syl}_p(E)$. But $|N_S(E)/E| \geqslant q^2$ in either case, a contradiction. Hence, we may assume that we are in case (iii) and $E \cap S' = Z(S)$. Let $e \in E \setminus Q_2$ so that $E \leq C_S(e)$, where $|C_S(e)| = q^4$. Then $Z(C_S(e))$ is a subgroup of E of index q centralized by $C_S(e)$ where $|C_S(e)E/E| = q$ and $C_S(e) \leq N_S(E) = Q_1$. By Lemma 2.3.10, $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E)) \cong \mathrm{SL}_2(q)$ and $\mathrm{Out}_{C_S(e)}(E) \in \mathrm{Syl}_p(E)$, and since $|N_S(E)/E| = q^2$, we have another contradiction. □

Proposition 4.5.10. *Suppose that E is an S-centric, S-radical subgroup of S, $S' \leq E$ and $q > 2$. Then $E \in \{Q_1, Q_2, S\}$.*

Proof. Since $S' \leq E$, we have that $Z(E) \leq Q_2$. Moreover, if $E \leq Q_2$, then using that E is S-centric, we conclude that $E = Q_2$. So we may suppose throughout the remainder of this proof that there is $e \in E \setminus Q_2$.

Suppose first that $Z(E) = Z(S)$ so that $S' \leq Z_2(E)$. Indeed, if $E \cap Q_2 > S'$, then it follows from the commutator formulas that $Z_2(E) = S'$ and S centralizes the chain $\{1\} \trianglelefteq Z(E) \trianglelefteq Z_2(S) \trianglelefteq E$, and since E is S-radical, we deduce that $E = S$. So if $Z(E) = Z(S)$, then $E \cap Q_2 = S'$.

In addition, suppose that $E' = Z(S)$. Consider $A \in \mathcal{A}(E)$. Since $S' \leq E$ and S' is elementary abelian, we infer that $|A| \geqslant 3n$. Moreover, there is $a \in A$ with $a \not\leq Q_2$, else $S' = J(E)$ and Q_2 centralizes the chain $\{1\} \trianglelefteq J(E) \trianglelefteq E$, a contradiction since E is S-radical. It follows that $A \leq C_S(a) \leq Q_1$, $|A| = q^3$ and $|A \cap S'| = q^2$. Then either $E = AS' \leq Q_1$; or $|E| > q^4$. In either case, it follows from Lemma 4.5.2 that $E \leq Q_1$ and then Q_1 centralizes the chain $\{1\} \trianglelefteq Z(E) \trianglelefteq E$. Since E is S-radical, $Q_1 \leq E$. Since $E \cap Q_2 = S'$, it follows from a consideration of orders that $E = Q_1$.

Suppose that $Z(S) = Z(E) < E'$. By Lemma 4.5.6, $C_E(E') \leq C_S(x)$ for some $x \in E' \setminus Z(E)$ and it follows that either $C_E(E') = S'$; or $C_E(E') \not\leq Q_2$ and $Z(C_E(E')) \leq S'$ has order q^2. In the former case, S centralizes the chain $\{1\} \trianglelefteq Z(E) \trianglelefteq C_E(E') \trianglelefteq E$, and since E is S-radical, $E = S$, a contradiction since $E \cap Q_2 = S'$. Therefore, $C_E(E') \not\leq Q_2$ and since $C_E(E') \cap Q_2 \leq E \cap Q_2 = S'$, we conclude that $|C_E(E')| \leqslant q^4$.

Let $A \in \mathcal{A}(C_E(E'))$ and suppose that $A \cap S' > Z(C_E(E'))$. Comparing with the commutator formulas, it follows that $A \leq C_S(A \cap S') = S'$ and so $A = S'$. Notice that if $S' = J(C_E(E'))$, then Q_2 centralizes the chain $\{1\} \trianglelefteq S' \trianglelefteq E$, a contradiction since E is S-radical. Thus, we may assume that there is $A \in \mathcal{A}(C_E(E))$ with $A \cap S' = Z(C_E(E'))$ and $|A| \geqslant q^3$. In particular, $C_E(E') = AS'$ and $|A| = q^3$. Then for $a \in A \setminus A \cap S'$, we infer that $A \leq C_S(a) \leq Q_1$ and so $C_E(E') \leq Q_1$. But now, since $S' \leq C_E(E')$, Q_1 centralizes the chain $\{1\} \trianglelefteq Z(E) \trianglelefteq C_E(E') \trianglelefteq E$, a contradiction since $|E| \leqslant q^5$, E is S-radical and $E' > Z(S)$.

Suppose now that $Z(S) < Z(E)$. Since $E \not\leq Q_2$, $Z(E) \leq S'$ and $E \leq C_S(x)$ for some $e \in Z(E) \setminus Z(S)$. Since E is S-centric, $Z(C_S(x)) \leq Z(E)$ and since $E \not\leq Q_2$, it follows from Lemma 4.5.6, that $Z(C_S(x)) = Z(E)$. Indeed, if $p = 2$, then $Z(E) = C_{Q_2}(E) = [Q_2, E]$ and Q_2 centralizes the chain $\{1\} \trianglelefteq Z(E) \trianglelefteq E$. Since E is S-radical, $Q_2 = J(E)$ is characteristic in E. Then, $[C_S(x), E] \leq J(E)$ and

$Z(E) = [J(S), C_S(x)]$ and $C_S(x)$ centralizes the chain $\{1\} \trianglelefteq Z(E) \trianglelefteq J(E) \trianglelefteq E$, and since E is S-radical, $E = C_S(x)$. Now, assuming $q > 2$, both $Z(S)$ and S' are characteristic subgroups of E by [Par76, Lemma 3.13]. Then S centralizes the chain $\{1\} \trianglelefteq Z(S) \trianglelefteq S' \trianglelefteq E$, a contradiction since E was assumed to be S-radical.

Suppose now that p is odd and $Z(C_S(x)) = Z(E)$. Let $A \in \mathcal{A}(E)$ such that $A \not\leq Q_2$. Then, there is $a \in A$ such that $|C_S(a)| = q^4$, $A \leq C_S(a) \cap C_S(x)$, $C_S(a) \leq Q_1$ and $Z(E) = C_S(a) \cap S'$. Now, $|C_S(x) \cap C_S(a)| = q^3$ and it follows that any elementary abelian subgroup of E not contained in Q_2 has order at most q^3. Since $E \cap Q_2$ is elementary abelian, it follows that either $J(E) = E \cap Q_2 \geq S'$, or $E \cap Q_2 = S'$ and there is $A \in \mathcal{A}(E)$ with $|A| = q^3$ and $A \cap S' = Z(E)$. In the latter case, it follows that $E = AS'$ has order q^4 and since $A \leq C_S(a) \leq Q_1$, we have that $E \leq Q_1$. Moreover, $E' = [A, S'] = Z(S)$ and Q_1 centralizes the chain $\{1\} \trianglelefteq E' \trianglelefteq E$, a contradiction since E is S-radical. Thus, $J(E) = E \cap Q_2$ and so Q_2 centralizes the chain $\{1\} \trianglelefteq J(E) \trianglelefteq E$, and since E is S-radical, $Q_2 = J(E)$. But then, since p is odd, $S' = [Q_2, E]Z(E)$, $Z(S) = [Q_2, E] \cap Z(E)$ and S centralizes the chain $\{1\} \trianglelefteq Z(S) \trianglelefteq S' \trianglelefteq E$, a contradiction since $Z(E) > Z(S)$ and E is S-radical. □

We now complete the classification of saturated fusion systems supported on a Sylow p-subgroup of $\mathrm{PSU}_4(p^n)$. When $q = p$ we get some exceptional behaviour, particularly when $p = 3$, and refer to [BFM19] and [Mon20] where these cases have already been treated. Hence, by Proposition 4.5.10, we may as well assume that $\mathcal{E}(\mathcal{F}) \subseteq \{Q_1, Q_2\}$.

As in earlier sections in this chapter, we endeavor to classify saturated fusion systems on S without the need for a $\mathcal{K}$-group hypothesis. When $p = 2$, since

$m_2(S/Q_i) > 1$, [Ben71] provides a list of groups with a strongly embedded subgroups, and so we focus more than the case where p is odd. Here, $Q_1/\Phi(Q_1)$ witnesses quadratic action by S, and we rely on results of Ho (although we believe it should be possible to find a more elementary proof) to show that $O^{p'}(\mathrm{Out}_{\mathcal{F}}(Q_1)) \cong \mathrm{SL}_2(q)$. With regards to Q_2, we come up short and rely on $\mathcal{K}$-group hypothesis to identify $O^{p'}(\mathrm{Out}_{\mathcal{F}}(Q_2))$ with $\mathrm{PSL}_2(q^2)$. We believe this can be achieved without using a $\mathcal{K}$-group hypothesis as follows:

By the conditions on $G := O^{p'}(\mathrm{Out}_{\mathcal{F}}(Q_2))$, we see quickly that $\mathrm{Syl}_p(G)$ is a TI-set for G. Then, using some appropriately chosen minimality condition, we should be able to prove that $G = \langle S, T\rangle$ and $C_{Q_2}(S) \cap C_{Q_2}(T) = \{1\}$ for any $S, T \in \mathrm{Syl}_p(G)$. Even better, $C_{Q_2}(S) \cap [Q_2, T] = \{1\}$ for all such S and T. Noticing that $|Q_2/C_{Q_2}(S)| = q^3$, we strive to show that $Q_2/C_{Q_2}(S) = [Q_2/C_{Q_2}(S), S] \cup \bigcup_{s\in S} C_{Q_2}(T^s)C_{Q_2}(S)/C_{Q_2}(S)$, where the intersection of any of the two subgroups in the union is $C_{Q_2}(S)$. Finally, we aim to show that $C_{Q_2}(S)$ and $C_{Q_2}(T)$ are the only centralizers of a Sylow p-subgroup of G contained in $C_{Q_2}(T)C_{Q_2}(S)$, for then we have a correspondence between Sylow p-subgroups of G and certain subgroups of Q_2 of order q. We are then in a position to recognize $\mathrm{PSL}_2(q^2)$ via a result of Hering, Kantor and Seitz which recognizes a split BN-pair of rank 1 in G [HKS72].

Finally, in the classification of fusion systems supported on S, we apply Corollary 4.1.4 using the Main Theorem when Q_1 and Q_2 are both essential and, as in earlier cases, we remark that this reduces to applying the main result from [DS85], which is independent of any $\mathcal{K}$-group hypothesis.

Theorem 4.5.11. *Let $\mathcal{F}$ be a saturated fusion system over a Sylow p-subgroup of* $\mathrm{PSU}_4(q)$ *for $q > 2$. Then one of the following occurs:*

(i) $\mathcal{F} = \mathcal{F}_S(S : \mathrm{Out}_{\mathcal{F}}(S))$;

(ii) $\mathcal{F} = \mathcal{F}_S(Q_1 : \mathrm{Out}_{\mathcal{F}}(Q_1))$ *where* $O^{p'}(\mathrm{Out}_{\mathcal{F}}(Q_1)) \cong \mathrm{SL}_2(q)$, *or* $q = p = 3$ *and* $\mathrm{Out}_{\mathcal{F}}(Q_1)$ *is determined in [BFM19];*

(iii) $\mathcal{F} = \mathcal{F}_S(Q_2 : \mathrm{Out}_{\mathcal{F}}(J(S)))$ *where* Q_2 *is an* $\Omega_4^-(q)$*-module for* $O^{p'}(\mathrm{Out}_{\mathcal{F}}(J(S)) \cong \mathrm{PSL}_2(q^2)$*;*

(iv) $\mathcal{F} = \mathcal{F}_S(G)$ *where* $G = \mathrm{Co}_2$, McL, $\mathrm{Aut}(\mathrm{McL})$, $\mathrm{PSU}_6(2)$ *or* $\mathrm{PSU}_6(2).2$ *and* $q = 3$*; or*

(v) $\mathcal{F} = \mathcal{F}_S(G)$ *where* $F^*(G) = O^{p'}(G) \cong \mathrm{PSU}_4(q)$.

Proof. If neither Q_1 nor Q_2 are essential then $\mathcal{F} = \mathcal{F}_S(S : \mathrm{Out}_{\mathcal{F}}(S))$ and (i) holds. Suppose that Q_1 is essential and assume first that $q = p$. If $p = 3$, then the action of $\mathrm{Out}_{\mathcal{F}}(Q_1)$ on Q_1 is determined completely in [BFM19] while if $p \geqslant 5$, then the action of $O^{p'}(\mathrm{Out}_{\mathcal{F}}(Q_1))$ is determined by [Mon20].

Suppose now that Q_1 is essential and $q > p$. If $p = 2$, then as $m_p(S/Q_1) > 1$, it follows from Proposition 3.2.7 that $O^{p'}(\mathrm{Out}_{\mathcal{F}}(Q_1)) \cong \mathrm{SL}_2(q)$. So suppose that p is odd. Let $T, P \in \mathrm{Syl}_p(O^{p'}(\mathrm{Out}_{\mathcal{F}}(Q_1)))$ and suppose that $1 \neq x \in T \cap P$. Notice that $Z(Q_1) = Z(S)$ so that $O^{p'}(\mathrm{Out}_{\mathcal{F}}(Q_1))$ acts trivially on $Z(Q_1)$. Then $[Q_1, T]Z(Q_1) = [Q_1, x]Z(Q_1) = [Q_1, P]Z(Q_1)$ and $[Q_1, T, T] \leq Z(Q_1) \geq [Q_1, P, P]$. It follows that $\langle P, T\rangle$ centralizes a series $\{1\} \trianglelefteq Z(Q_1) \trianglelefteq [Q_1, T]Z(Q_1) \trianglelefteq Q_1$ and by Lemma 2.1.9, $\langle T, P\rangle$ is a p-group. Since $T, P \in \mathrm{Syl}_p(O^{p'}(\mathrm{Out}_{\mathcal{F}}(Q_1)))$, we must have that $T = P$. Moreover, T acts quadratically on $Q_1/Z(Q_1) = Q_1/\Phi(Q_1)$ and so, by [Ho79, Theorem 1], $O^{p'}(\mathrm{Out}_{\mathcal{F}}(Q_1))$ is isomorphic to a p'-central extension of $\mathrm{PSL}_2(q)$. Then eliminating $\mathrm{PSL}_2(q)$ by Lemma 2.3.4 since T acts quadratically, we deduce that $O^{p'}(\mathrm{Out}_{\mathcal{F}}(Q_1)) \cong \mathrm{SL}_2(q)$. By Lemma 2.3.11 and since $T \in \mathrm{Syl}_p(O^p(O^{p'}(\mathrm{Out}_{\mathcal{F}}(Q_1))))$, we conclude that $Q_1/Z(Q_1)$ is a direct sum of two natural $\mathrm{SL}_2(q)$-modules.

Suppose that Q_2 is essential. Since S/Q_2 is elementary abelian of order q^2, it follows from Proposition 3.2.7 that $O^{2'}(\mathrm{Out}_{\mathcal{F}}(Q_2)) \cong \mathrm{PSL}_2(q^2)$. Then, since S does not act quadratically on Q_2 and Q_2 contains a non-central chief factor, by Lemma 2.3.12, we conclude that Q_2 is a natural $\Omega_4^-(q)$-module for $O^{2'}(\mathrm{Out}_{\mathcal{F}}(Q_2))$, as required.

If both Q_1 and Q_2 are essential, then by Proposition 3.1.13, $O_p(\mathcal{F}) \leq Q_1 \cap Q_2$ and $O_p(\mathcal{F})$ is normalized by $O^{p'}(\mathrm{Out}_{\mathcal{F}}(Q_2))$. Thus, $O_p(\mathcal{F}) = \{1\}$ and since Q_1 and Q_2 are characteristic in S and we satisfy the hypotheses of Corollary 4.1.4. $\square$

CHAPTER 5

RANK 2 AMALGAMS AND FUSION SYSTEMS

In this chapter, we introduce amalgams and manufacture a situation in which one may identify a rank 2 amalgam within a saturated fusion system. This amalgam data provides strong information about the fusion system and we observe that, in certain circumstances, proving uniqueness of the amalgam completely determines the fusion system. The majority of the work in this chapter is in investigating these rank 2 amalgams via the amalgam method. Although this analysis is in a purely group theoretic setting, the hypothesis we assume is motivated by fusion systems and determines a limited list of amalgams, all of which were previously recorded in the literature. This information is reflected in Theorem C, and then the Main Theorem and Corollary A are proved as consequences of Theorem C. Along the way, Proposition F and Proposition G are also proved and used as tools in the amalgam method. The chapter concludes with several identifications of finite simple groups from the garnered amalgam data provided in previous sections.

5.1 Amalgams in Fusion Systems

In this section, we introduce amalgams and demonstrate their connections with and applications to saturated fusion systems. We will only make use of elementary definitions and facts regarding amalgams as can be found in [DS85, Chapter 2].

Definition 5.1.1. An *amalgam* of rank n is a tuple $\mathcal{A} = \mathcal{A}(G_1, \ldots, G_n, B, \phi_1, \cdots, \phi_n)$ where B is a group, each G_i is a group and $\phi_i : B \to G_i$ is an injective group homomorphism. A group G is a *faithful completion* of $\mathcal{A}$ if there exists injective group homomorphisms $\psi_i : G_i \to G$ such that for all $i, j \in \{1, \ldots, n\}$, $\phi_i\psi_i = \phi_j\psi_j$, $G = \langle \mathrm{Im}(\psi_i) \rangle$ and no non-trivial subgroup of $B\phi_i\psi_i$ is normal in G. Under these circumstances, we identify $G_1, \ldots, G_n, B$ with their images in G and opt for the notation $\mathcal{A} = \mathcal{A}(G_1, \ldots, G_n, B)$.

For almost all the work in this thesis, we reduce to the case where the amalgam is of rank 2 and the groups G_1 and G_2 are finite groups. In this setting, we may always realize $\mathcal{A}$ in a faithful completion, namely the free amalgamated product of G_1 and G_2 over B, denoted $G_1 *_B G_2$. This completion is universal in that every faithful completion occurs as some quotient of this free amalgamated product. Generally, whenever we work in the setting of rank 2 amalgams we will opt to work in this free amalgamated product which we will often denote G and, in an abuse of terminology, refer to G as an amalgam. In particular, we may as well assume the following:

1. $G = \langle G_1, G_2 \rangle$, G_i is a finite group and $G_i < G$ for $i \in \{1, 2\}$;

2. no non-trivial subgroup of B is normal in G; and

3. $B = G_1 \cap G_2$.

Definition 5.1.2. Let $\mathcal{A} = \mathcal{A}(G_1, G_2, B, \phi_1, \phi_2)$ and $\mathcal{B} = \mathcal{B}(H_1, H_2, C, \psi_1, \psi_2)$ be two rank 2 amalgams. Then $\mathcal{A}$ and $\mathcal{B}$ are *isomorphic* if, up to permuting indices, there are isomorphisms $\theta_i : G_i \to H_i$ and $\xi : B \to C$ such that the following diagram commutes for $i \in \{1, 2\}$:

$$\begin{tikzcd} G_1 \arrow[d, "\theta_1"'] & B \arrow[l, "\phi_1"'] \arrow[r, "\phi_2"] \arrow[d, "\xi"'] & G_2 \arrow[d, "\theta_2"] \\ H_1 & C \arrow[l, "\psi_1"] \arrow[r, "\psi_2"'] & H_2 \end{tikzcd}$$

Often, for some finite group H arising as a faithful completion of some rank 2 amalgam $\mathcal{B}$, we will often say a completion G of $\mathcal{A}$ is *locally isomorphic* to H, by which we mean $\mathcal{A}$ is isomorphic to $\mathcal{B}$.

An important observation in this definition is that the faithful completions of two isomorphic amalgams coincide. In fact, two amalgams being isomorphic is equivalent to demanding that $G_1 *_B G_2 \cong H_1 *_C H_2$.

Say that $\mathcal{A} = \mathcal{A}(G_1, G_2, B)$ and $\mathcal{B} = \mathcal{B}(H_1, H_2, C)$ are *parabolic isomorphic* if, up to permuting indices, $G_i \cong H_i$ and $B \cong C$ as abstract groups.

We provide the following elementary example with regard to isomorphisms of amalgams.

Example 5.1.3. *For* $G = \mathrm{J}_2$*, there are two maximal subgroups* M_1, M_2 *containing* $N_G(S)$ *for* $S \in \mathrm{Syl}_2(G)$*. Furthermore,* $M_1/O_2(M_1) \cong \mathrm{SL}_2(4)$*,* $M_2/O_2(M_2) \cong \mathrm{Sym}(3) \times 3$ *and* $|N_G(S)/S| = 3$*. Thus,* G *gives rise to the amalgam* $\mathcal{A} := \mathcal{A}(M_1, M_2, N_G(S))$*.*

For $H = \mathrm{J}_3$ and $T \in \mathrm{Syl}_2(H)$, $S \cong T$ and H contains two maximal subgroups N_1, N_2 containing $N_G(T)$ such that $N_i \cong M_i$ for $i \in \{1, 2\}$. Thus, H gives rise to the amalgam $\mathcal{B} := \mathcal{B}(N_1, N_2, N_H(T))$.

Then $\mathcal{A}$ is isomorphic to $\mathcal{B}$.

Definition 5.1.4. Let $\mathcal{A} = \mathcal{A}(G_1, G_2, B)$ be an amalgam of rank 2. Then $\mathcal{A}$ is a *characteristic p amalgam of rank* 2 if the following hold for $i \in \{1, 2\}$:

(i) G_i is a finite group;

(ii) $\mathrm{Syl}_p(B) \subseteq \mathrm{Syl}_p(G_1) \cap \mathrm{Syl}_p(G_2)$; and

(iii) G_i is of characteristic p.

An important consideration for applications later in this thesis is whether $\mathrm{Syl}_p(B) \subseteq \mathrm{Syl}_p(G)$ where G is some faithful completion of some characteristic p amalgam of rank 2. This motivates the following definition.

Definition 5.1.5. Suppose that G is a faithful completion of the characteristic p amalgam $\mathcal{A}(G_1, G_2, B)$. Then G is a *Sylow completion* of $\mathcal{A}$ if $\mathrm{Syl}_p(B) \subseteq \mathrm{Syl}_p(G)$.

In the above definition, since G is not necessarily a finite group, we must define generally what a Sylow p-subgroup is. We say that P is a Sylow p-subgroup of a group G if every finite p-subgroup of G is conjugate in G to some subgroup of P.

The following theorem provides the connection between amalgams and fusion systems. Indeed, the original application of this theorem demonstrates that any saturated fusion system may be realized by a (possibly infinite) group.

Theorem 5.1.6. *Let p be a prime, G_1, G_2 and G_{12} be groups with $G_{12} \leq G_1 \cap G_2$. Assume that $S_1 \in \mathrm{Syl}_p(G_1)$ and $S_2 \in \mathrm{Syl}_p(G_{12}) \cap \mathrm{Syl}_p(G_2)$ with $S_2 \leq S_1$. Set*

$$G = G_1 *_{G_{12}} G_2$$

to be the free amalgamated product of G_1 and G_2 over G_{12}. Then $S_1 \in \mathrm{Syl}_p(G)$ and

$$\mathcal{F}_{S_1}(G) = \langle \mathcal{F}_{S_1}(G_1), \mathcal{F}_{S_2}(G_2) \rangle.$$

Proof. This is [Rob07, Theorem 1]. □

In other words, the above theorem implies that given two fusion systems which give rise to two rank 2 amalgams, and the data from these amalgams "generate" the fusion system, then provided that the amalgams are isomorphic, the fusion systems are isomorphic.

However, there are some key differences in the group theoretic applications of amalgams, and the fusion theoretic applications. Consider the configurations from Example 5.1.3. The two amalgams there, $\mathcal{A}$ and $\mathcal{B}$, are isomorphic. In this way, we can actually embed a copy of the 2-fusion system of J_2 inside the 2-fusion system of J_3, but the J_2 is certainly not a subgroup of J_3. Indeed, the 2-fusion system of J_3 contains an additional class of essential subgroups arising from different maximal subgroups of J_3 of shape $2^4 : (3 \times \mathrm{SL}_2(4))$ not involved in the amalgams.

Thus, there are some important considerations demonstrated in Example 5.1.3 that one should be aware of. One is that for a group G with two maximal subgroups M_1 and M_2 containing a Sylow p-subgroup of G, even though $G = \langle M_1, M_2 \rangle$ there are situations in which $\mathcal{F}_S(G) \neq \langle \mathcal{F}_S(M_1), \mathcal{F}_S(M_1) \rangle$. The second is that one must be very careful in choosing the “correct” completion when working with amalgams in the context of fusion systems. Indeed, most of the time, this often requires knowledge of the fusion systems, and in particular the essential subgroups, of the completions of the amalgam.

We now collect some results using the *amalgam method* which are relevant to this work. With the application to fusion systems in mind, we are particular interested in the case where the local action involves strongly p-embedded subgroups.

Definition 5.1.7. Let $\mathcal{A} := \mathcal{A}(G_1, G_2, G_{12})$ be a characteristic p amalgam of rank 2 such that there is $G_i^* \trianglelefteq G_i$ satisfying the following for $i \in \{1, 2\}$:

(i) $O_p(G_i) \leq G_i^*$ and $G_i = G_i^* G_{12}$;

(ii) $G_i^* \cap G_{12}$ is the normalizer of a Sylow p-subgroup of G_i^*; and

(iii) $G_i^*/O_p(G_i) \cong \mathrm{PSL}_2(p^n), \mathrm{SL}_2(p^n), \mathrm{PSU}_3(p^n), \mathrm{SU}_3(p^n), \mathrm{Sz}(2^n), \mathrm{Dih}(10), \mathrm{Ree}(3^n)$ or $\mathrm{Ree}(3)'$.

Then $\mathcal{A}$ is a *weak BN-pair of rank* 2. For G a faithful completion of $\mathcal{A}$, we say that G is a group with a weak BN-pair of rank 2.

We define the set of groups

$$\bigwedge = \{\mathrm{PSL}_3(q), \mathrm{PSp}_4(q), \mathrm{PSU}_4(q), \mathrm{PSU}_5(q), \mathrm{G}_2(q), {}^3\mathrm{D}_4(q), {}^2\mathrm{F}_4(2^n), \\ \mathrm{G}_2(2)', {}^2\mathrm{F}_4(2)', \mathrm{M}_{12}, \mathrm{J}_2, \mathrm{F}_3 \mid q = p^n, p \text{ a prime}\}$$

and associate a distinguished prime in each case. For ${}^2\mathrm{F}_4(2^n), \mathrm{G}_2(2)', {}^2\mathrm{F}_4(2)', \mathrm{M}_{12}, \mathrm{J}_2$ the prime is 2, for F_3 the prime is 3 and for the other cases, the prime is p where $q = p^n$.

For $X \in \bigwedge$, let $\mathrm{Aut}^0(X) = \mathrm{Aut}(X)$ unless $X = \mathrm{PSL}_3(q), \mathrm{PSp}_4(2^n), \mathrm{G}_2(3^n)$ in which case $\mathrm{Aut}^0(X)$ is group generated by all inner, diagonal and field automorphisms of X so that $\mathrm{Aut}^0(X)$ is of index 2 and $\mathrm{Aut}(X) = \langle \mathrm{Aut}^0(X), \phi\rangle$ where ϕ is a graph automorphism. Finally, define

$$\bigwedge^0 = \{Y \mid \mathrm{Inn}(X) \leq Y \leq \mathrm{Aut}^0(X), X \in \bigwedge\}.$$

For the remainder of this work, whenever we describe a group as being locally isomorphic to $Y \in \bigwedge^0$, we will always mean that Y is a faithful completion of the rank 2 amalgam given by amalgamating two non-conjugate maximal parabolic subgroups of Y which share a common Borel subgroup. It is straightforward to check that this amalgam is a weak BN-pair of rank 2.

Theorem 5.1.8. *Suppose that G is a group with a weak BN-pair of rank 2. Then one of the following holds:*

(i) *G is locally isomorphic to Y for some $Y \in \bigwedge^0$;*

(ii) *G is parabolic isomorphic to $\mathrm{G}_2(2)'$, J_2, $\mathrm{Aut}(\mathrm{J}_2)$, M_{12}, $\mathrm{Aut}(\mathrm{M}_{12})$ or F_3.*

Proof. This follows from [DS85, Theorem A], [Del88] and [Fan86]. □

For the following corollary, recall the model theorem Theorem 3.1.21 from Chapter 3.

Corollary 5.1.9. *Suppose that $\mathcal{F} = \langle \mathcal{F}_1, \mathcal{F}_2\rangle$ is a fusion system over the p-group S and assume that $\mathcal{F}_i$ is constrained and supported on S for $i \in \{1,2\}$, and $\mathcal{F}_i = N_{\mathcal{F}}(O_p(\mathcal{F}_i))$. Let G_i be a model for $\mathcal{F}_i$ arranged such that $S \in \mathrm{Syl}_p(G_i)$, and let G_{12} be the model for $\mathcal{F}_1 \cap \mathcal{F}_2$. If the amalgam $\mathcal{A} := \mathcal{A}(G_1, G_2, G_{12})$ extracted from*

$\mathcal{F}$ has a faithful completion which is locally isomorphic to Y for some $Y \in \bigwedge^0$ then either:

(i) *$\mathcal{F} \cong \mathcal{F}_S(Y)$; or*

(ii) *$\mathcal{A}$ is of type* F_3.

Proof. Suppose that $\mathcal{A}$ is not of type F_3. By Robinson's result, it is enough to show that $\mathcal{F}_S(Y) = \langle \mathcal{F}_S(G_1), \mathcal{F}_S(G_2) \rangle$ where G_1, G_2 are the relevant "maximal parabolic subgroups" of the groups described in Y. This follows immediately from the Alperin–Goldschmidt theorem and [GLS98, Corollary 3.1.6] when $F^*(Y)$ is a rank 2 group of Lie type, and we may employ the results in [AOV17] for the remaining cases when $p = 2$. □

Notice that all the candidates for $G_i^*/O_p(G_i)$ in the definition of a weak BN-pair of rank 2 have strongly p-embedded subgroups. Indeed, the fusion categories of groups which possess a weak BN-pair of rank 2 form the majority of the examples stemming from the hypothesis in the Main Theorem.

Another important class of amalgams which provide examples in the Main Theorem and Theorem C are *symplectic amalgams.*

Definition 5.1.10. Let $\mathcal{A} := \mathcal{A}(G_1, G_2, G_{12})$ be a characteristic p amalgam of rank 2. Then $\mathcal{A}$ is a *symplectic amalgam* if, up to interchanging G_1 and G_2, the following hold:

(i) $O^{p'}(G_1)/O_p(G_1) \cong \mathrm{SL}_2(p^n)$;

(ii) for $W := \langle ((O_p(G_1) \cap O_p(G_2))^{G_1})^{G_2} \rangle$, $G_2 = G_{12}W$ and $O^p(O^{p'}(G_2)) \leq W$;

(iii) for $S \in \mathrm{Syl}_p(G_{12})$, $G_{12} = N_{G_1}(S)$;

(iv) $\Omega(Z(S)) = \Omega(Z(O^{p'}(G_2)))$ for $S \in \mathrm{Syl}_p(G_{12})$; and

(v) for $Z_1 := \langle \Omega(Z(S))^{G_1} \rangle$, $Z_1 \leq O_p(G_2)$ and there is $x \in G_2$ such that $Z_1^x \not\leq O_p(G_1)$.

Theorem 5.1.11. *Suppose that $\mathcal{A} := \mathcal{A}(G_1, G_2, G_{12})$ is a symplectic amalgam such that $G_2/O_p(G_2)$ has a strongly p-embedded subgroup and for $S \in \mathrm{Syl}_p(G_{12})$, $G_{12} = N_{G_1}(S) = N_{G_2}(S)$. Assume further than G_i is a $\mathcal{K}$-group for $i \in \{1, 2\}$. Then one of the following holds, where $\mathcal{A}_k$ corresponds to the listing given in [PR12, Table 1.8]:*

(i) *$\mathcal{A}$ has a weak BN-pair of rank 2 of type ${}^3\mathrm{D}_4(p^n)$ ($\mathcal{A}_{27}$), $\mathrm{G}_2(p^n)$ ($\mathcal{A}_2$, $\mathcal{A}_6$ and $\mathcal{A}_{26}$ when $p \neq 3$), $\mathrm{G}_2(2)'$ ($\mathcal{A}_1$), J_2 ($\mathcal{A}_{41}$) or $\mathrm{Aut}(\mathrm{J}_2)$ ($\mathcal{A}_{41}^1$);*

(ii) *$p = 2$, $\mathcal{A} = \mathcal{A}_4$, $|S| = 2^6$, $O_2(L_2) \cong 2_+^{1+4}$ and $L_2/O_2(L_2) \cong (3 \times 3) : 2$;*

(iii) *$p = 5$, $\mathcal{A} = \mathcal{A}_{20}$, $|S| = 5^6$, $O_5(L_2) \cong 5_+^{1+4}$ and $L_2/O_5(L_2) \cong 2_-^{1+4}.5$;*

(iv) *$p = 5$, $\mathcal{A} = \mathcal{A}_{21}$, $|S| = 5^6$, $O_5(L_2) \cong 5_+^{1+4}$ and $L_2/O_5(L_2) \cong 2_-^{1+4}.\mathrm{Alt}(5)$;*

(v) *$p = 5$, $\mathcal{A} = \mathcal{A}_{46}$, $|S| = 5^6$, $O_5(L_2) \cong 5_+^{1+4}$ and $L_2/O_5(L_2) \cong 2 \cdot \mathrm{Alt}(6)$; or*

(vi) *$p = 7$, $\mathcal{A} = \mathcal{A}_{48}$, $|S| = 7^6$, $O_7(L_2) \cong 7_+^{1+4}$ and $L_2/O_7(L_2) \cong 2 \cdot \mathrm{Alt}(7)$.*

Proof. We apply the classification in [PR12] and upon inspection of the tables there, we need only rule out $\mathcal{A}_3$, $\mathcal{A}_5$ and $\mathcal{A}_{45}$ when $p = 3$; and $\mathcal{A}_{42}$ when $p = 2$. Set $Q_i := O_3(G_i)$ and $L_i := O^{3'}(L_i)$. With regards to $\mathcal{A}_{45}$, it is proved in [PR12, Theorem 11.4] that $N_{G_2}(S) \not\leq G_1$. In $\mathcal{A}_3$, we have that $L_2/O_3(L_2) \cong \mathrm{SL}_2(3)$ and $|S| = 3^6$. In particular, if $G_{12} = N_{G_1}(S) = N_{G_2}(S)$ then G has a weak BN-pair but comparing with the configurations in [DS85], we have a contradiction.

Suppose that we are in the situation of $\mathcal{A}_5$ so that L_2 is of shape $3.((3^2:Q_8)\times(3^2:Q_8)):3$. Furthermore, by [PR12, Lemma 6.21], we have that $Q_2 = \langle \Omega(Z(Q_1))^{G_2}\rangle$. Let K_2 be a Hall $2'$-subgroup of $L_2 \cap N_G(S)$. Then K_2 is elementary abelian of order 4. By hypothesis, K_2 normalizes Q_1 and so K_2 normalizes $\Omega(Z(Q_1))$. Moreover, K_2 centralizes $\Omega(Z(S)) = \Omega(Z(L_2)) = \Phi(Q_2)$ and since $|\Omega(Z(Q_1))/\Omega(Z(L_2))| = 3$ by [PR12, Lemma 6.21], it follows that there is $k \in K$ an involution which centralizes $\Omega(Z(Q_1))$. Since $\langle kQ_2\rangle \trianglelefteq G_2$, we infer that $\Omega(Z(S)) = [\langle kQ_2\rangle, \Omega(Z(Q_1))]^{G_2} = [\langle kQ_2\rangle, \langle \Omega(Z(Q_1))^{G_2}\rangle]$. But $Q_2 = \langle \Omega(Z(Q_1))^{G_2}\rangle$ by [PR12, Lemma 6.21] so that k centralizes $Q_2/\Phi(Q_2)$, a contradiction since G_2 is of characteristic 3.

In the situation of $\mathcal{A}_{42}$ when $p=2$, we have that $L_2/Q_2 \cong \mathrm{Alt}(5) \cong \mathrm{SL}_2(4)$ so that G has a weak BN-pair of rank 2. Since $|S| = 2^9$ in this case, comparing with [DS85], we have a contradiction. □

Remark. The symplectic amalgams $\mathcal{A}_3$, $\mathcal{A}_5$ and $\mathcal{A}_{45}$ where $G_2/O_3(G_2)$ has a strongly p-embedded subgroup have as example completions $\Omega_8^+(2):\mathrm{Sym}(3)$, $\mathrm{F}_4(2)$ and HN. Indeed, in these configurations $|S|$ is bounded and one can employ [PS21] to get a list of candidate fusion systems supported on S. It transpires that the only appropriate fusion systems supported on S are exactly the fusion categories of the above examples, but in each case there are three essentials, all normal in S, one of which is $\mathrm{Aut}_{\mathcal{F}}(S)$-invariant while the other two are fused under the action of $\mathrm{Aut}_{\mathcal{F}}(S)$.

Remark. In a later section, we come across an amalgam which satisfies almost all of the properties of $\mathcal{A}_{42}$. Indeed, this amalgam contains $\mathcal{A}_{42}$ as a subamalgam and we show that the fusion system supported from this configuration is the 2-fusion system of $\mathrm{PSp}_6(3)$. Indeed, $\mathrm{PSp}_6(3)$ is listed as an example completion of $\mathcal{A}_{42}$ in [PR12] and in $\mathrm{PSp}_6(3)$ itself, there is a choice of generating subgroups G_1, G_2 such that $(G_1, G_2, G_1 \cap G_2)$ is a symplectic amalgam. However, the fusion subsystem generated by the fusion systems of the groups G_1 and G_2 fails to generate the fusion system of $\mathrm{PSp}_6(3)$. In fact, such a subsystem fails to be saturated.

We now state the main hypothesis of this thesis with regard to fusion systems.

Hypothesis 5.1.12. $\mathcal{F}$ is a local $\mathcal{CK}$-system with $O_p(\mathcal{F}) = \{1\}$ and there are two $\mathrm{Aut}_{\mathcal{F}}(S)$-invariant maximally essential subgroups $E_1, E_2 \trianglelefteq S$ such that $\mathcal{F} = \langle N_{\mathcal{F}}(E_1), N_{\mathcal{F}}(E_2)\rangle$.

We now recognize a characteristic p amalgam of rank 2 in $\mathcal{F}$. Namely, we take the models G_1, G_2 and G_{12} of $N_{\mathcal{F}}(E_1)$, $N_{\mathcal{F}}(E_2)$ and $N_{\mathcal{F}}(S)$ and by Theorem 5.1.6, we have that $\mathcal{F} = \mathcal{F}_S(G)$ where $G = G_1 *_{G_{12}} G_2$, and we take the liberty of recognizing G_1, G_2 and G_{12} as subgroups of G.

We now have a hypothesis in purely amalgam theoretic terms. Indeed, G is a characteristic p amalgam of rank 2 such that, for $L_i := O^{p'}(G_i)$, $i \in \{1, 2\}$ and $\overline{L_i} := L_i/E_i$, applying Proposition 3.2.6 and Proposition 3.2.7, one of the following holds:

(i) $\overline{L_i}$ is isomorphic to rank 1 group of Lie type in characteristic p;

(ii) $(\overline{L_i}, p)$ is one of $(Z \cdot \mathrm{PSL}_3(4), 3)$, $(\mathrm{M}_{11}, 3)$, $(\mathrm{Sz}(32) : 5, 5)$, $({}^2\mathrm{F}_4(2)', 5)$, $(Z \cdot \mathrm{McL}, 5)$ or $(\mathrm{J}_4, 11)$, where $Z = Z(\overline{L_i})$ is a p'-group; or

(iii) $\overline{S}$ is cyclic or generalized quaternion and either $\overline{L_i} = N_{\overline{L_i}}(\overline{S})[O_{p'}(\overline{L_i}), \Omega(\overline{S})] = N_{\overline{L_i}}(\overline{S})\langle \Omega(\overline{S})^{\overline{L_i}}\rangle$ is p-solvable; or $\overline{L_i}/O_{p'}(\overline{L_i})$ is a non-abelian simple group, p is odd and $\overline{S}$ is cyclic.

Using the classifications of weak BN-pairs and symplectic amalgams, and treating the small cases using MAGMA (see [PS21]), we can identify a large proportion of the fusion systems under investigation. In an abuse of terminology, we will often say that $\mathcal{F}$ "has a weak BN-pair of rank 2" by which we mean that the amalgam determined by $\mathcal{F}$ is a weak BN-pair of rank 2.

In the following proposition, to verify that two of the fusion systems uncovered are exotic, the classification is invoked (see Section 3.3 and [PS18]). This is the only occasion in this work where we apply the classification in its full strength and not in an inductive context. Without the classification, outcome (iii) below would instead read "$\mathcal{F}$ is a simple fusion system on a Sylow 3-subgroup of F_3 which is not isomorphic to the 3-fusion category of F_3" and outcome (v) would read "$\mathcal{F}$ is a simple fusion system on a Sylow 7-subgroup of $\mathrm{G}_2(7)$ which is not isomorphic to 7-fusion category of $\mathrm{G}_2(7)$ or M."

Proposition 5.1.13. *Suppose that $\mathcal{F}$ satisfies Hypothesis 5.1.12. If the induced amalgam $\mathcal{A} = \mathcal{A}(G_1, G_2, G_{12})$ is a weak BN-pair of rank 2 or a symplectic amalgam satisfying the hypothesis of Theorem 5.1.11, then one of the following holds:*

(i) *$\mathcal{F} = \mathcal{F}_S(H)$, where $F^*(H)$ is isomorphic to a rank 2 simple group of Lie type in defining characteristic;*

(ii) *$\mathcal{F} = \mathcal{F}_S(H)$, where $F^*(H) \cong \mathrm{M}_{12}$ or J_2 and $p = 2$;*

(iii) *$\mathcal{F} = \mathcal{F}_S(H)$, where $H \cong \mathrm{G}_2(3)$ and $p = 2$;*

(iv) *$\mathcal{F}$ is a uniquely determined exotic system on a Sylow 3-subgroup of F_3;*

(v) *$\mathcal{F} = \mathcal{F}_S(H)$, where $F^*(H) \cong \mathrm{Ly}, \mathrm{HN}$ or B and $p = 5$; or*

(vi) *$\mathcal{F}$ is a uniquely determined exotic system on a Sylow 7-subgroup of $\mathrm{G}_2(7)$.*

Proof. Let $\mathcal{A}$ be the amalgam determined by $\mathcal{F}$ and G be the associated free amalgamated product. If $\mathcal{A}$ has a weak BN-pair of rank 2 which is determined up to local isomorphism then by Corollary 5.1.9, $\mathcal{F}$ satisfies part (i). If $p \in \{5, 7\}$ and $\mathcal{A}$ satisfies (iii)-(vi) of Theorem 5.1.11, then $|S| \leqslant p^6$ and $O^p(\mathcal{F}) = \mathcal{F}$. Then the result follows from the tables provided in [PS21] and the proof that $\mathcal{F}$ is exotic in outcome (v) is proved in [PS18]. Suppose that $p = 2$ and $\mathcal{A}$ is parabolic isomorphic to $\mathrm{G}_2(2)'$, M_{12} or J_2. Then $S = (S \cap O^2(G_1))(S \cap O^2(G_2))$ and it follows that $O^2(\mathcal{F}) = \mathcal{F}$. Moreover, by [AOV17] we have that that $O^{2'}(\mathcal{F})$ is isomorphic to $\mathrm{G}_2(2)'$, M_{12} or J_2 and these groups *tamely realize* $O^{2'}(\mathcal{F})$ in each case. In this context, this implies that $\mathcal{F} = \mathcal{F}_S(H)$ where $F^*(H) \cong \mathrm{G}_2(2)'$, M_{12} or J_2.

If $\mathcal{A}$ is parabolic isomorphic to $\mathrm{Aut}(\mathrm{M}_{12})$ or $\mathrm{Aut}(\mathrm{J}_2)$, then there is a subamalgam parabolic isomorphic to M_{12} or J_2 respectively. Moreover, considering this subamalgam in G, we obtain a subgroup $H \trianglelefteq G$ such that H is parabolic isomorphic to M_{12} or J_2. Applying the above, there exists a normal subsystem $\mathcal{H} = \mathcal{F}_{S \cap H}(H) \trianglelefteq \mathcal{F}$ such that $\mathcal{H}$ is isomorphic to the 2-fusion system of M_{12} or J_2. Utilizing the tameness of the 2-fusion systems of M_{12} or J_2 gives the result. Thus, we are left with the case where $\mathcal{A}$ is a symplectic amalgam with $|S| = 2^6$. It follows from [PR12, Lemma 6.21] that $S = (O^2(G_1) \cap S)(O^2(G_2) \cap S)$ so that $O^2(\mathcal{F}) = \mathcal{F}$ by [AKO11, Theorem I.7.4], and checking against the lists provided in [AOV17, Theorem 4.1], $\mathcal{F}$ is isomorphic to the 2-fusion system of $\mathrm{G}_2(3)$.

Finally, suppose that $\mathcal{A}$ is parabolic isomorphic to F_3. In particular, S is determined up to isomorphism. Then comparing with Section 3.3, we conclude that $\mathcal{F}$ is an simple exotic fusion system supported on a 3-group isomorphic to a Sylow 3-subgroup of F_3. $\square$

The bulk of configurations identified in the Main Theorem arise from groups which are completions of weak BN-pairs of rank 2 or symplectic amalgams. Indeed, the remaining cases are all "small" in various senses e.g. by the order of S, their "critical distance." Further to this, by [PS21] and [AOV17], the reduced fusion systems supported on S for (ii), (iii), (iv) and (v) and (vi) above are known; and the fusion systems supported on $T \in \mathrm{Syl}_p(F^*(G))$ in (i) are known in the case where $F^*(G) \cong \mathrm{PSL}_3(p^n), \mathrm{PSp}_4(p^n), \mathrm{G}_2(p^n)$, or $\mathrm{PSU}_4(p^n)$ by [Cle07], [HS19] and the work in Chapter 4.

5.2 The Amalgam Method

Hypothesis 5.1.12 along with Proposition 3.2.6 and Proposition 3.2.7 imply the following hypothesis, listed as Hypothesis B in the introduction, which we assume for the remainder of this chapter.

Hypothesis 5.2.1. $\mathcal{A} := (G_1, G_2, G_{12})$ is a characteristic p amalgam of rank 2 with faithful completion G satisfying the following:

(i) for $S \in \mathrm{Syl}_p(G_{12})$, $G_{12} = N_{G_1}(S) = N_{G_2}(S)$;

(ii) for $L_i := O^{p'}(G_i)$, $\overline{L_i} := L_i/O_p(G_i)$ has one of the following forms:

(a) $\overline{L_i}$ is isomorphic to rank 1 group of Lie type in characteristic p;

(b) $(\overline{L_i}, p)$ is one of $(Z \cdot \mathrm{PSL}_3(4), 3)$, $(\mathrm{M}_{11}, 3)$, $(\mathrm{Sz}(32):5, 5)$, $({}^2\mathrm{F}_4(2)', 5)$, $(Z \cdot \mathrm{McL}, 5)$ or $(\mathrm{J}_4, 11)$, where $Z = Z(\overline{L_i})$ is a p'-group; or

(c) $\overline{S}$ is cyclic or generalized quaternion and either $\overline{L_i} = \overline{S}[O_{p'}(\overline{L_i}), \Omega(\overline{S})]$ is p-solvable; or $\overline{L_i}/O_{p'}(\overline{L_i})$ is a non-abelian simple group, p is odd and $\overline{S}$ is cyclic.

From this point, our methodology is completely based in group theory and we only return to techniques in fusion systems for some identification arguments later. Indeed, for the amalgams considered, we can usually go as far as identifying the "shapes" of G_1 and G_2. We describe this below in the following theorem, presented in the introduction as Theorem C.

Theorem 5.2.2. *Suppose that $\mathcal{A} = \mathcal{A}(G_1, G_2, G_{12})$ satisfies Hypothesis 5.2.1. Then one of the following occurs:*

(i) *$\mathcal{A}$ is a weak BN-pair of rank 2;*

(ii) *$p = 2$, $\mathcal{A}$ is a symplectic amalgam, $|S| = 2^6$, $G_1/O_2(G_1) \cong \mathrm{Sym}(3)$ and $G_2/O_2(G_2) \cong (3 \times 3) : 2$;*

(iii) *$p = 2$, $\Omega(Z(S)) \trianglelefteq G_2$, $\langle (\Omega(Z(S))^{G_1})^{G_2} \rangle \not\leq O_2(G_1)$, $|S| = 2^9$, $O^{2'}(G_1)/O_2(G_1) \cong \mathrm{SU}_3(2)'$ and $O^{2'}(G_2)/O_2(G_2) \cong \mathrm{Alt}(5)$;*

(iv) *$p = 3$, $\Omega(Z(S)) \trianglelefteq G_2$, $\langle (\Omega(Z(S))^{G_1}) \rangle \not\leq O_2(G_2)$, $|S| \leqslant 3^7$ and $O_3(G_1) = \langle (\Omega(Z(S))^{G_1}) \rangle$ is cubic $2F$-module for $G_1/O_3(G_1)$; or*

(v) *$p = 5$ or 7, $\mathcal{A}$ is a symplectic amalgam and $|S| = p^6$.*

The aim is to prove Theorem 5.2.2 and then a combination of Proposition 5.1.13, [PS21] and [AOV17] yields the Main Theorem. Indeed, more information is given about the amalgams listed in (i)-(v) where they arise in the case analysis. It seems that more information may be extracted than what we have provided here, but with the application of fusion systems in mind and the available results classifying fusion systems supported on p-groups of small order, we stop short of completely describing G_1 and G_2 up to isomorphism, although this seems possible in most cases.

At various stages of the analysis, we refer to $\mathcal{F}$, $\mathcal{A}$ or G as being a minimal counterexample to the Main Theorem or Theorem 5.2.2 respectively. By this, we mean a counterexample in each case chosen such that $|G_1| + |G_2|$ is as small as possible.

We assume Hypothesis 5.2.1 and fix the following notation for this chapter. We let $G = G_1 *_{G_{12}} G_2$ and Γ be the (right) coset graph of G with respect to G_1 and G_2, with vertex set $V(\Gamma) = \{G_i g \mid g \in G, i \in \{1,2\}\}$ and $(G_i g, G_j h)$ an edge if $G_i g \neq G_j h$ and $G_i g \cap G_j h \neq \emptyset$ for $\{i,j\} = \{1,2\}$. It is clear that G operates on Γ by right multiplication. Throughout, we identify Γ with its set of vertices, let $d(\cdot,\cdot)$ to be the usual distance on Γ and observe the following notations.

Notation 5.2.3.

- For $\delta \in \Gamma$, $\Delta^{(n)}(\delta) = \{\lambda \in \Gamma \mid d(\delta,\lambda) \leqslant n\}$. In particular, we have that $\Delta^{(0)}(\delta) = \{\delta\}$ and we write $\Delta(\delta) := \Delta^{(1)}(\delta)$.
- For $\delta \in \Gamma$ and $\lambda \in \Delta(\delta)$, we let G_δ be the stabilizer in G of δ and $G_{\delta,\lambda}$ be the stabilizer in G of the edge $\{\delta,\lambda\}$.
- For $\delta \in \Gamma$, $G_\delta^{(n)}$ is the largest normal subgroup of G_δ which fixes $\Delta^{(n)}(\delta)$ element-wise. In particular, $G_\delta = G_\delta^{(0)}$.

The following propositions are elementary and their proofs may be found in [DS85, Chapter 3].

Proposition 5.2.4. *The following facts hold:*

(i) $G_{G_i g} = G_i^g$ *so that every vertex stabilizer is conjugate in G to either G_1 or G_2. In particular, G has finite vertex stabilizers.*

(ii) *Each edge stabilizer of Γ is conjugate in G to G_{12} in its action on Γ.*

(iii) *Γ is a tree.*

(iv) *G acts faithfully and edge transitively on Γ, but does not act vertex transitively.*

(v) *For each edge $\{\lambda_1, \lambda_2\}$, $G = \langle G_{\lambda_1}, G_{\lambda_2}\rangle$.*

(vi) *For $\delta \in \Gamma$ such that $G_\delta = G_i^g$, we have that $\Delta(\delta)$ and G_δ/G_{12}^g are equivalent as G_δ-sets. In particular, G_δ is transitive on $\Delta(\delta) \setminus \{\delta\}$.*

(vii) *G_δ is of characteristic p for all $\delta \in \Gamma$.*

(viii) *If δ and λ are adjacent vertices, then $\mathrm{Syl}_p(G_{\delta,\lambda}) \subseteq \mathrm{Syl}_p(G_\delta) \cap \mathrm{Syl}_p(G_\lambda)$.*

(ix) *If δ and λ are adjacent vertices, then for $S \in \mathrm{Syl}_p(G_{\delta,\lambda})$, $G_{\delta,\lambda} = N_{G_\delta}(S) = N_{G_\lambda}(S)$.*

The following notations will be used extensively throughout the rest of this work.

Notation 5.2.5. Set $\delta \in \Gamma$ to be an arbitrary vertex and $S \in \mathrm{Syl}_p(G_\delta)$.

- $L_\delta := O^{p'}(G_\delta)$.
- $Q_\delta := O_p(G_\delta) = O_p(L_\delta)$.
- $\overline{L_\delta} := L_\delta/Q_\delta$.
- $Z_\delta := \langle \Omega(Z(S))^{G_\delta}\rangle$.
- For $n \in \mathbb{N}$, $V_\delta^{(n)} := \langle Z_\lambda \mid d(\lambda, \delta) \leqslant n\rangle \trianglelefteq G_\delta$, with the additonal conventions $V_\delta^{(0)} = Z_\delta$ and $V_\delta := V_\delta^{(1)}$.
- $b_\delta := \min_{\lambda\in\Gamma}\{d(\delta, \lambda) \mid Z_\delta \not\leq G_\lambda^{(1)}\}$.
- $b := \min_{\delta\in\Gamma}\{b_\delta\}$.

We refer to b as the *critical distance* of the amalgam. Indeed, as G acts edge transitively on Γ it follows that $b = \min\{b_\delta, b_\lambda\}$ where δ and λ are any adjacent vertices in Γ. A *critical pair* is any pair (δ, λ) such that $Z_\delta \not\leq G_\lambda^{(1)}$ and $d(\delta, \lambda) = b$. This definition is not symmetric and so (λ, δ) is not necessarily a critical pair in this case.

It is clear from the definition that symplectic amalgams have critical distance 2. It is remarkable that in all the examples we uncover, $b \leqslant 5$ and if G does not have a weak BN-pair, then $b \leqslant 2$.

Proposition 5.2.6. *The following facts hold:*

(i) $b \geqslant 1$ *is finite.*

(ii) *We may choose* $\{\alpha, \beta\}$ *such that* $\{G_\alpha, G_\beta\} = \{G_1, G_2\}$ *and* $G_{\alpha,\beta} = G_{12} = N_G(S)$.

(iii) *If* $N \leq G_{\alpha,\beta}$, $N_{G_\alpha}(N)$ *operates transitively on* $\Delta(\alpha)$ *and* $N_{G_\beta}(N)$ *operates transitively on* $\Delta(\beta)$, *then* $N = 1$.

(iv) *For* $\delta \in \Gamma$, $\lambda \in \Delta(\delta)$ *and* $T \in \mathrm{Syl}_p(G_{\delta,\lambda})$, *no subgroup of* T *is normal in* $\langle L_\delta, L_\lambda \rangle$.

(v) *For* $\delta \in \Gamma$ *and* $\lambda \in \Delta(\delta)$, *there does not exist a non-trivial element* $g \in G_{\delta,\lambda}$ *with* $gQ_\delta/Q_\delta \in Z(L_\delta/Q_\delta)$ *and* $gQ_\lambda/Q_\lambda \in Z(L_\lambda/Q_\lambda)$.

(vi) *For* $\delta \in \Gamma$ *and* $\lambda \in \Delta(\delta)$, $V_\lambda^{(i)} = \langle (V_\delta^{(i-1)})^{G_\lambda} \rangle$.

For the remainder of this work, we will often fix a critical pair (α, α'). As Γ is a tree, we may set β to be the unique neighbour of α with $d(\beta, \alpha') = b - 1$. Then we label each vertex along the path from α to α' additively e.g. $\beta = \alpha + 1$, $\alpha' = \alpha + b$.

In this way we also see that β may be written as $\alpha' - b + 1$ and so we will often write vertices on the path from α' to α subtractively with respect to α'. The following diagram better explains the situation.

Lemma 5.2.7. *Let $\delta \in \Gamma$, (α, α') be a critical pair, $T \in \mathrm{Syl}_p(G_\alpha)$ and $S \in \mathrm{Syl}_p(G_{\alpha,\beta})$. Then*

(i) *$Q_\delta \leq G_\delta^{(1)}$;*

(ii) *$Z_{\alpha'} \leq G_\alpha$, $Z_\alpha \leq G_{\alpha'}$ and $[Z_\alpha, Z_{\alpha'}] \leq Z_\alpha \cap Z_{\alpha'}$;*

(iii) *$Z_\alpha \neq \Omega(Z(T))$; and*

(iv) *if $\Omega(Z(S))$ is centralized by $L \leq G_\beta$ such that L acts transitively on $\Delta(\beta)$, then $Z(L_\alpha) = \{1\}$.*

Proof. For all $\lambda \in \Delta(\delta)$, we have that $Q_\delta \leq T_\lambda \in \mathrm{Syl}_p(G_\lambda \cap G_\delta)$ and $Q_\delta \leq G_\lambda$. Since $Q_\delta \trianglelefteq G_\delta$, it follows immediately that $Q_\delta \leq G_\delta^{(1)}$. By the minimality of b, we have that $Z_{\alpha'} \leq G_\beta^{(1)} \leq G_\alpha$ and similarly $Z_\alpha \leq G_{\alpha'-1}^{(1)} \leq G_{\alpha'}$. In particular, Z_α normalizes $Z_{\alpha'}$ and vice versa, so that $[Z_\alpha, Z_{\alpha'}] \leq Z_\alpha \cap Z_{\alpha'}$.

Suppose that $Z_\alpha = \Omega(Z(T))$. Then $Z_\alpha = \Omega(Z(S))$ by the transitivity of G_α. By definition and minimality of b, $Z_\alpha \leq Z_\beta \leq G_{\alpha'}^{(1)}$, a contradiction. Finally, suppose that $\Omega(Z(S))$ is centralized by $L \leq G_\beta$ such that L acts transitively on $\Delta(\beta)$. Since Q_α is self-centralizing, it follows that $Z(L_\alpha)$ is a p-group and so $\Omega(Z(L_\alpha)) \leq \Omega(Z(S))$ and L centralizes $\Omega(Z(L_\alpha))$. Then Proposition 5.2.6 (iii) implies that $\Omega(Z(L_\alpha)) = \{1\}$, and so $Z(L_\alpha) = \{1\}$. $\square$

Lemma 5.2.8. *Suppose that $N \trianglelefteq G_\delta$ with N not p-closed and set $S \in \mathrm{Syl}_p(G_\delta)$. Then the following holds:*

(i) *If L_δ is not p-solvable, then $O^p(L_\delta) \leq N$.*

(ii) *If L_δ is p-solvable, then $K \leq NQ_\delta$, where $\overline{K}$ is the unique normal subgroup of $\overline{L_\delta}$ which is divisible by p and minimal with respect to this constraint.*

(iii) *$G_\delta = NN_{G_\delta}(S)$ and N is transitive on $\Delta(\delta)$.*

(iv) *For U/V any non-central chief factor for L_δ inside of Q_δ, we have that $Q_\delta \in \mathrm{Syl}_p(C_{L_\delta}(U/V))$.*

Proof. Suppose L_δ is not p-solvable and let $A \in \mathrm{Syl}_p(N)$. Notice that as N is not p-closed, $A \not\leq Q_\delta$ and since $\overline{L_\delta}$ has a strongly p-embedded subgroup, by Hypothesis 5.2.1 we have that $\widetilde{L_\delta} := \overline{L_\delta}/O_{p'}(\overline{L_\delta})$ is isomorphic to a non-abelian simple group; $\mathrm{Sz}(32):5$ or $\mathrm{Ree}(3)$. Suppose that either of the two latter cases occur. Then by Proposition 3.2.7, $\overline{L_\delta} \cong \mathrm{Sz}(32):5$ or $\mathrm{Ree}(3)$. It follows that $\overline{L_\delta} = \overline{\langle A^{L_\delta}\rangle}\overline{S}$ and so $L/\langle A^{L_\delta}\rangle$ is a p-group. Hence, $O^p(L_\delta) \leq \langle A^{L_\delta}\rangle \leq N$.

If $\widetilde{L_\delta}$ is a non-abelian simple group then $\widetilde{L_\delta} = \langle \widetilde{A}^{\widetilde{L_\delta}}\rangle$. In particular, $\overline{S} \leq \overline{\langle A^{L_\delta}\rangle}$ and so $S \leq \langle A^{L_\delta}\rangle Q_\delta \leq L_\delta$ and since $L_\delta = O^{p'}(L_\delta)$, $L_\delta = \langle A^{L_\delta}\rangle Q_\delta$. It then follows that $O^p(L_\delta) \leq \langle A^{L_\delta}\rangle \leq N$. Thus, we have proved (i).

By the Frattini argument $G_\delta = L_\delta N_{G_\delta}(S) = O^p(L_\delta)N_{G_\delta}(S) = \langle A^{G_\delta}\rangle N_{G_\delta}(S)$. Since $\langle A^{G_\delta}\rangle \leq N$, (iii) follows whenever L_δ is not p-solvable.

Suppose now that L_δ is p-solvable and let $\overline{K}$ be the unique minimal normal subgroup of $\overline{L_\delta}$ divisible by p. Again, we let $A \in \mathrm{Syl}_p(N)$ and remark that since N is not p-closed $A \not\leq Q_\delta$. Hence, $p \mid |\overline{N}|$ so that $\overline{K} \leq \overline{N}$ and $K \leq NQ_\delta$, completing the proof of (ii). By Proposition 3.2.6, $\overline{L_\delta} = \overline{SK} \leq \overline{N_{G_\delta}(S)N}$ so that $G_\delta = L_\delta N_{G_\delta}(S) \leq N_{G_\delta}(S)N \leq G_\delta$, completing the proof of (iii).

For (iv), choose any non-central chief factor U/V for L_δ inside Q_δ. Then U/V is a faithful, irreducible module for $L_\delta/C_{L_\delta}(U/V)$. Since $[Q_\delta, U] \trianglelefteq L_\delta$ and $[Q_\delta, U] < U$, $Q_\delta \leq C_{L_\delta}(U/V)$. Moreover, as $C_{L_\delta}(U/V)$ is normal in L_δ, we deduce that $O_p(C_{L_\delta}(U/V)) = Q_\delta$. If $C_{L_\delta}(U/V)$ is not p-closed, then $L_\delta = C_{L_\delta}(U/V)N_{L_\delta}(S)$ and it follows that U/V is irreducible for $N_{L_\delta}(S)$. But then $[U/V, S] = \{1\}$ from which it follows that $\{1\} = [U/V, \langle S^{L_\delta}\rangle] = [U/V, L_\delta]$, a contradiction. Hence, (iv). $\square$

Proposition 5.2.9. *For all $\delta \in \Gamma$ and $\lambda \in \Delta(\delta)$, $Q_\delta \not\leq Q_\lambda$.*

Proof. Suppose that there is $\delta \in \Gamma$ and $\lambda \in \Delta(\delta)$ with $Q_\delta \leq Q_\lambda$ and let $S \in \mathrm{Syl}_p(G_{\delta,\lambda})$. Then $J(Q_\lambda) \not\leq Q_\delta$ for otherwise, by Proposition 2.3.7 (iv), $J(Q_\lambda) = J(Q_\delta) \trianglelefteq \langle G_\lambda, G_\delta\rangle$. Furthermore, since $C_S(Q_\delta) \leq Q_\delta$, $\Omega(Z(Q_\lambda)) < \Omega(Z(Q_\delta))$. Let $V := \langle \Omega(Z(Q_\lambda))^{G_\delta}\rangle \leq \Omega(Z(Q_\delta))$ and choose $A \in \mathcal{A}(Q_\lambda) \setminus \mathcal{A}(Q_\delta)$. If $Q_\delta < C_S(V)$, then by Lemma 5.2.8 (iii), $G_\delta = \langle C_S(V)^{G_\delta}\rangle N_{G_\delta}(S) = C_{G_\delta}(V)N_{G_\delta}(S)$ normalizes $\Omega(Z(Q_\lambda))$, a contradiction. Hence, $Q_\delta = C_S(V)$.

By the choice of A, $|A| \geqslant |C_A(V)V| = |C_A(V)||V|/|V \cap C_A(V)| = |C_A(V)||V|/|V \cap A|$. Since $A = \Omega(C_S(A))$, we have that $A \cap V = C_V(A)$ and rearranging we conclude that $|A|/|C_A(V)| \geqslant |V|/|C_V(A)|$ and $A/C_A(V) \cong AQ_\delta/Q_\delta$ is an offender on the FF-module V. By Lemma 2.3.10, $L_\delta/C_{L_\delta}(V) \cong \mathrm{SL}_2(p^n)$ and $V/C_V(O^p(L_\delta))$ is a natural $\mathrm{SL}_2(q)$-module. But $Q_\lambda/Q_\delta < S/Q_\delta$ is a $G_{\lambda,\delta}$-invariant subgroup of S/Q_δ, a contradiction by Lemma 2.2.1 (vi). $\square$

Lemma 5.2.10. *Let $\delta \in \Gamma$, (α, α') be a critical pair and $S \in \mathrm{Syl}_p(G_{\alpha,\beta})$. Then*

(i) *$Q_\delta \in \mathrm{Syl}_p(G_\delta^{(1)})$ and $G_\delta^{(1)}/Q_\delta$ is centralized by L_δ/Q_δ;*

(ii) *either $Q_\delta \in \mathrm{Syl}_p(C_{L_\delta}(Z_\delta))$ or $Z_\delta = \Omega(Z(L_\delta))$;*

(iii) *$Z_\alpha \not\leq Q_{\alpha'}$; and*

(iv) *$C_S(Z_\alpha) = Q_\alpha$, and $C_{G_\alpha}(Z_\alpha)$ is p-closed and p-solvable.*

Proof. By Lemma 5.2.7 (i), we assume that $Q_\delta < T$ for $T \in \mathrm{Syl}_p(G_\delta^{(1)})$. Since $G_\delta^{(1)} \trianglelefteq G_\delta$ it follows that $O_p(G_\delta^{(1)}) = Q_\delta$ and so $G_\delta^{(1)}$ is not p-closed. But by Lemma 5.2.8 (iii), then $G_\delta^{(1)}$ would be transitive on $\Delta(\delta)$, a clear contradiction. Thus, $Q_\delta \in \mathrm{Syl}_p(G_\delta^{(1)})$. Letting $P \in \mathrm{Syl}_p(G_\delta)$, $[P, G_\delta^{(1)}] \leq P \cap G_\delta^{(1)} = Q_\delta$ so that $[L_\delta, G_\delta^{(1)}] \leq Q_\delta$, and so (i) holds.

If $Q_\delta \not\in \mathrm{Syl}_p(C_{L_\delta}(Z_\delta))$ then by Lemma 5.2.8 (iii), $G_\delta = C_{L_\delta}(Z_\delta)N_{G_\delta}(S)$ and so $Z_\delta = \langle \Omega(Z(S))^{G_\delta}\rangle = \Omega(Z(S))$. But then $\{1\} = [Z_\delta, S]^{G_\delta} = [Z_\delta, L_\delta]$ and so $Z_\delta \leq Z(L_\delta)$. Since Q_δ is self-centralizing, $Z(L_\delta)$ is a p-group and $Z_\delta = \Omega(Z(S)) = \Omega(Z(L_\delta))$, so that (ii) holds.

If $Z_\alpha \leq Q_{\alpha'}$ then $Z_\alpha \leq G_{\alpha'}^{(1)}$ a contradiction and so (iii) holds. Since $Z_\alpha \neq \Omega(Z(S))$ by Lemma 5.2.7 (iii), $C_S(Z_\alpha) = Q_\alpha \trianglelefteq C_{G_\alpha}(Z_\alpha)$ so that $C_{G_\alpha}(Z_\alpha)$ is p-closed and p-solvable. □

By the above lemma, we can reinterpret the minimal distance b as $b = \min_{\delta \in \Gamma}\{b_\delta\}$ where $b_\delta := \min_{\lambda \in \Gamma}\{d(\delta, \lambda) \mid Z_\delta \not\leq Q_\lambda\}$.

Lemma 5.2.11. *Let (α, α') be a critical pair. Then*

(i) *if $Z_{\alpha'} \leq Z(L_{\alpha'})$ then α is not conjugate to α'; and*

(ii) $C_{Z_\alpha}(Z_{\alpha'}) \neq Z_\alpha \cap Q_{\alpha'}$ *if and only if* $Z_{\alpha'} = \Omega(Z(L_{\alpha'}))$ *and* (α', α) *is not a critical pair.*

Proof. Suppose $Z_{\alpha'} \leq Z(L_{\alpha'})$. By Lemma 5.2.10 (ii), $Z_{\alpha'} = \Omega(Z(L_{\alpha'}))$. If α and α' were conjugate, then $Z_\alpha = \Omega(Z(L_\alpha))$, a contradiction to Lemma 5.2.7 (iii).

Suppose that $Z_{\alpha'} = \Omega(Z(L_{\alpha'}))$. Since $Z_\alpha \not\leq Q_{\alpha'}$ but $Z_\alpha \leq L_{\alpha'}$, we infer that $Z_\alpha = C_{Z_\alpha}(Z_{\alpha'}) \neq Z_\alpha \cap Q_{\alpha'}$. Suppose conversely that $C_{Z_\alpha}(Z_{\alpha'}) \neq Z_\alpha \cap Q_{\alpha'}$. Then $C_{L_{\alpha'}}(Z_{\alpha'})$ is not p-closed and by Lemma 5.2.10 (ii), we have that $Z_{\alpha'} = \Omega(Z(L_{\alpha'}))$. $\square$

Lemma 5.2.12. *Suppose that* $b > 2n$*. Then* $V_\delta^{(n)}$ *is abelian for all* $\delta \in \Gamma$*.*

Proof. Since $b > 2n$, for all $\lambda, \mu \in \Delta^{(n)}(\delta)$ we have that $Z_\lambda \leq G_\mu^{(1)}$ by the minimality of b. Thus, $Z_\lambda \leq Q_\mu$, Z_λ centralizes Z_μ and since $V_\delta^{(n)} = \langle Z_\mu \mid \mu \in \Delta^{(n)}(\delta)\rangle$, it follows that $V_\delta^{(n)}$ is abelian. $\square$

Lemma 5.2.13. $V_\lambda^{(n)}/[V_\lambda^{(n)}, Q_\lambda]$ *contains a non-central chief factor for* L_λ *for all* $n \geqslant 1$ *such that* $V_\lambda^{(n)} \leq Q_\lambda$*.*

Proof. Set $V_\mu^{(0)} = Z_\mu$ for all $\mu \in \Gamma$ and suppose that $O^p(L_\lambda)$ centralizes $V_\lambda^{(n)}/[V_\lambda^{(n)}, Q_\lambda]$. Observe that $V_\lambda^{(n)} = \langle (V_\mu^{(n-1)})^{L_\lambda}\rangle$ for $\mu \in \Delta(\lambda)$ so that $V_\mu^{(n-1)} \not\leq [V_\lambda^{(n)}, Q_\lambda] < V_\lambda^{(n)}$. Moreover, $V_\mu^{(n-1)}[V_\lambda^{(n)}, Q_\lambda] \trianglelefteq L_\lambda$ so that $V_\lambda^{(n)} = V_\mu^{(n-1)}[V_\lambda^{(n)}, Q_\lambda]$. Set $V_i := [V_\lambda^{(n)}, Q_\lambda; i]$. In particular, $V_0 = V_\lambda^{(n)}$ and $V_1 = [V_0, Q_\lambda] = [V_\mu^{(n-1)}, Q_\lambda]V_2$. Notice that $V_\lambda^{(n)} \neq V_\mu^{(n-1)}$ and let k be maximal such that $V_\lambda^{(n)} = V_\mu^{(n-1)}V_k$. Then $V_1 = [V_\mu^{(n-1)}, Q_\mu]V_{k+1} \leq V_\mu^{(n-1)}V_{k+1}$. But $V_\lambda^{(n)} = V_\mu^{(n-1)}V_1 = V_\mu^{(n-1)}V_{k+1}$, contradicting the maximal choice of k. Thus, $O^p(L_\lambda)$ does not centralize $V_\lambda^{(n)}/[V_\lambda^{(n)}, Q_\lambda]$, as required. $\square$

We will use the following lemma often in the amalgam method and without reference. Recall also that if $U, V \trianglelefteq G$ with $V < U$ then, in our setup and using coprime action, U/V does not contain a non-central chief factor for G if and only if $O^p(G)$ centralizes U/V.

Lemma 5.2.14. *For any $\lambda \in \Gamma$, $V_\lambda^{(n)}/V_\lambda^{(n-2)}$ contains a non-central chief factor for L_λ for all $n \geqslant 2$ such that $V_\lambda^{(n)} \leq Q_\lambda$.*

Proof. Assume that $V_\lambda^{(n)}/V_\lambda^{(n-2)}$ contains only central chief factors for L_λ so that $O^p(L_\lambda)$ centralizes $V_\lambda^{(n)}/V_\lambda^{(n-2)}$. Since $V_\lambda^{(n-2)} < V_\mu^{(n-1)} < V_\lambda^{(n)}$ for all $\mu \in \Delta(\lambda)$, we have that $V_\mu^{(n-1)} \trianglelefteq O^p(L_\lambda)G_{\lambda,\mu} = G_\lambda$ by a Frattini argument. But then $V_\mu^{(n-1)} \trianglelefteq \langle G_\mu, G_\lambda \rangle$, a contradiction. Thus, $V_\lambda^{(n)}/V_\lambda^{(n-2)}$ contains a non-central chief factor, as required. □

We now introduce some notation which is non-standard in the amalgam method and is tailored for our purposes.

Notation 5.2.15.

- If $Z_\delta \neq \Omega(Z(L_\delta))$, then $R_\delta = C_{L_\delta}(Z_\delta)$.
- If $Z_\delta = \Omega(Z(L_\delta))$ and $b > 1$, then $R_\delta = C_{L_\delta}(V_\delta/C_{V_\delta}(O^p(L_\delta))$.
- If $Z_\delta = \Omega(Z(L_\delta))$ and $b > 1$, then $C_\delta = C_{Q_\delta}(V_\delta)$.

Lemma 5.2.16. *Suppose that $Z_\delta = \Omega(Z(L_\delta))$, $b > 1$ and let $T \in \mathrm{Syl}_p(G_\delta)$. Then $R_\delta \cap T \leq Q_\delta$ and $C_T(V_\delta) = C_\delta$.*

Proof. Suppose for a contradiction, that $R_\delta \cap T \not\leq Q_\delta$. Then R_δ is not p-closed so that by Lemma 5.2.8 (iii), $G_\delta = R_\delta N_{G_\delta}(T)$. Let $\mu \in \Delta(\delta)$ with $T \in \mathrm{Syl}_p(G_{\delta,\mu})$. Then $Z_\mu \leq V_\delta$ so that $Z_\mu \trianglelefteq \langle G_\delta, G_\mu \rangle$, a contradiction.

Suppose now that $C_T(V_\delta) > Q_\delta$ so that $C_{G_\delta}(V_\delta)$ is not p-closed and is normal in G_δ. As above, by Lemma 5.2.8 (iii), we quickly get that $G_\delta = C_{G_\delta}(V_\delta)G_{\delta,\mu}$ normalizes Z_μ for $\mu \in \Delta(\delta)$ with $T \in \mathrm{Syl}_p(G_{\delta,\mu})$. Hence, the result. □

Lemma 5.2.17. *Suppose that $L_\delta/R_\delta \cong \mathrm{SL}_2(p^n)$, $Q_\delta \in \mathrm{Syl}_p(R_\delta)$ and $R_\delta \leq G_{\delta,\lambda}$ for some $\lambda \in \Delta(\delta)$. Then $\overline{L_\delta} \cong \mathrm{SL}_2(p^n)$.*

Proof. Since $R_\delta \leq G_{\delta,\lambda}$, we have that $[R_\delta, L_\delta] \leq [R_\delta, T]^{L_\delta} \leq \langle (R_\delta \cap T)^{L_\delta}\rangle = Q_\delta$ for $T \in \mathrm{Syl}_p(G_{\delta,\lambda})$. Hence, $\overline{R_\delta} \leq Z(\overline{L_\delta})$ is a p'-group. If $p^n > 3$, then as $L_\delta = O^{p'}(L_\delta)$, it follows from Lemma 2.2.1 (vii) that $\overline{L_\delta} \cong \mathrm{SL}_2(p^n)$.

If $L_\delta/R_\delta \cong \mathrm{Sym}(3)$ and $R_\delta \neq Q_\delta$, then $\overline{R_\delta}$ is a non-trivial 3-group since $L_\delta = O^{2'}(L_\delta)$ and for any prime $r \neq 2, 3$, $O_r(\overline{R_\delta})$ is complemented in $\overline{L_\delta}$. But now, since $\overline{R_\delta}$ is maximal and central in $O_3(\overline{L_\delta})$, $O_3(\overline{L_\delta})$ is abelian. By coprime action, $O_3(\overline{L_\delta}) = [O_3(\overline{L_\delta}), \overline{S}] \times C_{O_3(\overline{L_\delta})}(\overline{S})$ and $\overline{R_\delta}$ is complemented in $\overline{L_\delta}$ by $[O_3(\overline{L_\delta}), \overline{S}]\overline{S} \cong \mathrm{Sym}(3)$. Since $L_\delta = O^{2'}(L_\delta)$ the result follows.

If $L_\delta/R_\delta \cong \mathrm{SL}_2(3)$ then $\overline{R_\delta}$ is a non-trivial 2-group since $L_\delta = O^{3'}(L_\delta)$ and for any prime $r \neq 2, 3$, $O_r(\overline{R_\delta})$ is complemented in $\overline{L_\delta}$. Let A be a maximal subgroup of $\overline{R_\delta}$. Then $|O_2(\overline{L_\delta})/A| = 16$. By Gaschutz' theorem, we may assume that $\overline{R_\delta}/A$ is not complemented in $O_2(\overline{L_\delta})/A$. We see that $O_2(\overline{L_\delta})/A$ is a non-abelian group of order 16 with center of order at most 4. Checking the Small Groups Library in MAGMA for groups of order 48 with a quotient by a central involution isomorphic to $\mathrm{SL}_2(3)$ and a Sylow 2-subgroup satisfying the required properties, we have a contradiction. □

Lemma 5.2.18. *Suppose that $\delta \in \Gamma$, $Z_{\delta-1} = Z_{\delta+1}$, $Q_\delta \in \mathrm{Syl}_p(R_\delta)$ and $i \in \mathbb{N}$. If $Q_{\delta-1}Q_\delta \in \mathrm{Syl}_p(L_\delta)$, L_δ/R_δ is generated by any two distinct Sylow p-subgroups and $O^p(R_\delta)$ normalizes $V_{\delta-1}^{(i-1)}$, then $V_{\delta-1}^{(i-1)} = V_{\delta+1}^{(i-1)}$.*

Proof. Since $Q_{\delta-1}Q_\delta \in \mathrm{Syl}_p(L_\delta)$, if $Q_{\delta-1}R_\delta \neq Q_{\delta+1}R_\delta$, then $Z_{\delta+1} = Z_{\delta-1} \trianglelefteq L_\delta = \langle R_\delta, Q_{\delta-1}, Q_{\delta+1}\rangle$, a contradiction. Thus, $Q_{\delta-1}R_\delta = Q_{\delta+1}R_\delta$. As $Q_{\delta-1}Q_\delta \in \mathrm{Syl}_p(Q_{\delta-1}R_\delta)$, there is $r \in R_\delta$ such that $Q_{\delta-1}^rQ_\delta = (Q_{\delta-1}Q_\delta)^r = (Q_{\delta+1}Q_\delta) = Q_{\delta+1}Q_\delta$. Since $Q_{\delta-1}Q_\delta$ is the unique Sylow p-subgroup of $G_{\delta-1,\delta}$, it follows that $G_{\delta,\delta-1}^r = G_{\delta,\delta+1} = N_{G_\delta}(Q_\delta Q_{\delta+1})$. Set $\theta = (\delta-1)\cdot r \in \Delta(\delta)$. Then by properties of the graph, $G_{\delta,\delta+1} = G_{\delta,\delta-1}^r = G_{\delta,\delta-1\cdot r} = G_{\delta,\theta}$ and so $(\delta-1)\cdot r = \delta+1$. Since r acts as a graph automorphism on Γ, r preserves i neighbourhoods of vertices in the graph and it follows immediately that $V_{\delta-1\cdot r}^{(i-1)} = (V_{\delta-1}^{(i-1)})^r$ so that, as $V_{\delta-1}^{(i-1)}$ is normalized by $R_\delta = O^p(R_\delta)Q_\delta$, $V_{\delta+1}^{(i-1)} = V_{\delta-1}^{(i-1)}$, completing the proof. □

We record one further generic lemma concerning the action of R_γ for $\gamma \in \Gamma$.

Lemma 5.2.19. *Let $\gamma \in \Gamma$ and fix $\delta \in \Delta(\gamma)$. Then for $n < b$, $\langle V_\mu^{(n)} \mid Z_\mu = Z_\delta, \mu \in \Delta(\gamma)\rangle \trianglelefteq R_\gamma Q_\delta$.*

Proof. Set $U^\gamma := \langle V_\mu^{(n)} \mid Z_\mu = Z_\delta, \mu \in \Delta(\gamma)\rangle$ and let $r \in R_\gamma Q_\delta$. Since r is a graph automorphism, for $\mu \in \Delta(\gamma)$ such that $Z_\mu = Z_\delta$, $(V_\mu^{(n)})^r = V_{\mu\cdot r}^{(n)}$. But now, $Z_{\mu\cdot r} = Z_\mu^r = Z_\delta^r = Z_\delta$ and so $(V_\mu^{(n)})^r \leq U^\gamma$. Thus, $U^\gamma \trianglelefteq R_\gamma Q_\delta$, as required. □

As described in Section 2.1, we can guarantee cubic action on a faithful module for $\overline{L_\delta}$ for δ at least one of α, β. We use critical subgroups to achieve this and refer to Theorem 2.1.26 for their properties. The following proposition is listed as Proposition F in the introduction, and it is worth pointing out that it holds in much greater generality than in the hypotheses of this thesis.

Proposition 5.2.20. *There is $\lambda \in \Gamma$ such that there is a $\overline{G_\lambda}$-module V on which p'-elements of $\overline{G_\lambda}$ act faithfully and a p-subgroup C of $\overline{G_\lambda}$ such that $[V, C, C, C] = \{1\}$.*

Proof. Let $(\alpha, \dots, \alpha')$ be a path in Γ with (α, α') a critical pair. For each $\lambda \in (\alpha, \dots, \alpha')$, set K_λ to be a critical subgroup of Q_λ. Since $Z_\alpha \leq K_\alpha$, we must have that $K_\alpha \not\leq Q_{\alpha'}$. Set $c := \{\min(d(\mu, \lambda)) \mid K_\mu \not\leq Q_\lambda, \mu, \lambda \in (\alpha, \dots, \alpha')\}$. Choose a pair (μ, λ) such that $K_\mu \not\leq Q_\lambda$ and $d(\mu, \lambda) = c$. Then, by minimality of c, $K_\mu \leq G_\lambda$ but $K_\mu \not\leq Q_\lambda$ and from the definition of a critical subgroup, p'-elements of $\overline{G_\lambda}$ act faithfully on the $\overline{G_\lambda}$-module $K_\lambda/\Phi(K_\lambda)$. Moreover, again by minimality, K_λ normalizes K_μ so that $[K_\lambda, K_\mu, K_\mu, K_\mu] \leq [K_\mu, K_\mu, K_\mu] = \{1\}$, as required. □

Under the assumption that R_δ is p-solvable group which does not normalize a Sylow p-subgroup of L_δ, we are in a good position to apply Hall–Higman style arguments whenever $p \geqslant 5$. We get the following fact almost immediately from Corollary 2.3.24.

Corollary 5.2.21. *Suppose that $p \geqslant 5$, and $\overline{L_\alpha}$ and $\overline{L_\beta}$ have strongly p-embedded subgroups. Then, for some $\lambda \in \{\alpha, \beta\}$, one of the following holds:*

(i) *$p \geqslant 5$ is arbitrary and $\overline{L_\lambda} \cong \mathrm{PSL}_2(p^n), \mathrm{SL}_2(p^n), \mathrm{PSU}_3(p^n)$ or $\mathrm{SU}_3(p^n)$ for $n \in \mathbb{N}$; or*

(ii) *$p = 5$ and $\overline{L_\lambda} \cong 3 \cdot \mathrm{Alt}(6)$ or $3 \cdot \mathrm{Alt}(7)$.*

Proof. By Proposition 5.2.20, there is a p-element $x \in \overline{L_\lambda}$ which acts cubically on $K_\lambda/\Phi(K_\lambda)$. Suppose there is $y \in L_\lambda$ such that $[y, K_\lambda] \leq \Phi(K_\lambda)$. Since K_λ is a critical subgroup, by coprime action, y is a p-element so that $C_{L_\lambda} K_\lambda/\Phi(K_\lambda)$ is a normal p-subgroup. In particular, $\overline{L_\lambda}$ acts faithfully on $K_\lambda/\Phi(K_\lambda)$ and so we may apply Corollary 2.3.24 and the result holds. □

We now deal with the so called "pushing up" case of the amalgam method. The proof breaks up over a series of lemmas, culminating in Proposition 5.2.25 which was given as Proposition G in the introduction. Throughout, let $\lambda \in \varGamma$, $\mu \in \Delta(\lambda)$ and $S \in \mathrm{Syl}_p(G_{\lambda,\mu})$.

Lemma 5.2.22. *Suppose that $Q_\lambda \cap Q_\mu \trianglelefteq G_\lambda$. Then, writing $L := \langle Q_\mu^{G_\lambda} \rangle$, we have that $Q_\mu \in \mathrm{Syl}_p(L)$, $O_p(L) = Q_\mu \cap Q_\lambda$, $Z_\lambda/Z(L_\lambda)$ is a natural $\mathrm{SL}_2(q)$-module for L_λ/R_λ and no non-trivial characteristic subgroup of Q_μ is normal in L.*

Proof. Set $L := \langle Q_\mu^{G_\lambda} \rangle \trianglelefteq L_\lambda$ and let $V := Z_\lambda$ if $Z_\lambda \neq \Omega(Z(S))$, and $V := V_\lambda/C_{V_\lambda}(O^p(L_\lambda))$ if $Z_\lambda = \Omega(Z(S))$ and $b > 1$. Since $L \trianglelefteq L_\lambda$, we have that $C_L(O_p(L)) \leq O_p(L)$ and since $Q_\mu \not\leq Q_\lambda$, it follows by Lemma 5.2.8 that $L/O_p(L)$ has a strongly p-embedded subgroup and $L_\lambda = LS$ by Hypothesis 5.2.1. If $J(Q_\mu) \leq O_p(L)$, then $J(Q_\mu) \leq Q_\mu \cap Q_\lambda \leq Q_\mu$ and so, by Proposition 2.3.7 (iv), $J(Q_\mu) = J(Q_\mu \cap Q_\lambda) \trianglelefteq L_\lambda$, a contradiction.

Suppose first that $b = 1$ and $Z_\lambda = \Omega(Z(S))$. Then $Z_\lambda \leq Q_\mu$ and we may as well assume that $Z_\mu \not\leq Q_\lambda$. But then Z_μ centralizes $Q_\lambda/O_p(L)$ and $O_p(L)$. Since $\langle Z_\mu^{G_\lambda} \rangle$ contains elements of p'-order, using coprime action and that G_λ is of characteristic p, we have a contradiction. Now, if $V := Z_\lambda$, then $O_p(L) = C_{S\cap L}(V)$ and by Proposition 2.3.9 and Lemma 2.3.10, $L/C_L(V) \cong \mathrm{SL}_2(q)$. If $Z_\lambda = \Omega(Z(S))$, then $Q_\lambda \cap Q_\mu = C_\lambda$ and we may assume that μ belongs to a critical pair (μ, μ') with $d(\lambda, \mu) = b - 1$. Then b is odd, otherwise $\mu' - 1 \in \lambda^G$ and $Z_\mu \leq Q_{\mu'-1} \cap Q_{\mu'-2} = Q_{\mu'-1} \cap Q_{\mu'} \leq Q_{\mu'}$. Thus, $V_{\mu'} \cap Q_\lambda \leq C_\lambda$ and $V_\lambda \cap Q_{\mu'} \leq C_{\mu'}$. Without loss of generality, assume that $|V_{\mu'}/(V_{\mu'} \cap Q_\lambda)| \leqslant |V_\lambda/(V_\lambda \cap Q_{\mu'})|$. A straightforward calculation ensures that $V_\lambda Q_{\mu'}/Q_{\mu'}$ is an offender on $V_{\mu'}/[V_{\mu'}, Q_{\mu'}]$, $[V_{\mu'}, Q_{\mu'}] \leq C_{V_{\mu'}}(O^p(L_{\mu'}))$ and by Lemma 2.3.10, $L_{\mu'}/C_{L_{\mu'}}(V_{\mu'}/C_{V_{\mu'}}(O^p(L_{\mu'}))) \cong \mathrm{SL}_2(q)$.

Either way, it follows from Lemma 2.2.1 (vi) that $L_\lambda/C_{L_\lambda}(V) \cong L/C_L(V) \cong$

$\mathrm{SL}_2(q)$, $S = Q_\lambda Q_\mu$ and

$$Q_\mu O_p(L) = Q_\mu(Q_\lambda \cap L) = Q_\lambda Q_\mu \cap L = S \cap L \in \mathrm{Syl}_p(L).$$

Since $[O_p(L), Q_\mu] \leq [Q_\lambda, Q_\mu] \leq Q_\lambda \cap Q_\mu \leq O_p(L)$ it follows that $[O_p(L), L] = [O_p(L), \langle Q_\mu^{L_\lambda}\rangle] = [O_p(L), Q_\mu]^{L_\lambda} \leq Q_\lambda \cap Q_\mu$ and so $\widehat{L} := L/(Q_\lambda \cap Q_\mu)$ is a central extension of $L/O_p(L)$ by $\widehat{O_p(L)}$. But $Q_\mu \cap O_p(L) = Q_\mu \cap Q_\lambda$ and so $\widehat{Q_\mu}$ is complement to $\widehat{O_p(L)}$ in $\widehat{S \cap L}$. It follows by Gaschutz' theorem that there is a complement in $\widehat{L}$ to $\widehat{O_p(L)}$. Now, letting K_λ be a Hall p'-subgroup of $N_L(S \cap L)$, unless $q \in \{2, 3\}$, we deduce that $\widehat{Q_\mu} \leq [\widehat{S \cap L}, K_\lambda]$ is contained in a complement to $\widehat{O_p(L)}$ and since $L = \langle Q_\mu^{G_\lambda}\rangle$, it follows that $\widehat{O_p(L)} = \{1\}$ and $Q_\mu \in \mathrm{Syl}_p(L)$. If $q \in \{2, 3\}$, then $\widehat{L} \cong p \times \mathrm{SL}_2(p)$, $|\widehat{Q_\mu}| = p$ and one can check that $\langle \widehat{Q_\mu}^{\widehat{L}}\rangle \cong \mathrm{SL}_2(p)$, contradicting the initial definition of L. Thus $Q_\mu \in \mathrm{Syl}_p(L)$ and $O_p(L) = Q_\mu \cap Q_\lambda$. Since $L_\lambda = LQ_\lambda$, there is no non-trivial characteristic subgroup of Q_μ which is normal in L, for such a subgroup would then be normal in $\langle G_\lambda, G_\mu\rangle$.

It remains to show that $V := Z_\lambda$ so suppose that $Z_\lambda = \Omega(Z(S))$ and $V = V_\lambda/C_{V_\lambda}(O^p(L_\lambda))$. Moreover, $Z(L_\mu) = \{1\}$ by Lemma 5.2.7 (iv), $O_p(L) = C_\lambda$, $b > 1$ is odd and V_λ is abelian. Let R_L be the preimage in L of $O_{p'}(L/O_p(L))$ and suppose that R_L is not a p-group. Then $V_\lambda = [V_\lambda, R_L] \times C_{V_\lambda}(R_L)$ is an S-invariant decomposition, and since $Z_\lambda = \Omega(Z(S)) \leq C_{V_\lambda}(R_L)$, V_λ is centralized by R_L. Since V_λ is an FF-module for $L/O_p(L)$, unless $q = 2^n > 2$, using coprime action and Lemma 2.2.6 (v) we infer that $C_{V_\lambda}(R_L) = C_{V_\lambda}(O^p(L))$ so that Z_μ is centralized by L and normalized by $\langle L, G_\mu\rangle$, a contradiction.

Thus, we may reduce to the case where $p = 2$, $R_L = O_2(L)$ and $L/O_2(L) \cong \mathrm{SL}_2(2^n)$ for $n > 1$. Since $S \leq N_{G_\mu}(O_2(L))$, $[G_\mu : N_{G_\mu}(O_2(L))]$ is odd and applying [Ste86, Theorem 3], $V_\mu \trianglelefteq G = \langle L, G_\mu\rangle$, a contradiction. Therefore, $Z_\lambda \neq \Omega(Z(S))$ and

$V = Z_\lambda$. $\square$

Lemma 5.2.23. *Suppose that $Q_\lambda \cap Q_\mu \trianglelefteq L_\lambda$. Then $b > 1$ and, writing $L := \langle Q_\mu^{L_\lambda} \rangle$, $L/O_p(L) \cong L_\lambda/Q_\lambda \cong \mathrm{SL}_2(q)$, $b = 2$ and $O_p(L)$ contains a unique non-central chief factor for L. Moreover, there is $\lambda' \in \Delta(\mu)$ such that both (λ, λ') and (λ', λ) are critical pairs.*

Proof. Suppose that $b = 1$. Then $\Omega(Z(S)) \leq Q_\lambda \cap Q_\mu = O_p(L) \trianglelefteq G_\lambda$ and it follows from the definition of Z_λ that $Z_\lambda \leq O_p(L) \leq Q_\mu$. Thus, we may as well assume that $Z_\mu \not\leq Q_\lambda$. But then Z_μ centralizes $O_p(L)$ and so $O^p(L)$ centralizes $O_p(L)$, a contradiction since L is of characteristic p. Thus, we conclude that $b > 1$.

Suppose that (λ, δ) is not a critical pair for any $\delta \in \Gamma$. Then there is some μ' such that (μ, μ') is a critical pair and $d(\lambda, \mu') = b - 1$. Then $Z_\mu \neq \Omega(Z(S)) \neq Z_\lambda$, $C_{G_{\mu'}}(Z_{\mu'})$ is p-closed and $Z_{\mu'} \leq Q_{\mu+2} \cap Q_\lambda = Q_\lambda \cap Q_\mu$. But then, $[Z_\mu, Z_{\mu'}] = \{1\}$, a contradiction for then $Z_\mu \leq Q_{\mu'}$. Thus, we may assume λ belongs to a critical pair (λ, λ') with $d(\mu, \lambda') = d(\lambda, \lambda') - 1$. Suppose that b is odd. Then $Z_\lambda \leq Q_{\lambda'-1}$ and $\lambda' - 1 \in \lambda^G$. But then $Z_\lambda \leq Q_{\lambda'-1} \cap Q_{\lambda'-2} = Q_{\lambda'-1} \cap Q_{\lambda'} \leq Q_{\lambda'}$, a contradiction. Thus, b is even. Moreover, since $C_S(Z_\lambda)Q_\lambda \in \mathrm{Syl}_p(G_\lambda^{(1)})$ and $[Z_\lambda, Z_{\lambda'}] \neq \{1\}$, (λ', λ) is also a critical pair. Suppose that $b \geqslant 4$. Then $V_\lambda^{(2)} \leq O_p(L)$ and $V_\lambda^{(2)}/Z_\lambda$ contains a non-central chief factor. Thus, if $O_p(L)$ contains a unique non-central chief factor for L then $b = 2$.

Suppose that $O_p(L)$ contains more than one non-central chief factor within $O_p(L)$ and assume that p is odd. If $b = 2$, then $O_2(L) = Q_\lambda \cap Q_\mu = Z_\lambda(Q_\lambda \cap Q_\mu \cap Q_{\lambda'})$, a contradiction since $O_p(L)$ contains more than one non-central chief factor. Thus, we may assume that $b \geqslant 4$ and b is even. Set T_λ to be a Hall p'-subgroup of the preimage in L_λ of $Z(L_\lambda/R_\lambda)$. Note also that since p is odd, we may apply coprime action along with Lemma 2.2.6 (v) so that $Z_\lambda = [Z_\lambda, T_\lambda] \times C_{Z_\lambda}(T_\lambda) = [Z_\lambda, L_\lambda] \times Z(L_\lambda)$.

Choose $\lambda - 1 \in \Delta(\lambda)$ such that $\Omega(Z(L_{\lambda-1})) \neq \Omega(Z(L_\mu))$ and set $U = \langle V_\gamma | \Omega(Z(L_{\lambda-1})) = \Omega(Z(L_\gamma)), \gamma \in \Delta(\lambda)\rangle$. Let $r \in R_\lambda Q_{\lambda-1} \leq C_{L_\lambda}(\Omega(Z(L_{\lambda-1})))$. Since r is an automorphism of the graph, it follows that for $V_\gamma \leq U$, $V_\gamma^r = V_{\gamma\cdot r}$. But $\Omega(Z(L_{\gamma\cdot r})) = \Omega(Z(L_\gamma))^r = \Omega(Z(L_{\lambda-1}))^r = \Omega(Z(L_{\lambda-1}))$ and so $V_\gamma^r \leq U$ and $U \trianglelefteq R_\lambda Q_{\lambda-1}$. Note that if $U \leq Q_{\lambda'-2}$ then $U \leq Q_{\lambda'-2} \cap Q_{\lambda'-3} = Q_{\lambda'-2} \cap Q_{\lambda'-1} \leq Q_{\lambda'-1}$ and so, $U = Z_\lambda(U \cap Q_{\lambda'})$. Thus, $Z_{\lambda'}$ centralizes U/Z_λ and since $L_\lambda = \langle R_\lambda, Z_{\lambda'}, Q_{\lambda-1}\rangle$, it follows that $O^p(L_\lambda)$ centralizes U/Z_λ and so normalizes $V_{\lambda-1}$, a contradiction.

Therefore, $U \not\leq Q_{\lambda'-2}$ so that there is some $\lambda-2 \in \Delta^{(2)}(\lambda)$ such that $(\lambda-2, \lambda'-2)$ is also a critical pair. Since $Z_\lambda = [Z_\lambda, L_\lambda] \times \Omega(Z(L_\lambda))$, it suffices to prove that $[Z_\lambda, Z_{\lambda'}] = \Omega(Z(L_\mu)) = \Omega(Z(L_{\lambda'-1}))$ and that this holds for any critical pair, since then, as there $\lambda - 2 \in \Delta(\lambda - 1)$ with $(\lambda - 2, \lambda' - 2)$ a critical pair, $Z_\lambda = \Omega(Z_{L_{\lambda'-1}}) \times Z_{\lambda'-3} \times \Omega(Z(L_\lambda))$ which is contained in $Q_{\lambda'}$ since $b > 2$.

Suppose that $Z_\mu = \Omega(Z(S)) = \Omega(Z(L_\mu))$. In particular, $Z(L_\lambda) = \{1\}$ and Z_λ is irreducible. Since Z_λ is a natural $\mathrm{SL}_2(q)$-module, $Z_{\lambda'-1} = [Z_\lambda, Z_{\lambda'}] = Z_\mu$, as required.

Assume now that $Z_\mu \neq \Omega(Z(S))$. Then $Z_\lambda = [Z_\lambda, T_\lambda] \times C_{Z_\lambda}(T_\lambda)$, $[Z_\lambda, T_\lambda] = [Z_\lambda, L_\lambda]$ and $C_{Z_\lambda}(T_\lambda) = \Omega(Z(L_\lambda))$. Moreover, $[Z_\lambda, Z_{\lambda'}] = C_{[Z_\lambda, L_\lambda]}(S) = \Omega(Z(S)) \cap [Z_\lambda, L_\lambda]$. Since $\Omega(Z(S)) = \Omega(Z(L_\lambda)) \times \Omega(Z(L_\mu))$ and T_λ normalizes $\Omega(Z(L_\mu))$, we have that $\Omega(Z(L_\mu)) \geqslant [\Omega(Z(L_\mu)), T_\lambda] = [\Omega(Z(S)), T_\lambda] = \Omega(Z(S)) \cap [Z_\lambda, L_\lambda]$. Comparing orders, we conclude that $\Omega(Z(L_\mu)) = [\Omega(Z(S)), T_\lambda] = [Z_\lambda, Z_{\lambda'}]$. By symmetry, we have that $Z(L_{\lambda'-1}) = [Z_\lambda, Z_{\lambda'}]$, as required.

Suppose now that $p = 2$ and $O_2(L)$ contains more than one non-central chief factor within $O_2(L)$. Choose $1 < m < b/2$ minimal such that $V_\lambda^{(2m)} \leq Q_{\lambda'-2m}$. Notice by the minimal choice of m that $V_\lambda^{(2(m-k))} Q_{\lambda'-2(m-k)} \in \mathrm{Syl}_p(L_{\lambda'-2(m-k)})$ for all $k \leqslant m$.Then $V_\lambda^{(2m)} \leq Q_{\lambda'-2m} \cap Q_{\lambda'-2m-1} \leq Q_{\lambda'-2m+1}$ and, extending further, $V_\lambda^{(2m)} = V_\lambda^{(2m-2)}(V_\lambda^{(2m)} \cap Q_{\lambda'})$. But then, $O^p(L_\lambda)$ centralizes $V_\lambda^{(2m)}/V_\lambda^{(2m-2)}$, a contradiction. Thus, no such m exists. Even still an index q subgroup of $V_\lambda^{(2k)}/V_\lambda^{(2k-2)}$ is centralized by $Z_{\lambda'}$ for all $k < b/2$ and it follows that for all $1 < m < b/2$, $V_\lambda^{(2m)}/V_\lambda^{(2m-2)}$ contains a unique non-central chief factor and this factor is an FF-module for L_λ/Q_λ. Note that for R_1, R_2 the centralizers in $L/O_2(L)$ of distinct non-central chief factors in $V_\lambda^{(2m)}$ for $1 < m < b/2$, we deduce that R_1R_2/R_i is an odd order normal subgroup of $L_i/R_i \cong \mathrm{SL}_2(q)$ for $i \in \{1, 2\}$. Thus, unless $q = 2$, we have that $L/O_2(L)C_L(V_\lambda^{(2m)}) \cong \mathrm{SL}_2(q)$ and an application of the three subgroup lemma ensures that $L/O_2(L) \cong \mathrm{SL}_2(q)$.

Since no non-trivial characteristic subgroup of Q_β is normal in L, we may apply pushing up arguments from [Nil79, Theorem B] when $L/O_2(L) \cong \mathrm{SL}_2(q)$. Thus, Q_μ has class 2 and there is a unique non-central chief factor for L within $O_2(L)$. It is clear that $Z_\lambda/Z(L_\lambda)$ is the unique non-central chief factor for L inside $O_2(L)$ and is isomorphic to the natural module for $L/O_2(L) \cong \mathrm{SL}_2(q)$. Thus, $q = p = 2$ and since no non-trivial characteristic subgroup of Q_β is normal in L, we may apply [Gla71, Theorem 4.3] to see that Q_μ has nilpotency class 2 and exponent 4. Notice that if $b \geqslant 4$, then $V_\lambda^{(2)}$ is contained in Q_μ and $[V_\lambda^{(2)}, Q_\mu] \leq \Omega(Z(Q_\mu))$. But $\langle (\Omega(Z(Q_\mu))^L) \rangle$ is an FF-module for $L/O_2(L)$ by Proposition 2.3.9, and contains $[Z_\lambda, L_\lambda]$ as its unique non-central chief factor. Thus, it follows that $[V_\lambda^{(2)}, L] \leq Z_\lambda$ and $V_\mu \trianglelefteq \langle L, G_\mu \rangle$, a contradiction. Hence, we conclude that $b = 2$ so that $O_2(L)$ contains a unique non-central chief factor, as required. □

Lemma 5.2.24. *Suppose that $Q_\lambda \cap Q_\mu \trianglelefteq L_\lambda$. Then $Z_\mu \neq \Omega(Z(S))$.*

Proof. We suppose throughout that there is a unique non-central chief factor for L_λ contained in $Q_\mu \cap Q_\lambda$ and, as a consequence, that $L/O_p(L) \cong L_\lambda/Q_\lambda \cong \mathrm{SL}_2(q)$. Additionally, assume that $Z_\mu = \Omega(Z(S)) = \Omega(Z(L_\mu))$. Then $Z(L_\lambda) = \{1\}$ by Lemma 5.2.7 (iv). Hence, Z_λ is the unique non-central chief factor within $Q_\lambda \cap Q_\mu$. In particular, Z_λ is isomorphic to a natural $\mathrm{SL}_2(q)$-module and $[O^p(L_\lambda), Q_\lambda] = Z_\lambda$.

If $\Phi(Q_\lambda) \neq \{1\}$, then the irreducibility of Z_λ implies that $Z_\lambda \leq \langle (\Phi(Q_\lambda) \cap \Omega(Z(S)))^{L_\lambda} \rangle \leq \Phi(Q_\lambda)$. But then $O^p(L)$ acts trivially on $Q_\lambda/\Phi(Q_\lambda)$, a contradiction by coprime action. Thus, $\Phi(Q_\lambda) = \{1\}$ and Q_λ is elementary abelian. If p is odd or $q = 2$, then for T_λ the preimage in L_λ of $O_{p'}(\overline{L_\lambda})$, we have that $Q_\lambda = [Q_\lambda, T_\lambda] \times C_{Q_\lambda}(T_\lambda) = Z_\lambda \times C_{Q_\lambda}(T_\lambda)$ is an S-invariant decomposition and since $\Omega(Z(S)) \leq Z_\lambda$, we have that $C_{Q_\lambda}(T_\lambda) = \{1\}$ and $Q_\lambda = Z_\lambda$. But then $Z_\mu = Z_\lambda \cap Q_\mu = Q_\lambda \cap Q_\mu \trianglelefteq L_\lambda$, a contradiction.

If $q > 2$ is even, then since $S \leq N_{G_\mu}(O_2(L))$, we have that $[G : N_{G_\mu}(O_2(L))]$ is odd, applying [Ste86, Theorem 3], $V_\mu \trianglelefteq G = \langle L, G_\mu \rangle$, a contradiction. □

Proposition 5.2.25. *Let $S \in \mathrm{Syl}_p(G_\lambda \cap G_\mu)$ for $\lambda \in \varGamma$ and $\mu \in \Delta(\lambda)$. Then $Q_\lambda \cap Q_\mu$ is not normal in L_λ. Moreover, if $Z_\lambda Z_\mu \trianglelefteq L_\lambda$ then $Z_\mu = \Omega(Z(S)) \leq Z_\lambda$.*

Proof. Suppose that $Z_\lambda Z_\mu \trianglelefteq L_\lambda$ but $Z_\mu \neq \Omega(Z(S))$. By Lemma 5.2.10 (ii), we have that $C_S(Z_\mu) = Q_\mu$ and so $C_{Q_\lambda}(Z_\lambda Z_\mu) = Q_\lambda \cap C_S(Z_\mu) = Q_\lambda \cap Q_\mu$ and it follows that $Q_\lambda \cap Q_\mu \trianglelefteq L_\lambda$. Thus, we may suppose that $Q_\lambda \cap Q_\mu \trianglelefteq L_\lambda$, and derive a contradiction to complete the proof.

Under this assumption, Z_λ contains the unique non-central chief factor for L inside $Q_\mu \cap Q_\lambda$ and $Z_\mu \neq \Omega(Z(S))$. Moreover, $b = 2$ and there is $\lambda' \in \Delta(\mu)$ such that $Z_\lambda \not\leq Q_{\lambda'}$ and $Z_{\lambda'} \not\leq Q_\lambda$. Since $L_\lambda/Q_\lambda \cong \mathrm{SL}_2(q_\lambda)$ and $Z_\lambda/Z(L_\lambda)$ is a natural module, we get that $Q_\mu = (Q_{\lambda'} \cap Q_\mu \cap Q_\lambda)Z_\lambda Z_{\lambda'}$ and $Q_\lambda \cap Q_\mu = (Q_{\lambda'} \cap Q_\mu \cap$

$Q_\lambda)Z_\lambda$. Then $(Q_\lambda \cap Q_\mu)/\Phi(Q_{\lambda'} \cap Q_\mu \cap Q_\lambda)$ is elementary abelian and it follows that $\Phi(Q_\lambda \cap Q_\mu) = \Phi(Q_{\lambda'} \cap Q_\mu) = \Phi(Q_{\lambda'} \cap Q_\mu \cap Q_\lambda)$. Set $F := \Phi(Q_\lambda \cap Q_\mu)$. Since Q_λ contains a unique non-central chief factor for L_λ, we infer that F is centralized by $O^p(L)$ and as Q_μ has class 2, $F \leq Z(L)$. Let Z_μ^* be the preimage in Q_μ of $Z(Q_\mu/F)$. Since F is normal in both G_λ and $G_{\lambda'}$, we have that $Z_\mu^* \trianglelefteq \langle G_{\lambda,\mu}, G_{\mu,\lambda'}\rangle$. Moreover, since $Q_\mu = (Q_{\lambda'} \cap Q_\mu \cap Q_\lambda)Z_\lambda Z_{\lambda'}$, we have that $Q_\mu \cap Q_\lambda \cap Q_{\lambda'} \leq Z_\mu^*$. Since $[Z_\mu^*, Z_\lambda] \leq F \leq Z(L)$, we have that $Z_\mu^* \leq Q_\lambda$ and by symmetry, $Z_\mu^* = Q_\mu \cap Q_\lambda \cap Q_{\lambda'}$.

Suppose that p is odd and let $H_{\lambda,\mu}$ be a Hall p'-subgroup of $G_{\lambda,\mu} \cap L_\lambda$. By Lemma 2.2.1 (vi), $H_{\lambda,\mu}$ is cyclic of order $q_\lambda - 1$. Furthermore, $H_{\lambda,\mu}$ normalizes Q_μ, F and Z_μ^* and acts non-trivially on Q_μ/Z_μ^*. Now, for t_λ the unique involution in $H_{\lambda,\mu}$, t_λ centralizes $Q_\mu/Q_\lambda \cap Q_\mu$ and inverts $Q_\lambda \cap Q_\mu/Z_\mu^* = Z_\lambda Z_\mu^*/Z_\mu^*$. By coprime action, $Q_\mu/Z_\mu^* = Z_\lambda Z_\mu^*/Z_\mu^* \times C_{Q_\mu/Z_\mu^*}(t_\lambda)$ is a Q_μ-invariant decomposition. Since $[S, t_\lambda] \leq Q_\lambda \cap Q_\mu$ the previous decomposition is S-invariant. But then $[Q_\lambda, C_{Q_\mu/Z_\mu^*}(t_\lambda)] \leq (Q_\mu \cap Q_\lambda)/Z_\mu^* = Z_\lambda Z_\mu^*/Z_\mu^*$ and we deduce that Q_λ centralizes Q_μ/Z_μ^*. Hence, Q_λ normalizes $Q_{\lambda'} \cap Q_\mu$. Let $M = \langle Q_\lambda, Q_{\lambda'}, Q_\mu\rangle \leq G_\mu$. Then there is an $m \in M$ such that $(Q_\lambda Q_\mu)^m = Q_{\lambda'} Q_\mu$ and since $Q_{\lambda'} Q_\mu$ is the unique Sylow p-subgroup of $G_{\mu,\lambda'}$, it follows that $\lambda \cdot m = \lambda'$. But then $(Q_\lambda \cap Q_\mu)^m = Q_{\lambda'} \cap Q_\mu$ and as M normalizes $Q_{\lambda'} \cap Q_\mu$, we have that $Q_\mu \cap Q_\lambda = Q_\lambda \cap Q_\mu$, absurd since $Z_\lambda \leq Q_\lambda \cap Q_\mu$.

Suppose that $p = 2$. Since $(Q_\lambda \cap Q_\mu)/F$ and $(Q_\mu \cap Q_{\lambda'})/F$ are elementary abelian, by [PR12, Lemma 2.29], every involution in Q_μ/F is contained in $(Q_\lambda \cap Q_\mu)/F$ or $(Q_\mu \cap Q_{\lambda'})/F$. Indeed, for A any other elementary abelian subgroup of Q_μ/F and B the preimage of A in Q_μ, we must have that $B = (B \cap Q_\lambda) \cup (B \cap Q_{\lambda'})$. If $B \not\leq Q_\lambda$, then $F \cap Z_\lambda = C_{Z_\lambda}(B) = Z_\lambda \cap B$ and it follows that $B \cap Q_\lambda = F$. By symmetry, we have shown that $\mathcal{A}(Q_\mu/F) = \{(Q_\lambda \cap Q_\mu)/F, (Q_\mu \cap Q_{\lambda'})/F\}$.

Set $M = \langle Q_\lambda, Q_{\lambda'}, Q_\mu \rangle \leq G_\mu$ so that M normalizes Q_μ, Z_μ^* and F. Thus, all elements of M which do not normalize $Q_\mu \cap Q_\lambda$, conjugate $Q_\mu \cap Q_\lambda$ to $Q_\mu \cap Q_\lambda$, and vice versa. Thus all odd order elements normalize $Q_\mu \cap Q_\lambda$. There is an $m \in M$ such that $(Q_\lambda Q_\mu)^m = Q_{\lambda'} Q_\mu$ and since $Q_{\lambda'} Q_\mu$ is the unique Sylow 2-subgroup of $G_{\mu,\lambda'}$, it follows that $\lambda \cdot m = \lambda'$. Since $M = O^p(M) Q_\lambda Q_) \mu$, we may as well choose m of order coprime to p. But then $(Q_\lambda \cap Q_\mu)^m = Q_{\lambda'} \cap Q_\mu$ and as m normalizes $Q_{\lambda'} \cap Q_\mu$, we conclude that $Q_\mu \cap Q_\lambda = Q_\lambda \cap Q_\mu$, a final contradiction since $Z_\lambda \leq Q_\lambda \cap Q_\mu$. $\square$

We can now prove a result analogous to Lemma 5.2.14, instead working "down" through chief factors. Again, we will apply this lemma often and without reference throughout this chapter.

Lemma 5.2.26. *Let $\lambda \in \Gamma$ and $\mu \in \Delta(\lambda)$, $b > 1$ and $n \geqslant 2$. If $V_\lambda^{(n)} \leq Q_\lambda$, then $C_{Q_\lambda}(V_\lambda^{(n-2)})/C_{Q_\lambda}(V_\lambda^{(n)})$ contains a non-central chief factor for L_λ.*

Proof. Observe that as $V_\lambda^{(n)} \leq Q_\lambda$, we have that $Z(Q_\lambda) \leq C_{Q_\lambda}(V_\lambda^{(n)}) \leq C_{Q_\lambda}(V_\mu^{(n-1)}) \leq C_{Q_\lambda}(V_\lambda^{(n-2)})$. In particular, $C_{Q_\lambda}(V_\mu^{(n-1)})$ is non-trivial. If $C_{Q_\lambda}(V_\lambda^{(n-2)})/C_{Q_\lambda}(V_\lambda^{(n)})$ contains only central chief factors for L_λ, $O^p(L_\lambda)$ centralizes $C_{Q_\lambda}(V_\lambda^{(n-2)})/C_{Q_\lambda}(V_\lambda^{(n)})$ and normalizes $C_{Q_\lambda}(V_\mu^{(n-1)})$. Thus, $C_{Q_\lambda}(V_\mu^{(n-1)}) \trianglelefteq O^p(L_\lambda) G_{\lambda,\mu}) = G_\lambda$. In order to force a contradiction, we need only show that $C_{Q_\lambda}(V_\mu^{(n-1)}) = C_{Q_\mu}(V_\mu^{(n-1)})$.

Let $S \in \mathrm{Syl}_p(G_{\lambda,\mu})$. Since $n \geqslant 2$, $Z_\lambda \leq V_\lambda^{(n-2)}$ is centralized by $C_S(V_\mu^{(n-1)})$ and unless $n = 2$ and $V_\lambda^{(n-2)} = Z_\lambda = \Omega(Z(S))$, applying Lemma 5.2.10 (ii) and Lemma 5.2.16, we have that $C_S(V_\mu^{(n-1)}) \leq Q_\lambda \cap Q_\mu$ and $C_{Q_\lambda}(V_\mu^{(n-1)}) = C_{Q_\mu}(V_\mu^{(n-1)})$, as desired. If $V_\lambda^{(n-2)} = \Omega(Z(S))$, then $V_\mu^{(n-1)} = Z_\mu$ and $C_S(Z_\mu) = Q_\mu$. But then, $C_{Q_\lambda}(Z_\mu) = Q_\lambda \cap Q_\mu \trianglelefteq G_\lambda$, a contradiction by Proposition 5.2.25. $\square$

We will also makes use of the qrc-lemma, although where it is applied there are certainly more elementary arguments which would suffice. In this way, we do not use the lemma in its full capacity and instead, it serves as a way to reduce the length of some of our arguments. This lemma first appeared in [Ste92] but only for the prime 2. We use the extension to all primes presented in [Str06, Theorem 3].

Theorem 5.2.27 (qrc Lemma). *Let (H, M) be an amalgam such that both H, M are of characteristic p and contain a common Sylow p-subgroup. Set $Q_X := O_p(X)$ for $X \in \{H, M\}$, $Z = \langle \Omega(Z(S))^H \rangle$ and $V := \langle Z^M \rangle$. Suppose that M is p-minimal and $Q_H = C_S(Z)$. Then one of the following occurs:*

(i) *$Z \not\leq Q_M$;*

(ii) *Z is an FF-module for $H/C_H(Z)$;*

(iii) *the dual of Z is an FF-module for $H/C_H(Z)$;*

(iv) *Z is a $2F$-module with quadratic offender and V contains more than one non-central chief factor for M; or*

(v) *M has exactly one non-central chief factor in V, $Q_H \cap Q_M \trianglelefteq M$, $[V, O^p(M)] \leq Z(Q_M)$ and contains some non-trivial p-reduced module.*

Notice that case (v) of the qrc-lemma is ruled out in our analysis by Proposition 5.2.25 and in cases (ii) and (iii), Lemma 2.3.10 implies that $H/C_H(Z) \cong \mathrm{SL}_2(q)$, for q some power of p.

We will require some results on FF-modules for weak BN-pairs and other pushing up configurations in subamalgams.

Theorem 5.2.28. *Suppose that G satisfies Hypothesis 5.2.1 where L_α and L_β are p-solvable and let $S \in \mathrm{Syl}_p(L_\alpha) \cap \mathrm{Syl}_p(L_\beta)$. Assume that $G = \langle S^G \rangle$ and V is an FF-module for G such that $C_S(V) = \{1\}$. Then G has a weak BN-pair of rank 2 and is locally isomorphic to one of $\mathrm{SL}_3(p)$, $\mathrm{Sp}_4(p)$, or $\mathrm{G}_2(2)$. Moreover, if G is locally isomorphic to $G_2(2)$, then $G/C_G(V) \cong \mathrm{G}_2(2)$.*

Proof. If G has a weak BN-pair of rank 2 then this follows from [CD91, Theorem A, Theorem B, Corollary 1]. If G does not have a weak BN-pair of rank 2, comparing with Theorem 5.2.2, we see that $p = b = 2$, $L_\alpha/Q_\alpha \cong \mathrm{Sym}(3)$ and $L_\beta/Q_\beta \cong (3\times 3) : 2$. Moreover, there is $P_\beta \leq L_\beta$ such that P_β contains S, $P_\beta/Q_\beta \cong \mathrm{Sym}(3)$ and Q_β contains two non-central chief factors for P_β. Indeed, no non-trivial subgroup of S is normalized by both L_α and P_β and by [Fan86], (L_α, P_β, S) is locally isomorphic to M_{12}. Setting $X := \langle L_\alpha, P_\beta \rangle$ and applying [CD91], V is an FF-module for X upon restriction and applying [CD91, Lemma 3.12], we have a contradiction. □

Lemma 5.2.29. *Suppose that G is a minimal counterexample to Theorem 5.2.2, $\{\lambda, \delta\} = \{\alpha, \beta\}$ and the following conditions hold:*

(i) *$Z(Q_\alpha) = Z_\alpha$ is of order q^2 and $Z(Q_\beta) = Z_\beta = \Omega(Z(S))$ is of order q;*

(ii) *$L_\alpha/R_\alpha \cong \mathrm{SL}_2(q) \cong L_\beta/R_\beta$, and Z_α and $V_\beta/C_{V_\beta}(O^p(L_\beta))$ are natural $\mathrm{SL}_2(q)$-modules; and*

(iii) *there is a non-central chief factor U/W for G_λ such that, as an $\overline{L_\lambda}$-module, U/W is an FF-module, $C_{L_\lambda}(U/W) \neq R_\lambda$, and $C_{L_\lambda}(U/W) \cap R_\lambda$ normalizes $Q_\alpha \cap Q_\beta$.*

Then $q \in \{2, 3\}$ and one of the following holds:

(a) *there is* $H_\lambda \leq G_\lambda$ *containing* $G_{\alpha,\beta}$ *such that* $(H_\lambda, G_\delta, G_{\alpha,\beta})$ *is a weak BN-pair of rank* 2*,* $b \leqslant 5$ *and if* $b > 3$*, then* $(H_\lambda, G_\delta, G_{\alpha,\beta})$ *is parabolic isomorphic to* F_3 *and* $V_\alpha^{(2)}/Z_\alpha$ *is not acted on quadratically by* S*;*

(b) $p = 3$*,* $\lambda = \alpha$*, neither* $C_{L_\alpha}(U/W)$ *nor* R_α *normalizes* $Q_\alpha \cap Q_\beta$ *and there does not exist* $P_\alpha \leq L_\alpha$ *such that* $S(C_{L_\alpha}(U/W) \cap R_\alpha) \leq P_\alpha$*,* P_α *is* $G_{\alpha,\beta}$*-invariant,* $P_\alpha/C_{L_\alpha}(U/W) \cap R_\alpha \cong \mathrm{SL}_2(p)$*,* $L_\alpha = P_\alpha R_\alpha = P_\alpha C_{L_\alpha}(U/W)$ *and* $Q_\alpha \cap Q_\beta \ntrianglelefteq P_\alpha$*;*

(c) $\lambda = \beta$ *and neither* R_β *nor* $C_{L_\beta}(U/W)$ *normalizes* $V_\alpha^{(2)}$*; or*

(d) *there is* $H_\lambda \leq G_\lambda$ *containing* $G_{\alpha,\beta}$ *such that for* $X := \langle H_\lambda, G_\delta\rangle$ *and* $V := \langle Z_\beta^X\rangle$*, we have that* $V_\beta \leq V \leq S$*,* $C_S(V) \trianglelefteq X$ *and for* $\widetilde{X} := X/C_X(V)$*,* $\widetilde{X}$ *is locally isomorphic to* $\mathrm{SL}_3(p), \mathrm{Sp}_4(p)$ *or* $\mathrm{G}_2(2)$*; or* $p = 3$ *and there is an involution* x *in* $G_{\alpha,\beta}$ *such that* $\widetilde{X}/\widetilde{\langle x\rangle}$ *is locally isomorphic to* $\mathrm{PSp}_4(3)$*. Moreover, if* $\widetilde{Q}_\mu$ *contains more than one non-central chief factor for* $\widetilde{L}_\mu$ *where* $\mu \in \{\alpha, \beta\}$*, then* $\widetilde{Q}_\mu$ *contains two non-central chief factors and* $\widetilde{Q}_\nu$ *contains a unique non-central chief factor for* $\widetilde{L}_\nu$ *where* $\mu \neq \nu \in \{\alpha, \beta\}$*, and* $\widetilde{X} \cong \mathrm{G}_2(2)$*.*

Proof. It follows from (ii), (iii) and Lemma 2.3.10 that $L_\lambda/C_{L_\lambda}(U/W) \cong L_\lambda/R_\lambda \cong \mathrm{SL}_2(q)$ and $\mathrm{Syl}_p(C_{L_\lambda}(U/W)) = \mathrm{Syl}_p(R_\lambda) = \{Q_\lambda\}$. Thus, $C_{L_\lambda}(U/W)R_\lambda/Q_\lambda$ is a non-trivial normal p'-subgroup of L_λ/Q_λ. Assume that that $q \geqslant 4$ and $C_{L_\lambda}(U/W) \neq R_\lambda$. Then $C_{L_\lambda}(U/W)R_\lambda/C_{L_\lambda}(U/W) = Z(L_\lambda/C_{L_\lambda}(U/W))$ and $C_{L_\lambda}(U/W)R_\lambda/R_\lambda = Z(L_\lambda/R_\lambda)$. In particular, p is odd and $L_\lambda/C_{L_\lambda}(U/W) \cap R_\lambda$ is isomorphic to a central extension of $\mathrm{PSL}_2(q)$ by an elementary abelian group of order 4. Since $O^{p'}(L_\lambda) = L_\lambda$ and the p'-part of the Schur multiplier of $\mathrm{PSL}_2(q)$ is of order 2 by Lemma 2.2.1 (vii), we have a contradiction. Thus, we may assume that $q \in \{2, 3\}$ throughout so that G_α and G_β are p-solvable. By Lemma 2.3.14 (ii) and

Lemma 2.3.15 (ii), $L_\lambda/(C_{L_\lambda}(U/W)\cap R_\lambda) \cong (3\times 3):2$ if $p=2$, or $(Q_8\times Q_8):3$ if $p=3$.

Suppose that $p=2$. By Lemma 2.3.14 (iii), there are $P_1,\dots,P_4 \le L_\lambda$ such that $S(C_{L_\lambda}(U/W)\cap R_\lambda) \le P_i$ and $P_i/(C_{L_\lambda}(U/W)\cap R_\lambda) \cong \mathrm{Sym}(3)$. Indeed, $C_{L_\lambda}(U/W)S$ and $R_\lambda S$ are non-equal and satisfy this condition. Moreover, P_i is $G_{\alpha,\beta}$-invariant for all i. Since any two P_i generate L_λ, we may choose $P_\lambda = P_j \ne R_\lambda S$ such that $Q_\alpha\cap Q_\beta \not\trianglelefteq P_\lambda$ and $O^2(P_\lambda)$ does not centralize U/W. Set $H_\lambda := P_\lambda G_{\alpha,\beta}$, $X := \langle H_\lambda, G_\delta\rangle$ and $V := \langle Z_\beta^X\rangle$. By (i) and (ii), we have that $V_\beta \le V$.

Suppose that $p=3$. By Lemma 2.3.15 (iii), there is $P_1,\dots,P_5 \le L_\lambda$ such that $S(C_{L_\lambda}(U/W)\cap R_\lambda) \le P_i$ and $P_i/(C_{L_\lambda}(U/W)\cap R_\lambda) \cong \mathrm{SL}_2(3)$. Again, $C_{L_\lambda}(U/W)S$ and $R_\lambda S$ are non-equal and satisfy this condition, and for any $i\ne j$, $L_\lambda = \langle P_i, P_j\rangle$. Since $C_{L_\lambda}(U/W)S$ and $R_\lambda S$ are $G_{\alpha,\beta}$-invariant there is at least one other P_i which is $G_{\alpha,\beta}$-invariant. Notice that $R_\beta S$ normalizes $Q_\alpha\cap Q_\beta$ and as any two P_i generate, by Proposition 5.2.25 if $\lambda=\beta$ there is a choice of $P_\lambda = P_i$ such that $Q_\alpha\cap Q_\beta \not\trianglelefteq P_\lambda$, P_λ is $G_{\alpha,\beta}$-invariant and $O^3(P_\lambda)$ does not centralize U/W or V_β. If $\lambda=\alpha$, then unless outcome (c) holds, we may choose $P_\lambda = P_j \ne R_\lambda S$ such that $Q_\alpha\cap Q_\beta \not\trianglelefteq P_\lambda$ and $O^3(P_\lambda)$ does not centralize U/W. Again, we set $H_\lambda := P_\lambda G_{\alpha,\beta}$, $X := \langle H_\lambda, G_\delta\rangle$ and $V := \langle Z_\beta^X\rangle$, remarking that $V_\beta \le V$.

For $p=2$ or 3, $O_p(P_\lambda) = Q_\lambda$ and P_λ/Q_λ has a strongly p-embedded subgroup. Moreover, P_λ is of characteristic p, $C_S(V) \le C_\beta \le Q_\alpha\cap Q_\beta$ so that $C_S(V) = C_{Q_\alpha}(V) = C_{Q_\beta}(V) \trianglelefteq X$. If no non-trivial subgroup of $G_{\alpha,\beta}$ is normal in X, then X satisfies Hypothesis 5.2.1 and since both H_λ and G_δ are p-solvable, by minimality, $(H_\lambda, G_\delta, G_{\alpha,\beta})$ is a weak BN-pair of rank 2; or that $p=2$, X is a symplectic amalgam, $|S| = 2^6$ and exactly one of $\overline{H_\lambda}$ and $\overline{G_\delta}$ is isomorphic to $(3\times 3):2$. In the latter case, we get that Q_λ and Q_δ are non-abelian subgroups of order 2^5 and

$\overline{G_\delta}$ and $\overline{G_\lambda}$ are isomorphic to subgroups of $\mathrm{GL}_4(2)$. Moreover, for some $\gamma \in \{\lambda, \delta\}$, $|Q_\gamma/\Phi(Q_\gamma)| = 2^3$ so that $\overline{G_\gamma}$ is isomorphic to a subgroup of $\mathrm{GL}_3(2)$. One can check that this implies that $G = X$, a contradiction. If $(H_\lambda, G_\delta, G_{\alpha,\beta})$ is a weak BN-pair then we may associate a critical distance to it. Since $\langle (V_\delta^{(n)})^{H_\lambda}\rangle \leq \langle (V_\delta^{(n)})^{G_\lambda}\rangle$, it follows that the critical distance associated to $(H_\lambda, G_\delta, G_{\alpha,\beta})$ is greater than or equal to b. Comparing with the results in [DS85], we have that $b \leqslant 5$ and $b \leqslant 3$ unless $b = 5$, b is equal to the critical distance associated to $(H_\lambda, G_\delta, G_{\alpha,\beta})$ and $(H_\lambda, G_\delta, G_{\alpha,\beta})$ is parabolic isomorphic to F_3. That $V_\alpha^{(2)}/Z_\alpha$ is not acted on quadratically by S is a consequence of the structure of an F_3-type amalgam.

Hence, we may assume that some non-trivial subgroup of $G_{\alpha,\beta}$ is normal in X. Let K be the largest subgroup by inclusion satisfying this condition. Since S is the unique Sylow p-subgroup of $G_{\alpha,\beta}$, K normalizes S so that $O_p(K) = S \cap K \trianglelefteq X$. If $O_p(K) = \{1\}$, then K is a p'-group which is normal in G_δ, impossible since $F^*(G_\delta) = Q_\delta$ is self-centralizing in G_δ. Thus, there is a finite p-group which is normal in X. Since $O_p(K) \trianglelefteq S$, $Z_\beta \leq O_p(K)$. Then, by definition, $V \leq O_p(K)$. Indeed, as $[O_p(K), V] = [O_p(K), \langle Z_\beta^X\rangle] = \{1\}$, we conclude that $V \leq \Omega(Z(O_p(K)))$ and $O_p(K) \leq C_S(V)$. By an earlier observation, $C_S(V) \trianglelefteq X$ so that $C_S(V) = O_p(K)$.

Set $\widetilde{X} := X/C_X(V)$ so that $\widetilde{X} = \langle \widetilde{H_\lambda}, \widetilde{G_\delta}\rangle$ and $\widetilde{H_\lambda} \cong H_\lambda/C_{H_\lambda}(V)$ is a finite group. Additionally, $\widetilde{G_\delta} \cong G_\delta/C_{G_\delta}(V)$ is a finite group. Since $C_S(V) \in \mathrm{Syl}_p(C_{H_\lambda}(V) \cap C_{G_\delta}(V))$, $C_S(V) \leq C_\beta$ and H_λ does not normalize $Q_\alpha \cap Q_\beta$, we deduce that $\widetilde{Q_\lambda} = O_p(\widetilde{H_\lambda})$ and $\widetilde{H_\lambda}/\widetilde{Q_\lambda}$ has a strongly p-embedded subgroup. Similarly, $\widetilde{Q_\delta} = O_p(\widetilde{G_\delta})$ and $\widetilde{G_\delta}/\widetilde{Q_\delta}$ has a strongly p-embedded subgroup.

In order to show that the triple $(\widetilde{H_\lambda}, \widetilde{G_\delta}, \widetilde{G_{\alpha,\beta}})$ satisfies Hypothesis 5.2.1, we need to show that $\widetilde{H_\lambda}$ and $\widetilde{G_\delta}$ are of characteristic p, $\widetilde{G_{\alpha,\beta}} = \widetilde{H_\lambda} \cap \widetilde{G_\delta} = N_{\widetilde{H_\lambda}}(\widetilde{S}) = N_{\widetilde{G_\delta}}(\widetilde{S})$

and no non-trivial subgroup of $\widetilde{G_{\alpha,\beta}}$ is normal in both $\widetilde{H_\lambda}$ and $\widetilde{G}_\delta$. In the following, we often examine the "preimage in H_λ" of some subgroup of $\widetilde{H}_\lambda$, by which we mean the preimage in H_λ of the isomorphic image in $H_\lambda/C_{H_\lambda}(V)$.

Notice that if $\widetilde{H}_\lambda$ is not of characteristic p then $F^*(\widetilde{H}_\lambda) \neq \widetilde{Q}_\lambda$. Then, as $\widetilde{H}_\lambda$ is p-solvable, $\widetilde{H}_\lambda$ is not of characteristic p then $O_{p'}(\widetilde{H}_\lambda) \neq \{1\}$ so that for $\mathcal{C}_\lambda$ the preimage in H_λ of $O_{p'}(\widetilde{H}_\lambda)$, $[\mathcal{C}_\lambda, Q_\lambda, V] = \{1\}$. For $r \in \mathcal{C}_\lambda$ of order coprime to p, it follows from the A$\times$B-lemma that if r centralizes $C_V(Q_\lambda)$, then $\widetilde{r} = 1$. Since Q_λ is self-centralizing in S, we have that $C_V(Q_\lambda) \leq Z(Q_\lambda)$. Similarly, if $\widetilde{G}_\delta$ is not of characteristic p, defining $\mathcal{C}_\delta$ analogously, by the A$\times$B-lemma we need only show $\mathcal{C}_\delta$ centralizes $C_V(Q_\delta) \leq Z(Q_\delta)$.

Suppose that $\lambda = \beta$. Then $|Z(Q_\beta)| = p$ and so, either $\widetilde{H}_\beta$ is of characteristic p; or $p = 3$, $|\widetilde{\mathcal{C}_\beta}| = 2$ and $\mathcal{C}_\beta$ acts non-trivially on Z_β. In the latter case, $\widetilde{\mathcal{C}_\beta} \leq Z(\widetilde{H_\beta})$ so that $[\mathcal{C}_\beta, S] \leq C_{H_\beta}(V)$. Moreover, by coprime action, we have that $V = [V, \mathcal{C}_\beta] \times C_V(\mathcal{C}_\beta)$ is an S-invariant decomposition and as $\widetilde{\mathcal{C}_\beta}$ acts non-trivially on Z_β, it follows that $V = [V, \mathcal{C}_\beta]$ is inverted by $\widetilde{\mathcal{C}_\beta}$. By the Frattini argument, $\mathcal{C}_\beta S = C_{H_\beta}(V)S(G_{\alpha,\beta} \cap \mathcal{C}_\beta)$ and we may as well assume that there is $x \in G_{\alpha,\beta} \cap \mathcal{C}_\beta$ such that $\widetilde{\langle x \rangle} = \widetilde{\mathcal{C}_\beta}$. But then $[x, Q_\alpha] \leq [x, S] \leq C_S(V)$ and as $x \in G_{\alpha,\beta} \leq G_\alpha$, $\widetilde{G}_\alpha$ is not of characteristic p.

Consider $\mathcal{C}_\alpha$, the preimage in G_α of $O_{p'}(\widetilde{G}_\alpha)$. If $\widetilde{G_\alpha}$ is not of characteristic p, then applying the A$\times$B-lemma, $\mathcal{C}_\alpha \cap C_{G_\alpha}(Z_\alpha) \leq C_{G_\alpha}(V)$ and $\widetilde{\mathcal{C}_\alpha}$ is isomorphic to a normal p'-subgroup of $\mathrm{GL}_2(p)$.

Suppose that $|\widetilde{\mathcal{C}_\alpha}| = 3$ if $p = 2$, or $\widetilde{\mathcal{C}_\alpha} \cong Q_8$ if $p = 3$. Noticing that $[S, C_G(Z_\alpha)] \leq [L_\alpha, C_{G_\alpha}(Z_\alpha)] \leq R_\alpha$, by the Frattini argument, $C_{G_\alpha}(Z_\alpha)G_{\alpha,\beta} = R_\alpha G_{\alpha,\beta}$ and $G_\alpha = R_\alpha G_{\alpha,\beta}\mathcal{C}_\alpha$. By Proposition 5.2.25, since $\mathcal{C}_\alpha G_{\alpha,\beta}$ normalizes $Q_\alpha \cap Q_\beta$, it remains to prove that R_α normalizes $Q_\alpha \cap Q_\beta$ to get a contradiction.

Assume that R_α does not normalize $Q_\alpha \cap Q_\beta$ and let $M_\alpha := C_{G_\alpha}(Z_\alpha)G_{\alpha,\beta}$. Then, $C_{G_\alpha}(Z_\alpha) \not\leq G_{\alpha,\beta}$ so that $Q_\alpha = O_p(M_\alpha)$. Reapplying the A×B-lemma yields $\widetilde{M_\alpha \cap \mathcal{C}_\alpha} = \{1\}$ if $p = 2$ and $|\widetilde{M_\alpha \cap \mathcal{C}_\alpha}| \leqslant 2$ if $p = 3$. In the latter case, suppose that $\widetilde{M_\alpha \cap \mathcal{C}_\alpha}$ is non-trivial and choose $x \in M_\alpha \cap \mathcal{C}_\alpha$ with $[x, V] \neq \{1\}$. Indeed, $\widetilde{\langle x \rangle} = \widetilde{M_\alpha \cap \mathcal{C}_\alpha}$ is central in $\widetilde{M_\alpha}$. It follows that $[x, S] \leq C_{M_\alpha}(V)$. Now, by the Frattini argument, $(\mathcal{C}_\alpha \cap M_\alpha)S = C_{M_\alpha}(V)S(G_{\alpha,\beta} \cap \mathcal{C}_\alpha)$ and we may as well assume that $x \in G_{\alpha,\beta}$ so that $[x, S] \leq C_S(V)$. But then $[x, Q_\beta] \leq C_X(V)$ and so $\widetilde{H_\beta}$ is not of characteristic 3. Indeed, we can arrange that $\langle x \rangle C_{H_\beta}(V) = \mathcal{C}_\beta$.

Now, we may form $M_\alpha^* := C_{G_\alpha}(Z_\alpha)(L_\beta \cap G_{\alpha,\beta})$ and $H_\beta^* := (H_\beta \cap L_\beta)(M_\alpha^* \cap G_{\alpha,\beta})$ and arguing as above, we infer that $\widetilde{M_\alpha^*}$ and $\widetilde{H_\beta^*}$ are both of characteristic p. Moreover, by construction and since R_α does not normalize $Q_\alpha \cap Q_\beta$, we deduce that $\widetilde{Q}_\alpha = O_p(\widetilde{M_\alpha^*})$ and $\widetilde{M_\alpha^*}/\widetilde{Q}_\alpha$ has a strongly p-embedded subgroup. Similarly, $\widetilde{Q}_\beta = O_p(\widetilde{H_\beta^*})$ and $\widetilde{H_\beta^*}/\widetilde{Q}_\beta$ also has a strongly p-embedded subgroup. Set $Y := \langle M_\alpha^*, H_\beta^* \rangle$ and write $G_{\alpha,\beta}^* := M_\alpha^* \cap G_{\alpha,\beta}$

Since $\widetilde{S} = \widetilde{Q}_\alpha \widetilde{Q}_\beta$, it is easily checked that $\widetilde{G_{\alpha,\beta}^*} = N_{\widetilde{M_\alpha^*}}(\widetilde{S}) = N_{\widetilde{H_\beta^*}}(\widetilde{S}) = \widetilde{M_\alpha^*} \cap \widetilde{H_\beta^*}$. Suppose there exists $K^* \leq \widetilde{G_{\alpha,\beta}^*}$ such that $K^* \trianglelefteq \langle \widetilde{M_\alpha^*}, \widetilde{H_\beta^*} \rangle = \widetilde{Y}$. Since $\widetilde{M_\alpha^*}$ and $\widetilde{H_\beta^*}$ are both of characteristic p, we may assume that K^* is not a p'-group, and since $K^* \leq \widetilde{G_{\alpha,\beta}^*}$, $O_p(K^*) = K^* \cap \widetilde{S} \neq \{1\}$. Let K_α denote the preimage of $O_p(K^*)$ in M_α^* and K_β denote the preimage of $O_p(K^*)$ in H_β^*. Then, $T_\alpha := Q_\alpha \cap K_\alpha$ is a normal p-subgroup of M_α^* and, likewise, $T_\beta := Q_\beta \cap K_\beta$ is a normal p-subgroup of H_β^*. Since $\widetilde{T_\alpha T_\beta} = \widetilde{T_\alpha} = \widetilde{T_\beta}$, a comparison of orders yields $T_\alpha T_\beta = T_\alpha = T_\beta \trianglelefteq Y$. Moreover, $T_\alpha > C_S(V)$ and as Y is normalized by $G_{\alpha,\beta}$, T_α is normalized by $G_{\alpha,\beta}$. But now, $G_\alpha = G_{\alpha,\beta}\mathcal{C}_\alpha M_\alpha^*$ and as $\mathcal{C}_\alpha$ centralizes $Q_\alpha/C_S(V)$, $T_\alpha \trianglelefteq \langle G_\alpha, H_\beta \rangle = X$, a contradiction since $C_S(V)$ is the largest p-subgroup of $G_{\alpha,\beta}$ which is normalized by X. Hence, the triple $(\widetilde{M^*_\alpha}, \widetilde{H^*_\beta}, \widetilde{G^*_{\alpha\beta}})$ satisfies Hypothesis 5.2.1.

Since $C_S(V) \leq Q_\alpha \cap Q_\beta$ and $C_S(V)$ is the largest subgroup of S which is normal in Y, we have that $J(S) \not\leq C_S(V)$ and a elementary calculation yields that $\Omega(Z(C_S(V)))$ is an FF-module for $\widetilde{Y}$. Moreover, by construction, $Y = \langle S^Y \rangle$ and, by minimality and since $\widetilde{M^*_\alpha}$ and $\widetilde{H^*_\beta}$ are p-solvable, $\widetilde{Y}$ is locally isomorphic to one of $\mathrm{SL}_3(p)$, $\mathrm{Sp}_4(p)$ or $\mathrm{G}_2(2)$. Moreover, $V_\alpha^{(2)} \leq V$ so that $C_S(V) \leq C_{Q_\alpha}(V_\alpha^{(2)})$. If $\widetilde{Y}$ is locally isomorphic to $\mathrm{SL}_3(p)$, then C_β is the largest normal subgroup of H_β contained in $Q_\alpha \cap Q_\beta$, it follows that $C_\beta \leq C_S(V) \leq C_{Q_\alpha}(V_\alpha^{(2)})$, a contradiction for then $C_\beta \trianglelefteq \langle G_\alpha, G_\beta \rangle$.

If $\widetilde{Y}$ is locally isomorphic to $\mathrm{Sp}_4(p)$, then it follows that $|\widetilde{C_\beta}| \leqslant p$. We may as well assume that $C_S(V) = C_{Q_\alpha}(V_\alpha^{(2)})$ has index p in C_β, else we obtain a contradiction as before. Since $C_S(V) \trianglelefteq X$ and $G_\beta = \langle H_\beta, R_\beta \rangle = \langle H_\beta, C_{L_\beta}(U/W) \rangle$, it follows that neither R_β nor $C_{L_\beta}(U/W)$ normalizes $V_\alpha^{(2)}$ and conclusion (c) holds. If $\widetilde{Y} \cong \mathrm{G}_2(2)$, then one can calculate in a similar manner that $C_S(V) = C_{Q_\alpha}(V_\alpha^{(2)})$ and again we retrieve outcome (c).

Therefore, if $\lambda = \beta$ and $\widetilde{G_\alpha}$ is not of characteristic p, then $p = 3$ and $|\widetilde{\mathcal{C}_\alpha}| = 2$. Then $[\widetilde{S}, \widetilde{\mathcal{C}_\alpha}] = \{1\}$ and, again applying the Frattini argument, we have that $\mathcal{C}_\alpha S = C_{G_\alpha}(V) S (G_{\alpha,\beta} \cap \mathcal{C}_\alpha)$. Choose $x \in G_{\alpha,\beta} \cap \mathcal{C}_\alpha$ with $[x, V] \neq \{1\}$ so that $\widetilde{\langle x \rangle} = \widetilde{\mathcal{C}_\alpha}$. Indeed, $[x, S] \leq C_S(V)$ and it follows that $\widetilde{H_\beta}$ is not of characteristic 3. Hence, we may have that $\widetilde{H_\beta}$ is not of characteristic 3 if and only if $\widetilde{G_\alpha}$ is not of characteristic 3. Moreover, there is $x \in G_{\alpha,\beta}$ such that $\widetilde{\langle x \rangle} = \widetilde{\mathcal{C}_\alpha} = \widetilde{\mathcal{C}_\beta}$.

If $\widetilde{G_\alpha}$ is not of characteristic p, then set $\widehat{X} := \widetilde{X}/\widetilde{\langle x \rangle}$ so that both $\widehat{H_\beta}$ and $\widehat{G_\alpha}$ are of characteristic 3. Moreover, $\widehat{L_\alpha}/\widehat{R_\alpha} \cong \mathrm{PSL}_2(3)$ and $\widehat{O^{p'}(H_\beta)}/(R_\beta \widehat{\cap O^{p'}}(H_\beta)) \cong \mathrm{SL}_2(3)$. As in the construction of $\widetilde{Y}$ above, it is easily checked that $\widehat{G_{\alpha,\beta}} =$

$N_{\widehat{G_\alpha}}(\widehat{S}) = N_{\widehat{H_\beta}}(\widehat{S}) = \widehat{G_\alpha} \cap \widehat{H_\beta}$ and no non-trivial subgroup of $\widehat{G_{\alpha,\beta}}$ is normal in $\widehat{X}$. Thus, by minimality, the triple $(\widehat{G_\alpha}, \widehat{H_\beta}, \widehat{G_{\alpha,\beta}})$ is a weak BN-pair. Indeed, $\widehat{L_\alpha} = O^{3'}(\widehat{G_\alpha})$ and $\widehat{L_\alpha} \cong \mathrm{PSL}_2(3)$ or $\mathrm{SL}_2(3)$. If $\widehat{L_\alpha} \cong \mathrm{SL}_2(3)$, then a Sylow 2-subgroup of $\widehat{L_\alpha}$ is of order 16, and arguing as in Lemma 5.2.17, we force a contradiction. Thus, $\widehat{L_\alpha} \cong \mathrm{PSL}_2(3)$ and $\widehat{X}$ is locally isomorphic to $\mathrm{PSp}_4(3)$. Then, using that C_β is the largest normal subgroup of H_β which is contained in $Q_\alpha \cap Q_\beta$ and $C_{Q_\alpha}(V_\alpha^{(2)})$ is the largest subgroup of C_β normal in G_α, it follows that $C_S(V) = C_{Q_\alpha}(V_\alpha^{(2)}) \trianglelefteq X$. Since $G_\beta = \langle H_\beta, R_\beta\rangle = \langle H_\beta, C_{L_\beta}(U/W)\rangle$, it follows that neither R_β nor $C_{L_\beta}(U/W)$ normalizes $V_\alpha^{(2)}$ and conclusion (c) holds. Thus, we may as well assume that whenever $\lambda = \beta$, $\widetilde{X}$ satisfies Hypothesis 5.2.1 and acts faithfully on V.

Suppose now that $\lambda = \alpha$ so that $H_\alpha/C_{H_\alpha}(Z_\alpha)$ is isomorphic to a subgroup of $\mathrm{GL}_2(p)$. If $\widetilde{H_\alpha}$ is not of characteristic p then, by the A×B-lemma, $\mathcal{C}_\alpha \not\leq C_{H_\alpha}(Z_\alpha)$ and so $\mathcal{C}_\alpha C_{H_\alpha}(Z_\alpha)/C_{H_\alpha}(Z_\alpha)$ is isomorphic to a normal p'-subgroup of $\mathrm{GL}_2(p)$. If $p = 2$ or $|\mathcal{C}_\alpha C_{H_\alpha}(Z_\alpha)/C_{H_\alpha}(Z_\alpha)| > 2$ and $p = 3$, using the Frattini argument it follows that $H_\alpha = C_{P_\alpha}(Z_\alpha)\mathcal{C}_\alpha G_{\alpha,\beta} = (R_\alpha \cap C_{L_\alpha}(U/W))\mathcal{C}_\alpha G_{\alpha,\beta}$ which normalizes $Q_\alpha \cap Q_\beta$, a contradiction. Thus, $p = 3$ and $|\widetilde{\mathcal{C}_\alpha}| = 2$ so that $[\mathcal{C}_\alpha, S] \leq C_X(V)$. Additionally, by coprime action, $V = [V, \mathcal{C}_\alpha] \times C_V(\mathcal{C}_\alpha)$ and as $\widetilde{\mathcal{C}_\alpha}$ does not centralize Z_β we deduce that $V = [V, \mathcal{C}_\alpha]$ is inverted by $\widetilde{\mathcal{C}_\alpha}$. Then, by the Frattini argument, $S\mathcal{C}_\alpha = SC_{H_\alpha}(V)(G_{\alpha,\beta} \cap \mathcal{C}_\alpha)$ and we may choose $x \in G_{\alpha,\beta} \cap \mathcal{C}_\alpha$ with $[x, V] \neq \{1\}$ so that $\widetilde{\langle x\rangle} = \widetilde{\mathcal{C}_\alpha}$ and $[x, S] \leq C_S(V)$. It follows that $\widetilde{G_\beta}$ is not of characteristic 3.

If $\widetilde{G_\beta}$ is not of characteristic p then, by the A×B-lemma, $\mathcal{C}_\beta$ does not centralize Z_β. In particular, $p = 3$ and $|\widetilde{\mathcal{C}_\beta}| = 2$. Then applying coprime action, $\widetilde{\mathcal{C}_\beta}$ inverts V and we see that there is $x \in G_{\alpha,\beta}$ with $\widetilde{\langle x\rangle} = \widetilde{\mathcal{C}_\alpha} = \widetilde{\mathcal{C}_\beta}$. Hence, $\widetilde{H_\alpha}$ is of characteristic p if and only if $\widetilde{G_\beta}$ is of characteristic p.

If $\widetilde{G_\beta}$ is not of characteristic p, then set $\widehat{X} := \widetilde{X}/\widetilde{\langle x \rangle}$ so that $\widehat{H_\alpha}$ and $\widehat{G_\beta}$ are both of characteristic 3, $\widehat{O^{p'}(H_\alpha)}/O^{p'}\widehat{(H_\alpha \cap R_\alpha} \cong \mathrm{PSL}_2(3)$ and $\widehat{L}_\beta/\widehat{R}_\beta \cong \mathrm{SL}_2(3)$. As in the above, it quickly follows that $\widehat{X}$ satisfies Hypothesis 5.2.1 and by minimality, the triple $(\widehat{H_\alpha}, \widehat{G_\beta}, \widehat{G_{\alpha,\beta}})$ is a weak BN-pair of rank 2. Indeed, $\widehat{O^{p'}(H_\alpha)} \cong \mathrm{PSL}_2(3)$ and $\widehat{X}$ is locally isomorphic to $\mathrm{PSp}_4(3)$, and the outstanding case in (d) is satisfied. We may as well assume that whenever $\lambda = \alpha$, $\widetilde{X}$ has satisfies Hypothesis 5.2.1 and acts faithfully on V.

Finally, for either $\lambda = \alpha$ or $\lambda = \beta$, $\widetilde{X}$ satisfies Hypothesis 5.2.1 and acts faithfully on V. Moreover, since $J(S) \not\leq C_S(V)$ an elementary argument (as in the proof of Proposition 2.3.9) implies that V is an FF-module for $\widetilde{X}$. By minimality, $\widetilde{X}$ satisfies Hypothesis 5.2.1 and since both $\widetilde{H}_\lambda$ and $\widetilde{G}_\delta$ are p-solvable, $\widetilde{X}$ is determined by Theorem 5.2.28. Counting the number of non-central chief factors in amalgams locally isomorphic to $\mathrm{SL}_3(p), \mathrm{Sp}_4(p)$ or $\mathrm{G}_2(2)$ (as can be gleaned from [DS85]), outcome (d) is satisfied. $\square$

The hypothesis of Lemma 5.2.29 exhibit a common situation we encounter in the work ahead: where $Z_\beta = Z(Q_\beta)$ is of order p, and both $Z(Q_\alpha) = Z_\alpha$ and $V_\beta/C_{V_\beta}(O^p(L_\beta))$ are natural $\mathrm{SL}_2(p)$-modules for $L_\alpha/R_\alpha \cong \mathrm{SL}_2(p) \cong L_\beta/R_\beta$. Upon first glance, it seems that we have very little control over the action of R_λ for $\lambda \in \{\alpha, \beta\}$. Throughout this chapter we strive to force situations in which the full hypotheses of Lemma 5.2.29 are satisfied. In applying Lemma 5.2.29, the outcomes there will often force contradictions and the conclusion we draw is that $O^p(R_\lambda)$ centralizes U/W, as described in Lemma 5.2.29 (iii). In this situation, Lemma 5.2.18 becomes a powerful tool in dispelling a large number of cases. Motivated by this, we make the following hypothesis and record a large number of lemmas controlling the actions of R_λ for $\lambda \in \varGamma$.

Hypothesis 5.2.30. The following conditions hold:

(i) $Z(Q_\alpha) = Z_\alpha$ is of order p^2 and $Z(Q_\beta) = Z_\beta = \Omega(Z(S))$ is of order p; and

(ii) $L_\alpha/R_\alpha \cong \mathrm{SL}_2(p) \cong L_\beta/R_\beta$, and Z_α and $V_\beta/C_{V_\beta}(O^p(L_\beta))$ are natural $\mathrm{SL}_2(p)$-modules.

As a first consequence of this hypothesis, we make the following observation, gaining control over the order of V_β and the number of non-central chief factors in $V_\alpha^{(2)}$.

Lemma 5.2.31. *Suppose that $b > 2$ and Hypothesis 5.2.30 is satisfied. Then, for $\lambda \in \alpha^G$ and $\delta \in \Delta(\lambda)$, exactly one of the following occurs:*

(i) *$|V_\delta| = p^3$ and $[V_\lambda^{(2)}, Q_\lambda] = Z_\lambda$; or*

(ii) *$C_{V_\delta}(O^p(L_\delta)) \neq Z_\delta$, $|V_\delta| = p^4$ and for $V^\lambda := \langle C_{V_\delta}(O^p(L_\delta))^{G_\lambda}\rangle$, both $V_\lambda^{(2)}/V^\lambda$ and V^λ/Z_λ contain a non-central chief factor for L_λ, $[V^\lambda, Q_\lambda] = Z_\lambda$, $[V_\lambda^{(2)}, Q_\lambda] = V^\lambda$ and $V^\lambda V_\delta \ntrianglelefteq L_\delta$. Moreover, whenever $Z_{\delta+1}C_{V_\delta}(O^p(L_\delta)) = Z_{\delta-1}C_{V_\delta}(O^p(L_\delta))$ for $\delta \in \Gamma^G$, we have that $Z_{\delta+1} = Z_{\delta-1}$.*

Proof. Suppose first that $|V_\delta| = p^3$. Then $[Q_\lambda, V_\lambda^{(2)}] = [Q_\lambda, V_\delta]^{G_\lambda} = Z_\lambda$ and the result holds. So assume now that $C_{V_\delta}(O^p(L_\delta)) \neq Z_\delta$. In particular, since $Z_\delta = Z(Q_\delta)$, Q_δ does not centralize $C_{V_\delta}(O^p(L_\delta))$. By coprime action, $V_\delta/Z_\delta = [V_\delta/Z_\delta, O^p(L_\delta)] \times C_{V_\delta/Z_\delta}(O^p(L_\delta))$. Set V^δ to be the preimage in V_δ of $[V_\delta/Z_\delta, O^p(L_\delta)]$ so that V^δ/Z_δ is a natural $\mathrm{SL}_2(p)$-module and $|V^\delta| = p^3$. Notice that $Z_\lambda V^\delta$ is normalized by L_δ and from the definition of V_δ, $V_\delta = Z_\lambda V^\delta$ has order p^4 and $|C_{V_\delta}(O^p(L_\delta))| = p^2$. Letting $V^\lambda := \langle C_{V_\delta}(O^p(L_\delta))^{G_\lambda}\rangle$, we have that $[Q_\lambda, V^\lambda] \leq Z_\lambda$.

If Q_λ centralizes V^λ, then $Q_\lambda \cap Q_\delta = C_{Q_\delta}(C_{V_\delta}(O^p(L_\delta))) \trianglelefteq L_\delta$, a contradiction by Proposition 5.2.25. Thus, $[Q_\lambda, V^\lambda] = Z_\lambda < V^\lambda \leq V_\lambda^{(2)}$.

Assume that $[V_\lambda^{(2)}, Q_\lambda] = Z_\lambda$. This is the case whenever $V_\delta \leq V^\lambda$. Then $Z_\delta \leq [V_\delta, Q_\lambda] = [V^\delta, Q_\lambda] \leq Z_\lambda$ and since $O^p(L_\delta)$ acts non-trivially on V^δ/Z_δ, it follows that $Z_\lambda \leq V^\delta$ so that $V_\delta = V^\delta$, a contradiction. Thus, we conclude that $V_\delta \not\leq V^\lambda$, $V_\delta \cap V^\lambda = [V_\delta, Q_\lambda]Z_\lambda = C_{V_\delta}(O^p(L_\delta))Z_\lambda$ and $[V_\lambda^{(2)}, Q_\lambda] = V^\lambda$.

Suppose that V^λ/Z_λ does not contain a non-central chief factor for L_λ. Then L_λ normalizes $Z_\lambda C_{V_\delta}(O^p(L_\delta))$ and $[Q_\lambda, Z_\lambda C_{V_\delta}(O^p(L_\delta))] \trianglelefteq L_\lambda$. But $[Q_\lambda, Z_\lambda C_{V_\delta}(O^p(L_\delta))] \leq Z_\delta$ and so Q_λ centralizes $C_{V_\delta}(O^p(L_\delta))$. Hence, $Q_\lambda \cap Q_\delta = C_{Q_\delta}(C_{V_\delta}(O^p(L_\delta))) \trianglelefteq L_\delta$, a contradiction by Proposition 5.2.25. Thus, V^λ/Z_λ contains a non-central chief factor for L_λ. Since $[V_\lambda^{(2)}, Q_\lambda] = V^\lambda$, it follows immediately from Lemma 5.2.13 that $V_\lambda^{(2)}/V^\lambda$ contains a non-central chief factor for L_λ.

Suppose that $Z_\lambda C_{V_\delta}(O^p(L_\delta)) = Z_\mu C_{V_\delta}(O^p(L_\delta))$ for some $\mu \in \Delta(\delta)$. Since $|Z_\lambda C_{V_\delta}(O^p(L_\delta))| = p^3$ and $Z_\lambda| = p^2$, if $Z_\lambda \neq Z_\mu$, then $Z_\lambda C_{V_\delta}(O^p(L_\delta)) = Z_\lambda Z_\mu$. Suppose that $Z_\lambda \neq Z_\mu$, so that $Z_\lambda Z_\mu = Z_\lambda C_{V_\delta}(O^p(L_\delta)) = Z_\mu C_{V_\delta}(O^p(L_\delta))$ is normalized by $Q_\lambda R_\delta$ and $Q_\mu R_\delta$. If $Q_\lambda R_\delta \neq Q_\mu R_\delta$ then $Z_\lambda C_{V_\delta}(O^p(L_\delta)) \trianglelefteq L_\delta = \langle Q_\lambda, Q_\mu, R_\delta \rangle$, and from the definition of V_δ, $V_\delta = Z_\lambda C_{V_\delta}(O^p(L_\delta))$ is centralized by Q_λ, a contradiction by Lemma 5.2.16. Thus, $Q_\lambda R_\delta = Q_\mu R_\delta$. Then, there is $r \in R_\delta$ such that $Q_\lambda^r Q_\delta = (Q_\lambda Q_\delta)^r = (Q_\mu Q_\delta)^r = Q_\mu^r Q_\delta$ and we may as well pick r of order coprime to p. Moreover, since $O^p(R_\delta)$ centralizes Q_δ/C_δ, it follows that $Q_\lambda \in \mathrm{Syl}_p(Q_\lambda O^p(R_\delta))$. But then $Q_\mu \in \mathrm{Syl}_p(Q_\lambda O^p(R_\delta))$. Since r centralizes Q_δ/C_δ we conclude that $Q_\lambda \cap Q_\delta = Q_\mu \cap Q_\delta = C_{Q_\delta}(C_{V_\delta}(O^p(L_\delta))) \trianglelefteq L_\delta$, a contradiction by Proposition 5.2.25.

It remains to prove that $V^\lambda V_\delta \ntrianglelefteq L_\delta$ so suppose for a contradiction that $V^\lambda V_\delta \trianglelefteq L_\delta$. Since $Q_\lambda \cap Q_\delta \ntrianglelefteq L_\delta$ by Proposition 5.2.25, there is $\mu \in \Delta(\mu)$ such that $Q_\delta = (Q_\lambda \cap Q_\delta)(Q_\mu \cap Q_\delta)$. Moreover, as $V^\lambda V_\delta \trianglelefteq L_\delta$, $V^\lambda V_\delta = V^\mu V^\delta$. Now,

$$Z_\delta \leq [Q_\delta, V^\lambda V_\delta] = [Q_\lambda \cap Q_\delta, V^\lambda V^\delta][Q_\mu \cap Q_\delta, V^\mu V^\delta] \leq Z_\lambda Z_\mu$$

and $[Q_\delta, V^\lambda V_\delta] \trianglelefteq L_\delta$. If $[Q_\delta, V^\lambda V_\delta] = Z_\delta$, then $[Q_\delta, V^\lambda] \leq Z_\lambda$ and V^λ/Z_λ does not contain a non-central chief factor, a contradiction. If $Z_\lambda Z_\mu \trianglelefteq L_\delta$, then $V_\delta = Z_\lambda Z_\mu$ is of order p^3, another contradiction. Thus, $[Q_\delta, V^\lambda V_\delta]$ is of order p^2 and it follows from the structure of V_δ that $[Q_\delta, V^\lambda V_\delta] = C_{V_\delta}(O^p(L_\delta)) \leq Z_\mu Z_\lambda$. But then $Z_\lambda Z_\mu = Z_\lambda C_{V_\delta}(O^p(L_\delta)) = Z_\mu C_{V_\delta}(O^p(L_\delta))$ so that $Z_\lambda = Z_\mu$. But then $Q_\delta = (Q_\lambda \cap Q_\delta)(Q_\mu \cap Q_\delta)$ centralizes Z_λ, a contradiction by Lemma 5.2.10 (iv). $\square$

Lemma 5.2.32. *Suppose that $b > 3$ and Hypothesis 5.2.30 is satisfied. If Z_α, V^α/Z_α and $V_\alpha^{(2)}/V^\alpha$ are FF-modules or trivial modules for $\overline{L_\alpha}$, then $R_\alpha = C_{L_\alpha}(V_\alpha^{(2)})Q_\alpha$.*

Proof. Of the configurations described in Theorem 5.2.2 which satisfy $b > 2$, all satisfy $R_\alpha = Q_\alpha$ and so we may assume throughout that G is a minimal counterexample to Theorem 5.2.2 such that $R_\alpha \neq C_{L_\alpha}(V_\alpha^{(2)})Q_\alpha$.

Suppose first that $|V_\beta| \neq p^3$ so that V^α/Z_α contains a non-central chief factor for L_α. Since $L_\alpha/R_\alpha \cong \mathrm{SL}_2(p)$ and $Q_\alpha \in \mathrm{Syl}_p(R_\alpha)$, $|S/Q_\alpha| = p$ and by Lemma 2.3.10, $L_\alpha/C_{L_\alpha}(V^\alpha/Z_\alpha) \cong L_\alpha/C_{L_\alpha}(V_\alpha^{(2)})/V^\alpha) \cong \mathrm{SL}_2(p)$. Thus, if $C_{L_\alpha}(V^\alpha/Z_\alpha) \neq R_\alpha$ a standard calculation yields $L_\alpha/C_{L_\alpha}(V^\alpha)Q_\alpha$ is a central extension of $\mathrm{PSL}_2(p)$ by a fours group, or that $p \in \{2,3\}$. Since $L_\alpha = O^{p'}(L_\alpha)$ and the 2-part of the Schur multiplier has order 2 when $p \geqslant 5$, we deduce that $p \in \{2,3\}$. Moreover, if $p = 3$ and $R_\alpha C_{L_\alpha}(V^\alpha/Z_\alpha)S < L_\alpha$, then $|R_\alpha C_{L_\alpha}(V^\alpha/Z_\alpha)/R_\alpha| = 2$,

$|L_\alpha/C_{L_\alpha}(V^\alpha)Q_\alpha| = 2^4.3$ and Lemma 2.3.15 (ii) gives a contradiction. Hence, if $C_{L_\alpha}(V^\alpha/Z_\alpha) \neq R_\alpha$ then $L_\alpha = R_\alpha C_{L_\alpha}(V^\alpha/Z_\alpha)S$. But now, $C_{L_\alpha}(V^\alpha/Z_\alpha)$ normalizes $Z_\alpha C_{V_\beta}(O^p(L_\beta))$ and so normalizes $[Z_\alpha C_{V_\beta}(O^p(L_\beta)), Q_\alpha] = Z_\beta$, a contradiction for then $Z_\beta \trianglelefteq L_\alpha$. Thus, $C_{L_\alpha}(V^\alpha/Z_\alpha) = R_\alpha$. Similarly, considering $C_{L_\alpha}(V_\alpha^{(2)}/V^\alpha)$, we have that $V_\beta V^\alpha \trianglelefteq C_{L_\alpha}(V_\alpha^{(2)}/V^\alpha)$ and so $Z_\alpha C_{V_\beta}(O^p(L_\beta)) = Z_\alpha[V_\beta, Q_\alpha] = [V_\beta V^\alpha, Q_\alpha] \trianglelefteq C_{L_\alpha}(V_\alpha^{(2)}/V^\alpha)$. Then $[Z_\alpha C_{V_\beta}(O^p(L_\beta)), Q_\alpha] = Z_\beta$ is normalized by $C_{L_\alpha}(V_\alpha^{(2)}/V^\alpha)$ and, as above, we conclude that $C_{L_\alpha}(V_\alpha^{(2)}/V^\alpha) = R_\alpha$ and the result holds.

Hence, we may assume that $|V_\beta| = p^3$ throughout. Since Hypothesis 5.2.30 is satisfied, $V_\alpha^{(2)}/Z_\alpha$ is an FF-module and $C_{L_\alpha}(V_\alpha^{(2)}/Z_\alpha) \cap R_\alpha = C_{L_\alpha}(V_\alpha^{(2)})$ centralizes $Q_\alpha/C_{Q_\alpha}(V_\alpha^{(2)})$ and so normalizes $Q_\alpha \cap Q_\beta > C_{Q_\alpha}(V_\alpha^{(2)}$, we apply Lemma 5.2.29, taking $\lambda = \alpha$. As $b > 3$ and $V_\alpha^{(2)}/Z_\alpha$ is an FF-module (so admits quadratic action), so that outcome (a) does not hold. Since $\lambda = \alpha$ outcome (c) does not hold.

Suppose (d) holds. Then, by construction, $\langle V_\beta^{H_\alpha} \rangle = \langle V_\beta^{G_\alpha} \rangle = V_\alpha^{(2)}$ from which it follows that $V_\beta^{(3)} \leq V := \langle Z_\beta^X \rangle$ and the images of both Q_β/C_β and $C_\beta/C_{Q_\beta}(V_\beta^{(3)})$ in $\widetilde{L}_\beta$ contain a non-central chief factor for $\widetilde{L}_\beta$. By Lemma 5.2.29, $\widetilde{X} \cong \mathrm{G}_2(2)$. It follows from the structure of $\mathrm{G}_2(2)$ that $|Q_\alpha/C_\beta| = 2^2$ and $|Q_\alpha/C_{Q_\alpha}(V_\alpha^{(2)})| = 2^4$ and $|\widetilde{C_{Q_\alpha}(V_\alpha^{(2)})}| = 2$. Then, $C_S(V) = C_{Q_\beta}(V_\beta^{(3)}) \trianglelefteq X$. By Lemma 2.3.14 (iii), there are four non-equal subgroups of $L_\alpha/C_{L_\alpha}(V_\alpha^{(2)})Q_\alpha \cong (3 \times 3) : 2$ isomorphic to $\mathrm{Sym}(3)$, and so there is $H_\alpha^* \neq H_\alpha$ such that $S \in H_\alpha^*$, $O^2(H_\alpha^*)$ acts non-trivially on $V_\alpha^{(2)}/Z_\alpha$ and Z_α and $G_\alpha = \langle H_\alpha, H_\alpha^* \rangle$. If H_α^* does not normalize $Q_\alpha \cap Q_\beta$, then setting X^* for the subgroup of G obtained from employing the method in Lemma 5.2.29 with H_α^* instead of H_α, it follows from the work above that X^* also satisfies outcome (d) and for $V^* := \langle Z_\beta^{X^*} \rangle$, $C_S(V) = C_S(V^*) = C_{Q_\beta}(V_\beta^{(3)}) \trianglelefteq \langle H_\alpha, H_\alpha^* \rangle = G_\alpha$, a contradiction. Hence, H_α^* normalizes $Q_\alpha \cap Q_\beta$. Choose τ in

$C_{L_\alpha}(V_\alpha^{(2)}/Z_\alpha) \setminus C_{L_\alpha}(V_\alpha^{(2)})$. Then τ normalizes V_β so normalizes $C_\beta = C_{Q_\alpha}(V_\beta)$, and $G_\alpha = \langle \tau, H_\alpha^* \rangle$. If τ centralizes Q_α/C_β, then τ normalizes $Q_\alpha \cap Q_\beta$ so that G_α normalizes $Q_\alpha \cap Q_\beta$, a contradiction by Proposition 5.2.25. Thus, τ acts non-trivially on Q_α/C_β. Now, $[O^2(H_\alpha^*), \tau] \leq C_{G_\alpha}(V_\alpha^{(2)})$ and as $O^2(H_\alpha^*)$ normalizes $Q_\alpha \cap Q_\beta$, $O^2(H_\alpha^*)$ normalizes $(Q_\alpha \cap Q_\beta)^\tau$. But then H_α^* normalizes $C_\beta = Q_\alpha \cap Q_\beta \cap Q_\beta^\tau$ and so $G_\alpha = \langle \tau, H_\alpha^* \rangle$ normalizes C_β, another contradiction.

Thus, we may assume that outcome (b) of Lemma 5.2.29 holds so that $p = 3$ and neither R_α nor $C_{L_\alpha}(V_\alpha^{(2)}/Z_\alpha)$ normalizes $Q_\alpha \cap Q_\beta$. Indeed, for the subgroup H_α as constructed in Lemma 5.2.29, we have that $Q_\alpha \cap Q_\beta \trianglelefteq H_\alpha$. Now, $C_{L_\alpha}(V_\alpha^{(2)}/Z_\alpha)$ normalizes C_β and we may assume that it acts non-trivially on Q_α/C_β for otherwise $Q_\alpha \cap Q_\beta \trianglelefteq G_\alpha = \langle H_\alpha, C_{L_\alpha}(V_\alpha^{(2)}/Z_\alpha) \rangle$, a contradiction by Proposition 5.2.25. Furthermore, $[O^3(O^{3'}(H_\alpha)), C_{L_\alpha}(V_\alpha^{(2)}/Z_\alpha)] \leq C_{L_\alpha}(V_\alpha^{(2)})G_\alpha^{(1)}$ and as H_α normalizes $Q_\alpha \cap Q_\beta$ and $O^3(C_{L_\alpha}(V_\alpha^{(2)}))$ centralizes Q_α/C_β, it follows that for any $r \in C_{L_\alpha}(V_\alpha^{(2)}/Z_\alpha)$ of order coprime to p which does not normalize $Q_\alpha \cap Q_\beta$, $O^3(O^{3'}(H_\alpha))$ normalizes $(Q_\alpha \cap Q_\beta)^r$ and H_α normalizes $C_\beta = Q_\alpha \cap Q_\beta \cap Q_\beta^r$, a final contradiction for then $C_\beta \trianglelefteq G_\alpha = \langle H_\alpha, C_{L_\alpha}(V_\alpha^{(2)}/Z_\alpha) \rangle$. □

Lemma 5.2.33. *Suppose that $b > 5$ and Hypothesis 5.2.30 is satisfied. If $O^p(R_\alpha)$ centralizes $V_\alpha^{(2)}$ and $V_\alpha^{(4)}/V_\alpha^{(2)}$ contains a unique non-central chief factor which, as a $\mathrm{GF}(p)\overline{L_\alpha}$-module, is an FF-module then $O^p(R_\alpha)$ centralizes $V_\alpha^{(4)}$.*

Proof. Since none of the configurations described in Theorem 5.2.2 have $b > 5$, we may assume that G is a minimal counterexample such that $O^p(R_\alpha)$ does not centralize $V_\alpha^{(4)}/V_\alpha^{(2)}$, $V_\alpha^{(4)}/V_\alpha^{(2)}$ contains a unique non-central chief factor and $O^p(R_\alpha)$ centralizes $V_\alpha^{(2)}$. Since $O^p(R_\alpha)$ centralizes $V_\alpha^{(2)}$, an application of the three subgroup lemma implies that $O^p(R_\alpha)$ centralizes $Q_\alpha/C_{Q_\alpha}(V_\alpha^{(2)})$ and $C_{Q_\alpha}(V_\alpha^{(2)}) \leq Q_\alpha \cap Q_\beta$, $Q_\alpha \cap Q_\beta \trianglelefteq R_\alpha$.

We may apply Lemma 5.2.29 with $\lambda = \alpha$. Since $b > 5$, (a) is not satisfied. Indeed, as $\lambda = \alpha$ and R_α normalizes $Q_\alpha \cap Q_\beta$, we suppose that conclusion (d) is satisfied. For X as constructed in Lemma 5.2.29, we have that $V_\beta^{(5)} \leq V := \langle Z_\beta^X \rangle$ and the images in $\widetilde{L}_\beta$ of Q_β/C_β, $C_\beta/C_{Q_\beta}(V_\beta^{(3)})$ and $C_{Q_\beta}(V_\beta^{(3)})/C_{Q_\beta}(V_\beta^{(5)})$ all contain a non-central chief factor for $\widetilde{L}_\beta$, a contradiction by Lemma 5.2.29. □

Lemma 5.2.34. *Suppose that $b > 3$ and Hypothesis 5.2.30 is satisfied. If $V_\beta^{(3)}/V_\beta$ contains a unique non-central chief factor which, as a* $\mathrm{GF}(p)\overline{L_\beta}$*-module, is an FF-module, then $O^p(R_\beta)$ centralizes $V_\beta^{(3)}$.*

Proof. Since the only configuration in Theorem 5.2.2 which satisfies $b > 3$ (where G is parabolic isomorphic to F_3) satisfies $[O^p(R_\beta), V_\beta^{(3)}] = \{1\}$, we may assume that G is a minimal counterexample such that $O^p(R_\beta)$ does not centralize $V_\beta^{(3)}$. Since $O^p(R_\beta)$ centralizes V_β, the three subgroup lemma implies that $O^p(R_\beta)$ centralizes Q_β/C_β so that R_β normalizes $Q_\alpha \cap Q_\beta$. Thus, the hypotheses of Lemma 5.2.29 are satisfied with $\lambda = \beta$. Since $C_{L_\beta}(V_\beta^{(3)}/V_\beta)$ normalizes $V_\alpha^{(2)}$ and $\lambda = \beta$, conclusions (b) and (c) are not satisfied. As $b > 3$, if outcome (a) is satisfied then $b = 5$ and $(G_\alpha, H_\beta, G_{\alpha,\beta})$ is parabolic isomorphic to F_3 and $H_\beta/Q_\beta \cong \mathrm{GL}_2(3)$. Then S is determined up to isomorphism. Indeed, as $V_\beta = \langle Z_\alpha^{G_\beta} \rangle = \langle Z_\alpha^{H_\beta} \rangle = Z_3(S)$, $Q_\beta = C_S(Z_3(S)/Z(S))$ is uniquely determined in S, and so is uniquely determined up to isomorphism. But then one can check (e.g. employing MAGMA) that $\Phi(Q_\beta) = C_\beta$ has index 9 in Q_β, and as $\overline{G_\beta}$ acts faithfully on $Q_\beta/\Phi(Q_\beta)$, $\overline{G_\beta} = \overline{H_\beta} \cong \mathrm{GL}_2(3)$ and $G_\beta = H_\beta$, a contradiction.

Hence, we are left with conclusion (d). But then $V_\alpha^{(4)} \leq V := \langle Z_\beta^X \rangle$ and the images of $Q_\alpha/C_{Q_\alpha}(V_\alpha^{(2)})$ and $C_{Q_\alpha}(V_\alpha^{(2)})/C_{Q_\alpha}(V_\alpha^{(4)})$ in $\widetilde{L}_\alpha$ both contain a non-central chief factor for $\widetilde{L}_\alpha$. Moreover, the images of Q_β/C_β and $C_\beta/C_{Q_\beta}(V_\beta^{(3)})$ also a contain non-central chief factor for $\widetilde{L}_\beta$, and we have a contradiction. □

Lemma 5.2.35. *Suppose that $b > 5$ and Hypothesis 5.2.30 is satisfied. If $V_\beta^{(5)}/V_\beta^{(3)}$ contains a unique non-central chief factor which, as a $\overline{L_\beta}$-module, is an FF-module and $O^p(R_\beta)$ centralizes $V_\beta^{(3)}$, then $[O^p(R_\beta), V_\beta^{(5)}] = \{1\}$.*

Proof. Since none of the configurations in Theorem 5.2.2 satisfy $b > 5$, we may assume the G is a minimal counterexample to Theorem 5.2.2 with $[O^p(R_\beta), V_\beta^{(3)}] = \{1\}$ and $[O^p(R_\beta), V_\beta^{(5)}] \neq \{1\}$. Since $O^p(R_\beta)$ centralizes V_β, $O^p(R_\beta)$ centralizes Q_β/C_β so that R_β normalizes $Q_\alpha \cap Q_\beta$ we may apply Lemma 5.2.29 with $\lambda = \beta$. Since $O^p(R_\beta)$ normalizes $V_\alpha^{(2)}$ and $b > 5$, we are in case (d) of Lemma 5.2.29. Then, $V_\alpha^{(6)} \leq V := \langle Z_\beta^X \rangle$ and the image of $Q_\alpha/C_{Q_\alpha}(V_\alpha^{(6)})$ in $\widetilde{L}_\alpha$ contains at least three non-central chief factors for $\widetilde{L}_\alpha$, a contradiction. □

5.3 $Z_{\alpha'} \not\leq Q_\alpha$

Throughout this section, we assume Hypothesis 5.2.1. In addition, within this section we suppose that $Z_{\alpha'} \not\leq Q_\alpha$ for a chosen critical pair (α, α'). By Lemma 5.2.10 (iv), this condition is equivalent to $[Z_\alpha, Z_{\alpha'}] \neq \{1\}$. We set $S \in \mathrm{Syl}_p(G_{\alpha,\beta})$ throughout.

Lemma 5.3.1. *(α', α) is also a critical pair, $C_{Z_\alpha}(Z_{\alpha'}) = Z_\alpha \cap Q_{\alpha'}$ and $C_{Z_{\alpha'}}(Z_\alpha) = Z_{\alpha'} \cap Q_\alpha$.*

Proof. Since $Z_{\alpha'} \not\leq Q_\alpha$ we have that both (α, α') and (α', α) are critical pairs. In particular, all the results we prove in this section hold upon interchanging α and α'. By Lemma 5.2.11, $C_{Z_\alpha}(Z_{\alpha'}) = Z_\alpha \cap Q_{\alpha'}$. □

Lemma 5.3.2. *For $\lambda \in \{\alpha, \alpha'\}$, $Z_\lambda/\Omega(Z(L_\lambda))$ is natural $\mathrm{SL}_2(q)$-module for $L_\lambda/R_\lambda \cong \mathrm{SL}_2(q)$. Moreover, $S = Z_{\alpha'}Q_\alpha \in \mathrm{Syl}_p(G_{\alpha,\beta})$, $Z_\alpha Q_{\alpha'} \in \mathrm{Syl}_p(G_{\alpha',\alpha'-1})$*

and if $q > p$, then $R_\lambda = Q_\lambda$.

Proof. Without loss of generality, assume that $|Z_\alpha Q_{\alpha'}/Q_{\alpha'}| \leqslant |Z_{\alpha'}Q_\alpha/Q_\alpha|$. By Lemma 5.3.1, we have that

$$|Z_\alpha/C_{Z_\alpha}(Z_{\alpha'})| = |Z_\alpha/Z_\alpha \cap Q_{\alpha'}| = |Z_\alpha Q_{\alpha'}/Q_{\alpha'}|$$
$$\leqslant |Z_{\alpha'}Q_\alpha/Q_\alpha| = |Z_{\alpha'}/Z_{\alpha'} \cap Q_\alpha| = |Z_{\alpha'}/C_{Z_{\alpha'}}(Z_\alpha)|.$$

Thus, $Z_{\alpha'}$ is a non-trivial offender on Z_α, and Z_α is an FF-module for $L_\alpha/C_{L_\alpha}(Z_\alpha)$. Since $\overline{L_\alpha}$ has a strongly p-embedded subgroup, by Lemma 2.3.10, we conclude that $L_\alpha/R_\alpha \cong \mathrm{SL}_2(q)$ and $Z_\alpha/\Omega(Z(L_\alpha))$ is a natural $\mathrm{SL}_2(q)$-module.

Since $L_\alpha/R_\alpha \cong \mathrm{SL}_2(q)$ and $Z_\alpha/\Omega(Z(L_\alpha))$ is a natural $\mathrm{SL}_2(q)$-module, we infer that $q = |Z_\alpha/C_{Z_\alpha}(Z_{\alpha'})| \leqslant |Z_{\alpha'}/C_{Z_{\alpha'}}(Z_\alpha)| = |Z_{\alpha'}Q_\alpha/Q_\alpha| \leqslant q$. In particular, by a symmetric argument, $Z_{\alpha'}/\Omega(Z(L_{\alpha'}))$ is also a natural module for $L_{\alpha'}/R_{\alpha'} \cong \mathrm{SL}_2(q)$. It follows immediately that $Z_{\alpha'}Q_\alpha \in \mathrm{Syl}_p(G_{\alpha,\beta})$ and $Z_\alpha Q_{\alpha'} \in \mathrm{Syl}_p(G_{\alpha',\alpha'-1})$. By Proposition 3.2.7, whenever $q > p$ and $\lambda \in \{\alpha, \alpha'\}$, $\overline{R_\lambda} \leq Z(\overline{L_\lambda})$ and since $\mathrm{PSL}_2(q)$ is perfect and the p'-part of its Schur multiplier is order 2 whenever $q \geqslant 4$, using $L_\lambda = O^{p'}(L_\lambda)$ gives $\overline{L_\lambda} \cong \mathrm{SL}_2(q)$ and $R_\lambda = Q_\lambda$. □

In the following proposition, we divide the analysis of the case $[Z_\alpha, Z_{\alpha'}] \neq \{1\}$ into two subcases. The remainder of this section is split into two subsections dealing with each of these subcases individually.

Proposition 5.3.3. *One of the following holds:*

(i) *b is even and $Z_\beta = \Omega(Z(S)) = \Omega(Z(L_\beta))$; or*

(ii) *$Z_\beta \neq \Omega(Z(S))$ and for $\lambda \in \{\alpha, \beta\}$, $Z_\lambda/\Omega(Z(L_\lambda))$ is a natural* $\mathrm{SL}_2(q_\lambda)$*-module for L_λ/R_λ.*

Proof. Notice that if $Z_\beta = \Omega(Z(S))$ then $\{1\} = [Z_\beta, S]^{G_\beta} = [Z_\beta, \langle S^{G_\beta}\rangle] = [Z_\beta, L_\beta]$ so that $Z_\beta = \Omega(Z(L_\beta))$. Since $Z_{\alpha'}$ is not centralized by $Z_\alpha \leq L_{\alpha'}$, it follows immediately in this case that b is even.

Suppose that $Z_\beta \neq \Omega(Z(S))$. If $b = 1$, the result follows immediately from Lemma 5.3.2 replacing α' by β and so we may assume that $b > 1$. Assume that $V_\alpha \leq Q_{\alpha'-1}$. In particular, $V_\alpha \leq Z_\alpha Q_{\alpha'} \in \mathrm{Syl}_p(L_{\alpha'})$ by Lemma 5.3.2. Thus, $[V_\alpha, Z_{\alpha'}] \leq [Z_\alpha, Z_{\alpha'}] \leq Z_\alpha$ so that $[V_\alpha, O^p(L_\alpha)] \leq Z_\alpha$ and $Z_\alpha Z_\beta \trianglelefteq L_\alpha$, a contradiction by Proposition 5.2.25. Thus, there is $\alpha - 1 \in \Delta(\alpha)$ with $Z_{\alpha-1} \not\leq Q_{\alpha'-1}$. Then $(\alpha-1, \alpha'-1)$ is a critical pair and since $Z_\alpha \neq \Omega(Z(S)) \neq Z_\beta$, by Lemma 5.2.10 (ii), we conclude that $[Z_{\alpha-1}, Z_{\alpha'-1}] \neq \{1\}$ and Lemma 5.3.2 gives the result. □

5.3.1 $Z_\beta \neq \Omega(Z(S))$

We first consider the case where $[Z_\alpha, Z_{\alpha'}] \neq \{1\}$ and $Z_\beta \neq \Omega(Z(S))$. Under these hypotheses, and using the symmetry in α and α', it is not hard to show that every $\gamma \in \Gamma$ belongs to some critical pair . The main work in this subsection is then to show that $R_\gamma = Q_\gamma$ and $\overline{L_\gamma} \cong \mathrm{SL}_2(q)$, for then, all examples we obtain arise from weak BN-pairs of rank 2 and G is determined by [DS85].

As hinted at in Lemma 5.3.2, there is a clear distinction between the cases where $p \in \{2, 3\}$ and $p \geqslant 5$ due to solvability of $\mathrm{SL}_2(p)$ when $p \in \{2, 3\}$. Throughout this subsection, and the subsections to come, this dichotomy will become a prominent theme.

Lemma 5.3.4. *Suppose that $Z_\beta \neq \Omega(Z(S))$, $b > 1$ and for $\lambda \in \{\alpha, \beta\}$, $Z_\lambda/\Omega(Z(L_\lambda))$ is a natural $\mathrm{SL}_2(q_\lambda)$-module for L_λ/R_λ. Then the following hold:*

(i) *$V_\alpha \not\leq Q_{\alpha'-1}$ and there is a critical pair $(\alpha-1, \alpha'-1)$ with $[Z_{\alpha-1}, Z_{\alpha'-1}] \neq \{1\}$ and $V_{\alpha-1} \not\leq Q_{\alpha'-2}$;*

(ii) *V_λ/Z_λ and Z_λ are FF-modules for $\overline{L_\lambda}$;*

(iii) *$q_\alpha = q_\beta$; and*

(iv) *unless $q_\lambda \in \{2, 3\}$, $R_\lambda = C_{L_\lambda}(V_\lambda/Z_\lambda)$ and $L_\lambda/C_{L_\lambda}(V_\lambda)Q_\lambda \cong \mathrm{SL}_2(q_\lambda)$.*

Proof. By the minimality of b, $V_\alpha \leq Q_{\alpha'-2}$. Suppose that $V_\alpha \leq Q_{\alpha'-1} \leq Z_\alpha Q_{\alpha'}$. Then $[V_\alpha, Z_{\alpha'}] = [Z_\alpha, Z_{\alpha'}] \leq Z_\alpha$. In particular, since $Z_{\alpha'} \not\leq Q_\alpha$, $[V_\alpha, O^p(L_\alpha)] \leq Z_\alpha$. Hence, $Z_\beta Z_\alpha \trianglelefteq L_\alpha$, a contradiction to Proposition 5.2.25. Thus, we assume that $V_\alpha \not\leq Q_{\alpha'-1}$. In particular, there is some $\alpha - 1 \in \Delta(\alpha)$ such that $(\alpha - 1, \alpha' - 1)$ is a critical pair with $[Z_{\alpha-1}, Z_{\alpha'-1}] \neq \{1\}$. We may assume that $V_{\alpha-1} \not\leq Q_{\alpha'-2}$ else we arrive at a similar contradiction as the above. Hence (i) holds.

Suppose first that b was odd. Then, by Lemma 5.3.2, Proposition 5.3.3 and as α' is conjugate to β, $L_\beta/R_\beta \cong \mathrm{SL}_2(q_\beta)$ and $q_\beta = q_{\alpha'} = q_\alpha$ and (iii) holds in this case. Now suppose that b is even so $\alpha' - 1$ is conjugate to β. In either case, we observe that $V_\alpha \cap Q_{\alpha'-1} = Z_\alpha(V_\alpha \cap Q_{\alpha'})$ has index at most q_β in V_α and is centralized, modulo Z_α, by $Z_{\alpha'}$. Furthermore, since $Z_\alpha Z_\beta \ntrianglelefteq L_\alpha$, it follows from Lemma 5.2.8 (iii) that $Q_\alpha \in \mathrm{Syl}_p(C_{L_\alpha}(V_\alpha/Z_\alpha))$ and by Lemma 2.3.10, we have that $q_\alpha \leqslant q_\beta$. But then $(\alpha - 1, \alpha' - 1)$ is also a critical pair with $V_{\alpha-1} \cap Q_{\alpha'-2} = Z_{\alpha-1}(V_{\alpha-1} \cap Z_{\alpha'-1})$ a subgroup of $V_{\alpha-1}$ of index at most q_α and applying the same reasoning as before alongside Lemma 5.2.8 (iii), we deduce that $Q_\beta \in \mathrm{Syl}_p(C_{L_\beta}(V_\beta/Z_\beta))$ and using

Lemma 2.3.10 we see that $q_{\alpha-1} = q_\beta \leqslant q_\alpha$. Thus, $q_\alpha = q_\beta$ and V_λ/Z_λ is an FF-module for $\overline{L_\lambda}$ for all $\lambda \in \Gamma$, and (ii) and (iii) hold.

It remains to prove (iv). By Lemma 2.3.10, for all $\lambda \in \Gamma$, $L_\lambda/C_{L_\lambda}(V_\lambda/Z_\lambda) \cong L_\lambda/R_\lambda \cong \mathrm{SL}_2(q_\lambda)$. Suppose that $q_\lambda \not\in \{2,3\}$ and assume that $C_{L_\lambda}(V_\lambda/Z_\lambda) \neq R_\lambda$. Since $\{Q_\lambda\} = \mathrm{Syl}_p(C_{L_\lambda}(V_\lambda/Z_\lambda)) = \mathrm{Syl}_p(R_\lambda)$, we infer that $\overline{R_\lambda C_{L_\lambda}(V_\lambda/Z_\lambda)}$ is a group of order coprime to p and we see immediately that p is odd, $C_{L_\lambda}(V_\lambda/Z_\lambda)R_\lambda/R_\lambda = Z(L_\lambda/R_\lambda)$ and $C_{L_\lambda}(V_\lambda/Z_\lambda)R_\lambda/C_{L_\lambda}(V_\lambda/Z_\lambda) = Z(L_\lambda/C_{L_\lambda}(V_\lambda/Z_\lambda))$. Thus, $L_\lambda/(C_{L_\lambda}(V_\lambda/Z_\lambda) \cap R_\lambda)$ is isomorphic to a central extension of $\mathrm{PSL}_2(q_\lambda)$ by an elementary abelian group of order 4. Since $L_\lambda = O^{p'}(L_\lambda)$ and the 2-part of the Schur multiplier of $\mathrm{PSL}_2(q)$ is of order 2 by Lemma 2.2.1 (vii) when p is odd, we have a contradiction. Thus, we shown that, unless $q_\lambda \in \{2,3\}$, $C_{L_\lambda}(V_\lambda/Z_\lambda) = R_\lambda$ and (iv) is proved. $\square$

Lemma 5.3.5. *Suppose that for $Z_\beta \neq \Omega(Z(S))$ and for $\lambda \in \{\alpha,\beta\}$, $Z_\lambda/\Omega(Z(L_\lambda))$ is a natural $\mathrm{SL}_2(q_\lambda)$-module for L_λ/R_λ. Then $b \leqslant 2$.*

Proof. Assume throughout that $b > 2$ so that V_λ is abelian for all $\lambda \in \Gamma$. For $\delta \in \Gamma$ and $\nu \in \Gamma$, set $S_{\delta,\nu} \in \mathrm{Syl}_p(G_{\delta,\nu})$ and $Z_{\delta,\nu} := \Omega(Z(S_{\delta,\nu}))$. Choose $\mu \in \Delta(\alpha'-1)$ such that $Z_{\mu,\alpha'-1} \neq Z_{\alpha'-1,\alpha'-2}$. Thus we know, $Z_{\alpha'-1} = Z_{\mu,\alpha'-1}Z_{\alpha'-1,\alpha'-2}$. Then, using Lemma 5.3.4 (i), as $V_\alpha \not\leq Q_{\alpha'-1}$ and V_α centralizes $Z_{\alpha'-1,\alpha'-2}$, we have that $L_{\alpha'-1} = \langle Q_\mu, R_{\alpha'-1}, V_\alpha\rangle$.

Set $U_{\alpha'-1,\mu} := \langle Z_\delta \mid Z_{\mu,\alpha'-1} = Z_{\delta,\alpha'-1}, \delta \in \Delta(\alpha'-1)\rangle$. Let $r \in R_{\alpha'-1}Q_\mu$. Since r is an automorphism of the graph, it follows that for Z_δ with $Z_{\mu,\alpha'-1} = Z_{\delta,\alpha'-1}$ and $\delta \in \Delta(\alpha'-1)$, we have that $Z_\delta^r = Z_{\delta\cdot r}$ and $\{\delta, \alpha'-1\}\cdot r = \{\delta\cdot r, \alpha'-1\}$. Since $S_{\delta,\alpha'-1}$ is the unique Sylow p-subgroup of $G_{\delta,\alpha'-1}$, it follows that $Z_{\delta,\alpha'-1}^r = Z_{\delta\cdot r,\alpha'-1}$. Since $R_{\alpha'-1}Q_\mu$ normalizes $Z_{\delta,\alpha'-1}$, we have that $Z_{\delta\cdot r,\alpha'-1} = Z_{\mu,\alpha'-1}$ so that $Z_{\delta\cdot r} \leq U_{\alpha'-1,\mu}$.

Thus, $U_{\alpha'-1,\mu} \trianglelefteq R_{\alpha'-1}Q_\mu$.

Suppose that $U_{\alpha'-1,\mu} \le Q_\alpha$. By Lemma 5.3.4 (i), there is $\alpha-1 \in \Delta(\alpha)$ such that $Z_{\alpha-1} \not\le Q_{\alpha'-1}$ and $Z_{\alpha'-1} \not\le Q_{\alpha-1}$. Moreover, we have that $L_{\alpha'-1} = \langle Q_\mu, R_{\alpha'-1}, Z_{\alpha-1}\rangle$. Then, $U_{\alpha'-1,\mu} = Z_{\alpha'-1}(U_{\alpha'-1,\mu} \cap Q_{\alpha-1})$ is centralized, modulo $Z_{\alpha'-1}$, by $Z_{\alpha-1}$ so that $U_{\alpha'-1,\mu} \trianglelefteq L_{\alpha'-1} = \langle Q_\mu, R_{\alpha'-1}, Z_{\alpha-1}\rangle$. Since $Z_{\alpha-1}$ centralizes $U_{\alpha'-1,\mu}/Z_{\alpha'-1}$, $O^p(L_{\alpha'-1})$ centralizes $U_{\alpha'-1,\mu}/Z_{\alpha'-1}$ and $Z_\mu Z_{\alpha'-1} \trianglelefteq L_{\alpha'-1}$, a contradiction by Proposition 5.2.25. Thus, $U_{\alpha'-1,\mu} \not\le Q_\alpha$.

Hence, there is $\delta \in \Delta(\alpha'-1)$ with $Z_{\delta,\alpha'-1} = Z_{\mu,\alpha'-1} \ne Z_{\alpha'-1,\alpha'-2}$, $L_{\alpha'-1} = \langle Q_\delta, R_{\alpha'-1}, V_\alpha\rangle$ and (α,δ) a critical pair. We may as well assume that $\delta = \alpha'$ and $Z_{\alpha',\alpha'-1} \ne Z_{\alpha'-1,\alpha'-2}$. By Lemma 5.3.1, Lemma 5.3.4 applies to α' in place of α. Then $V_{\alpha'} \not\le Q_\beta$ and there is $\alpha'+1 \in \Delta(\alpha')$ with $(\alpha'+1,\beta)$ a critical pair satisfying $Z_{\alpha'+1} \not\le Q_\beta$ and $Z_\beta \not\le Q_{\alpha'+1}$. Choose $\mu^* \in \Delta(\alpha')$ such that $Z_{\mu^*,\alpha'} \ne Z_{\alpha',\alpha'-1}$ so that $Z_{\alpha'} = Z_{\mu^*,\alpha'}Z_{\alpha',\alpha'-1}$. Then, as $Z_\alpha \not\le Q_{\alpha'}$ and Z_α centralizes $Z_{\alpha',\alpha'-1}$, we have that $L_{\alpha'} = \langle Z_\alpha, Q_{\mu^*}, R_{\alpha'}\rangle$. Forming U_{α',μ^*} in an analogous way to $U_{\alpha'-1,\mu}$, we see that $U_{\alpha',\mu^*} \trianglelefteq R_{\alpha'}Q_{\mu^*}$ and $U_{\alpha',\mu^*} \not\le Q_\beta$. Thus, there is some δ^* with $Z_{\delta^*,\alpha'} \ne Z_{\alpha',\alpha'-1}$, $L_{\alpha'} = \langle Q_{\delta^*}, R_{\alpha'}, Z_\alpha\rangle$ and (β,δ^*) a critical pair. We may as well take $\mu^* = \alpha'+1$ so that $L_{\alpha'} = \langle Z_\alpha, Q_{\alpha'+1}, R_{\alpha'}\rangle$ and $Z_{\alpha'+1,\alpha'} \ne Z_{\alpha',\alpha'-1}$

Now, let $R := [Z_\beta, Z_{\alpha'+1}] \le Z_\beta \cap Z_{\alpha'+1}$. Then R is centralized by $Z_\beta Q_{\alpha'+1} \in \mathrm{Syl}_p(G_{\alpha'+1,\alpha'})$ so that $R \le Z_{\alpha'+1,\alpha'}$. Since $b > 1$, Z_α centralizes $R \le Z_\beta$ and so R is centralized by $L_{\alpha'} = \langle Q_{\alpha'+1}, R_{\alpha'}, Z_\alpha\rangle$ and $R \le Z(L_{\alpha'}) \le Z_{\alpha',\alpha'-1}$. But $R \le Z_\beta \le V_\alpha$ and since $b > 2$, V_α is abelian so centralizes R. In particular, R is centralized by $L_{\alpha'-1} = \langle V_\alpha, R_{\alpha'-1}, Q_{\alpha'}\rangle$. But then $R \trianglelefteq \langle L_{\alpha'}, L_{\alpha'-1}\rangle$, a final contradiction. □

Proposition 5.3.6. *Suppose that $Z_\beta \ne \Omega(Z(S))$, $b = 2$ and for $\lambda \in \{\alpha,\beta\}$, $Z_\lambda/\Omega(Z(L_\lambda))$ is a natural $\mathrm{SL}_2(q_\lambda)$-module for $L_\lambda/R_\lambda \cong \mathrm{SL}_2(q_\lambda)$. Then $p = 3$ and*

G is locally isomorphic to H where $F^*(H) \cong \mathrm{G}_2(3^n)$.

Proof. Since $b > 1$, by Lemma 5.3.4 (iii), we have that $q_\alpha = q_\beta$ and $V_\alpha \not\leq Q_\beta$. But then $Q_\alpha = V_\alpha(Q_\alpha \cap Q_{\alpha'})$ and it follows that $O^p(L_\alpha)$ centralizes Q_α/V_α. In particular, V_α contains all non-central chief factors for L_α within Q_α, and consequently $C_{L_\alpha}(V_\alpha)$ is a p-group. By Lemma 5.3.4 (i), there is $\alpha-1 \in \Delta(\alpha)$ such that $(\alpha-1, \beta)$ is a critical pair with $[Z_{\alpha-1}, Z_\beta] \neq \{1\}$ and applying Lemma 5.3.4 (ii) again, $C_{L_{\alpha-1}}(V_{\alpha-1})$ is a p-group. By Lemma 5.3.4 (iv), unless $q_\alpha \in \{2,3\}$, we conclude that $\overline{L_\alpha} \cong \overline{L_\beta} \cong \mathrm{SL}_2(q_\alpha)$ and G has a weak BN-pair of rank 2. Comparing with [DS85], the result holds.

Hence, we assume that $q_\alpha = q_\beta \in \{2,3\}$ and for $\lambda \in \{\alpha,\beta\}$, V_λ/Z_λ and Z_λ are FF-modules for $\overline{L_\lambda}$. Moreover, for some $\delta \in \{\alpha,\beta\}$, we assume that $C_{L_\delta}(V_\delta/Z_\delta) \neq R_\delta$ and $\overline{L_\delta} \not\cong \mathrm{SL}_2(p)$. By Lemma 2.3.14 (ii) and Lemma 2.3.15 (ii), $\overline{L_\delta} \cong (3 \times 3) : 2$ or $(Q_8 \times Q_8) : 3$ for $p = 2$ or 3 respectively. Since $O^p(L_\delta)$ centralizes Q_δ/V_δ we have that $C_{L_\delta}(V_\delta/Z_\delta)$ normalizes $Q_\alpha \cap Q_\beta$.

If $p = 2$, by Lemma 2.3.14 (iii), we may choose $P_\alpha \leq L_\alpha$ such that $\overline{P_\alpha} \cong \mathrm{Sym}(3)$, $\Omega(Z(S)) \ntrianglelefteq P_\alpha$ and $Q_\alpha \cap Q_\beta \ntrianglelefteq P_\alpha$. If $\overline{L_\alpha} \cong \mathrm{Sym}(3)$ then $L_\alpha = P_\alpha$, and if $\overline{L_\alpha} \cong (3 \times 3) : 2$, then as there are two choices for P_α, both are $G_{\alpha,\beta}$-invariant and neither normalizes $Q_\alpha \cap Q_\beta$. For such a P_α, set $H_\alpha = P_\alpha G_{\alpha,\beta}$. We make an analogous choice for $H_\beta \leq G_\beta$ and observe that $P_\lambda = O^{2'}(H_\lambda)$ for $\lambda \in \{\alpha,\beta\}$.

If $p = 3$, by Lemma 2.3.15 (iii), we may choose $P_\alpha \leq L_\alpha$ such that $\overline{P_\alpha} \cong \mathrm{SL}_2(3)$, $\Omega(Z(S)) \ntrianglelefteq P_\alpha$ and $Q_\alpha \cap Q_\beta \ntrianglelefteq P_\alpha$. If $\overline{L_\alpha} \cong \mathrm{SL}_2(3)$ then $L_\alpha = P_\alpha$, and if $\overline{L_\alpha} \cong (Q_8 \times Q_8) : 3$, then there are three choices for P_α. Since all contain S, there is at least one choice such that P_α is $G_{\alpha,\beta}$-invariant and does not normalize $Q_\alpha \cap Q_\beta$. For this P_α, set $H_\alpha = P_\alpha G_{\alpha,\beta}$ and choose H_β in a similar fashion. Again, observe that $P_\lambda = O^{2'}(H_\lambda)$ for $\lambda \in \{\alpha, \beta\}$.

Set $X := \langle H_\alpha, H_\beta \rangle$ and suppose that there is $\{1\} \neq Q \leq S$ with $Q \trianglelefteq X$. Then $Q \leq O_p(H_\alpha) \cap O_p(H_\beta) = Q_\alpha \cap Q_\beta$. Suppose $\Omega(Z(S)) \not\leq Q$. Then $V_\beta = \langle \langle \Omega(Z(S))^{H_\alpha} \rangle^{H_\beta} \rangle$ centralizes Q and since Q is normal in H_α, $[O^p(P_\alpha), Q] \leq [V_\beta, Q]^{H_\alpha} = \{1\}$. Considering the action of $V_\alpha = \langle \langle \Omega(Z(S))^{H_\beta} \rangle^{H_\alpha} \rangle$ on Q yields $[O^p(P_\beta), Q] = \{1\}$. But $Q \trianglelefteq S$ and so $Q \cap \Omega(Z(S))$ is non-trivial and centralized by $G = \langle H_\alpha, R_\alpha, H_\beta, R_\beta \rangle$, a contradiction. Hence, $\Omega(Z(S)) \leq Q$. But then $Q \geq V_\beta = \langle \langle \Omega(Z(S))^{H_\alpha} \rangle^{H_\beta} \rangle \not\leq Q_\alpha$, a contradiction.

Thus, any subgroup of $G_{\alpha,\beta}$ which is normal in X is a p'-group. Such a subgroup would be contained in H_λ and so would centralize Q_λ for $\lambda \in \{\alpha, \beta\}$. Since $S \leq H_\lambda \leq G_\lambda$, we have that H_λ is of characteristic p, $C_{H_\lambda}(Q_\lambda) \leq Q_\lambda$ and no non-trivial subgroup of $G_{\alpha,\beta}$ is normal in X. Moreover, $\overline{P_\alpha} \cong \overline{P_\alpha} \cong \mathrm{SL}_2(p)$ and X has a weak BN-pair of rank 2. For $\lambda \in \{\alpha, \beta\}$, since Q_λ contains precisely two non-central chief factors for P_λ, and neither P_α nor P_β normalizes $\Omega(Z(S))$, by [DS85], X is locally isomorphic to $\mathrm{G}_2(3)$ and S is isomorphic to a Sylow 3-subgroup of $\mathrm{G}_2(3)$. Then Q_α and Q_β are distinguished up to isomorphism. Noticing that [PS18, Lemma 7.8] applies in this situation independent of any fusion system hypothesis, it follows that for $\lambda \in \{\alpha, \beta\}$, $\overline{G_\lambda}$ is isomorphic to a subgroup of $\mathrm{GL}_2(3)$, a contradiction to the assumption that $\overline{L_\delta} \not\cong \mathrm{SL}_2(p)$. Thus, we conclude that G has a weak BN-pair of rank 2 and the result follows upon comparison with [DS85]. □

Remark. The graph automorphism of $\mathrm{G}_2(3)$ normalizes $S \in \mathrm{Syl}_3(\mathrm{G}_2(3))$ and fuses Q_α and Q_β, and so Hypothesis 5.2.1 only allows for groups locally isomorphic to $\mathrm{G}_2(3^n)$ decorated by field automorphisms.

Proposition 5.3.7. *Suppose that $Z_\beta \neq \Omega(Z(S))$ and for $\lambda \in \{\alpha, \beta\}$, $Z_\lambda/\Omega(Z(L_\lambda))$*

is a natural $\mathrm{SL}_2(q_\lambda)$*-module for* L_λ/R_λ*. Then* G *is locally isomorphic to* H *where* $(F^*(H), p)$ *is one of* $(\mathrm{PSL}_3(p^n), p)$*,* $(\mathrm{PSp}_4(2^n), 2)$ *or* $(\mathrm{G}_2(3^n), 3)$*.*

Proof. By Lemma 5.3.5 and Proposition 5.3.6, we may suppose that $b = 1$. Then, $Z_\alpha \not\leq Q_\beta$, $Z_\beta \not\leq Q_\alpha$, $Q_\alpha = Z_\alpha(Q_\alpha \cap Q_\beta)$ and $Q_\beta = Z_\beta(Q_\alpha \cap Q_\beta)$. In particular, $\Phi(Q_\alpha) = \Phi(Q_\alpha \cap Q_\beta) = \Phi(Q_\beta)$ is trivial and so both Q_α and Q_β are elementary abelian. For $\lambda \in \{\alpha, \beta\}$, by coprime action we have that $Q_\lambda = [Q_\lambda, R_\lambda] \times C_{Q_\lambda}(R_\lambda)$ is an S-invariant decomposition. But $\Omega(Z(S)) \leq Z_\lambda \leq C_{Q_\lambda}(R_\lambda)$ and since $[Q_\alpha, R_\lambda] \trianglelefteq S$, we must have that $[Q_\alpha, R_\lambda] = \{1\}$. It follows that R_λ centralizes Q_λ and, as G_λ is of characteristic p, $Q_\lambda = R_\lambda$. Thus, G has a weak BN-pair of rank 2 and is determined by [DS85], hence the result. □

Remark. Similarly to the $\mathrm{G}_2(3^n)$ example, the graph automorphisms for $\mathrm{PSL}_3(p^n)$ and $\mathrm{PSp}_4(2^n)$ fuse Q_α and Q_β and are not permitted by the hypothesis.

5.3.2 $Z_\beta = \Omega(Z(S))$

Given Proposition 5.3.3, we may assume in this subsection that b is even and $Z_\beta = \Omega(Z(S))$. The general aim will be to demonstrate that $b = 2$ and $\overline{L_\alpha} \cong \mathrm{SL}_2(q)$ for then, it will quickly follow that the amalgam is symplectic and we may apply the classification in [PR12]. We are able to show that, in all the cases considered, $b = 2$. However, at the end of this section we uncover a configuration where $R_\alpha \neq Q_\alpha$.

Lemma 5.3.8. *Let* $\alpha - 1 \in \Delta(\alpha) \setminus \{\beta\}$ *with* $Z_{\alpha-1} \neq Z_\beta$*. Then* $\Omega(Z(L_\alpha)) = \{1\}$*,* $Z_\alpha = Z_\beta \times Z_{\alpha-1}$ *is a natural* $\mathrm{SL}_2(q)$*-module,* $Q_\beta \in \mathrm{Syl}_p(R_\beta)$ *and* $[Z_\alpha, Z_{\alpha'}] = Z_{\alpha'-1} = Z_\alpha \cap Q_{\alpha'} = Z_\beta = [V_\beta, Q_\beta]$*.*

Proof. Since L_β is transitive on $\Delta(\beta)$ and centralizes $Z_\beta = \Omega(Z(S))$, by

Lemma 5.2.7 (iv), we have that $Z(L_\alpha) = \{1\}$. Then, by Lemma 5.3.2, Z_α is a natural $\mathrm{SL}_2(q)$-module for $L_\alpha/R_\alpha \cong \mathrm{SL}_2(q)$.

Now, $[Z_\alpha, S] = [Z_\alpha, Z_{\alpha'}Q_\alpha] = [Z_\alpha, Z_{\alpha'}] = \Omega(Z(S)) = Z_\beta$. Thus, $[V_\beta, Q_\beta] = [\langle Z_\alpha^{G_\beta}\rangle, Q_\beta] = Z_\beta \leq C_{V_\beta}(O^p(L_\beta))$ and so $Q_\beta \leq R_\beta$. By Lemma 5.2.16, we have that $Q_\beta \in \mathrm{Syl}_p(R_\beta)$.

By considering $[Z_{\alpha'}, Z_\alpha Q_{\alpha'}]$ and again employing Lemma 5.3.2, we deduce that, for $T \in \mathrm{Syl}_p(G_{\alpha',\alpha'-1})$, $[Z_{\alpha'}, Z_\alpha] = \Omega(Z(T)) = Z_{\alpha'-1}$. Then $Z_\beta = Z_{\alpha'-1} \leq Q_{\alpha'}$ and it follows immediately that $Z_\beta = Z_\alpha \cap Q_{\alpha'}$. By properties of natural $\mathrm{SL}_2(q)$-modules, $Z_\alpha = Z_\beta \times Z_\beta^x = Z_\beta \times Z_{\beta\cdot x}$ for $x \in L_\alpha \backslash G_{\alpha,\beta}R_\alpha$. In particular, we may choose $\alpha-1 \in \Delta(\alpha)$ conjugate to β by an element of $L_\alpha \setminus G_{\alpha,\beta}R_\alpha$ so that $Z_\alpha = Z_\beta \times Z_{\alpha-1}$. $\square$

Proposition 5.3.9. *Suppose that $b > 2$. Then $L_\beta/R_\beta \cong \mathrm{SL}_2(p) \cong L_\alpha/R_\alpha$ and both Z_α and $V_\beta/C_{V_\beta}(O^p(L_\beta))$ are natural modules.*

Proof. Suppose first that $m_p(S/Q_\alpha) \geqslant 2$ so that $R_\alpha = Q_\alpha$ and $\overline{L_\alpha} \cong \mathrm{SL}_2(q)$ for $q > p$. If $b = 4$ then $L_{\alpha+2} = \langle Q_\beta, Q_{\alpha'-1}\rangle$ normalizes $Z_\beta = Z_{\alpha'-1}$, a contradiction. Hence, $b > 4$ and $V_\alpha^{(2)}$ is abelian. If $V_\alpha^{(2)} \not\leq Q_{\alpha'-2}$, then there is a critical pair $(\alpha-2, \alpha'-2)$ and by Lemma 5.3.8, $Z_{\alpha'-3} = Z_{\alpha-1}$. But then $Z_\alpha = Z_{\alpha-1} \times Z_\beta = Z_{\alpha'-2}$ and since $b > 2$, we have a contradiction. If $V_\alpha^{(2)} \leq Q_{\alpha'-1}$, then since $Z_\alpha Q_{\alpha'} \in \mathrm{Syl}_p(L_{\alpha'})$, $V_\alpha^{(2)} = Z_\alpha(V_\alpha^{(2)} \cap Q_{\alpha'})$ and $Z_{\alpha'}$ centralizes $V_\alpha^{(2)}/Z_\alpha$. But then $O^p(L_\alpha)$ centralizes $V_\alpha^{(2)}/Z_\alpha$ and $V_\beta \trianglelefteq L_\alpha$, a contradiction. Hence, there is $\alpha - 1 \in \Delta(\alpha)$ such that $V_{\alpha-1}$ acts non-trivially on $V_{\alpha'-1}/Z_{\alpha'-1}$. Notice that $[V_{\alpha'-1}, V_{\alpha-1}, V_{\alpha-1}] \leq [V_\alpha^{(2)}, V_\alpha^{(2)}] = \{1\}$. Hence, $V_{\alpha'-1}/Z_{\alpha'-1}$ is a quadratic module and by Lemma 2.3.5, we may assume that $m_p(S/Q_\beta) = 1$, else applying Lemma 2.3.5, $\overline{L_\beta}$ is a rank 1 group of Lie type, G has a weak BN-pair of rank 2 and a comparison with [DS85] gives a contradiction. But since $b > 2$, we have that $V_\alpha^{(2)} \cap Q_{\alpha'-1} = Z_\alpha(V_\alpha^{(2)} \cap Q_{\alpha'})$

is an index p subgroup of $V_\alpha^{(2)}$ which is centralized by $Z_{\alpha'}$, modulo Z_α, and as $m_p(S/Q_\alpha) \geqslant 2$ and $V_\alpha^{(2)}/Z_\alpha$ is not centralized by $O^p(L_\alpha)$, we have a contradiction.

Hence, $m_p(S/Q_\alpha) = 1$, $L_\alpha/R_\alpha \cong \mathrm{SL}_2(p)$ and Z_α is a natural $\mathrm{SL}_2(p)$-module. Set $U_{\alpha,\alpha-1} := \langle V_\lambda \mid Z_\lambda = Z_{\alpha-1}, \lambda \in \Delta(\alpha)\rangle$ for a fixed subgroup $Z_{\alpha-1} \neq Z_\beta$. Then by Lemma 5.2.19, $U_{\alpha,\alpha-1} \trianglelefteq R_\alpha Q_{\alpha-1}$. If $U_{\alpha,\alpha-1} \not\leq Q_{\alpha'-2}$, there there is some $V_{\alpha-1}$ with $(\alpha-2, \alpha'-2)$ a critical pair and $Z_{\alpha-1} \neq Z_\beta$. But then $Z_\alpha = Z_{\alpha-1} \times Z_\beta \leq V_{\alpha'-1} \leq Q_{\alpha'}$, a contradiction since $b > 2$. Suppose that $U_{\alpha,\alpha-1} \leq Q_{\alpha'-1}$ so that $[Z_{\alpha'}, U_{\alpha,\alpha-1}] = [Z_{\alpha'}, Z_\alpha(U_{\alpha,\alpha-1} \cap Q_{\alpha'})] \leq Z_\alpha \leq U_{\alpha,\alpha-1}$. Then $U_{\alpha,\alpha-1} \trianglelefteq L_\alpha = \langle R_\alpha, Z_{\alpha'}, Q_{\alpha-1}\rangle$. Since $Z_{\alpha'}$ centralizes $U_{\alpha,\alpha-1}/Z_\alpha$, $[O^p(L_\alpha), V_{\alpha-1}] \leq [O^p(L_\alpha), U_{\alpha,\alpha-1}] = Z_\alpha \leq V_{\alpha-1}$. In particular, $V_{\alpha-1} \trianglelefteq \langle G_\alpha, G_{\alpha-1}\rangle$, a contradiction.

Thus, $U_{\alpha,\alpha-1} \leq Q_{\alpha'-2}$, $U_{\alpha,\alpha-1} \not\leq Q_{\alpha'-1}$ and we may choose $V_{\alpha-1} \not\leq Q_{\alpha'-1}$ with $Z_{\alpha-1} \neq Z_\beta$. Notice that $[V_{\alpha'-1}, V_{\alpha-1}, V_{\alpha-1}] \leq [V_\alpha^{(2)}, V_\alpha^{(2)}] \leq Z_\alpha$ since $b \geqslant 4$. Since $Z_\alpha \not\leq V_{\alpha'-1}$, we must have that $[V_{\alpha'-1}, V_{\alpha-1}, V_{\alpha-1}] \leq Z_\beta = Z_{\alpha'-1}$. In particular, $V_{\alpha-1}$ acts quadratically on $V_{\alpha'-1}/Z_{\alpha'-1}$. If $V_{\alpha'-1} \cap Q_\alpha \leq Q_{\alpha-1}$, then $[V_{\alpha'-1} \cap Q_\alpha, V_{\alpha-1}] \leq Z_{\alpha-1}$. But if $Z_{\alpha-1} \leq V_{\alpha'-1}$, then $Z_\alpha \leq V_{\alpha'-1} \leq Q_{\alpha'}$ and so $[V_{\alpha'-1} \cap Q_\alpha, V_{\alpha-1}] = \{1\}$. Since $m_p(S/Q_\alpha) = 1$, $V_{\alpha-1}$ centralizes an index p subgroup of $V_{\alpha'-1}$ and the result holds. So assume that $V_{\alpha'-1} \cap Q_\alpha \not\leq Q_{\alpha-1}$. Notice that $[V_{\alpha-1}, V_{\alpha'-1} \cap Q_\alpha, V_{\alpha'-1} \cap Q_\alpha] \leq [V_{\alpha'-1}, V_{\alpha'-1}] = \{1\}$, and so $V_{\alpha'-1} \cap Q_\alpha$ acts quadratically on $V_{\alpha-1}$.

Observe that $Z(Q_\alpha) \leq Q_{\alpha'-1}$ else $Z(Q_\alpha)$ centralizes $V_{\alpha'-1} \cap Q_\alpha$, $V_{\alpha'-1}/Z_{\alpha'-1}$ is an FF-module and the result holds by Lemma 2.3.10. Then $Z(Q_\alpha) = Z_\alpha(Z(Q_\alpha) \cap Q_{\alpha'})$ and $O^p(L_\alpha)$ centralizes $Z(Q_\alpha)/Z_\alpha$. Then, by coprime action and using that $Z_\beta \leq Z_\alpha = [Z(Q_\alpha), O^p(L_\alpha)]$, it follows that $Z(Q_\alpha) = Z_\alpha$. Define $\mathcal{U}_{\alpha,\alpha-1} := [U_{\alpha,\alpha-1}, Q_\alpha; i]Z_\alpha$ with i chosen minimally so that $[U_{\alpha,\alpha-1}, Q_\alpha; i+1] \leq Z_\alpha$. Then $[V_{\alpha'-1} \cap Q_\alpha, \mathcal{U}_{\alpha,\alpha-1}] \leq Z_\alpha \cap Q_{\alpha'} = Z_\beta = Z_{\alpha'-1}$ since $Z_\alpha \not\leq V_{\alpha'-1}$. If $\mathcal{U}_{\alpha,\alpha-1} \not\leq Q_{\alpha'-1}$, then $V_{\alpha'-1}/Z_{\alpha'-1}$ is an FF-module, and the result follows. Thus, $\mathcal{U}_{\alpha,\alpha-1} \leq Q_{\alpha'-1}$ so that $\mathcal{U}_{\alpha,\alpha-1} = Z_\alpha(\mathcal{U}_{\alpha,\alpha-1} \cap Q_{\alpha'})$ and, as $U_{\alpha,\alpha-1}$ is normalized by $R_\alpha Q_{\alpha-1}$, $\mathcal{U}_{\alpha,\alpha-1} \trianglelefteq L_\alpha = \langle R_\alpha, Z_{\alpha'}, Q_{\alpha-1}\rangle$ and $[\mathcal{U}_{\alpha,\alpha-1}, Q_\alpha] = Z_\alpha$. But $Z_{\alpha'}$ centralizes $\mathcal{U}_{\alpha,\alpha-1}/Z_\alpha$, so that $O^p(L_\alpha)$ centralizes $\mathcal{U}_{\alpha,\alpha-1}/Z_\alpha$ and $\mathcal{U}_{\alpha,\alpha-1} = [V_{\alpha-1}, Q_\alpha; i]Z_\alpha = [V_\lambda, Q_\alpha; i]Z_\alpha$ for $\lambda \in \Delta(\alpha)$.

Suppose first that $m_p(S/Q_\beta) \geqslant 2$, so that by Lemma 2.3.5 and Proposition 3.2.7, $\overline{L_{\alpha'-1}}$ is a central extension of a rank 1 group of Lie type. Since $V_{\alpha'-1} \cap Q_\alpha$ acts quadratically on $V_{\alpha-1}$, $V_{\alpha'-1} \cap Q_\alpha \cap Q_{\alpha-1}$ has index at most pq_β in $V_{\alpha'-1}$, where $q_\beta := |\Omega(Z(S/Q_\beta))|$ by [DS85, (5.9)]. Since $V_{\alpha'-1} \cap Q_\alpha \cap Q_{\alpha-1}$ is centralized by $V_{\alpha-1}$, we have that $|V_{\alpha'-1}/C_{V_{\alpha'-1}}(O^p(L_{\alpha'-1}))| \leqslant (pq_\beta)^d$ where d is the number of conjugates of $V_{\alpha-1}Q_{\alpha'-1}/Q_{\alpha'-1}$ required to generate $\overline{L_{\alpha'-1}}$. By Lemma 2.3.4, $\overline{L_{\alpha'-1}} \not\cong \mathrm{Ree}(3^n)$ and if p is odd, then $\overline{L_{\alpha'-1}} \not\cong \mathrm{PSL}_2(p^n)$.

If $\overline{L_{\alpha'-1}} \cong \mathrm{Sz}(2^n)$ then by Lemma 2.2.3 (iii), (vi), $d = 3$, $q_\beta = 2^n > 2$ and $|V_{\alpha'-1}/C_{V_{\alpha'-1}}(O^p(L_{\alpha'-1}))| \leqslant 2^{3+3n}$. Since the minimal degree of a non-trivial GF(2)-representation for $\mathrm{Sz}(2^n)$ is $4n$, as $n > 1$ is odd by Lemma 2.2.3 (i), we deduce that $n = 3$, $|(V_{\alpha'-1} \cap Q_\alpha)Q_{\alpha-1}/Q_{\alpha-1}| = 8$ and $V_{\alpha'-1}/C_{V_{\alpha'-1}}(O^p(L_{\alpha'-1}))$ is a natural Sz(8)-module. By conjugacy, $V_{\alpha-1}/C_{V_{\alpha-1}}(O^p(L_{\alpha-1}))$ is also a natural Sz(8)-module and as $V_{\alpha-1} \cap Q_{\alpha'-1}$ has index at most 8 and $[V_{\alpha-1} \cap Q_{\alpha'-1}, V_{\alpha'-1} \cap Q_\alpha] = Z_{\alpha'-1} = Z_\beta$ is of order 2, one can calculate (e.g. using MAGMA) that we have a contradiction.

If $\overline{L_{\alpha'-1}} \cong (\mathrm{P})\mathrm{SU}_3(p^n)$ then by Lemma 2.2.2 (i),(ii), (vi) and (vii), $d = 4$, $q_\beta = p^n > 2$ and $|V_{\alpha'-1}/C_{V_{\alpha'-1}}(O^p(L_{\alpha'-1}))| \leqslant p^{4+4n}$. Since the minimal degree of a non-trivial GF(p)-representation for $(\mathrm{P})\mathrm{SU}_3(p^n)$ is $6n$, we deduce that $n \leqslant 2$. Moreover, unless $p^n \in \{4, 9\}$ we have that $d = 3$ by Lemma 2.2.2 (vi) so that $|V_{\alpha'-1}/C_{V_{\alpha'-1}}(O^p(L_{\alpha'-1}))| \leqslant p^{3+3n}$. In this scenario, we conclude that $n = 1$ and

$V_{\alpha'-1}/C_{V_{\alpha'-1}}(O^p(L_{\alpha'-1}))$ is a natural $\mathrm{SU}_3(p)$-module for $\overline{L_{\alpha'-1}} \cong \mathrm{SU}_3(p)$. But then, $Z_{\alpha'}C_{V_{\alpha'-1}}(O^p(L_{\alpha'-1}))/C_{V_{\alpha'-1}}(O^p(L_{\alpha'-1}))$ is a $G_{\alpha',\alpha'-1}$-invariant subgroup of order p, and we have a contradiction by Lemma 2.2.13 (iii). If $p^n \in \{4,9\}$ then $V_{\alpha'-1}/C_{V_{\alpha'-1}}(O^p(L_{\alpha'-1}))$ is a natural $\mathrm{SU}_3(p^2)$-module of order p^{12}. Again, $Z_{\alpha'}C_{V_{\alpha'-1}}(O^p(L_{\alpha'-1}))/C_{V_{\alpha'-1}}(O^p(L_{\alpha'-1}))$ is a $G_{\alpha',\alpha'-1}$-invariant subgroup of order p, and we have a contradiction by Lemma 2.2.13 (iii).

If $\overline{L_{\alpha'-1}} \cong \mathrm{SL}_2(p^n)$, then $n > 1$. If, in addition, $(V_{\alpha'-1} \cap Q_\alpha)Q_{\alpha-1} \in \mathrm{Syl}_p(L_{\alpha-1})$ then, by Lemma 2.3.11, $V_{\alpha-1}/C_{V_{\alpha-1}}(O^p(L_{\alpha-1}))$ is a direct sum of natural $\mathrm{SL}_2(p^n)$-modules. Since $Z_\alpha C_{V_{\alpha-1}}(O^p(L_{\alpha-1}))/C_{V_{\alpha-1}}(O^p(L_{\alpha-1}))$ has order p and is $G_{\alpha,\alpha-1}$-invariant, comparing with Lemma 2.2.6 (vi), we have a contradiction.

Thus, we may assume that $\overline{L_{\alpha'-1}} \cong \mathrm{SL}_2(p^n)$, $n > 1$ and $(V_{\alpha'-1} \cap Q_\alpha)Q_{\alpha-1} \notin \mathrm{Syl}_p(L_{\alpha-1})$. Then $V_{\alpha'-1} \cap Q_\alpha \cap Q_{\alpha-1}$ has index at most q_β in $V_{\alpha'-1}$ and is centralized by $V_{\alpha-1}$. Unless $p^n = 9$ or $|V_{\alpha-1}Q_{\alpha'-1}/Q_{\alpha'-1}| = 2$, by Lemma 2.2.1 (iii), (iv), $\overline{L_{\alpha'-1}}$ is generated by two conjugates of $V_{\alpha-1}Q_{\alpha'-1}/Q_{\alpha'-1}$ and so $|V_{\alpha'-1}/C_{V_{\alpha'-1}}(O^p(L_{\alpha'-1}))| \leqslant q_\beta^2$. Since $V_{\alpha'-1}/C_{V_{\alpha'-1}}(O^p(L_{\alpha'-1}))$ contains a non-central chief factor, $V_{\alpha'-1}/C_{V_{\alpha'-1}}(O^p(L_{\alpha'-1}))$ is a quadratic irreducible module of order q_β^2. Since $|Z_{\alpha'}/Z_{\alpha'-1}| = p$ and $Z_{\alpha'} \not\leq C_{V_{\alpha'-1}}(O^p(L_{\alpha'-1}))$, there is a $G_{\alpha',\alpha'-1}$-invariant subgroup of $V_{\alpha'-1}/C_{V_{\alpha'-1}}(O^p(L_{\alpha'-1}))$ of order p. Then by Lemma 2.3.12 and writing $V := V_{\alpha'-1}/C_{V_{\alpha'-1}}(O^p(L_{\alpha'-1}))$, we have that $C_V(S)$ has order p and V admits quadratic action so that V is natural $\Omega_4^-(p)$-module. Moreover, applying Lemma 2.2.9 (b) and observing that $V_{\alpha-1}$ acts quadratically on $V_{\alpha'-1}/Z_{\alpha'-1}$, we infer that $p = 2$. But then, $|V_{\alpha-1}Q_{\alpha'-1}/Q_{\alpha'-1}| = 2$, a contradiction to the assumption.

We now suppose that $p^n = 9$ or $|V_{\alpha-1}Q_{\alpha'-1}/Q_{\alpha'-1}| = 2$ so that three conjugates of $V_{\alpha-1}Q_{\alpha'-1}/Q_{\alpha'-1}$ generate $\overline{L_{\alpha'-1}}$ and $|V_{\alpha'-1}/C_{V_{\alpha'-1}}(O^p(L_{\alpha'-1}))| \leqslant q_\beta^3$. Using that $Z_{\alpha'}/Z_{\alpha'-1}$ is $G_{\alpha,\beta}$-invariant and of order p and $V_{\alpha-1}$ acts quadratically, again applying Lemma 2.3.12 we deduce that $V_{\alpha'-1}/C_{V_{\alpha'-1}}(O^2(L_{\alpha'-1}))$ is a natural $\Omega_4^-(2)$-module for $\overline{L_{\alpha'-1}} \cong \mathrm{PSL}_2(4)$. Then, for $V := V_{\alpha'-1}/Z_{\alpha'-1}$, by Lemma 2.2.10 $V = [V, O^2(L_{\alpha'-1})] \times C_V(O^2(L_{\alpha'-1}))$ where $[V, O^2(L_{\alpha'-1})]$ is a natural $\Omega_4^-(2)$-module. By conjugacy and applying Lemma 2.2.9 (ii), $[V_{\alpha-1}, Q_\alpha, Q_\alpha, Q_\alpha] \leq Z_{\alpha-1}$. If $[V_{\alpha-1}, Q_\alpha, Q_\alpha, Q_\alpha] = Z_{\alpha-1}$ then $\mathcal{U}_{\alpha,\alpha-1} = [V_{\alpha-1}, Q_\alpha, Q_\alpha]Z_\alpha$ is normal in L_α. But then $Z_{\alpha-1} = [\mathcal{U}_{\alpha,\alpha-1}, Q_\alpha] \trianglelefteq L_\alpha$, a contradiction. Thus, $[V_{\alpha-1}, Q_\alpha, Q_\alpha] \leq Z(Q_\alpha) \cap V_{\alpha-1} = Z_\alpha$ and $\mathcal{U}_{\alpha,\alpha-1} = [V_{\alpha-1}, Q_\alpha]Z_\alpha \trianglelefteq L_\alpha$. Then, by conjugacy, $[V_{\alpha'-1}, Q_{\alpha'-2}]Z_{\alpha'-2} \trianglelefteq L_{\alpha'-2}$ and $[V_{\alpha'-1}, Q_{\alpha'-2}]Z_{\alpha'-2} = [V_{\alpha'-3}, Q_{\alpha'-2}]Z_{\alpha'-2} \leq Q_{\alpha-1}$. Since $Z_{\alpha-1} \not\leq [V_{\alpha'-1}, Q_{\alpha'-2}]Z_{\alpha'-2}$, we conclude that $[V_{\alpha'-1}, Q_{\alpha'-2}]Z_{\alpha'-2}$ is centralized by $V_{\alpha-1}$, a contradiction to the structure of $V_{\alpha'-1}/C_{V_{\alpha'-1}}(O^2(L_{\alpha'-1}))$ by Lemma 2.2.9 (iii), (iv).

Thus, we have shown that $m_p(S/Q_\alpha) = m_p(S/Q_\beta) = 1$. Since $V_{\alpha'-1} \cap Q_\alpha \cap Q_{\alpha-1}$ has index p^2 and is centralized by $V_{\alpha-1}$, $L_{\alpha'-1}/R_{\alpha'-1}$ and $V_{\alpha'-1}/C_{V_{\alpha'-1}}(O^p(L_{\alpha'-1}))$ are determined by Proposition 2.3.19. Since $Z_{\alpha'}C_{V_{\alpha'-1}}(O^p(L_{\alpha'-1}))/C_{V_{\alpha'-1}}(O^p(L_{\alpha'-1}))$ has order p and is $G_{\alpha',\alpha'-1}$-invariant, and $V_{\alpha'-1} = \langle Z_{\alpha'}^{L_{\alpha'-1}} \rangle$, by Lemma 2.3.22 we have that $L_{\alpha'-1}/R_{\alpha'-1} \cong \mathrm{Sz}(2), \mathrm{Dih}(10)$, $(3 \times 3) : 2$ or $(3 \times 3) : 4$. In particular, using coprime action, it follows that for $V := V_{\alpha'-1}/Z_{\alpha'-1}$, $V = [V, O^2(L_{\alpha'-1})] \times C_V(O^2(L_{\alpha'-1}))$ where $[V, O^2(L_{\alpha'-1})]$ is irreducible and $|C_V(O^2(L_{\alpha'-1}))| = 2$.

Suppose that $L_{\alpha'-1}/R_{\alpha'-1} \cong \mathrm{Sz}(2)$ or $(3 \times 3) : 4$. Then, by Lemma 2.2.14 (iii) and Lemma 2.3.21 (iii), $[V, Q_{\alpha'}; 3] \neq \{1\} = [V, Q_{\alpha'}; 4]$ and, by conjugacy, we infer that $[V_{\alpha-1}, Q_\alpha; 4] \leq Z_{\alpha-1}$. Then, as above, it quickly follows that $[V_{\alpha-1}, Q_\alpha; 4] = \{1\}$, $\mathcal{U}_{\alpha,\alpha-1} = [V_{\alpha-1}, Q_\alpha, Q_\alpha]Z_\alpha$ and $Z_\alpha = [V_{\alpha-1}, Q_\alpha; 3]$. Moreover, we deduce that $[U_{\alpha,\alpha-1}, Q_\alpha] \not\leq Q_{\alpha'-1}$, else $[U_{\alpha,\alpha-1}, Q_\alpha] = Z_\alpha([U_{\alpha,\alpha-1}, Q_\alpha] \cap Q_{\alpha'})$ is centralized, modulo Z_α, by $Z_{\alpha'}$ from which we have that $[V_{\alpha-1}, Q_\alpha]Z_\alpha \trianglelefteq L_\alpha$. But then, by conjugacy, $[V_{\alpha'-1}, Q_{\alpha'-2}]Z_{\alpha'-2} = [V_{\alpha'-3}, Q_{\alpha'-2}]Z_{\alpha'-2}$ is centralized by $V_{\alpha-1}$, contradicting Lemma 2.2.14 (ii) and Lemma 2.3.21 (ii). If $[V_\alpha^{(2)}, Q_\alpha] \not\leq Q_{\alpha'-2}$, then as $\Phi(V_\alpha^{(2)}) \leq Z_\alpha \leq Q_{\alpha'-1}$, $V_\alpha^{(2)} \cap Q_{\alpha'-2} = [U_{\alpha,\alpha-1}, Q_\alpha](V_\alpha^{(2)} \cap Q_{\alpha'-2} \cap Q_{\alpha'-1})$ so that $V_\alpha^{(2)} = [V_\alpha^{(2)}, Q_\alpha](V_\alpha^{(2)} \cap Q_{\alpha'})$ and $V_\alpha^{(2)}/[V_\alpha^{(2)}, Q_\alpha]$ is centralized by $O^p(L_\alpha)$, a contradiction by Lemma 5.2.13. Thus, as $\Phi(U_{\alpha,\alpha-1}) \leq \Phi(V_\alpha^{(2)}) \leq Z_\alpha \leq Q_{\alpha'-1}$, $U_{\alpha,\alpha-1}[V_\alpha^{(2)}, Q_\alpha] = [V_\alpha^{(2)}, Q_\alpha](U_{\alpha,\alpha-1}[V_\alpha^{(2)}, Q_\alpha] \cap Q_{\alpha'})$ and $U_{\alpha,\alpha-1}[V_\alpha^{(2)}, Q_\alpha] \trianglelefteq L_\alpha$. In particular, $V_\alpha^{(2)} = V_{\alpha-1}[V_\alpha^{(2)}, Q_\alpha]$ from which it follows that $[Q_{\alpha-1}, V_\alpha^{(2)}] \leq [V_\alpha^{(2)}, Q_\alpha]$ and $O^p(L_\alpha)$ centralizes $V_\alpha^{(2)}/[V_\alpha^{(2)}, Q_\alpha]$, and a contradiction is again provided by Lemma 5.2.13.

Suppose that $L_{\alpha'-1}/R_{\alpha'-1} \cong \mathrm{Dih}(10)$ or $(3 \times 3) : 2$. Then, applying Lemma 2.2.14 (ii) and Lemma 2.3.14 (v), and using that $P/Q_{\alpha-1} = \Omega(P/Q_{\alpha-1})$ where $P \in \mathrm{Syl}_2(G_{\alpha,\alpha-1})$, $[V_{\alpha-1}, Q_\alpha, Q_\alpha] \leq Z_{\alpha-1}$. If $[V_{\alpha-1}, Q_\alpha] \leq Z(Q_\alpha)$, then as $|Z_\alpha/Z_\beta| = 2$, $Z_\alpha \neq Z(Q_\alpha)$ and we have a contradiction. Thus, $[V_\alpha^{(2)}, Q_\alpha, Q_\alpha] = Z_\alpha$ and $\mathcal{U}_{\alpha,\alpha-1} = [U_{\alpha,\alpha-1}, Q_\alpha]$. In particular, since $Z_\alpha \not\leq V_{\alpha'-1}$, it follows that $\mathcal{U}_{\alpha,\alpha-1} = [U_{\alpha,\alpha-1}, Q_\alpha] \leq Q_{\alpha'-1}$, else $[U_{\alpha,\alpha-1}, Q_\alpha, V_{\alpha'-1} \cap Q_\alpha] = Z_\beta = Z_{\alpha'-1}$ and $V_{\alpha'-1}/Z_{\alpha'-1}$ is an FF-module. Thus, $\mathcal{U}_{\alpha,\alpha-1} = [V_{\alpha-1}, Q_\alpha]Z_\alpha \trianglelefteq L_\alpha$. But then $Z_{\alpha-1} = [V_{\alpha-1}, Q_\alpha, Q_\alpha] \trianglelefteq L_\alpha$, a final contradiction. $\square$

Before continuing, observe that we may now assume that whenever $b > 2$, both L_α/R_α and L_β/R_β are isomorphic to $\mathrm{SL}_2(p)$. Throughout this section, under these conditions and given a module V on which $\overline{L_\gamma}$ acts, for any $\gamma \in \mathit{\Gamma}$, we will often utilize coprime action. By this, we mean that when $p \geqslant 5$, taking T_γ to be the preimage in $\overline{L_\gamma}$ of $Z(L_\gamma/R_\gamma)$, we have that $V = [V, T] \times C_V(T)$. Indeed, if V is an FF-module for $\overline{L_\gamma}$, then this leads to a splitting $V = [V, \overline{L_\gamma}] \times C_V(\overline{L_\gamma})$. If $p \in \{2, 3\}$, since $\overline{L_\gamma}$ is solvable, we automatically have the conclusion $V = [V, O^p(\overline{L_\gamma})] \times C_V(O^p(\overline{L_\gamma}))$. Without explaining this each time it is used, we will generally just refer to "coprime action" and hope that it is clear in each instance where the conclusions we draw come from.

Lemma 5.3.10. *Suppose that $b > 2$. Then $Z_\beta = Z(Q_\beta)$ and $Z_\alpha = Z(Q_\alpha)$.*

Proof. By minimality of b, and using that b is even, we infer that $Z(Q_\alpha) \leq Q_\lambda$ for all $\lambda \in \Delta^{(b-2)}(\alpha)$. In particular, $Z(Q_\alpha) \leq Q_{\alpha'-2}$. If $Z(Q_\alpha) \not\leq Q_{\alpha'-1}$ then as $[Z(Q_\alpha), V_{\alpha'-1}, V_{\alpha'-1}] \leq [V_{\alpha'-1}, V_{\alpha'-1}] = \{1\}$, $[Z(Q_\alpha), V_{\alpha'-1}]$ is centralized by $V_{\alpha'-1}Q_\alpha \in \mathrm{Syl}_p(L_\alpha)$ and has exponent p. Thus, $[Z(Q_\alpha), V_{\alpha'-1}] \leq \Omega(Z(S)) = Z_\beta = Z_{\alpha'-1}$, a contradiction for otherwise $O^p(L_{\alpha'-1})$ centralizes $V_{\alpha'-1}$. Thus, $Z(Q_\alpha) \leq Q_{\alpha'-1}$ so that $Z(Q_\alpha) = Z_\alpha(Z(Q_\alpha) \cap Q_{\alpha'})$, $Z_{\alpha'}$ centralizes $Z(Q_\alpha)/Z_\alpha$ and $O^p(L_\alpha)$ centralizes $Z(Q_\alpha)/Z_\alpha$. Since $Z_\beta \leq Z_\alpha$ an application of coprime action yields $Z(Q_\alpha) = [Z(Q_\alpha), O^p(L_\alpha)] = Z_\alpha$, as desired. As a consequence, using that Q_α is self-centralizing, $Z(S)$ has exponent p.

Let $\alpha-1 \in \Delta(\alpha)$ such that $Z_{\alpha-1} \neq Z_\beta$, $V_{\alpha-1} \leq Q_{\alpha'-2}$ and $V_{\alpha-1} \not\leq Q_{\alpha'-1}$, as chosen in Proposition 5.3.9. By minimality of b, and using that b is even, we have that $Z(Q_{\alpha'-1}) \leq Q_\lambda$ for all $\lambda \in \Delta^{(b-1)}(\alpha)$. In particular, $Z(Q_{\alpha'-1}) \leq Q_\alpha$.

If $Z(Q_{\alpha'-1}) \not\leq Q_{\alpha-1}$ then $Z(Q_{\alpha'-1})Q_{\alpha-1} \in \mathrm{Syl}_p(L_{\alpha-1})$. Again, using minimality of b, we infer that $Z(Q_{\alpha-1}) \leq Q_{\alpha'-2}$ so that $[Z(Q_{\alpha'-1}), Z(Q_{\alpha-1})] \leq Z(Q_{\alpha'-1}) \cap Z(Q_{\alpha-1})$. Thus, $[Z(Q_{\alpha'-1}), Z(Q_{\alpha-1})]$ is centralized by $Z(Q_{\alpha'-1})Q_{\alpha-1} \in \mathrm{Syl}_p(L_{\alpha-1})$. Then, $[Z(Q_{\alpha'-1}), Z(Q_{\alpha-1})] \leq Z_{\alpha-1}$ and as $Z_{\alpha-1} \not\leq Z(Q_{\alpha'-1})$, $[Z(Q_{\alpha'-1}), Z(Q_{\alpha-1})] = \{1\}$ and $Z(Q_{\alpha-1})$ is centralized by $Z(Q_{\alpha'-1})Q_{\alpha-1} \in \mathrm{Syl}_p(L_{\alpha-1})$. But then $Z(Q_{\alpha-1}) = Z_{\alpha-1}$ and by conjugacy, $Z(Q_{\alpha'-1}) = Z_{\alpha'-1} \leq Z_{\alpha'-2} \leq Q_{\alpha-1}$, a contradiction.

Thus, $Z(Q_{\alpha'-1}) \leq Q_{\alpha-1}$ and so, $[Z(Q_{\alpha'-1}), V_{\alpha-1}] \leq Z_{\alpha-1} \cap Z(Q_{\alpha'-1})$. Since $Z_{\alpha-1}$ does not centralize $Z_{\alpha'}$, we deduce that $[Z(Q_{\alpha'-1}), V_{\alpha-1}] = \{1\}$. But then $Z(Q_{\alpha'-1})$

is centralized by $V_{\alpha-1}Q_{\alpha'-1} \in \mathrm{Syl}_p(L_{\alpha'-1})$ and $Z(Q_{\alpha'-1}) = Z_{\alpha'-1}$, as required. $\square$

Combining Proposition 5.3.9 and Lemma 5.3.10, we now satisfy Hypothesis 5.2.30. Thus, whenever b and the non-central chief factors in $V_\lambda^{(n)}$ satisfy the necessary requirements for $\lambda \in \{\alpha, \beta\}$ and various values of n, we may freely apply the results contained between Lemma 5.2.31 and Lemma 5.2.35.

Lemma 5.3.11. *Suppose that $b > 2$. Then $|V_\beta| = p^3$ and $[V_\alpha^{(2)}, Q_\alpha] = Z_\alpha$.*

Proof. If $V_\alpha^{(2)} \leq Q_{\alpha'-2}$, then $Z_\alpha(V_\alpha^{(2)} \cap Q_{\alpha'})$ has index p in $V_\alpha^{(2)}$ so that $V_\alpha^{(2)}/Z_\alpha$ has a unique non-central chief factor. Then the result holds by Lemma 5.2.31. Thus, we suppose that $V_\alpha^{(2)} \not\leq Q_{\alpha'-2}$. Then there is $\alpha-2$ such that $(\alpha-2, \alpha'-2)$ is a critical pair and by Lemma 5.3.8, we have that $Z_{\alpha-1} = Z_{\alpha'-3}$. Since $b > 2$ and $Z_\beta Z_{\alpha-1} \leq Z_\alpha \cap Z_{\alpha'-2}$, it follows that $Z_\beta = Z_{\alpha-1} = Z_{\alpha'-3} = Z_{\alpha'-1}$. If $|V_\beta| \neq p^3$, since $Z_\alpha(V_\alpha^{(2)} \cap Q_{\alpha'-2} \cap Q_{\alpha'-1})$ has index at most p^2 in $V_\alpha^{(2)}$ and by Lemma 5.2.32, $O^p(R_\alpha)$ centralizes $V_\alpha^{(2)}$. By Lemma 5.2.18, $Z_{\alpha-2} \leq V_{\alpha-1} = V_\beta \leq Q_{\alpha'-2}$, a contradiction. $\square$

Lemma 5.3.12. *$b \neq 4$.*

Proof. Since none of the conclusions of Theorem 5.2.2 have $b = 4$, we may suppose that G is a minimal counterexample with $b = 4$. Suppose that $V_\alpha^{(2)} \leq Q_{\alpha'-2}$. Then $V_\alpha^{(2)} \cap Q_{\alpha'-1} = Z_\alpha(V_\alpha^{(2)} \cap Q_{\alpha'})$ is an index p subgroup of $V_\alpha^{(2)}$ which is centralized, modulo Z_α, by $Z_{\alpha'}$. Thus, $V_\alpha^{(2)}/Z_\alpha$ is an FF-module for $\overline{L_\alpha}$. Then Lemma 5.2.32 implies that $O^p(R_\alpha)$ centralizes $V_\alpha^{(2)}$ and since $Z_{\alpha'-1} = Z_\beta$, Lemma 5.2.18 implies that $Z_\alpha \leq V_\beta = V_{\alpha'-1} \leq Q_{\alpha'}$, a contradiction. We have a similar contradiction if $V_\alpha^{(2)} \cap Q_{\alpha'-2} \leq Q_{\alpha'-1}$.

Thus, $V_\alpha^{(2)} \not\leq Q_{\alpha'-2}$ and $V_\alpha^{(2)} \cap Q_{\alpha'-2} \not\leq Q_{\alpha'-1}$. In particular, $V_\alpha^{(2)}$ is non-abelian and $Z_\alpha \leq \Phi(Q_\alpha)$. Suppose that $r \in L_\alpha$ is of order coprime to p and centralizes $V_\alpha^{(2)}$. Then, by the three subgroup lemma, r centralizes $Q_\alpha/C_{Q_\alpha}(V_\alpha^{(2)})$. Since $C_{Q_\alpha}(V_\alpha^{(2)}) \leq Q_{\alpha'-2}$ and $V_\alpha^{(2)} \cap Q_{\alpha'-2} \not\leq Q_{\alpha'-1}$, we have that $Z_{\alpha'}$ centralizes $C_{Q_\alpha}(V_\alpha^{(2)})V_\alpha^{(2)}/V_\alpha^{(2)}$ so that $O^p(L_\alpha)$ centralizes $C_{Q_\alpha}(V_\alpha^{(2)})V_\alpha^{(2)}/V_\alpha^{(2)}$. By coprime action, r centralizes Q_α, and so $r = 1$. Thus, every p'-element of L_α acts faithfully on $V_\alpha^{(2)}/\Phi(V_\alpha^{(2)})$.

Now, $Z_\alpha(V_\alpha^{(2)} \cap \dots \cap Q_{\alpha'})$ has index p^2 in $V_\alpha^{(2)}$ so that $V_\alpha^{(2)}/Z_\alpha$ is a 2F-module for $\overline{L_\alpha}$. Furthermore,

$$[V_\alpha^{(2)}, V_{\alpha'-1}, V_{\alpha'-1}] \leq [V_\alpha^{(2)}, V_{\alpha'-2}^{(2)}, V_{\alpha'-1}] \leq [Q_{\alpha'-1}, V_{\alpha'-1}] = Z_{\alpha'-1} = Z_\beta$$

and $V_\alpha^{(2)}/Z_\alpha$ is a faithful quadratic 2F-module for $\overline{L_\alpha}$. Then $\overline{L_\alpha}$ is determined by Lemma 2.3.10 and Proposition 2.3.19 and since $\overline{L_\alpha}$ has a quotient isomorphic to $\mathrm{SL}_2(p)$, we have that $\overline{L_\alpha} \cong \mathrm{SL}_2(p), \mathrm{SU}_3(2)', (3 \times 3) : 2$ or $(Q_8 \times Q_8) : 3$. Notice that V_β/Z_α is of order p and is not contained in $C_{V_\alpha^{(2)}/Z_\alpha}(O^p(L_\alpha))$. Setting $V := V_\alpha^{(2)}/Z_\alpha$ there is a $G_{\alpha,\beta}$-invariant subgroup of $V/C_V(O^p(L_\alpha))$ of order p which generates V and by Lemma 2.3.22, we have that $\overline{L_\alpha} \cong (3 \times 3) : 2$. Moreover, since $V_\alpha^{(2)}/Z_\alpha$ contains two non-central chief factors for L_α, for $U_\alpha := [V_\alpha^{(2)}, L_\alpha]$, we have that $Z_{\alpha'-2} = Z_{\alpha'-1}[U_\alpha \cap Q_{\alpha'-2}, V_{\alpha'-1}] \leq U_\alpha$ so that $V_\beta \leq U_\alpha$, $V_\alpha^{(2)} = U_\alpha$ and $|V_\alpha^{(2)}/Z_\alpha| = 2^4$.

Let $P_\alpha \leq L_\alpha$ with $S \leq P_\alpha$, $P_\alpha/Q_\alpha \cong \mathrm{Sym}(3)$, $L_\alpha = P_\alpha R_\alpha$ and $O_3(\overline{P_\alpha}) \trianglelefteq \overline{L_\alpha}$. Then P_α is $G_{\alpha,\beta}$-invariant and upon showing that no non-trivial subgroup of S is normalized by both P_α and G_β, then triple $(P_\alpha G_{\alpha,\beta}, G_\beta, G_{\alpha,\beta})$ satisfies Hypothesis 5.2.1. To this end, suppose that Q is non-trivial subgroup of S normalized by P_α and G_β. Then $Z_\beta \leq Q$ so that $Z_\beta \leq \Omega(Z(Q))$. Taking consecutive normal closure, we deduce that $V_\beta \leq \Omega(Z(Q))$ and $\Omega(Z(Q))/Z_\alpha$ contains of the non-central L_α-chief factors contained in $V_\alpha^{(2)}/Z_\alpha$. Write W for the preimage in $V_\alpha^{(2)}$ of this non-central chief factor, noting that by the definition of $V_\alpha^{(2)}$, $W \cap V_\beta = Z_\alpha$. However, $WV_\beta \leq \Omega(Z(Q))$ and $[W, V_\beta] = \{1\}$ so that $W \leq Q_{\alpha'-2}$ and $[W, V_{\alpha'-2}] \leq Z_{\alpha'-2} \cap W = Z_\beta = Z_{\alpha'-1}$ and $W = Z_\alpha(W \cap Q_{\alpha'})$. Then W contains no non-central chief factor for L_α, a contradiction. Thus, $Q = \{1\}$ and $(P_\alpha G_{\alpha,\beta}, G_\beta, G_{\alpha,\beta})$ satisfies Hypothesis 5.2.1. Assuming that G is a minimal counterexample to Theorem 5.2.2, we conclude that $P_\alpha/Q_\alpha \cong \mathrm{Sym}(3) \cong \overline{L_\beta}$ and $(P_\alpha G_{\alpha,\beta}, G_\beta, G_{\alpha,\beta})$ is a weak BN-pair of rank 2. By [DS85], $|S| \leqslant 2^7$ and since $|V_\alpha^{(2)}| = 2^6$ and $Q_\alpha/V_\alpha^{(2)}$ contains a non-central chief factor for L_α, we have a contradiction. □

Lemma 5.3.13. *Suppose that $b > 2$. Then the following hold:*

(i) $V_\alpha^{(2)} \leq Q_{\alpha'-2}$ *but* $V_\alpha^{(2)} \not\leq Q_{\alpha'-1}$*;*

(ii) $[V_\alpha^{(2)}, Q_\alpha] = Z_\alpha$ *and* $|V_\beta| = p^3$*;*

(iii) $O^p(R_\alpha)$ *centralizes* $V_\alpha^{(2)}$ *and* $V_\alpha^{(2)}/Z_\alpha$ *is a faithful FF-module for* $L_\alpha/R_\alpha \cong \mathrm{SL}_2(p)$*;*

(iv) $b \geqslant 8$*; and*

(v) $Z_{\alpha'-2} \leq V_\alpha^{(2)} \leq Z(V_\alpha^{(4)})$*.*

Proof. By Lemma 5.3.12, we have that $b > 4$ so that $V_\alpha^{(2)}$ is abelian. Moreover, (ii) holds by Lemma 5.3.11. Suppose first that $V_\alpha^{(2)} \not\leq Q_{\alpha'-2}$ so that there is a critical pair $(\alpha - 2, \alpha' - 2)$ such that $[Z_{\alpha-2}, Z_{\alpha'-2}] = Z_{\alpha-1} = Z_{\alpha'-3}$. Since $b > 2$, $Z_\alpha \neq Z_{\alpha'-2}$ and $Z_{\alpha-1} = Z_\beta$. Now, $[V_\alpha^{(2)} \cap Q_{\alpha'-2}, V_{\alpha'-1}] \leq Z_{\alpha'-2} \cap V_\alpha^{(2)}$. Since $V_\alpha^{(2)}$ is abelian and $V_\alpha^{(2)} \not\leq Q_{\alpha'-2}$, $Z_{\alpha'-2} \not\leq V_\alpha^{(2)}$. But $Z_{\alpha'-1} \leq V_\alpha^{(2)}$ and so it follows that $[V_\alpha^{(2)} \cap Q_{\alpha'-2}, V_{\alpha'-1}] \leq Z_{\alpha'-1}$ and $V_\alpha^{(2)} \cap Q_{\alpha'-2} \leq Q_{\alpha'-1}$. Then $V_\alpha^{(2)}/Z_\alpha$ is an FF-module and by Lemma 5.2.32, $O^p(R_\alpha)$ centralizes $V_\alpha^{(2)}$. But then by Lemma 5.2.18, $Z_{\alpha-2} \leq V_{\alpha-1} = V_\beta \leq Q_{\alpha'-2}$, a contradiction since $(\alpha-2, \alpha'-2)$ is a critical pair.

Thus, $V_\alpha^{(2)} \leq Q_{\alpha'-2}$. If $V_\alpha^{(2)} \leq Q_{\alpha'-1}$, then $V_\alpha^{(2)} = Z_\alpha(V_\alpha^{(2)} \cap Q_{\alpha'})$ and $O^p(L_\alpha)$ would centralize $V_\alpha^{(2)}/Z_\alpha$, a contradiction, and so (i) holds. Now, it follows that $V_\alpha^{(2)}/Z_\alpha$ is an FF-module and by Lemma 5.2.32, $O^p(R_\alpha)$ centralizes $V_\alpha^{(2)}$ and (iii) holds.

Since $V_\alpha^{(2)} \not\leq Q_{\alpha'-1}$, we infer that $Z_{\alpha'-2} = [V_\alpha^{(2)}, V_{\alpha'-1}]Z_{\alpha'-1} \leq V_\alpha^{(2)}$. If $b \geqslant 8$, then $V_\alpha^{(2)} \leq Z(V_\alpha^{(4)})$ and (v) holds, and so we may assume that $b = 6$ for the remainder of the proof. Notice that if $Z_{\alpha'-1} = Z_{\alpha'-3}$, it follows from Lemma 5.2.18 that $Z_{\alpha'} \leq V_{\alpha'-1} = V_{\alpha'-3} \leq Q_\alpha$, a contradiction. Since $Z_\beta = Z_{\alpha'-1} \neq Z_{\alpha'-3}$ and $b = 6$, we have that $Z_{\alpha'-2} = Z_{\alpha+2}$. Let $\alpha - 1 \in \Delta(\alpha)$ such that $V_{\alpha-1} \not\leq Q_{\alpha'-1}$ and $Z_{\alpha-1} \neq Z_\beta$, chosen as in Proposition 5.3.9. We have that $V_{\alpha'-1}^{(3)} \leq Q_{\alpha+2}$ since $V_{\alpha'-1}^{(3)}$ centralizes $Z_{\alpha+2} = Z_{\alpha'-2} \leq V_{\alpha'-1}$. Then $V_{\alpha'-1}^{(3)} \cap Q_\beta = V_{\alpha'-1}(V_{\alpha'-1}^{(3)} \cap Q_\alpha)$ and

$$[V_{\alpha-1}, V_{\alpha'-1}^{(3)} \cap Q_\alpha] \leq [V_\alpha^{(2)}, V_{\alpha'-1}^{(3)} \cap Q_\alpha] \leq Z_\alpha \cap V_{\alpha'-1}^{(3)} = Z_\beta = Z_{\alpha'-1}.$$

In particular, $V_{\alpha'-1}^{(3)}/V_{\alpha'-1}$ contains a unique non-central chief factor $L_{\alpha'-1}$, which as a $\mathrm{GF}(p)\overline{L_{\alpha'-1}}$-module is isomorphic to a natural $\mathrm{SL}_2(p)$-module. Thus, we may apply Lemma 5.2.34 so that $O^p(R_{\alpha'-1})$ acts trivially on $V_{\alpha'-1}^{(3)}$. Since $Z_{\alpha+2} = Z_{\alpha'-2}$, it follows from Lemma 5.2.18 that $Z_\alpha \leq V_{\alpha'-2}^{(2)} = V_{\alpha+2}^{(2)} \leq Q_\alpha$, an obvious contradiction. Thus, $b \geqslant 8$ and the lemma holds. □

Lemma 5.3.14. $b = 2$.

Proof. We may suppose that $b \geqslant 8$ by Lemma 5.3.13. Suppose first that $V_\alpha^{(4)} \not\leq$

$Q_{\alpha'-4}$. Since $Z_{\alpha'-3} \leq Z_{\alpha'-2} \leq Z(V_\alpha^{(4)})$ is centralized by $V_\alpha^{(4)}$, it follows that $Z_{\alpha'-3} = Z_{\alpha'-5}$ and by Lemma 5.2.18, we have that $V_{\alpha'-3} = V_{\alpha'-5}$. Now, $[V_\alpha^{(4)} \cap Q_{\alpha'-4}, V_{\alpha'-3}] = [V_\alpha^{(4)} \cap Q_{\alpha'-4}, V_{\alpha'-5}] \leq Z_{\alpha'-5} = Z_{\alpha'-3}$ and so $V_\alpha^{(4)} \cap Q_{\alpha'-4} \leq Q_{\alpha'-3}$. Since $V_\alpha^{(4)}$ centralizes $Z_{\alpha'-2}$, we deduce that $V_\alpha^{(4)} \cap Q_{\alpha'-4} = V_\alpha^{(2)}(V_\alpha^{(4)} \cap Q_{\alpha'-4} \cap Q_{\alpha'-1})$ and so $V_\alpha^{(4)}/V_\alpha^{(2)}$ contains a unique non-central chief factor for L_α. Now, by Lemma 5.2.33 and Lemma 5.2.18, since $Z_{\alpha'-3} = Z_{\alpha'-5}$ we conclude that $Z_{\alpha'} \leq V_{\alpha'-3}^{(3)} = V_{\alpha'-5}^{(3)} \leq Q_\alpha$, a contradiction.

Therefore, we continue assuming that $V_\alpha^{(4)} \leq Q_{\alpha'-4}$. Then $V_\alpha^{(4)} \cap Q_{\alpha'-3}$ centralizes $Z_{\alpha'-2}$ and we may assume that $V_\alpha^{(4)} \not\leq Q_{\alpha'-3}$, else $V_\alpha^{(4)} = V_\alpha^{(2)}(V_\alpha^{(2)} \cap Q_{\alpha'-1})$ and $O^p(L_\alpha)$ centralizes $V_\alpha^{(4)}/V_\alpha^{(2)}$. Since $|V_{\alpha'-3}| = p^3$, $V_\alpha^{(4)} \not\leq Q_{\alpha'-3}$ and $V_\alpha^{(4)}$ centralizes $Z_{\alpha'-2}$, by Lemma 5.2.16 $V_{\alpha'-3} \neq Z_{\alpha'-2}Z_{\alpha'-4}$ and so, $Z_{\alpha'-2} = Z_{\alpha'-4}$. If $O^p(R_\beta)$ centralizes $V_\beta^{(3)}$ then applying Lemma 5.2.18 to $Z_{\alpha'-2} = Z_{\alpha'-4}$ yields $Z_{\alpha'} \leq V_{\alpha'-2}^{(2)} = V_{\alpha'-4}^{(2)} \leq Q_\alpha$, a contradiction. Thus, to obtain a final contradiction, by Lemma 5.2.34, it suffices to show that $V_{\alpha'-1}^{(3)}/V_{\alpha'-1}$ contains a unique non-central chief factor for $L_{\alpha'-1}$ which, as a $\mathrm{GF}(p)\overline{L_{\alpha'-1}}$-module, is an FF-module.

By the symmetry in the hypothesis of (α, α') and (α', α), we may assume that $Z_{\alpha+2} = Z_{\alpha+4}$. Let $\alpha - 1 \in \Delta(\alpha)$ such that $V_{\alpha-1} \not\leq Q_{\alpha'-1}$ and $Z_{\alpha-1} \neq Z_\beta$, as in Proposition 5.3.9. Then $V_{\alpha'-1}^{(3)}$ centralizes $Z_{\alpha+2}$ so that $V_{\alpha'-1}^{(3)} \leq Q_{\alpha+2}$, $V_{\alpha'-1}^{(3)} \cap Q_\beta = V_{\alpha'-1}(V_{\alpha'-1}^{(3)} \cap Q_\alpha)$ and

$$[V_{\alpha-1}, V_{\alpha'-1}^{(3)} \cap Q_\alpha] \leq [V_\alpha^{(2)}, V_{\alpha'-1}^{(3)} \cap Q_\alpha] \leq Z_\alpha \cap V_{\alpha'-1}^{(3)} = Z_\beta = Z_{\alpha'-1}.$$

In particular, either $O^p(L_{\alpha'-1})$ centralizes $V_{\alpha'-1}^{(3)}/V_{\alpha'-1}$ or $V_{\alpha'-1}^{(3)}/V_{\alpha'-1}$ contains a unique non-central chief factor for $L_{\alpha'-1}$, and the result holds. □

Proposition 5.3.15. *Suppose $p \geqslant 5$. Then $R_\alpha = Q_\alpha$, G is a symplectic amalgam* *and one of the following holds:*

(i) *G is locally isomorphic to H where $F^*(H) \cong \mathrm{G}_2(p^n)$;*

(ii) *G is locally isomorphic to H where $F^*(H) \cong {}^3\mathrm{D}_4(p^n)$;*

(iii) *$p = 5$, $|S| = 5^6$, $Q_\beta \cong 5_+^{1+4}$ and $\overline{L_\beta} \cong 2_-^{1+4}.5$;*

(iv) *$p = 5$, $|S| = 5^6$, $Q_\beta \cong 5_+^{1+4}$ and $\overline{L_\beta} \cong 2_-^{1+4}.\mathrm{Alt}(5)$;*

(v) *$p = 5$, $|S| = 5^6$, $Q_\beta \cong 5_+^{1+4}$ and $\overline{L_\beta} \cong 2 \cdot \mathrm{Alt}(6)$; or*

(vi) *$p = 7$, $|S| = 7^6$, $Q_\beta \cong 7_+^{1+4}$ and $\overline{L_\beta} \cong 2 \cdot \mathrm{Alt}(7)$.*

Proof. By Lemma 5.3.14, we have that $b = 2$. Note that $Q_\alpha \cap Q_\beta = Z_\alpha(Q_\alpha \cap Q_\beta \cap Q_{\alpha'})$. Since $Z_{\alpha'} \leq Q_\beta$, it follows that $[Q_\alpha, Z_{\alpha'}, Z_{\alpha'}, Z_{\alpha'}] = \{1\}$ and by the Hall–Higman Theorem, $O^p(R_\alpha)$ centralizes Q_α and since Q_α is self-centralizing, $R_\alpha = Q_\alpha$ and $\overline{L_\alpha} \cong \mathrm{SL}_2(q)$.

We now intend to show that the amalgam is symplectic. We immediately satisfy condition (i) in the definition of a symplectic amalgam. We have that $W := \langle (Q_\alpha \cap Q_\beta)^{L_\alpha} \rangle \not\leq Q_\beta$, for otherwise $W = Q_\alpha \cap Q_\beta \trianglelefteq L_\alpha$, a contradiction by Proposition 5.2.25. Therefore, by Lemma 5.2.8 (iii), we have that $G_\beta = \langle W^{L_\beta} \rangle N_{G_\beta}(S)$, satisfying condition (ii). From our hypothesis, we automatically satisfy condition (iii). By Proposition 5.3.3, we satisfy condition (iv). Since $b = 2$ and $d(\alpha, \beta) = 1$, we have that $Z_\alpha \leq Q_\beta$. Moreover, by hypothesis and the symmetry between α and α' we have that $Z_\alpha \not\leq Q_{\alpha'} = Q_\alpha^x$ for some $x \in G_\beta$. Hence, G is a symplectic amalgam and the result holds by Theorem 5.1.11. □

Thus, we have reduced to the case where $b = 2$ and $p \in \{2, 3\}$. Since Proposition 5.3.9 only applied to the cases where $b > 2$, we have no knowledge of the structure of $\overline{L_\beta}$ or V_β. As intimated earlier, we attempt to show that $R_\alpha = Q_\alpha$ and apply the results in [PR12]. Then Proposition 5.1.13 completes the analysis of this case for fusion systems.

Proposition 5.3.16. *Suppose that $p \in \{2,3\}$, $b = 2$ and $m_p(S/Q_\beta) = 1$. Then $R_\alpha = Q_\alpha$, $|S| \leqslant 2^6$, G is a symplectic amalgam and one of the following holds:*

(i) *G has a weak BN-pair of rank 2 and G is locally isomorphic to H where $F^*(H) \cong \mathrm{G}_2(2)'$; or*

(ii) *$p = 2$, $|S| = 2^6$, $Q_\beta \cong 2^{1+4}_+$ and $\overline{L_\beta} \cong (3 \times 3) : 2$.*

Proof. If $R_\alpha = Q_\alpha$, then $\overline{L_\alpha} \cong \mathrm{SL}_2(q)$ and similarly to Proposition 5.3.15, G is a symplectic amalgam and the result holds after comparing with the tables listed in [PR12] and an application of [DS85] and [Fan86]. Hence, $\overline{L_\alpha} \not\cong \mathrm{SL}_2(q)$ so that $R_\alpha \neq Q_\alpha$ and by Lemma 5.3.2, $L_\alpha/R_\alpha \cong \mathrm{SL}_2(p)$. If Q_α is elementary abelian, then applying coprime action, we have that $Q_\alpha = [Q_\alpha, R_\alpha] \times C_{Q_\alpha}(R_\alpha)$ is an S-invariant decomposition. But $Z_\beta \leq Z_\alpha \leq C_{Q_\alpha}(R_\alpha)$ from which it follows that $Q_\alpha = C_{Q_\alpha}(R_\alpha)$ and $R_\alpha = Q_\alpha$, a contradiction. Thus, $[V_\beta, Q_\beta] = Z_\beta \leq Z_\alpha \leq \Phi(Q_\alpha)$.

If S/Q_β is cyclic then $\Phi(Q_\alpha)(Q_\alpha \cap Q_\beta)$ is an index p subgroup of Q_α and since $V_\beta \not\leq Q_\alpha$ and $[V_\beta, Q_\alpha \cap Q_\beta] \leq Z_\alpha \leq \Phi(Q_\alpha)$, it follows that $Q_\alpha/\Phi(Q_\alpha)$ contains a unique non-central chief factor for L_α which is isomorphic to an FF-module for $\overline{L_\alpha} \cong \mathrm{SL}_2(p)$, a contradiction.

Hence, we may assume that $p = 2$ and S/Q_β is generalized quaternion. Set $L := \langle V_\beta, V_\beta^x \rangle Q_\alpha$ with $x \in L_\alpha$ chosen such that $Z_\beta^x \neq Z_\beta$ and $x^2 \in G_{\alpha,\beta}$. In particular, $LR_\alpha = L_\alpha$. Write $\alpha - 1 = \beta^x$. Then, as $[Q_\beta, V_\beta] = Z_\beta \leq Z_\alpha$, $(Q_\beta \cap Q_\alpha \cap Q_{\alpha-1})/Z_\alpha$ is centralized by $O^2(O^{2'}(L))$. Since $S = V_\beta Q_\alpha$ normalizes $Q_\beta \cap Q_\alpha \cap Q_{\alpha-1}$, if

$Q_\beta \cap Q_\alpha \cap Q_{\alpha-1}$ is not elementary abelian then $Z_\beta \leq \Phi(Q_\beta \cap Q_\alpha \cap Q_{\alpha-1})$ and the choice of L yields that $Z_\alpha \leq \Phi(Q_\beta \cap Q_\alpha \cap Q_{\alpha-1})$, a contradiction. Thus, $Q_\beta \cap Q_\alpha \cap Q_{\alpha-1}$ is elementary abelian.

Suppose that $V_\beta \cap Q_\alpha \leq Q_{\alpha-1}$. Then $V_\beta \cap Q_\alpha$ is an elementary abelian subgroup of V_β of index 2. As V_β is non-abelian, $|V_\beta/Z(V_\beta)| = 4$ and since $|S/Q_\beta| \neq 2$, we must have that $[O^2(L_\beta), V_\beta] \leq Z(V_\beta)$. But then $Z_\alpha Z(V_\beta) \trianglelefteq L_\beta$ and it follows from the definition of V_β that $V_\beta = Z_\alpha Z(V_\beta)$ is abelian, a contradiction.

Let $V \leq Q_\beta$ be a normal subgroup of S which does not contain Z_α. Since $O^2(O^{2'}(L))$ centralizes $(Q_\beta \cap Q_\alpha \cap Q_{\alpha-1})/Z_\alpha$ and $S = V_\beta Q_\alpha$, $O^{2'}(L)$ normalizes $(V \cap Q_\alpha \cap Q_{\alpha-1})Z_\alpha$. Then $[Q_\alpha, V \cap Q_\alpha \cap Q_{\alpha-1}] = [Q_\alpha, (V \cap Q_\alpha \cap Q_{\alpha-1})Z_\alpha] \trianglelefteq O^{2'}(L)$. If $Z_\beta \leq [Q_\alpha, V \cap Q_\alpha \cap Q_{\alpha-1}]$, then by the construction of L, $Z_\alpha \leq [Q_\alpha, V \cap Q_\alpha \cap Q_{\alpha-1}] \leq V$, a contradiction. Thus, $[Q_\alpha, V \cap Q_\alpha \cap Q_{\alpha-1}] = \{1\}$ and $V \cap Q_\alpha \cap Q_{\alpha-1} \leq Z(Q_\alpha)$. Now, if $Z(Q_\alpha) \not\leq Q_\beta$, then $Z(Q_\alpha)$ centralizes $Q_\alpha \cap Q_\beta$, an index 2 subgroup of Q_β. Since $|S/Q_\beta| \neq 2$, this is a contradiction, and so $Z(Q_\alpha) = Z_\alpha(Z(Q_\alpha) \cap Q_{\alpha'})$ and since $Z_\beta \leq Z_\alpha = [Z(Q_\alpha), O^2(O^{2'}(L))]$, it follows from coprime action that $Z(Q_\alpha) = Z_\alpha$. Therefore, since $Z_\alpha \not\leq V$, $V \cap Q_\alpha \cap Q_{\alpha-1} = Z_\beta$.

Now, $[V_\beta, V_\beta] = Z_\beta \leq Q_{\alpha-1}$ and so $(V_\beta \cap Q_\alpha)Q_{\alpha-1}/Q_{\alpha-1}$ is elementary abelian and since $m_p(S/Q_\beta) = 1$, $|(V_\beta \cap Q_\alpha)Q_{\alpha-1}/Q_{\alpha-1}| = 2$. By coprime action, $V_\beta/Z_\beta = [V_\beta/Z_\beta, O^2(L_\beta)] \times C_{V_\beta/Z_\beta}(O^2(L_\beta))$ and for V^β the preimage in V_β of $[V_\beta/Z_\beta, O^2(L_\beta)]$, we deduce that $V_\beta = V^\beta Z_\alpha$. In particular, V^β has index at most 2 in V_β.

Suppose first that $V^\beta \neq V_\beta$. Since $Z_\alpha \not\leq V^\beta$, we have that $V^\beta \cap Q_\alpha \cap Q_{\alpha-1} = Z_\beta$. Since V^β has index 2 in V_β, Z_β has index 2 in $V_\beta \cap Q_\alpha \cap Q_{\alpha-1}$, from which it follows that $V_\beta \cap Q_\alpha \cap Q_{\alpha-1} = Z_\alpha$. In particular, V_β/Z_β has order at most 8 and L_β/R_β embeds into $\mathrm{GL}_3(2)$. But a Sylow 2-subgroup of $\mathrm{GL}_3(2)$ is dihedral of order 8, and so we have a contradiction.

Suppose that $V^\beta = V_\beta$. Since $Z(V_\beta)$ centralizes Z_α, $Z(V_\beta) \leq Q_\alpha$ and since $Z_\alpha \not\leq Z(V_\beta)$, $Z_\beta = Z(V_\beta) \cap Q_{\alpha-1}$ has index at most 2 in $Z(V_\beta)$. Again, $O^2(L_\beta)$ centralizes $Z(V_\beta)$ and as $V_\beta = V^\beta$, we have that $Z(V_\beta) = Z_\beta$. In particular, V_β is extraspecial and since $V_\beta \cap Q_\alpha \cap Q_{\alpha-1}$ has index 4 in V_β and is elementary abelian, $V_\beta \cong 2^{1+4}_+$. Comparing with [Win72], we conclude that $\mathrm{Out}(V_\beta) \cong \mathrm{Sym}(3) \wr 2$ and as L_β/R_β acts faithfully on V_β and has generalized quaternion Sylow 2-subgroups, we have a contradiction. □

Proposition 5.3.17. *Suppose that $p \in \{2,3\}$, $b = 2$ and $m_p(S/Q_\beta) > 1$. Then one of the following holds:*

(i) *$R_\alpha = Q_\alpha$, G has a weak BN-pair of rank 2, and either G is locally isomorphic to H where $(F^*(H), p)$ is $(\mathrm{G}_2(2^n), 2)$ or $({}^3\mathrm{D}_4(p^a), p)$, or $p = 2$ and G is parabolic isomorphic to J_2 or $\mathrm{Aut}(\mathrm{J}_2)$; or*

(ii) *$p = 2$, $|S| = 2^9$, $\overline{L_\beta} \cong \mathrm{Alt}(5)$, $Q_\beta \cong 2^{1+6}_+$, $V_\beta = O^2(L_\beta)$, V_β/Z_β is a natural $\Omega_4^-(2)$-module for $\overline{L_\beta}$, $\overline{L_\alpha} \cong \mathrm{SU}_3(2)'$, Q_α is a special 2-group of shape 2^{2+6} and Q_α/Z_α is a natural $\mathrm{SU}_3(2)$-module.*

Proof. Suppose that $R_\alpha = Q_\alpha$. Then, as in Proposition 5.3.15, G is a symplectic amalgam and the result follows from Theorem 5.1.11 and Proposition 5.1.13. Indeed, the amalgams presented in [PR12] satisfying the above hypothesis are either weak BN-pairs of rank 2 (and (i) holds by [DS85]); or $\mathcal{A}_{42}$ when $p = 2$. In the latter case, $\mathrm{PSp}_6(3)$ is listed as an example completion. But comparing with the list of maximal subgroups in [Con+85], for $G \cong \mathrm{PSp}_6(3)$, $\overline{L_\alpha} \cong 2^{2+6} : \mathrm{SU}_3(2)'$ and from the perspective of this work, $R_\alpha \neq Q_\alpha$. Either way, we assume throughout this proof that $R_\alpha \neq Q_\alpha$ with the goal of showing that G has "the same" structural properties as $\mathcal{A}_{42}$ in [PR12] in order to satisfy outcome (ii).

Since $R_\alpha \neq Q_\alpha$, we have that $L_\alpha/R_\alpha \cong \mathrm{SL}_2(p)$. As in Proposition 5.3.16, if Q_α is elementary abelian then an application of coprime action implies that $R_\alpha = Q_\alpha$, a contradiction to the initial assumption. Again, as in Proposition 5.3.16, we set $L := \langle V_\beta, V_\beta^x\rangle Q_\alpha$ with $x \in L_\alpha$ chosen such that $LR_\alpha = L_\alpha$ and $x^2 \in G_{\alpha,\beta}$ and write $\alpha - 1 = \beta^x$. Then $Q_\beta \cap Q_\alpha \cap Q_{\alpha-1}$ is elementary abelian, $V_\beta \cap Q_\alpha \not\leq Q_{\alpha-1}$ and for any $V \leq Q_\beta$ which is normal in S and does not contain Z_α, we must have that $V \cap Q_\alpha \cap Q_{\alpha-1} = Z_\beta$.

Now, $V_\beta \cap Q_\alpha \cap Q_{\alpha-1}$ contains Z_α so is normalized by L. By construction, $V_\beta \cap Q_\alpha \cap Q_{\alpha-1} = V_{\alpha-1} \cap Q_\alpha \cap Q_\beta = V_\beta \cap Q_\alpha \cap V_{\alpha-1}$. In particular, $V_\beta \cap Q_\alpha \cap V_{\alpha-1}$ is an elementary abelian subgroup of index $r_\beta p$ in V_β, where $r_\beta = |(V_\beta \cap Q_\alpha)Q_{\alpha-1}/Q_{\alpha-1}|$.

Since $Z_\alpha \leq V_\beta$, we have that $Z(V_\beta) \leq Q_\alpha$ and as $Z_\alpha \not\leq Z(V_\beta)$, we have that $Z(V_\beta) \cap Q_{\alpha-1} = Z_\beta$. Choose V^β minimally with respect to inclusion such that $V^\beta \trianglelefteq L_\beta$ and V^β/Z_β contains a non-central chief factor for L_β. If $V_\beta \neq V^\beta$, then $Z_\alpha \not\leq V^\beta$ and $V^\beta \cap Q_\alpha \cap Q_{\alpha-1} = Z_\beta$. Then, by conjugacy, $[V^\beta \cap Q_\alpha, V^{\alpha-1} \cap Q_\alpha] \leq V^\beta \cap Q_\alpha \cap V^{\alpha-1} \leq Z_\beta \cap Z_{\alpha-1} = \{1\}$. But V^β contains a non-central chief factor for L_β and as $m_p(S/Q_\beta) > 1$ and $V^\beta \cap Q_\alpha$ has index p in V^β, we must have that $V^{\alpha-1} \cap Q_\alpha \leq Q_\beta$. Thus, $[V^{\alpha-1}, Q_\alpha] \leq V^{\alpha-1} \cap Q_\alpha \cap Q_\beta = Z_{\alpha-1} \leq Z_\alpha$. Since $Z_\alpha \leq \Phi(Q_\alpha)$ and L contains elements of p'-order, $O^p(L)$ does not centralize Q_α/Z_α and we infer that $V^{\alpha-1} \leq Q_\alpha$ so that $V^{\alpha-1} = Z_{\alpha-1}$, a contradiction since $V^{\alpha-1}/Z_{\alpha-1}$ contains a non-central chief factor for $L_{\alpha-1}$. Thus, $V^\beta = V_\beta = [V_\beta, O^p(L_\beta)] \leq O^p(L_\beta)$, $Z(V_\beta)$ contains no non-central chief factors so that $C_{V_\beta}(O^p(L_\beta)) = Z(V_\beta)$ and by Lemma 2.3.2, $V_\beta/Z(V_\beta)$ is irreducible as an $\overline{L_\beta}$-module.

Again, we remark that $Z_\beta = \Phi(V_\beta) \leq Q_\alpha$. We aim to show that $Z(V_\beta) \leq Q_{\alpha-1}$ so that $[V_\beta, V_\beta] = \Phi(V_\beta) = Z(V_\beta) = Z_\beta$ is of order p and V_β is an extraspecial group. Towards this, we suppose that $Z(V_\beta) \not\leq Q_{\alpha-1}$. Then the action of L implies that $Z(V_{\alpha-1}) \not\leq Q_\beta$. Set $V := V_\beta/Z(V_\beta)$ throughout.

Suppose that $\overline{L_{\alpha-1}} \cong \mathrm{M}_{11}, \mathrm{Ree}(3)$ or a central extension of $\mathrm{PSL}_3(4)$ and $p = 3$. It follows that $Z(V_\beta)(V_\beta \cap Q_\alpha \cap V_{\alpha-1})$ has index at most p^2 in V_β and is centralized by $Z(V_{\alpha-1})$. If $\overline{L_\beta} \cong \mathrm{M}_{11}$ then there is $x \in L_\beta$ such that for $J := \langle Z(V_{\alpha-1}), Z(V_{\alpha-1})^x, Q_\beta\rangle$, $\overline{J} \cong \mathrm{PSL}_2(11)$ and J centralizes a subgroup of V of index at most 3^4. Since 11 does not divide $|\mathrm{GL}_4(3)|$, J centralizes V, a contradiction since $\overline{J}$ contains a non-trivial 3-element. If $\overline{L_\beta} \cong \mathrm{PSL}_3(4)$, then there is $x \in L_\beta$ such that $L_\beta = \langle Z(V_{\alpha-1}), Z(V_{\alpha-1})^x, Q_\beta\rangle$ so that $|V| \leqslant 3^4$. Since 7 divides $|L_\beta|$ but $|\mathrm{GL}_4(3)|$ is not divisible by 7, we have that L_β centralizes V, another contradiction.

Suppose now that $\overline{L_{\alpha-1}} \cong \mathrm{Sz}(2^n)$ for $n \geqslant 3$. Since $Z_\beta \leq Q_{\alpha-1}$, $(V_\beta \cap Q_\alpha)Q_{\alpha-1}/Q_{\alpha-1}$ is elementary abelian and it follows that $r_\beta \leq 2^n$ and that the index of $Z(V_\beta)(V_\beta \cap Q_\alpha \cap V_{\alpha-1})$ in V_β is at most r_β. Moreover, $\overline{L_\beta}$ may be generated by three conjugates of an involution by Lemma 2.2.3 (vi) from which it follows that V has order at most $r_\beta^3 \leqslant 2^{3n}$. Since the minimal degree of a non-trivial GF(2)-representation of $\mathrm{Sz}(2^n)$ is $4n$, we have a contradiction.

Thus, $\overline{L_{\alpha-1}} \cong (\mathrm{P})\mathrm{SU}_3(p^n)$, $(\mathrm{P})\mathrm{SL}_2(p^n)$ or $\mathrm{Ree}(3^n)$. Suppose that $|Z(V_\beta)Q_{\alpha-1}/Q_{\alpha-1}| \geqslant p^2$. Using the action of L, we infer that $|Z(V_{\alpha-1})Q_\beta/Q_\beta| \geqslant p^2$. Then by Lemma 2.2.1 (iv), (v), Lemma 2.2.2 (viii) and Lemma 2.2.4 (vi), $\overline{L_\beta}$ is generated by $3, 2$ or 3 conjugates of $Z(V_{\alpha-1})Q_\beta/Q_\beta$ for $(\mathrm{P})\mathrm{SU}_3(p^n)$, $(\mathrm{P})\mathrm{SL}_2(p^n)$ or $\mathrm{Ree}(3^n)$ respectively. Moreover, $r_\beta \leq p^{2n}, p^n$ or 3^{2n} respectively and so the index of $Z(V_\beta)(V_\beta \cap Q_\alpha \cap V_{\alpha-1})$ in V_β is strictly less than p^{2n}, p^n or 3^{2n}. Applying a similarly methodology as above, we conclude that V has order strictly less than p^{6n}, p^{2n}

or 3^{6n} and since the relevant minimal degrees of non-trivial GF(p)-representations are $6n$, $2n$ and $7n$, we have a contradiction.

Thus, we deduce that $|Z(V_\beta)Q_{\alpha-1}/Q_{\alpha-1}| = p$ so that $|Z(V_\beta)| = p^2$. In particular, $C_S(Z(V_\beta))$ has index p in S so that $V_{\alpha-1} \cap C_{Q_\alpha}(Z(V_\beta))$ has index at most p^2 in $V_{\alpha-1}$ and is centralized by $Z(V_\beta)$. Suppose that $\overline{L_{\alpha-1}} \cong (\mathrm{P})\mathrm{SU}_3(p^n)$. Then Lemma 2.2.2 (vi), (vii) implies that $\overline{L_\beta}$ is generated by four conjugates of $Z(V_{\alpha-1})Q_\beta/Q_\beta$ from which we conclude that $|V| \leqslant p^8$. Since the minimal degree of a GF(p)-module is $6n$, the only possibility is that $p^n = 3$. In this case, Lemma 2.2.2 (vi) implies that $\overline{L_\beta}$ is generated by three conjugates of $Z(V_{\alpha-1})Q_\beta/Q_\beta$ so that $|V| = 3^6$ and V is a natural $\mathrm{SU}_3(3)$-module. But $V_\beta \cap Q_\alpha$ is $G_{\alpha,\beta}$-invariant, contains $Z(V_\beta)$ and has index 3 in V_β contradicting Lemma 2.2.13 (iii).

Suppose now that $\overline{L_{\alpha-1}} \cong \mathrm{Ree}(3^n)$ for $n \geqslant 1$ and $|Z(V_\beta)Q_{\alpha-1}/Q_{\alpha-1}| = 3$. Then by Lemma 2.2.4 (vi), $\overline{L_\beta}$ is generated by at most three conjugates of $Z(V_{\alpha-1})Q_\beta/Q_\beta$ from which it follows that $|V| \leq 3^6$. Since the minimal degree of a non-trivial GF(3)-representation for $\mathrm{Ree}(3^n)$ is $7n$, we have a contradiction.

Assume that $\overline{L_{\alpha-1}} \cong (\mathrm{P})\mathrm{SL}_2(p^n)$ for $n > 1$ and $|Z(V_\beta)Q_{\alpha-1}/Q_{\alpha-1}| = p$. Then Lemma 2.2.1 (iv), (v) implies that $\overline{L_\beta}$ is generated by three conjugates of $Z(V_{\alpha-1})Q_\beta/Q_\beta$ from which it follows that $|V| \leq p^6$. It follows from Lemma 2.3.12 that $n = 2$, V is irreducible and V is either a natural $\mathrm{SL}_2(p^2)$-module, a natural $\Omega_3(p^2)$-module, or a natural $\Omega_4^-(p)$-module. Using that $V_\beta \cap Q_\alpha$ is a $G_{\alpha,\beta}$-invariant subgroup of V_β of index p which contains $[V_\beta, S]Z(V_\beta)$, V is a natural $\Omega_4^-(p)$-module. Moreover, as $Q_\beta = V_\beta(Q_\beta \cap Q_{\alpha-1})$ and $[Q_{\alpha-1}, Z(V_{\alpha-1})] \leq Z_{\alpha-1} \leq V_\beta$, it follows that $O^p(L_\beta)$ centralizes Q_β/V_β so that V contains the unique non-central chief factor for L_β within Q_β, and $\overline{L_\beta} \cong \mathrm{PSL}_2(p^2)$. Applying Lemma 2.2.10 to V_β/Z_β, if $p = 2$ then it follows that $V^\beta \neq V_\beta$, a contradiction;

while if $p = 3$, then by Lemma 2.2.10, $[V_\beta/Z_\beta, S, S]$ is 2-dimensional as a $\mathrm{GF}(3)S$-module and it follows from the structure of a natural $\Omega_4^-(3)$-module described in Lemma 2.2.9 that $Z_\alpha Z(V_\beta) = [V_\beta, S, S] = [V_\beta, V_{\alpha-1} \cap Q_\alpha, V_{\alpha-1} \cap Q_\alpha] \leq V_{\alpha-1}$, a contradiction.

Thus, $Z(V_\beta) \leq Q_{\alpha-1}$ and by a previous observation, $Z(V_\beta) = Z_\beta = \Phi(V_\beta)$ is of order p and V_β is an extraspecial group. Moreover, $V_\beta \cap Q_\alpha \cap Q_\beta$ has index pr_β in V_β and is elementary abelian. Suppose that $|V_\beta| = p^{2r+1}$. Then $|V_\beta \cap Q_\alpha \cap Q_\beta| = p^{2r+1}/pr_\beta$ and since the maximal abelian subgroups of V_β have order p^{r+1}, we deduce that $p^{2r}/r_\beta \leqslant p^{r+1}$ and $p^{r-1} \leqslant r_\beta$. We reiterate that if V_β/Z_β contains a unique non-central chief factor, then V_β/Z_β is irreducible.

Suppose that $\overline{L_\beta} \cong (\mathrm{P})\mathrm{SU}_3(p^n)$ so that $r_\beta \leq p^{2n}$. In particular, $r-1 \leqslant 2n$ and so $|V_\beta| \leqslant p^{4n+3}$. But then $|V_\beta/Z_\beta| \leqslant p^{4n+2}$ and since the minimal degree of a $\mathrm{GF}(p)$-representation on $\overline{L_\beta}$ is $6n$, we conclude that $n = 1$, $p = 3$ and V_β/Z_β is a natural module for $\overline{L_\beta}$. But then $V_\beta \cap Q_\alpha$ is a $G_{\alpha,\beta}$-invariant subgroup of index 3, and we have a contradiction by Lemma 2.2.13 (iii). Suppose that $\overline{L_\beta} \cong \mathrm{Ree}(3^n)$. Then $r-1 \leqslant 2n$ and so $|V_\beta/Z_\beta| \leqslant p^{4n+2}$, a contradiction since the minimal degree of a $\mathrm{GF}(3)$-representation on $\overline{L_\beta}$ is $7n$. If $\overline{L_\beta} \cong \mathrm{Sz}(2^n)$, then $r_\beta \leq p^n$ and so $r-1 \leqslant n$ and $|V_\beta| \leqslant 2^{2n+3}$. Then $|V_\beta/Z_\beta| \leqslant 2^{2n+2}$, a contradiction since the minimal degree of a $\mathrm{GF}(2)$-representation on $\overline{L_\beta}$ is $4n$ and $n > 1$.

Hence, we may suppose that S/Q_β is elementary abelian of order p^n and $n > 1$. Then $|V_\beta/Z_\beta| \leqslant p^{2n+2}$. If $n \geqslant 3$, then $\overline{L_\beta} \cong \mathrm{PSL}_2(p^n)$ or $\mathrm{SL}_2(p^n)$. Moreover, $|V_\beta/Z_\beta| < p^{3n}$ and so V_β/Z_β is irreducible and described by Lemma 2.3.12. In particular, V_β/Z_β is not a natural $\Omega_3(p^n)$-module. Since $V_\beta \cap Q_\alpha$ is a $G_{\alpha,\beta}$-invariant subgroup of index p, V_β/Z_β is not a natural $\mathrm{SL}_2(p^n)$-module or a natural $\Omega_4^-(p^{n/2})$-module. If V_β/Z_β is a triality module, then $n = 3a$ for some $a \geqslant 1$.

Then $|V_\beta/Z_\beta| = p^{6a+2} \geqslant p^{8a}$ from which it follows that $a = 1$, V_β/Z_β is irreducible and $|V_\beta| = p^9$. Now, $C_\beta \leq Q_\alpha$. Moreover, since $C_{\alpha-1}(V_\beta \cap Q_\alpha \cap V_{\alpha-1})$ has index at most p^4 in $C_{\alpha-1}V_{\alpha-1}$ and is centralized by C_β, $C_\beta \leq Q_{\alpha-1}$ by Lemma 2.2.11 (iii). Since $Z_\alpha \not\leq C_\beta$, we have that $C_\beta = Z_\beta$, $Q_\beta = V_\beta$ and $|S| = p^{12}$. We may assume that G is a minimal counterexample to Theorem 5.2.2 and we let $X = \langle R_\alpha G_{\alpha,\beta}, G_\beta \rangle$ and Q be the largest subgroup of S normal in X, so that $Z_\beta \leq Q$ as $Z_\beta \trianglelefteq X$. Note that if $R_\alpha \leq G_{\alpha,\beta}$ then by Lemma 5.2.17, $R_\alpha = Q_\alpha$, a contradiction. Thus, $R_\alpha G_{\alpha,\beta}/Q_\alpha$ has a strongly p-embedded subgroup and $Q \leq Q_\alpha$. Then, as $Q \trianglelefteq L_\beta$ and $Q \leq Q_\alpha \cap Q_\beta$, we have that $Z_\beta \leq Q \leq C_\beta = Z_\beta$. Now, X/Q satisfies Hypothesis 5.2.1 and is a $b = 1$ type amalgam with $|S/Q| = p^{11}$. Comparing with Theorem 5.2.2, since G was an assumed minimal counterexample, no such examples exist.

Hence, we may suppose that S/Q_β is elementary abelian of order p^2 so that $|V_\beta/Z_\beta| \leqslant p^6$. Then $O^3(\overline{L_\beta}) \not\cong \mathrm{PSL}_2(8)$ since the minimal degree of a GF(3)-representation is 7. If $\overline{L_\beta}$ is isomorphic to a central extension of $\mathrm{PSL}_3(4)$ then V_β/Z_β is irreducible and one can check that since Z_α/Z_β is $G_{\alpha,\beta}$-invariant and of order 3, and $V_\beta \cap Q_\alpha$ is $G_{\alpha,\beta}$-invariant and index 3, we get a contradiction. If $\overline{L_\beta} \cong \mathrm{M}_{11}$, then using that V_β/Z_β is irreducible, we conclude that $|V_\beta/Z_\beta| = 3^5$, and $|V_\beta| = 3^6$, a contradiction since V_β is extraspecial.

Thus, $\overline{L_\beta} \cong \mathrm{SL}_2(p^2)$ or $\mathrm{PSL}_2(p^2)$ and V_β/Z_β is described by Lemma 2.3.12. Since $V_\beta \cap Q_\alpha$ is a $G_{\alpha\beta}$-invariant subgroup of index p containing $[V_\beta, S]$ Lemma 2.3.12 implies that V_β/Z_β is a natural $\Omega_4^-(p)$-module and $|V_\beta| = p^5$. Now, as L_β/C_β embeds in the automorphism group of V_β, we infer that $Q_\beta = V_\beta C_\beta$. Moreover, using [Win72], if $p = 2$ then $\overline{L_\beta} \cong \mathrm{Out}(V_\beta) \cong \Omega_4^-(2)$ and $V_\beta \cong Q_8 * D_8 \cong 2_-^{1+4}$; and if $p = 3$ then V_β has exponent 3 and $\overline{L_\beta}$ is isomorphic to a subgroup of $\mathrm{Sp}_4(3)$.

Suppose that $p = 3$ and let $K \in \mathrm{Syl}_2(L_\beta)$. Since $\overline{L_\beta} \cong \mathrm{PSL}_2(9)$, $K \cong \mathrm{Dih}(8)$. Letting $1 \neq i \in Z(K)$, we have that $|C_{V_\beta/Z_\beta}(i)| = 9$ and by coprime action $V_\beta = C_{V_\beta}(i)[V_\beta, i]$. Since $[V_\beta, V_\beta] \leq C_{V_\beta}(i)$ it follows from the three subgroup lemma that $[[V_\beta, i], C_{V_\beta}(i)] = \{1\}$ and since $|[V_\beta, i]| \leqslant 3^3$, it follows that $Z_\beta = C_{V_\beta}(i) \cap [V_\beta, i]$ and $C_{V_\beta}(i) \cong [V_\beta, i] \cong 3_+^{1+2}$. Since $i \leq Z(K)$, K normalizes $[V_\beta, i]$ and since $Z_\beta = Z(L_\beta)$, K acts trivially on $Z_\beta = Z([V_\beta, i])$ and by [Win72], K embeds into $\mathrm{Sp}_2(3) \cong \mathrm{SL}_2(3)$. But $\mathrm{SL}_2(3)$ has quaternion Sylow 2-subgroups, a contradiction.

Thus, we have shown that $p = 2$. Now, $Z_\alpha \not\leq C_\beta$ and so $Z_\beta = C_\beta \cap Q_{\alpha-1}$ has index at most 4 in C_β and $|C_\beta| \leqslant 8$. Since $Z(C_\beta)$ is centralized by $L_\beta = O^2(L_\beta)C_\beta$ and Q_α is self centralizing, $Z(C_\beta) \leq Z(Q_\alpha) = Z_\alpha$. Thus, $Z(C_\beta) = Z_\beta$ and as $|C_\beta| \leqslant 8$, either $C_\beta = Z_\beta$, or $C_\beta \cong Q_8$ or $\mathrm{Dih}(8)$. If $C_\beta = Z_\beta$ then we have that $Q_\beta = V_\beta \cong 2_-^{1+4}$, $|S| = 2^7$ and $|Q_\alpha| = 2^6$. Since $Z_\alpha \leq \Phi(Q_\alpha)$ and $R_\alpha \neq Q_\alpha$, we have that $Z_\alpha = \Phi(Q_\alpha)$ and Q_α/Z_α is a faithful quadratic 2F-module for $\overline{L_\alpha}$. As $L_\alpha/R_\alpha \cong \mathrm{Sym}(3)$, using Lemma 2.3.10 and Proposition 2.3.19, it follows that $\overline{L_\alpha} \cong (3 \times 3) : 2$. Now, for every subgroup Z of Z_α of order 2, is easy to check that Q_α/Z is an extraspecial group. In the language of Beisiegel [Bei77], Q_α is an ultraspecial 2-group of order 2^6. Checking in MAGMA utilizing the Small Groups library, the automorphism groups of all such groups have 3-part at most 9. Since there is $r \in (L_\beta \cap G_{\alpha,\beta})$ a 3-element centralizing Z_α by Lemma 2.2.9 (v), $r \in G_\alpha \setminus L_\alpha$ and a Sylow 3-subgroup of $\overline{G_\alpha}$ has order at least 27, and as $\overline{G_\alpha}$ acts faithfully on Q_α, we have a contradiction.

Thus, C_β is non-abelian of order 8. Furthermore, $|S| = 2^9$ and if Q_α/Z_α is a natural $\mathrm{SU}_3(2)$-module for $\overline{L_\alpha} \cong \mathrm{SU}_3(2)'$, then since C_β is $G_{\alpha,\beta}$-invariant, there is a 3-element in $L_\alpha \cap G_{\alpha,\beta}$ which acts non-trivially on C_β so that $C_\beta \cong Q_8$ and $Q_\beta = 2_+^{1+6}$. Thus, to complete the proof, it suffices to show that Q_α/Z_α is a natural

$SU_3(2)$-module. Now, $Q_\alpha \cap Q_\beta = Z_\alpha(Q_\alpha \cap Q_{\alpha'})$ has index 4 in Q_α and, modulo Z_α, is centralized by $Z_{\alpha'}$. It is clear that $Z_{\alpha'}$ acts quadratically on Q_α/Z_α and, since $Z_\alpha \leq \Phi(Q_\alpha)$ and $R_\alpha \neq Q_\alpha$, $\overline{L_\alpha}$ is determined by Proposition 2.3.19. Since $L_\alpha/R_\alpha \cong \mathrm{Sym}(3)$, we need only rule out the case where $\overline{L_\alpha} \cong (3 \times 3) : 2$.

Assume that $\overline{L_\alpha} \cong (3 \times 3) : 2$ and $|C_\beta| = 8$. Observe that $Q_\alpha = (Q_\alpha \cap Q_\beta)(Q_\alpha \cap Q_{\alpha-1}) = (V_\beta \cap Q_\alpha)(V_{\alpha-1} \cap Q_\beta)(Q_\beta \cap Q_\alpha \cap Q_{\alpha-1})$. Then, $V_\beta \cap Q_\alpha \cap Q_{\alpha-1} = V_{\alpha-1} \cap Q_\alpha \cap Q_\beta = Z_\alpha$, and it follows that $Z_\alpha = \Phi(Q_\alpha)$. By coprime action, we have that $Q_\alpha/Z_\alpha = [Q_\alpha/Z_\alpha, O^2(L_\alpha)] \times C_{Q_\alpha/Z_\alpha}(O^2(L_\alpha))$ where $|[Q_\alpha/Z_\alpha, O^2(L_\alpha)]| = 2^4$. Taking Q_α^* to be the preimage in Q_α of $[Q_\alpha/Z_\alpha, O^2(L_\alpha)]$, form $S^* = V_\beta Q_\alpha^*$ and $L_\lambda^* = \langle (S^*)^{L_\lambda}$ for $\lambda \in \{\alpha, \beta\}$. It is clear that $S^* \in \mathrm{Syl}_2(L_\lambda^*)$, $V_\beta = O_2(L_\beta^*)$ and $Q_\alpha^* = O_2(L_\alpha^*)$, and $L_\lambda^*/O_2(L_\lambda^*) \cong \overline{L_\lambda}$ for $\lambda \in \{\alpha, \beta\}$. Then for K a Hall $2'$-subgroup of $G_{\alpha,\beta}$, we conclude that $(L_\alpha^* K, L_\beta^* K, S^* K)$ satisfies Hypothesis 5.2.1 and since G is a minimal counterexample, comparing with Theorem 5.2.2, we have a contradiction. □

Corollary 5.3.18. *Suppose that outcome (ii) in Proposition 5.3.17 holds and G is obtained from a fusion system satisfying Hypothesis 5.1.12. Then $\mathcal{F}$ is isomorphic to the 2-fusion system of* $\mathrm{PSp}_6(3)$*.*

Proof. Since $Q_\alpha \in \mathrm{Syl}_2(O^2(L_\alpha))$ and $V_\beta \leq S \cap O^2(L_\beta)$ is not contained in Q_α, it follows that $O^2(O^{2'}(\mathcal{F})) = O^{2'}(\mathcal{F})$ so $O^{2'}(\mathcal{F})$ is reduced. Comparing with the lists in [AOV17], it follows that $O^{2'}(\mathcal{F})$ is isomorphic to the 2-fusion system of $\mathrm{PSp}_6(3)$. Furthermore, by [AOV17, Proposition 6.4], the only fusion system supported on a Sylow 2-subgroup of $\mathrm{PSp}_6(3)$ with $O_2(\mathcal{F}) = \{1\}$ is the fusion category of $\mathrm{PSp}_6(3)$. Thus, $\mathcal{F} = O^{2'}(\mathcal{F})$ and the result holds. □

In summary, in this section we have proved the following:

Theorem 5.3.19. *Suppose that $\mathcal{A} = \mathcal{A}(G_\alpha, G_\beta, G_{\alpha,\beta})$ is an amalgam satisfying Hypothesis 5.2.1. If $Z_{\alpha'} \not\leq Q_\alpha$, then one of the following holds:*

(i) *$\mathcal{A}$ is a weak BN-pair of rank 2;*

(ii) *$\mathcal{A}$ is a symplectic amalgam; or*

(iii) *$p = 2$, $|S| = 2^9$, $\overline{L_\beta} \cong \mathrm{Alt}(5)$, $Q_\beta \cong 2_-^{1+6}$, $V_\beta = O^2(L_\beta)$, V_β/Z_β is a natural $\Omega_4^-(2)$-module for $\overline{L_\beta}$, $\overline{L_\alpha} \cong \mathrm{SU}_3(2)'$, Q_α is a special 2-group of shape 2^{2+6} and Q_α/Z_α is a natural $\mathrm{SU}_3(2)$-module.*

Consequently, if $\mathcal{A}$ is obtained from a fusion system satisfying Hypothesis 5.1.12, then $\mathcal{F}$ is not a counterexample to the Main Theorem.

5.4 $Z_{\alpha'} \leq Q_\alpha$

We now begin the second half of our analysis, where $Z_{\alpha'} \leq Q_\alpha$ so that $[Z_\alpha, Z_{\alpha'}] = \{1\}$.

Lemma 5.4.1. *The following hold:*

(i) *$Z_\beta = \Omega(Z(S)) = \Omega(Z(L_\beta))$ and b is odd; and*

(ii) *$Z(L_\alpha) = \{1\}$.*

Proof. Since $Z_{\alpha'} \leq Q_\alpha$ we have that $\{1\} = [Z_\alpha, Z_{\alpha'}]$. Then, for $T \in \mathrm{Syl}_p(G_{\alpha',\alpha'-1})$, as $Z_\alpha \not\leq Q_{\alpha'}$, $Q_{\alpha'} < C_T(Z_{\alpha'})$ and by Lemma 5.2.10 (ii), we get that $Z_{\alpha'} = \Omega(Z(T)) = \Omega(Z(L_{\alpha'}))$. By Lemma 5.2.7 (iii), $Z_\alpha \not\leq \Omega(Z(L_\alpha))$ and so α and α' are not conjugate. Thus, α' is conjugate to β, b is odd and $Z_\beta = \Omega(Z(S)) =$

$\Omega(Z(L_\beta)))$. Since L_β acts transitively on $\Delta(\beta)$, by Lemma 5.2.7 (iv), we conclude that $Z(L_\alpha) = \{1\}$. □

Lemma 5.4.2. *Suppose that $b > 1$. Then V_β is abelian, $\{1\} \neq [V_\beta, V_{\alpha'}] \leq V_{\alpha'} \cap V_\beta$ and V_β acts quadratically on $V_{\alpha'}$.*

Proof. Since $Z_\alpha \leq V_\beta$ and $Z_\alpha \not\leq Q_{\alpha'}$ it follows that that $V_\beta \not\leq C_{L_{\alpha'}}(V_{\alpha'})$. By minimality of b, $V_\beta \leq Q_{\alpha'-1} \leq L_{\alpha'}$ and so $\{1\} \neq [V_\beta, V_{\alpha'}] \leq V_{\alpha'}$. Again, by minimality of b, $V_{\alpha'} \leq Q_{\alpha+2} \leq L_\beta$ and so $[V_\beta, V_{\alpha'}] \leq V_{\alpha'} \cap V_\beta$. Since V_β is abelian, $[V_{\alpha'}, V_\beta, V_\beta] = \{1\}$, completing the proof. □

Lemma 5.4.3. *Suppose that $b > 1$ and let U/V to be any non-central chief factor for $L_{\alpha'}$ inside of $V_{\alpha'}$. If p is an odd prime then for $\widetilde{L_{\alpha'}} := L_{\alpha'}/C_{L_{\alpha'}}(U/V)$, we have one of the following:*

(i) $p = 3$, $\widetilde{L_{\alpha'}} \cong 2 \cdot \mathrm{Alt}(5)$ *and* $T = Z_\alpha Q_{\alpha'} \in \mathrm{Syl}_p(L_{\alpha'})$*;*

(ii) $p = 3$, $\widetilde{L_{\alpha'}} \cong 2^{1+4}_- \cdot \mathrm{Alt}(5)$ *and* $T = Z_\alpha Q_{\alpha'} \in \mathrm{Syl}_p(L_{\alpha'})$*;*

(iii) $p \geqslant 3$ *is arbitrary,* $\widetilde{L_{\alpha'}} \cong \mathrm{SL}_2(p)$ *and* $T = Z_\alpha Q_{\alpha'} \in \mathrm{Syl}_p(L_{\alpha'})$*;*

(iv) $p \geqslant 3$ *is arbitrary and* $\overline{L_{\alpha'}} \cong \mathrm{SL}_2(p^{a+1})$ *or* $\mathrm{(P)SU}_3(p^a)$ *for* $a \geqslant 1$.

Proof. Suppose that p is an odd prime. Since $[V_{\alpha'}, V_\beta, V_\beta] = \{1\}$ and $Z_\alpha \not\leq Q_{\alpha'}$ we deduce that $[U/V, Z_\alpha, Z_\alpha] = \{1\} \neq [U/V, Z_\alpha]$, so $\langle (\widetilde{Z_\alpha})^{L_{\alpha'}} \rangle$ is as determined in Lemma 2.3.5. In particular, if $m_p(T/Q_{\alpha'}) \geqslant 2$, then $\langle (\widetilde{Z_\alpha})^{L_{\alpha'}} \rangle = \widetilde{L_{\alpha'}} \cong \mathrm{SL}_2(p^{a+1})$ or $\mathrm{(P)SU}_3(p^a)$ for $a \geqslant 1$. Additionally, in this case, by Proposition 3.2.7, we have that $O_{p'}(\overline{L_{\alpha'}}) \leq Z(\overline{L_{\alpha'}})$ and so $\widetilde{L_{\alpha'}} = \overline{L_{\alpha'}}$. If $m_p(T/Q_{\alpha'}) = 1$ and $\langle (\widetilde{Z_\alpha})^{L_{\alpha'}} \rangle$ is not p-solvable then $\widetilde{L_{\alpha'}}$ is not p-solvable and by Lemma 2.3.5, $\langle (\widetilde{Z_\alpha})^{L_{\alpha'}} \rangle = \widetilde{L_{\alpha'}} \cong 2 \cdot \mathrm{Alt}(5)$ or $2^{1+4}_-.\mathrm{Alt}(5)$ if $p = 3$; or $\mathrm{SL}_2(p)$ if $p \geqslant 5$.

Finally, suppose that $m_p(T/Q_{\alpha'}) = 1$ and $\langle (\widetilde{Z}_\alpha)^{L_{\alpha'}} \rangle$ is p-solvable. Then $p = 3$, $\widetilde{S}$ is cyclic and $\widetilde{N} := \langle (\widetilde{Z}_\alpha)^{L_{\alpha'}} \rangle \cong \mathrm{SL}(2,3)$. Then $\widetilde{S}$ normalizes $\widetilde{N}$ and centralizes a Sylow 3-subgroup of $\widetilde{N}$, from which it follows that $\widetilde{S}$ centralizes $\widetilde{N}$. Thus, $\overline{S} \cong \widetilde{S} = (\widetilde{S} \cap \widetilde{N}) \times C_{\widetilde{S}}(\widetilde{N})$. Since $\overline{S}$ is cyclic, $C_{\widetilde{S}}(\widetilde{N}) = \{1\}$, $|\overline{S}| = 3$ and $\widetilde{L}_{\alpha'} = \langle (\widetilde{Z}_\alpha)^{L_{\alpha'}} \rangle \cong \mathrm{SL}_2(3)$. $\square$

Lemma 5.4.4. *Suppose that $b > 1$, $C_{V_\beta}(V_{\alpha'}) = V_\beta \cap Q_{\alpha'}$ and $V_{\alpha'} \leq Q_\beta$. Then both Z_α and $V_\beta/C_{V_\beta}(O^p(L_\beta))$ are natural $\mathrm{SL}_2(p)$-modules for $L_\alpha/R_\alpha \cong \mathrm{SL}_2(p)$ and $L_\beta/R_\beta \cong \mathrm{SL}_2(p)$ respectively. Moreover, $[Q_\beta, V_\beta] = Z_\beta = [V_{\alpha'}, V_\beta] \leq V_{\alpha'} \cap V_\beta$ and $Q_\beta \in \mathrm{Syl}_p(R_\beta)$.*

Proof. Suppose that $C_{V_\beta}(V_{\alpha'}) = V_\beta \cap Q_{\alpha'}$ and $V_{\alpha'} \leq Q_\beta$. Note, that if $V_{\alpha'} \leq Q_\alpha$, then $[Z_\alpha, V_{\alpha'}] = \{1\}$ and $Z_\alpha \leq Q_{\alpha'}$, a contradiction. Additionally, $[Z_\alpha, V_{\alpha'}, V_{\alpha'}] \leq [V_\beta, V_{\alpha'}, V_{\alpha'}] = \{1\}$ and it follows that both Z_α and $V_{\alpha'}$ admit quadratic action. Hence, by Lemma 2.3.5, if $m_p(S/Q_\beta) > 1 < m_p(S/Q_\alpha)$ then both $\overline{L_\alpha}$ and $\overline{L_\beta}$ are groups of Lie type and G has a weak BN-pair. Then G is determined by [DS85], and no configurations occur.

Notice that $Z_\alpha \cap Q_{\alpha'} = C_{Z_\alpha}(V_{\alpha'})$ and that $V_{\alpha'} \cap Q_\alpha \leq C_{V_{\alpha'}}(Z_\alpha)$. If $m_p(S/Q_\beta) = 1$, then it follows that an index p subgroup of Z_α is centralized by $V_{\alpha'}$. Then by Lemma 2.3.10 and as $Z(L_\alpha) = \{1\}$, Z_α is a natural $\mathrm{SL}_2(p)$-module for $L_\alpha/R_\alpha \cong \mathrm{SL}_2(p)$ and $|S/Q_\alpha| = p$. But then an index p subgroup of $V_{\alpha'}$ is centralized by Z_α and $V_{\alpha'}/C_{V_{\alpha'}}(O^p(L_{\alpha'}))$ is natural $\mathrm{SL}_2(p)$-module for $L_{\alpha'}/R_{\alpha'} \cong \mathrm{SL}_2(p)$. We reach a similar conclusion assuming that $m_p(S/Q_\alpha) = 1$. Then $[Z_\alpha, Q_\beta] = Z_\beta$ so that $[V_\beta, Q_\beta] = Z_\beta$ is of order p, and by Lemma 5.2.16, $Q_\beta \in \mathrm{Syl}_p(R_\beta)$. Since $\{1\} \neq [V_{\alpha'}, V_\beta] \leq [Q_\beta, V_\beta]$, we conclude that $Z_\beta = [V_{\alpha'}, V_\beta] \leq V_{\alpha'} \cap V_\beta$. $\square$

Lemma 5.4.5. *Suppose that $b > 1$, $C_{V_\beta}(V_{\alpha'}) = V_\beta \cap Q_{\alpha'}$ and $V_{\alpha'} \not\leq Q_\beta$. Then $Q_\beta \in \mathrm{Syl}_p(R_\beta)$, $L_\alpha/R_\alpha \cong \mathrm{SL}_2(p) \cong L_\beta/R_\beta$ and both Z_α and $V_\beta/C_{V_\beta}(O^p(L_\beta))$ are* *natural* $\mathrm{SL}_2(p)$*-modules.*

Proof. Assume that $C_{V_\beta}(V_{\alpha'}) = V_\beta \cap Q_{\alpha'}$ and $V_{\alpha'} \not\leq Q_\beta$. Suppose first that $|V_\beta/C_{V_\beta}(V_{\alpha'})| = |V_\beta Q_{\alpha'}/Q_{\alpha'}| = p$. Then by Lemma 2.3.10, $V_\beta/C_{V_\beta}(O^p(L_\beta))$ is a natural $\mathrm{SL}_2(p)$-module for $L_\beta/R_\beta \cong \mathrm{SL}_2(p)$. Since $Q_\alpha \cap Q_\beta \ntrianglelefteq L_\beta$ by Proposition 5.2.25, $Q_\beta \cap O^p(L_\beta) \not\leq Q_\alpha$ and $Z_\alpha \cap C_{V_\beta}(O^p(L_\beta))$ is centralized by $Q_\beta \cap O^p(L_\beta)$. Now, $V_\beta \neq Z_\alpha C_{V_\beta}(O^p(L_\beta))$, for otherwise Q_α centralizes $V_\beta C_{V_\beta}(O^p(L_\beta))$ and $O^p(L_\beta)$ centralizes V_β, and so $Z_\alpha \cap C_{V_\beta}(O^p(L_\beta))$ has index p in Z_α. Thus, Z_α is an FF-module and by Lemma 2.3.10, using that $Z(L_\alpha) = \{1\}$, Z_α is a natural $\mathrm{SL}_2(p)$-module for $L_\alpha/R_\alpha \cong \mathrm{SL}_2(p)$. Then, $[Q_\beta, V_\beta] = [Q_\beta, Z_\alpha]^{G_\beta} = Z_\beta \leq C_{V_\beta}(O^p(L_\beta))$ and by Lemma 5.2.16, $Q_\beta \in \mathrm{Syl}_p(R_\beta)$ and the result holds.

Thus, $|V_\beta Q_{\alpha'}/Q_{\alpha'}| \geqslant p^2$ and as V_β is elementary abelian, $m_p(S/Q_\beta) \geqslant 2$. If G has weak BN-pair of rank 2, then comparing with [DS85], we have that $m_p(S/Q_\beta) = \{1\}$ whenever $b > 2$. Hence, we may assume that $m_p(S/Q_\alpha) = \{1\}$ by Proposition 3.2.7 and Lemma 2.3.5. Since V_β is a quadratic module for $\overline{L_\beta}$, by Lemma 2.3.5, $\overline{L_\beta}$ is a rank 1 group of Lie type, but not a Ree group. In particular, $\overline{L_\beta}$ is p-minimal and applying the qrc lemma, we either deduce that Z_α is (dual to) an FF-module for $L_\alpha/R_\alpha \cong \mathrm{SL}_2(p)$ so that Z_α is a natural $\mathrm{SL}_2(p)$-module for $L_\alpha/R_\alpha \cong \mathrm{SL}_2(p)$; or V_β contains more than one non-central chief factor for L_β.

Suppose first that $|V_{\alpha'}Q_\beta/Q_\beta| \geqslant p^2$. If $\overline{L_\beta} \cong (\mathrm{P})\mathrm{SL}_2(p^n)$ or $\mathrm{Sz}(2^n)$, then by Lemma 2.2.1 (iv),(v) and Lemma 2.2.3 (vi), at most three conjugates of $V_{\alpha'}Q_\beta/Q_\beta$ generate $\overline{L_\beta}$ and as $V_\beta Q_{\alpha'}/Q_{\alpha'}$ is of exponent p, we infer that $|V_\beta/C_{V_\beta}(O^p(L_\beta))| \leqslant p^{3n}$. Since the minimal degree of a $\mathrm{GF}(2)$ representation for $\mathrm{Sz}(2^n)$ is $4n$, we deduce that $\overline{L_\beta} \cong (\mathrm{P})\mathrm{SL}_2(p^n)$, and in this case, two conjugates suffice to generate and $|V_\beta/C_{V_\beta}(O^p(L_\beta))| \leqslant p^{2n}$. Then by Lemma 2.3.12, $|V_\beta/C_{V_\beta}(O^p(L_\beta))| = p^{2n}$,

$|V_\beta Q_{\alpha'}/Q_{\alpha'}| = p^n$ and $V_\beta Q_{\alpha'} \in \mathrm{Syl}_p(L_{\alpha'})$. But V_β acts quadratically on $V_{\alpha'}$ and by Lemma 2.3.11, $V_\beta/C_{V_\beta}(O^p(L_\beta))$ is a natural $\mathrm{SL}_2(p^n)$-module. Since $n > 2$ and $Z_\alpha C_{V_\beta}(O^p(L_\beta))/C_{V_\beta}(O^p(L_\beta))$ is a $G_{\alpha,\beta}$-invariant subgroup of order p, we have a contradiction by Lemma 2.2.6 (vi).

If $\overline{L_\beta} \cong \mathrm{SU}_3(p^n)$ then, by Lemma 2.2.2 (vi), three conjugates of $V_{\alpha'}Q_\beta/Q_\beta$ generate $\overline{L_\beta}$ and as V_β is elementary abelian, $|V_\beta Q_{\alpha'}/Q_{\alpha'}| \leqslant p^{2n}$. But the minimal degree of a $\mathrm{GF}(p)$ representation for $\overline{L_\beta}$ is $6n$ and so $V_\beta/C_{V_\beta}(O^p(L_\beta))$ is a natural $\mathrm{SU}_3(p^n)$-module of order p^{6n} and $|V_\beta Q_{\alpha'}/Q_{\alpha'}| = p^{2n}$, impossible since V_β acts quadratically on $V_{\alpha'}$.

Finally, we assume that $|V_{\alpha'}Q_\beta/Q_\beta| = p$. If $C_{V_{\alpha'}}(V_\beta) = V_{\alpha'} \cap Q_\beta$ then by Lemma 2.3.10, $L_{\alpha'}/R_{\alpha'} \cong \mathrm{SL}_2(p)$, impossible since $|V_\beta Q_{\alpha'}/Q_{\alpha'}| \geqslant p^2$. Since $V_\beta \cap Q_{\alpha'}$ centralizes $V_{\alpha'}$, we may as well assume that $V_{\alpha'} \cap Q_\beta \not\leq Q_\alpha$ and $V_{\alpha'} \cap Q_\beta$ acts quadratically on Z_α. Thus, $m_p(S/Q_\alpha) = 1$, for otherwise, as Z_α is a quadratic module, by Lemma 2.3.5 and Proposition 3.2.7, $\overline{L_\alpha}$ would be isomorphic to a rank 1 group of Lie type and G would have a weak BN-pair of rank 2. Then G would be determined by [DS85], wherein there are no examples.

Thus, $V_{\alpha'} \cap Q_\beta \cap Q_\alpha$ is an index p^2 subgroup of $V_{\alpha'}$ which is centralized by Z_α. If $\overline{L_{\alpha'}} \cong (\mathrm{P})\mathrm{SL}_2(p^n)$ or $\mathrm{Sz}(2^n)$, then by Lemma 2.2.1 (iv), (v) and Lemma 2.2.3 (vi), at most three conjugates of $Z_\alpha Q_{\alpha'}/Q_{\alpha'}$ generate $\overline{L_{\alpha'}}$ and $|V_{\alpha'}/C_{V_{\alpha'}}(O^p(L_{\alpha'}))| \leqslant p^6$. Considering minimal degrees of representations, we infer that $\overline{L_\beta} \cong (\mathrm{P})\mathrm{SL}_2(p^n)$ where $n \in \{2, 3\}$ and, by conjugacy, $V_\beta/C_{V_\beta}(O^p(L_\beta))$ contains a unique non-central chief factor for L_β. But now, $|V_\beta Q_{\alpha'}/Q_{\alpha'}| \geqslant p^2$ and acts quadratically on $V_{\alpha'}$ and applying Lemma 2.3.12, $V_\beta/C_{V_\beta}(O^p(L_\beta))$ is a natural $\mathrm{SL}_2(p^2)$-module for $\overline{L_\beta}$. Applying the qrc lemma since $\overline{L_\beta}$ is p-minimal, outcome (ii) or (iii) holds so that Z_α is (dual to) an FF-module and by Lemma 2.3.10, Z_α is natural $\mathrm{SL}_2(p)$-module for $L_\alpha/R_\alpha \cong \mathrm{SL}_2(p)$. But then $Z_\alpha C_{V_\beta}(O^p(L_\beta))/C_{V_\beta}(O^p(L_\beta))$ is a $G_{\alpha,\beta}$-invariant subgroup of order p in $V_\beta/C_{V_\beta}(O^p(L_\beta))$, impossible by Lemma 2.2.6 (vi).

Assume now that $\overline{L_\beta} \cong \mathrm{SU}_3(p^n)$ so that by Lemma 2.2.2 (vi), (vii), at most four conjugates generate $Z_\alpha Q_{\alpha'}/Q_{\alpha'}$ generate $\overline{L_{\alpha'}}$ and $|V_{\alpha'}/C_{V_{\alpha'}}(O^p(L_{\alpha'}))| \leqslant p^8$. Using that $m_p(S/Q_\beta) \geqslant 2$ and the minimal degree of a $\mathrm{GF}(p)$ representation for $\overline{L_{\alpha'}}$ is $6n$, we infer that $\overline{L_\beta} \cong \mathrm{SU}_3(p)$ for p an odd prime. But in this case, again applying Lemma 2.2.2 (vi), three conjugates suffice to generate and so $V_{\alpha'}/C_{V_{\alpha'}}(O^p(L_{\alpha'}))$ is a natural $\mathrm{SU}_3(p)$-module of order p^6. Now, $|V_\beta Q_{\alpha'}/Q_{\alpha'}| = p^2$ and as V_β acts quadratically on $V_{\alpha'}$, and we have a final contradiction. $\square$

We now prove the "converse" to the above statements.

Lemma 5.4.6. *If $b > 1$ and both $V_\beta/C_{V_\beta}(O^p(L_\beta))$ and Z_α are natural $\mathrm{SL}_2(p)$-modules, then $C_{V_\beta}(V_{\alpha'}) = V_\beta \cap Q_{\alpha'}$.*

Proof. Since $|Z_\alpha| = p^2$ and $V_\beta = [V_\beta, O^p(L_\beta)]Z_\alpha$, as in the proof of Lemma 5.2.31, we may assume that $|V_\beta| = p^3$ or $|V_\beta| = p^4$. Suppose first that $|V_\beta| = p^3$. Then $Z_{\alpha+2} = V_\beta \cap Q_{\alpha'}$ centralizes $V_{\alpha'}$ and the result holds. Hence, we may assume that $|V_\beta| = p^4$.

If $V_{\alpha'} \not\leq Q_\beta$, then $[V_{\alpha'}, V_\beta] \not\leq Z_{\alpha+2}$ for otherwise $Z_{\alpha+2}Z_{\alpha+2}^g$ is of order p^3 and normalized by $L_\beta = \langle V_{\alpha'}, V_{\alpha'}^g, R_\beta\rangle$ for some appropriately chosen $g \in L_\beta$, contrary to the definition of V_β. Thus, $Z_{\alpha+2}[V_{\alpha'}, V_\beta] = V_\beta \cap Q_{\alpha'}$ is of order p^3 and centralizes $V_{\alpha'}$, as desired.

Assume now that $V_{\alpha'} \leq Q_\beta$ so that $[V_{\alpha'}, V_\beta] = Z_\beta$. Then, if $[V_\beta \cap Q_{\alpha'}, V_{\alpha'}] \neq \{1\}$, $[V_\beta \cap Q_{\alpha'}, V_{\alpha'}] \leq Z_\beta \cap Z_{\alpha'}$ so that $Z_\beta = Z_{\alpha'}$. But then, $V_\beta \not\leq Q_{\alpha'}$ and V_β centralizes $V_{\alpha'}/Z_{\alpha'}$, a contradiction since $O^p(L_{\alpha'})$ acts non-trivially on $V_{\alpha'}$. Hence,

$[V_\beta \cap Q_{\alpha'}, V_{\alpha'}] = \{1\}$ and $C_{V_\beta}(V_{\alpha'}) = V_\beta \cap Q_{\alpha'}$, as desired. $\square$

Lemma 5.4.7. *If $b > 1$ and $V_\beta/C_{V_\beta}(O^p(L_\beta))$ is a natural $\mathrm{SL}_2(p)$-module for $L_\beta/R_\beta \cong \mathrm{SL}_2(p)$, then $C_{V_\beta}(V_{\alpha'}) = V_\beta \cap Q_{\alpha'}$.*

Proof. Suppose that $V_\beta/C_{V_\beta}(O^p(L_\beta))$ is a natural $\mathrm{SL}_2(p)$-module. Since $Q_\alpha \cap Q_\beta \ntrianglelefteq L_\alpha$, $Q_\beta \cap O^p(L_\beta)$ is not contained in Q_α. Since $Z_\alpha \not\leq C_{V_\beta}(O^p(L_\beta))$ either $Q_\beta \cap O^p(L_\beta)$ centralizes an index p subgroup of Z_α so that $L_\alpha/R_\alpha \cong \mathrm{SL}_2(p)$ with Z_α the natural module; or $V_\beta = Z_\alpha C_{V_\beta}(O^p(L_\beta))$. In the former case, the result follows from Lemma 5.4.6 while in the latter case, $[V_\beta, Q_\alpha] \leq C_{V_\beta}(O^p(L_\beta))$ so that V_β is centralized by $O^p(L_\beta)$, a contradiction. $\square$

5.4.1 $C_{V_\beta}(V_{\alpha'}) < V_\beta \cap Q_{\alpha'}$

The hypothesis for this subsection is $b > 1$ and $C_{V_\beta}(V_{\alpha'}) < V_\beta \cap Q_{\alpha'}$. Notice that as $C_T(V_{\alpha'}) \leq Q_{\alpha'}$, this condition is equivalent to $[V_\beta \cap Q_{\alpha'}, V_{\alpha'}] \neq \{1\}$. Thus, for some $\alpha' + 1 \in \Delta(\alpha')$, we have that $[V_\beta \cap Q_{\alpha'}, Z_{\alpha'+1}] \neq \{1\}$. We fix a particular $\alpha' + 1 \in \Delta(\alpha')$ for the remainder of this subsection. Since b is odd, $\alpha' + 1$ is conjugate to α and $V_\beta \cap Q_{\alpha'} \not\leq Q_{\alpha'+1}$. Furthermore, $[Z_{\alpha'+1}, V_\beta \cap Q_{\alpha'}, V_\beta \cap Q_{\alpha'}] \leq [V_{\alpha'}, V_\beta, V_\beta] = \{1\}$ so that both $Z_{\alpha'+1}$ and $V_{\alpha'}$ admit quadratic action. Throughout, we set $H := [V_\beta \cap Q_{\alpha'}, V_{\alpha'}]$.

Lemma 5.4.8. *Suppose that $C_{V_\beta}(V_{\alpha'}) < V_\beta \cap Q_{\alpha'}$. Then $m_p(S/Q_\beta) = 1$; or G is locally isomorphic to H where $F^*(H) \cong {}^2\mathrm{F}_4(2^{2a+1})$ and $a \geqslant 1$.*

Proof. Suppose that $m_p(S/Q_\beta) > 1$. Since $V_{\alpha'}$ admits quadratic action by V_β, we have that $\overline{L_\beta} \cong \overline{L_{\alpha'}} \cong \mathrm{Sz}(2^{2a+1}), \mathrm{SL}_2(p^{a+1})$ or $(\mathrm{P})\mathrm{SU}_3(p^r)$ for $a \geqslant 1$ and $p^r > 2$. If $m_p(S/Q_\alpha) \neq 1$, since $Z_{\alpha'+1}$ admits quadratic action by $V_\beta \cap Q_{\alpha'}$ and $\overline{L_\alpha} \cong \overline{L_{\alpha'+1}}$, it follows that both $\overline{L_\alpha}$ and $\overline{L_\beta}$ are isomorphic to groups of Lie type of rank 1. Thus, G has a weak BN-pair and, using the results in [DS85], no configurations exists for p odd and G is locally isomorphic to some automorphism group of ${}^2\mathrm{F}_4(q)$ for $q > 2$, whenever $p = 2$. Thus, we assume that $m_p(S/Q_\alpha) = 1$. Now, L_β is p-minimal and the hypotheses of the qrc lemma are satisfied. If case (v) of the qrc lemma occurs, then $Q_\alpha \cap Q_\beta \trianglelefteq L_\beta$. But then, upon conjugating, $V_\beta \cap Q_{\alpha'} \leq Q_{\alpha'-1} \cap Q_{\alpha'} = Q_\lambda \cap Q_{\alpha'}$ for all $\lambda \in \Delta(\alpha')$ and so $H = \{1\}$. Since $b > 1$, case (i) of the qrc lemma is not satisfied.

Now, $V_{\alpha'}$ acts quadratically on V_β so that $q_{\alpha'} := |V_{\alpha'}Q_\beta/Q_\beta| \leqslant |\Omega(Z(S/Q_\beta))|$ by [DS85, (5.10)], and so, $V_{\alpha'} \cap Q_\beta \cap Q_\alpha$ has index at most $pq_{\alpha'}$ in $V_{\alpha'}$ and is centralized by Z_α. Then $|V_{\alpha'}/C_{V_{\alpha'}}(O^p(L_{\alpha'}))| \leqslant (pq_{\alpha'})^d$ where d is the number of conjugates of $Z_\alpha Q_{\alpha'}/Q_{\alpha'}$ required to generate $\overline{L_{\alpha'}}$.

If $\overline{L_{\alpha'}} \cong \mathrm{Sz}(2^n)$ then by Lemma 2.2.3 (iii), (vi), $d = 3$, $q_{\alpha'} = 2^n > 2$ and $|V_{\alpha'}/C_{V_{\alpha'}}(O^p(L_{\alpha'}))| \leqslant 2^{3+3n}$. Since the minimal degree of a non-trivial GF(2)-representation for $\mathrm{Sz}(2^n)$ is $4n$, as $n > 1$ is odd by Lemma 2.2.3 (i), we have that $n = 3$, $|V_{\alpha'}Q_\beta/Q_\beta| = 8$ and $V_{\alpha'}/C_{V_{\alpha'}}(O^p(L_{\alpha'}))$ is a natural Sz(8)-module. In particular, $V_{\alpha'}$ contains a unique non-central chief factor for $L_{\alpha'}$ so that outcomes (ii) or (iii) of the qrc lemma holds and Z_α is (dual to) an FF-module for L_α/R_α. By Lemma 2.3.10, Z_α is a natural $\mathrm{SL}_2(p)$-module for $L_\alpha/R_\alpha \cong \mathrm{SL}_2(p)$. But then, $Z_\alpha C_{V_\beta}(O^p(L_\beta))/C_{V_\beta}(O^p(L_\beta))$ is of order 2 and normalized by $G_{\alpha,\beta}$, a contradiction by Lemma 2.2.14 (iv).

If $\overline{L_{\alpha'}} \cong (\mathrm{P})\mathrm{SU}_3(p^n)$ then by Lemma 2.2.2 (i),(ii), (vi) and (vii), $d = 4$, $q_{\alpha'} = p^n > 2$ and $|V_{\alpha'}/C_{V_{\alpha'}}(O^p(L_{\alpha'}))| \leqslant p^{4+4n}$. Since the minimal degree of a non-trivial GF(p)-representation for $(\mathrm{P})\mathrm{SU}_3(p^n)$ is $6n$, we infer that $n \leqslant 2$ and $V_{\alpha'}$ contains a unique non-central chief factor for $L_{\alpha'}$. Then, outcomes (ii) or

(iii) of the qrc lemma holds and Z_α is (dual to) an FF-module for L_α/R_α. By Lemma 2.3.10, Z_α is a natural $\mathrm{SL}_2(p)$-module for $L_\alpha/R_\alpha \cong \mathrm{SL}_2(p)$. If $p^n \notin \{4, 9\}$ we have that $d = 3$ by Lemma 2.2.2 (vi) so that $|V_{\alpha'}/C_{V_{\alpha'}}(O^p(L_{\alpha'}))| \leqslant p^{3+3n}$. In this scenario, $n = 1$ and $V_{\alpha'}/C_{V_{\alpha'}}(O^p(L_{\alpha'}))$ is a natural $\mathrm{SU}_3(p)$-module for $\overline{L_{\alpha'}} \cong \mathrm{SU}_3(p)$. But then, $Z_{\alpha'-1}C_{V_{\alpha'}}(O^p(L_{\alpha'}))/C_{V_{\alpha'}}(O^p(L_{\alpha'}))$ is a $G_{\alpha',\alpha'-1}$-invariant subgroup of order p, and we have a contradiction by Lemma 2.2.13 (iii). If $p^n \in \{4, 9\}$ then $V_{\alpha'}/C_{V_{\alpha'}}(O^p(L_{\alpha'}))$ is a natural $\mathrm{SU}_3(p^2)$-module of order p^{12}. Again, $Z_{\alpha'-1}C_{V_{\alpha'}}(O^p(L_{\alpha'}))/C_{V_{\alpha'}}(O^p(L_{\alpha'}))$ is a $G_{\alpha',\alpha'-1}$-invariant subgroup of order p, and we have a contradiction by Lemma 2.2.13 (iii).

Thus, $\overline{L_{\alpha'}} \cong \mathrm{SL}_2(q)$ so that $\overline{L_{\alpha'}}$ is generated by at most 3 conjugates of $Z_\alpha Q_{\alpha'}/Q_{\alpha'}$ from which it follows that $|V_{\alpha'}/C_{V_{\alpha'}}(O^p(L_{\alpha'}))| \leqslant p^3 q_{\alpha'}^3$. Note that if $q_{\alpha'} = q$ then by Lemma 2.3.11, $V_\beta/C_{V_\beta}(O^p(L_\beta))$ is a direct sum of natural $\mathrm{SL}_2(q)$-modules, and as an index pq subgroup of $V_{\alpha'}$ is centralized by Z_α with $p < q$, $V_{\alpha'}/C_{V_{\alpha'}}(O^p(L_{\alpha'}))$ is a natural $\mathrm{SL}_2(q)$-module for $\overline{L_{\alpha'}}$. As above, outcome (ii) or (iii) in the statement of the qrc lemma holds, Z_α is a natural $\mathrm{SL}_2(p)$-module for $L_\alpha/R_\alpha \cong \mathrm{SL}_2(p)$ and we have a contradiction as $Z_\alpha C_{V_\beta}(O^p(L_\beta))/C_{V_\beta}(O^p(L_\beta))$ is of order $p < q$ and normalized by $G_{\alpha,\beta}$. Thus, $|V_{\alpha'}/C_{V_{\alpha'}}(O^p(L_{\alpha'}))| \leqslant q_{\alpha'}^3$ and applying Lemma 2.3.12, we have that $V_{\alpha'}$ contains a unique non-central chief factor for $L_{\alpha'}$ and outcome (ii) or (iii) of the qrc lemma holds. Again, Z_α is a natural $\mathrm{SL}_2(p)$-module for L_α/R_α and $Z_\alpha C_{V_\beta}(O^p(L_\beta))/C_{V_\beta}(O^p(L_\beta))$ is of order $p < q$ and normalized by $G_{\alpha,\beta}$. Since $q > p$ and V_β acts quadratically on $V_{\alpha'}$, again by Lemma 2.3.12, we see that $V_{\alpha'}/C_{V_{\alpha'}}(O^p(L_{\alpha'}))$ is a natural $\Omega_4^-(2)$-module for $\overline{L_{\alpha'}} \cong \mathrm{PSL}_2(4)$. Notice that as Z_α is a natural $\mathrm{SL}_2(p)$-module, $[V_\beta, Q_\beta] = [Z_\alpha, Q_\beta]^{G_\beta} = Z_\beta$.

Suppose that $b > 3$. Then $V_\beta^{(3)}$ centralizes $Z_{\alpha'} = [V_\beta \cap Q_{\alpha'}, V_{\alpha'}] \leq V_\beta$ and if $Z_{\alpha'} \neq Z_{\alpha'-2}$, then $V_\beta^{(3)}$ centralizes $Z_{\alpha'-1} = Z_{\alpha'} \times Z_{\alpha'-2}$. But then, $V_\beta^{(3)} \leq Q_{\alpha'-2}$, for otherwise $L_{\alpha'-2} = \langle V_\beta^{(3)}, Q_{\alpha'-1}, Q_{\alpha'-2}\rangle$ normalizes $Z_{\alpha'-1}$, a contradiction. It follows that $V_\beta^{(3)} \le Q_{\alpha'-1}$ and $V_\beta(V_\beta^{(3)} \cap Q_{\alpha'})$ has index at most p in $V_\beta^{(3)}$. Since $\overline{L_\beta} \cong \mathrm{PSL}_2(4)$ and $[V_\beta^{(3)} \cap Q_{\alpha'}, V_{\alpha'}] = Z_{\alpha'} \le V_\beta$, we deduce that $O^2(L_\beta)$ centralizes $V_\beta^{(3)}/V_\beta$, a contradiction. Thus, we conclude that $Z_{\alpha'} = Z_{\alpha'-2}$. Using Lemma 5.4.4 and Lemma 5.4.5, we may assume that every critical pair satisfies the same hypothesis as (α, α'). Suppose that $V_\beta^{(3)} \not\le Q_{\alpha'-2}$ so that there is a critical pair $(\beta-3, \alpha'-2)$. Arguing as above, we have that $Z_{\alpha'-2} = Z_{\alpha'-4}$. Continuing along the critical path, this would eventually imply that $Z_{\alpha'} = \cdots = Z_\beta$. But then $[V_\beta \cap Q_{\alpha'}, V_{\alpha'}] = Z_\beta$ and since $V_\beta \cap Q_{\alpha'}$ has index 2 in V_β, this yields a contradiction. We may as well assume that (α, α') is a critical pair with $V_\beta^{(3)} \le Q_{\alpha'-2}$ and $Z_{\alpha'} = Z_{\alpha'-2}$. If $b > 5$, then $V_\beta^{(3)}$ is elementary abelian so that $[V_{\alpha'}, V_\beta^{(3)} \cap Q_{\alpha'-1}, V_\beta^{(3)} \cap Q_{\alpha'-1}] \le [V_\beta^{(3)}, V_\beta^{(3)}] = \{1\}$. It follows that $V_\beta^{(3)} \cap Q_{\alpha'-1} = V_\beta(V_\beta^{(3)} \cap Q_{\alpha'})$ has index 2 in $V_\beta^{(3)}$ and as $Z_{\alpha'} \le V_\beta$ and $\overline{L_\beta} \cong \mathrm{PSL}_2(4)$, we have that $O^2(L_\beta)$ centralizes $V_\beta^{(3)}/V_\beta$, a contradiction. If $b = 5$, then using that $(\alpha'+1, \beta)$ is a critical pair, by the above, we conclude that $Z_\beta = Z_{\alpha+3} = Z_{\alpha'-2}$ so that $Z_\beta = Z_{\alpha'}$, and we obtain a contradiction as before.

Thus, we may assume that $b = 3$. By Lemma 2.2.10, we have that $V_\beta/Z_\beta = [V_\beta/Z_\beta, O^2(L_\beta)] \times C_{V_\beta/Z_\beta}(O^2(L_\beta))$. Set V^β to be the preimage in V_β of $[V_\beta/Z_\beta, O^2(L_\beta)]$ so that V^β contains a non-central chief factor for L_β. It follows that $Z_{\alpha'} = [V^\beta \cap Q_{\alpha'}, V_{\alpha'}] \le V^\beta$ so that $Z_{\alpha'-1} = Z_\beta \times Z_{\alpha'} \le V^\beta$. Since V_β is the normal closure of $Z_{\alpha'-1}$ in G_β, we deduce that $V^\beta = V_\beta$, V_β/Z_β is irreducible and $|V_\beta| = 2^5$. Then, $|[V_\beta, V_{\alpha'}]| = 8$ and $Z_{\alpha'-1} = [V_\beta, Q_{\alpha'-1}, Q_{\alpha'-1}] \le [V_\beta, V_{\alpha'}] = V_\beta \cap V_{\alpha'}$. In addition, it follows from Lemma 2.2.9 (v) that a Sylow 3-subgroup of $L_\beta \cap G_{\beta,\alpha'-1}$ acts irreducibly on $[V_\beta, Q_{\alpha'-1}]/Z_{\alpha'-1}$ so that $[V_\beta, V_{\alpha'}]^{L_\beta \cap G_{\beta,\alpha'-1}} = [V_\beta, Q_{\alpha'-1}]$. In particular, since $[V_\beta, V_{\alpha'}] \le [V_{\alpha'-1}^{(2)}, V_{\alpha'-1}^{(2)}]$ and $[V_{\alpha'-1}^{(2)}, V_{\alpha'-1}^{(2)}]$ is $G_{\beta,\alpha'-1}$-invariant, we must have that $[V_{\alpha'-1}^{(2)}, V_{\alpha'-1}^{(2)}] = [V_{\alpha'-1}^{(2)}, Q_{\alpha'-1}]$, and the same holds for α upon conjugating.

Suppose that $[V_{\alpha'-1}^{(2)}, V_{\alpha'-1}^{(2)}]/Z_{\alpha'-1}$ does not contain a non-central chief factor for $L_{\alpha'-1}$. Then $[V_{\alpha'-1}^{(2)}, V_{\alpha'-1}^{(2)}] = [V_\beta, Q_{\alpha'-1}] = [V_{\alpha'}, Q_{\alpha'-1}] \leq V_\beta \cap V_{\alpha'}$, a contradiction, since $[V_\beta, V_{\alpha'}]$ is of order 8. Thus, $[V_{\alpha'-1}^{(2)}, V_{\alpha'-1}^{(2)}]/Z_{\alpha'-1}$ contains a non-central chief factor and again, the same result holds upon conjugating to α. Notice that if $Z_{\alpha'} \leq [V_\alpha^{(2)}, V_\alpha^{(2)}]$, then $Z_{\alpha'-1} \leq [V_\alpha^{(2)}, V_\alpha^{(2)}]$. Since $\overline{L_\beta} \cong \mathrm{PSL}_2(4)$, $|\Delta(\beta) \setminus \{\beta\}| = 5$ and S/Q_β acts transitively on $\Delta(\beta) \setminus \{\alpha, \beta\}$. Then $V_\beta = Z_\alpha \langle Z_{\alpha'-1}^S \rangle \leq [V_\alpha^{(2)}, V_\alpha^{(2)}]$, a contradiction to the definition of $V_\alpha^{(2)}$.

Now, $|[V_{\alpha'}, Q_{\alpha'-1}]| = 2^4$ and $[V_{\alpha'}, Q_{\alpha'-1}] \leq [Q_{\alpha'-1}, Q_{\alpha'-1}] \leq Q_\beta$ from which it follows that $[V_{\alpha'}, Q_{\alpha'-1}] = V_{\alpha'} \cap Q_\beta$. Thus, $[V_\alpha^{(2)}, V_\alpha^{(2)}] \leq Q_\beta$ and $[[V_\alpha^{(2)}, V_\alpha^{(2)}] \cap Q_{\alpha'-1}, V_{\alpha'} \cap Q_\beta] \leq Z_{\alpha'-1} \cap [V_\alpha^{(2)}, V_\alpha^{(2)}] = Z_\beta$. Hence, $[V_\alpha^{(2)}, V_\alpha^{(2)}]/Z_\alpha$ is an FF-module for $\overline{L_\alpha}$. By coprime action, and writing $V := [V_\alpha^{(2)}, V_\alpha^{(2)}]$, we have that $V/Z_\alpha = [V/Z_\alpha, O^2(L_\alpha)] \times C_{V/Z_\alpha}(O^2(L_\alpha))$ and by Lemma 2.3.10, $[V/Z_\alpha, O^2(L_\alpha)]$ is a natural $\mathrm{SL}_2(2)$-module for $L_\alpha/C_{L_\alpha}(V/Z_\alpha) \cong \mathrm{SL}_2(2)$. Moreover, $(V_\beta \cap V)/Z_\alpha$ is of order 4 and since $V/Z_\alpha \neq C_{V/Z_\alpha}(O^2(L_\alpha))((V_\beta \cap V)/Z_\alpha)$, otherwise Q_β centralizes $V/C_{V/Z_\alpha}(O^2(L_\alpha))$, and $(V_\beta \cap V)/Z_\alpha \not\leq C_{V/Z_\alpha}(O^2(L_\alpha))$, we must have that $V_\beta/Z_\alpha \cap C_{V/Z_\alpha}(O^2(L_\alpha))$ is of order 2. Taking the preimage in V_β and quotienting by Z_β, it follows that there is a $G_{\alpha,\beta}$-invariant subgroup of $[V_\beta/Z_\beta, Q_\alpha]$ which contains Z_α/Z_β and is of order 4. Since the 3-element in $L_\beta \cap G_{\alpha,\beta}$ acts irreducibly on $[V_\beta, Q_\alpha]/Z_\alpha$ by Lemma 2.2.9 (v), we have a contradiction. □

From this point on, we assume that $m_p(S/Q_\beta) = 1$. In particular, if p is odd then by Lemma 2.3.5, $|S/Q_\beta| = p$. The following lemma, along with its proof, appeared earlier as Proposition 2.3.19 and Lemma 2.3.22. We recall it here as it will be applied liberally throughout this subsection.

Lemma 5.4.9. *For $\gamma \in \Gamma$, $G := \overline{L_\gamma}$ and $S \in \mathrm{Syl}_p(G)$, assume that V is a faithful* $\mathrm{GF}(p)G$*-module with $C_V(O^p(G)) = \{1\}$ and $V = \langle C_V(S)^G \rangle$. If there is a p-element $1 \neq x \in G$ such that $[V, x, x] = \{1\}$ and $|V/C_V(x)| = p^2$ then, setting $L := \langle x^G \rangle$, one of the following holds:*

(i) *p is odd, $G = L \cong \mathrm{(P)SU}_3(p)$ and V is the natural module;*

(ii) *p is arbitrary, $G \cong \mathrm{SL}_2(p^2)$ and V is the natural module;*

(iii) *$p = 2$, $G = L \cong \mathrm{PSL}_2(4)$ and V is a natural $\Omega_4^-(2)$-module;*

(iv) *$p = 3$, $G = L \cong 2 \cdot \mathrm{Alt}(5)$ or $2_-^{1+4}.\mathrm{Alt}(5)$ and V is the unique irreducible quadratic $2F$-module of dimension 4;*

(v) *p is arbitrary, $G = L \cong \mathrm{SL}_2(p)$ and V is the direct sum of two natural $\mathrm{SL}_2(p)$-modules;*

(vi) *$p = 2$, $L \cong \mathrm{SU}_3(2)'$, G is isomorphic to a subgroup of $\mathrm{SU}_3(2)$ which contains $\mathrm{SU}_3(2)'$ and V is a natural $\mathrm{SU}_3(2)$-module viewed as an irreducible $\mathrm{GF}(2)G$-module by restriction;*

(vii) *$p = 2$, $L \cong \mathrm{Dih}(10)$, $G \cong \mathrm{Dih}(10)$ or $\mathrm{Sz}(2)$ and V is a natural $\mathrm{Sz}(2)$-module viewed as an irreducible $\mathrm{GF}(2)G$-module by restriction;*

(viii) *$p = 3$, $G = L \cong (Q_8 \times Q_8) : 3$ and $V = V_1 \times V_2$ where V_i is a natural $\mathrm{SL}_2(3)$-module for $G/C_G(V_i) \cong \mathrm{SL}_2(3)$;*

(ix) *$p = 2$, $G = L \cong (3 \times 3) : 2$ and $V = V_1 \times V_2$ where V_i is a natural $\mathrm{SL}_2(2)$-module for $G/C_G(V_i) \cong \mathrm{Sym}(3)$; or*

(x) *$p = 2$, $L \cong (3 \times 3) : 2$, $G \cong (3 \times 3) : 4$, V is irreducible as a $\mathrm{GF}(2)G$-module and $V|_L = V_1 \times V_2$ where V_i is a natural $\mathrm{SL}_2(2)$-module for $L/C_L(V_i) \cong \mathrm{Sym}(3)$.*

Moreover, if V is generated by a $N_G(S)$-invariant subspace of order p then (G, V) satisfies outcome (iii), (vii) (ix) or (x).

Lemma 5.4.10. *Suppose that $C_{V_\beta}(V_{\alpha'}) < V_\beta \cap Q_{\alpha'}$. Then $L_\alpha/R_\alpha \not\cong \mathrm{SL}_2(p^2)$, $V_{\alpha'} \not\leq Q_\beta$ and either:*

(i) *$L_\alpha/R_\alpha \cong \mathrm{SL}_2(p)$ and Z_α is a natural $\mathrm{SL}_2(p)$-module; or*

(ii) *$Z_\alpha \cap Q_{\alpha'} \not\leq Q_{\alpha'+1}$ and $Z_{\alpha'+1} \cap Q_\beta \not\leq Q_\alpha$, and there is $x \in S \setminus Q_\alpha$ such that $[V, x, x] = \{1\}$, $|Z_\alpha/C_{Z_\alpha}(x)| = p^2$ and both L_α/R_α and Z_α are determined by Lemma 5.4.9.*

Proof. Suppose that $V_{\alpha'} \leq Q_\beta$. If $Z_{\alpha'+1}$ is a natural $\mathrm{SL}_2(q)$-module then $Z_{\alpha'} = [Z_{\alpha'+1}, V_\beta \cap Q_{\alpha'}] = H \leq [V_\beta, V_{\alpha'}] \leq Z_\beta$ and $Z_{\alpha'} = Z_\beta$. But then $[V_\beta, V_{\alpha'}] = Z_{\alpha'}$ and $O^p(L_{\alpha'})$ centralizes $V_{\alpha'}/Z_{\alpha'}$, a contradiction. Thus, as $Z(L_{\alpha'+1}) = \{1\}$, $Z_{\alpha'+1}$ is not an FF-module for $\overline{L_{\alpha'+1}}$ and, by conjugation, Z_α is not an FF-module for $\overline{L_\alpha}$. If $[Z_\alpha \cap Q_{\alpha'}, V_{\alpha'}] = \{1\}$ then, as $m_p(S/Q_\beta) = 1$, $|Z_\alpha/C_{Z_\alpha}(V_{\alpha'})| \leqslant |Z_\alpha/Z_\alpha \cap Q_{\alpha'}| = p$ and Z_α is an FF-module, a contradiction. Without loss of generality we may assume that $Z_\alpha \cap Q_{\alpha'} \not\leq Q_{\alpha'+1}$. Suppose that $|(Z_\alpha \cap Q_{\alpha'})Q_{\alpha'+1}/Q_{\alpha'+1}| \geqslant |Z_{\alpha'+1}Q_\alpha/Q_\alpha|$. Then,

$$\begin{aligned}
|Z_{\alpha'+1}/C_{Z_{\alpha'+1}}(Z_\alpha \cap Q_{\alpha'})| &\leqslant |Z_{\alpha'+1}/C_{Z_{\alpha'+1}}(Z_\alpha)| \\
&= |Z_{\alpha'+1}Q_\alpha/Q_\alpha| \\
&\leqslant |(Z_\alpha \cap Q_{\alpha'})Q_{\alpha'+1}/Q_{\alpha'+1}| \\
&= |(Z_\alpha \cap Q_{\alpha'})/C_{Z_\alpha \cap Q_{\alpha'}}(Z_{\alpha'+1})|,
\end{aligned}$$

a contradiction since $Z_{\alpha'+1}$ was assumed not to be an FF-module. So assume now that $|(Z_\alpha \cap Q_{\alpha'})Q_{\alpha'+1}/Q_{\alpha'+1}| < |Z_{\alpha'+1}Q_\alpha/Q_\alpha|$. In particular, since $m_p(S/Q_{\alpha'}) = 1$, we deduce that $|Z_\alpha/Z_\alpha \cap Q_{\alpha'+1}| \leqslant |Z_{\alpha'+1}Q_\alpha/Q_\alpha|$ and by a similar calculation as before, $Z_{\alpha'+1}Q_\alpha/Q_\alpha$ is an offender on Z_α, a contradiction since Z_α is not an FF-module. Thus, $V_{\alpha'} \not\leq Q_\beta$.

Suppose that $V_{\alpha'} \cap Q_\beta \leq Q_\alpha$. Since $m_p(S/Q_\beta) = 1$, it follows that $V_{\alpha'} \cap Q_\beta$ is of index p in $V_{\alpha'}$ and is centralized by Z_α. In particular, $V_{\alpha'}$ contains a unique non-central chief factor, $L_{\alpha'}/R_{\alpha'} \cong \mathrm{SL}_2(p)$ and $V_{\alpha'}/C_{V_{\alpha'}}(O^p(L_{\alpha'}))$ is a natural $\mathrm{SL}_2(p)$-module. Since $Z_\alpha \not\leq C_{V_\beta}(O^p(L_\beta))$, it follows that $Z_\alpha/C_{Z_\alpha}(O^p(L_\beta))$ is of order p. Since $Q_\alpha \cap Q_\beta \not\trianglelefteq L_\beta$ by Proposition 5.2.25, $Q_\beta \cap O^p(L_\beta)$ is not contained in Q_α and centralizes a subgroup of index p in Z_α. It follows that Z_α is the natural module for $L_\alpha/R_\alpha \cong \mathrm{SL}_2(p)$.

Hence, $V_{\alpha'} \cap Q_\beta \not\leq Q_\alpha$. If $Z_\alpha \cap Q_{\alpha'} \leq Q_\lambda$ for all $\lambda \in \Delta(\alpha')$, then $[Z_\alpha \cap Q_{\alpha'}, V_{\alpha'}] = \{1\}$ and since $V_{\alpha'} \cap Q_\beta$ acts non-trivially on Z_α and $m_p(S/Q_{\alpha'}) = 1$, as above, Z_α is a natural module for $L_\alpha/R_\alpha \cong \mathrm{SL}_2(p)$ and (i) is satisfied. Suppose now that $Z_\alpha \cap Q_{\alpha'} \not\leq Q_\delta$ for some $\delta \in \Delta(\alpha')$ and $Z_\delta \cap Q_\beta \leq Q_\alpha$. Then $p \leqslant |Z_\delta/C_{Z_\delta}(Z_\alpha \cap Q_{\alpha'})| \leqslant |Z_\delta/Z_\delta \cap Q_\beta \cap Q_\alpha| = |Z_\delta Q_\beta/Q_\beta| = p$ and Z_δ is an FF-module. By conjugation, Z_α is a natural $\mathrm{SL}_2(p)$-module and $Z_\alpha \cap Q_{\alpha'} = Z_\beta$ centralizes Z_δ and so $Z_\alpha \cap Q_{\alpha'} \leq Q_\delta$, a contradiction.

Thus, we now suppose that $Z_\alpha \cap Q_{\alpha'} \not\leq Q_\delta$ and $Z_\delta \cap Q_\beta \not\leq Q_\alpha$. Since $Z_\alpha \cap Q_{\alpha'} \leq V_\beta \cap Q_{\alpha'} \not\leq Q_\delta$, without loss of generality, we may as well relabel $\alpha'+1$ and assume that $\delta = \alpha'+1$. Thus, $Z_\alpha \cap Q_{\alpha'} \not\leq Q_{\alpha'+1}$ and $Z_{\alpha'+1} \cap Q_\beta \not\leq Q_\alpha$.

Now,

$$\begin{aligned}|Z_\alpha/C_{Z_\alpha}(Z_{\alpha'+1}\cap Q_\beta)| &\leqslant |Z_\alpha/C_{Z_\alpha}(Z_{\alpha'+1})|\\ &\leqslant p|(Z_\alpha\cap Q_{\alpha'})Q_{\alpha'+1}/Q_{\alpha'+1}|\\ &= p|(Z_\alpha\cap Q_{\alpha'})/C_{Z_\alpha\cap Q_{\alpha'}}(Z_{\alpha'+1})|\end{aligned}$$

and

$$\begin{aligned}|Z_{\alpha'+1}/C_{Z_{\alpha'+1}}(Z_\alpha\cap Q_{\alpha'})| &\leqslant |Z_{\alpha'+1}/C_{Z_{\alpha'+1}}(Z_\alpha)|\\ &\leqslant p|(Z_{\alpha'+1}\cap Q_\beta)Q_\alpha/Q_\alpha|\\ &= p|(Z_{\alpha'+1}\cap Q_\beta)/C_{Z_{\alpha'+1}\cap Q_\beta}(Z_\alpha)|.\end{aligned}$$

If $|Z_\alpha/C_{Z_\alpha}(Z_{\alpha'+1}\cap Q_\beta)| \neq |Z_{\alpha'+1}/C_{Z_{\alpha'+1}}(Z_\alpha\cap Q_{\alpha'})|$, then, by conjugacy, one can calculate that Z_α is an FF-module for $L_\alpha/R_\alpha \cong \mathrm{SL}_2(q)$. But then, $|Z_\alpha/C_{Z_\alpha}(Z_{\alpha'+1}\cap Q_\beta)| = |S/Q_\alpha| = |Z_{\alpha'+1}/C_{Z_{\alpha'+1}}(Z_\alpha\cap Q_{\alpha'})|$, a contradiction. Thus, $|Z_\alpha/C_{Z_\alpha}(Z_{\alpha'+1}\cap Q_\beta)| = |Z_{\alpha'+1}/C_{Z_{\alpha'+1}}(Z_\alpha\cap Q_{\alpha'})|$. If $m_p(S/Q_\alpha)=1$, then we may as well assume that $|Z_\alpha/C_{Z_\alpha}(Z_{\alpha'+1}\cap Q_\beta)| = p^2$ and the result holds by Lemma 5.4.9. So suppose that $m_p(S/Q_\alpha)\geqslant 2$. Then, as Z_α is a quadratic module, $\overline{L_\alpha}$ is a group of Lie type. By [DS85, (5.12)], unless $|Z_\alpha/C_{Z_\alpha}(Z_{\alpha'+1}\cap Q_\beta)| \leqslant p^2$, $\overline{L_\alpha}\cong \mathrm{SL}_2(q)$ for some $q>p$ and Z_α is a natural $\mathrm{SL}_2(q)$-module. But then, as $q>p$, we conclude that $[Z_{\alpha'+1}\cap Q_\beta, Z_\alpha\cap Q_{\alpha'}] = Z_\beta = Z_{\alpha'} = H$. But then, by Lemma 2.3.10, $V_\beta/C_{V_\beta}(O^p(L_\beta))$ is a natural module for $L_\beta/R_\beta\cong \mathrm{SL}_2(p)$ and since $|Z_\alpha/C_{Z_\alpha}(O^p(L_\beta))| = |Z_\alpha/Z_\beta| > p$, we have a contradiction.

Thus,$|Z_\alpha/C_{Z_\alpha}(Z_{\alpha'+1}\cap Q_\beta)| = p^2$ and Z_α is determined by Lemma 5.4.9. To complete the proof we need only show that $\overline{L_\alpha}\ncong \mathrm{SL}_2(p^2)$. We obtain contradiction as above in the case that $\overline{L_\alpha}\cong \mathrm{SL}_2(p^2)$ with Z_α an associated natural $\mathrm{SL}_2(p^2)$-module. Hence, $p = 2$ and Z_α is a natural $\Omega_4^-(2)$-module for $\overline{L_\alpha} \cong \mathrm{PSL}_2(4)$. Since $Q_\alpha \cap Q_\beta \ntrianglelefteq L_\beta$ by Proposition 5.2.25, $Q_\beta \cap O^2(L_\beta) \not\leq Q_\alpha$ so that $S = Q_\alpha(Q_\beta \cap O^2(L_\beta))$. In particular, $Z_\alpha \cap C_{V_\beta}(O^2(L_\beta)) = Z_\beta$. Now, $V_{\alpha'} \cap Q_\beta$, acts quadratically on Z_α and so $|(V_{\alpha'} \cap Q_\beta)Q_\alpha/Q_\alpha| = 2$. Moreover, $|[Z_{\alpha'+1}, Q_{\alpha'}]| = 2^3$ and since $|Z_\alpha \cap Q_\beta \cap Q_\alpha| = 4$ and $V_{\alpha'}/[V_{\alpha'}, Q_{\alpha'}]$ contains a non-central chief factor by Lemma 5.2.13, we have that $|S/Q_\beta| = 2$ and $V_{\alpha'}/[V_{\alpha'}, Q_{\alpha'}]$ is an FF-module for $\overline{L_{\alpha'}}$. Since $Z_\alpha \cap C_{V_\beta}(O^2(L_\beta)) = Z_\beta$, we may as well assume that $[V_{\alpha'}, Q_{\alpha'}]/Z_{\alpha'}$ has a non-central chief factor, and so it too is an FF-module. But then, again since $Z_\alpha \cap C_{V_\beta}(O^2(L_\beta)) = Z_\beta$, we conclude that $[V_{\alpha'}, Q_{\alpha'}] = [Z_{\alpha'+1}, Q_{\alpha'}]C_{[V_{\alpha'},Q_{\alpha'}]}(O^2(L_{\alpha'}))$, a contradiction for then $Q_{\alpha'+1}$ centralizes $[V_{\alpha'}, Q_{\alpha'}]/C_{[V_{\alpha'},Q_{\alpha'}]}(O^2(L_{\alpha'}))$. Hence, the result. □

Lemma 5.4.11. *Suppose that $C_{V_\beta}(V_{\alpha'}) < V_\beta \cap Q_{\alpha'}$. Then either Z_α is a natural module for $L_\alpha/R_\alpha \cong \mathrm{SL}_2(p)$ or the following holds:*

(i) *$S = Q_\alpha Q_\beta$;*

(ii) *$|S/Q_\alpha| = p$;*

(iii) *$L_\alpha/R_\alpha \in \{\mathrm{SL}_2(p), \mathrm{SU}_3(2)', \mathrm{Dih}(10), (3 \times 3) : 2, (Q_8 \times Q_8) : 3, 2 \cdot \mathrm{Alt}(5), 2_-^{1+4}.\mathrm{Alt}(5)\}$;*

(iv) *$H = [Q_{\alpha'}, V_{\alpha'}] \leq Z_{\alpha'}$;*

(v) *$Q_\beta \in \mathrm{Syl}_p(R_\beta)$;*

(vi) *$|Z_\alpha/Z_\beta| = p^2$; and*

(vii) *unless $L_\alpha/R_\alpha \cong \mathrm{SU}_3(2)'$ and $H < Z_{\alpha'}$, we have that $H = Z_{\alpha'}$ and $Z_\alpha = Z_\beta \times Z_{\alpha-1}$ for some $\alpha - 1 \in \Delta(\alpha - 1)$.*

Proof. Since this result holds in all the relevant cases in Theorem 5.2.2, we may assume that G is a minimal counterexample to the lemma. We assume throughout that Z_α is not a natural module for $L_\alpha/R_\alpha \cong \mathrm{SL}_2(p)$ and so Z_α is determined by Lemma 5.4.10 (ii). Suppose first that $S \neq Q_\alpha Q_\beta$. Since $m_p(S/Q_\beta) = 1$ and V_β admits quadratic action, it follows from Lemma 2.3.5 that $p = 2$, and then by Lemma 5.4.9 and Lemma 5.4.10, $m_2(S/Q_\alpha) = 1$. For $\mu \in \{\alpha, \beta\}$, let O_μ be the preimage in L_μ of $O_{2'}(\overline{L_\mu})$ and $L_\mu^* := O_\mu Q_\alpha Q_\beta$. Then $L_\mu^* \trianglelefteq L_\mu$ and L_μ^* has index at least 2 and at most 4 in L_μ. Set K to be a Hall $2'$-subgroup of $G_{\alpha,\beta}$ and set $G_\mu^* := L_\mu^* K$. Then G_μ^* has index at least 2 and at most 4 in G_μ, and is normal in G_μ. Moreover, for $X = \langle G_\alpha^*, G_\beta^* \rangle$, X is normalized by $G_{\alpha,\beta}$ and $G = \langle X, G_{\alpha,\beta} \rangle$. Thus, the subgroup of S which is normal in X is also normal in G and so is trivial. Hence, any subgroup of $G_{\alpha,\beta} \cap X$ which is normal in X is a p'-group and we can arrange that it is contained in $K \leq G_\mu^*$, a contradiction since G_μ^* is of characteristic p. Thus, the amalgam $(G_\alpha^*, G_\beta^*, KQ_\alpha Q_\beta)$ satisfies Hypothesis 5.2.1. Since G_α^* and G_β^* are solvable, by minimality, $(G_\alpha^*, G_\beta^*, KQ_\alpha Q_\beta)$ is a weak BN-pair; or X is a symplectic amalgam with $|S| = 2^6$. In all cases, for some $\mu \in \{\alpha, \beta\}$, we infer that $\overline{L_\mu^*} \cong \mathrm{Sym}(3)$. But then, it follows that $\overline{L_\mu} \cong \mathrm{Sym}(3) \times R$, where R is a 2-group, a contradiction since $m_p(S/Q_\mu) = 1$. Hence, $S = Q_\alpha Q_\beta$ and (i) is proved.

Since $m_p(S/Q_\beta) = 1$ and $V_\beta \cap Q_{\alpha'}$ acts quadratically on $Z_{\alpha'+1}$, by [DS85, (5.9)], we deduce that $V_\beta \cap Q_{\alpha'} \cap Q_{\alpha'+1}$ has index at most p^2 in V_β and V_β contains at most two non-central chief factors for L_β. By Lemma 5.2.13, $V_\beta/[V_\beta, Q_\beta]$ contains a non-central chief factor. Suppose that $[V_\beta, Q_\beta]$ also contains a non-central chief factor. Then it follows that $U := V_\beta/[V_\beta, Q_\beta]$ is an FF-module for $\overline{L_\beta}$ and so V_β/C is a natural $\mathrm{SL}_2(p)$-module for $L_\beta/C_{L_\beta}(U)$, where C is the preimage in V_β of $C_U(O^p(L_\beta))$. Since $C \trianglelefteq L_\beta$, it follows from the definition of V_β that $Z_\alpha \not\leq C$. If $V_\beta = Z_\alpha C$ then $[Q_\alpha, V_\beta] \leq C$ and $O^p(L_\beta)$ centralizes U, a contradiction. Since

V_β/C has order p^2, $Z_\alpha \cap C$ is $G_{\alpha,\beta}$-invariant of index p in Z_α. In particular, $L_\alpha/R_\alpha \not\cong (\mathrm{P})\mathrm{SU}_3(p), \mathrm{SU}_3(2)'$ or $\mathrm{SU}_3(2)'.2$.

Since $[V_\beta, Q_\beta]$ contains a non-central chief factor, $[V_\beta, Q_\beta] \not\leq Z_\beta$ and it follows from Lemma 5.4.10 that $L_\alpha/R_\alpha \cong \mathrm{Sz}(2)$ or $(3\times 3):4$. Then $[Z_\alpha, Q_\beta, Q_\beta] \neq Z_\beta$. Since $[V_\beta, Q_\beta]$ contains only one non-central chief factor, either $[V_\beta, Q_\beta, Q_\beta] \leq C_{V_\beta}(O^2(L_\beta))$ or that $O^2(L_\beta)$ centralizes $[V_\beta, Q_\beta]/[V_\beta, Q_\beta, Q_\beta]$. Suppose the latter. Since $V_\beta = \langle Z_\alpha^{L_\beta}\rangle$, it follows that $[V_\beta, Q_\beta] = [Z_\alpha, Q_\beta][V_\beta, Q_\beta, Q_\beta]$. But then $[V_\beta, Q_\beta, Q_\beta] = [Z_\alpha, Q_\beta, Q_\beta][V_\beta, Q_\beta, Q_\beta, Q_\beta] = [Z_\alpha, Q_\beta, Q_\beta]Z_\beta = [Z_\alpha, Q_\beta, Q_\beta]$. Then Q_α centralizes $[V_\beta, Q_\beta, Q_\beta]$ and so $O^2(L_\beta)$ centralizes $[V_\beta, Q_\beta] = [Z_\alpha, Q_\beta]$, a contradiction.

Suppose now that $[V_\beta, Q_\beta, Q_\beta] \leq C_{V_\beta}(O^2(L_\beta))$. Then $[V_\beta, Q_\beta, Q_\beta] = [Z_\gamma, Q_\beta, Q_\beta]$ for all $\gamma \in \Delta(\beta)$. Let $L_\beta^* := C_{L_\beta}([V_\beta, Q_\beta, Q_\beta])$. Since $|Q_\beta/Q_\beta \cap Q_\alpha| = 4$, $Q_\alpha \cap Q_\beta \not\trianglelefteq L_\beta$, $S = Q_\alpha Q_\beta$ and $\langle Q_\gamma \mid \gamma \in \Delta(\beta)\rangle \leq L_\beta^*$, L_β^* has index at most 2 in L_β and $L_\beta^*/O_2(L_\beta^*) \cong L_\beta/Q_\beta$. Set $S^* = L_\beta^* \cap S$ so that $Q_\alpha \leq S^*$ and notice that if $S = S^*$, then $L_\beta^* = L_\beta$ and S centralizes $[Z_\alpha, Q_\beta, Q_\beta]$, contradicting the structure of Z_α. Thus, L_β^* and S^* have index exactly 2 in L_β and S respectively. Set $L_\alpha^* := \langle (S^*)^{G_\alpha}\rangle$. Then, L_α^* has index 2 in L_α and $L_\alpha^*/R_\alpha \cong (3\times 3):2$ or $\mathrm{Dih}(10)$. Setting K to be a Hall $2'$-subgroup of $G_{\alpha,\beta}$ and $G_\mu^* := L_\mu^* K$ for $\mu \in \{\alpha, \beta\}$, we have that G_μ^* has index 2 in G_μ and the amalgam $X^* := (G_\alpha^*, G_\beta^*, KS^*)$ satisfies Hypothesis 5.2.1. Since $L_\alpha^*/R_\alpha \cong \mathrm{Dih}(10)$ or $(3\times 3):2$, comparing with the amalgams in Theorem 5.2.2, we have a contradiction.

Hence, we assume that $[V_\beta, Q_\beta, O^p(L_\beta)] = \{1\}$ and $Q_\beta \leq R_\beta$. Then by Lemma 5.2.16, $Q_\beta \in \mathrm{Syl}_p(R_\beta)$. Moreover, by Lemma 5.4.7, $V_\beta/[V_\beta, Q_\beta]$ is a 2F-module for $\overline{L_\beta}$, but not an FF-module. Since $Q_\beta \cap O^p(L_\beta) \not\leq Q_\alpha$, it follows that $Q_\beta \cap O^p(L_\beta)$ centralizes $[Z_\gamma, Q_\beta] = [V_\beta, Q_\beta]$ for all $\gamma \in \Delta(\beta)$.

Suppose first that $|S/Q_\alpha| > p$. Since $S = Q_\alpha Q_\beta$ and $Q_\beta \cap O^p(L_\beta)$ centralizes $[Z_\alpha, Q_\beta]$, $L_\alpha/R_\alpha \not\cong \mathrm{Sz}(2)$ or $(3 \times 3) : 4$. Set $Q_\beta^* := \langle (Q_\alpha \cap Q_\beta)^{G_\beta} \rangle$. Then Q_β^* centralizes $[Z_\alpha, Q_\beta]$. If $S = Q_\beta^* Q_\alpha$ then S centralizes $[Z_\alpha, Q_\beta]$. However, since $|S/Q_\alpha| > p$, comparing with the list in Lemma 5.4.9, we have a contradiction. So $Q_\beta^* < Q_\beta$ and $Q_\beta^* Q_\alpha < S$. Then, $Q_\beta^* Q_\alpha$ is a proper $G_{\alpha,\beta}$-invariant subgroup of S/Q_α, from which it follows that $L_\alpha/R_\alpha \cong \mathrm{(P)SU}_3(p)$ or $\mathrm{SU}_3(2)'.2$. Since Q_β^* centralizes $[Z_\alpha, Q_\beta]$, $|Q_\beta^* Q_\alpha/Q_\alpha| = p$.

If $p = 2$, then as $m_2(S/Q_\beta) = 1$, L_β is solvable. Set $L_\beta^* := C_{L_\beta}([V_\beta, Q_\beta])$. Then, $Q_\beta^* \leq O_2(L_\beta^*)$ and since $O_2(L_\beta^*)$ is $G_{\alpha,\beta}$-invariant and centralizes $[Z_\alpha, Q_\beta]$, $|O_2(L_\beta^*)Q_\alpha/Q_\alpha| = 2$ and $Q_\beta^* = O_2(L_\beta^*)$. Moreover, $S^* := Q_\alpha Q_\beta^* = S \cap L_\beta^* \in \mathrm{Syl}_2(L_\beta^*)$. Setting $L_\alpha^* := \langle (S^*)^{G_\alpha} \rangle$, we have that $L_\alpha^* \trianglelefteq G_\alpha$ and $S^* \in \mathrm{Syl}_2(L_\alpha^*)$. For $\mu \in \{\alpha, \beta\}$, set $G_\mu^* := L_\mu^* K$, where K is a Hall $2'$-subgroup of $G_{\alpha,\beta}$. Then the amalgam $X := (G_\alpha^*, G_\beta^*, S^* K)$ satisfies Hypothesis 5.2.1 and since L_{α^*}/R_α is isomorphic to a proper subgroup of $\mathrm{SU}_3(2)$, we have a contradiction.

Thus, p is odd and $L_\alpha/R_\alpha \cong \mathrm{(P)SU}_3(p)$. But then $m_p(S/Q_\alpha) = 2$ so that $R_\alpha = Q_\alpha$ by Proposition 3.2.7, and $H = Z_{\alpha'} \leq V_\beta$. Moreover, since $V_\beta/[V_\beta, Q_\beta]$ is a 2F-module for $\overline{L_\beta}$ and $m_p(S/Q_\beta) = 1$, by Lemma 5.4.9, we deduce that $L_\beta/R_\beta \cong \mathrm{SL}_2(p)$, $(Q_8 \times Q_8) : 3$, $2 \cdot \mathrm{Alt}(5)$ or $2_-^{1+4}.\mathrm{Alt}(5)$ with the latter three only occurring when $p = 3$. In particular, $|S/Q_\beta| = p$.

Suppose first that $b > 3$. If $V_\alpha^{(2)} \cap Q_{\alpha'-2} \not\leq Q_{\alpha'-1}$, then as $Z_{\alpha'}$ is centralized by $Q_{\alpha'}, Q_{\alpha'-1}$ and $V_\alpha^{(2)}$, we have that $Z_{\alpha'} \leq Z(L_{\alpha'-1})$, a contradiction. Thus, $V_\alpha^{(2)} \cap Q_{\alpha'-2} = Z_\alpha(V_\alpha^{(2)} \cap \cdots \cap Q_{\alpha'})$ has index at most p in $V_\alpha^{(2)}$. Moreover, $Z_{\alpha'+1} \cap Q_\beta$ normalizes $V_\alpha^{(2)}$ and $[Z_{\alpha'+1} \cap Q_\beta, V_\alpha^{(2)}, V_\alpha^{(2)}] = \{1\}$. But $Z_{\alpha'+1} \cap Q_\beta$ has order p^5 and it follows that $V_\alpha^{(2)} \cap \cdots \cap Q_{\alpha'} = (Z_\alpha \cap Q_{\alpha'})(V_\alpha^{(2)} \cap \cdots \cap Q_{\alpha'+1})$ and $Z_{\alpha'+1} \cap Q_\beta$ centralizes an index p subgroup of $V_\alpha^{(2)}/Z_\alpha$. Since $\overline{L_\alpha} \cong \mathrm{(P)SU}_3(p)$, this is a contradiction.

Suppose now that $b = 3$. Then $L_{\alpha'-1} = \langle Q_{\alpha'}, Q_\beta, Q_{\alpha'-1}\rangle$ centralizes $Z_{\alpha'} \cap Z_\beta$ and so, $Z_{\alpha'} \cap Z_\beta = \{1\}$. Then $[Z_{\alpha'+1} \cap Q_\beta, Z_\alpha \cap Q_{\alpha'}] \leq Z_{\alpha'} \cap Z_\beta = \{1\}$ and since $m_p(S/Q_\beta) = 1$, an index p subgroup of Z_α is centralized and Z_α is an FF-module, a contradiction.

Thus, $|S/Q_\alpha| = p$ and, as $Q_\alpha \cap Q_\beta \ntrianglelefteq L_\beta$ by Proposition 5.2.25, $Q_\beta = (Q_\alpha \cap Q_\beta)(Q_\gamma \cap Q_\beta)$ for some $\gamma \in \Delta(\beta)$. Thus, $L_\beta = \langle Q_\gamma \mid \gamma \in \Delta(\beta)\rangle$ centralizes $[V_\beta, Q_\beta]$ and $[V_\beta, Q_\beta] \leq Z_\beta$. The remaining properties follow from Lemma 5.4.9 and may be checked in MAGMA. $\square$

Lemma 5.4.12. *Suppose that $C_{V_\beta}(V_{\alpha'}) < V_\beta \cap Q_{\alpha'}$. If Z_α is a natural $\mathrm{SL}_2(p)$-module for $L_\alpha/R_\alpha \cong \mathrm{SL}_2(p)$ then for $V := V_\beta/C_{V_\beta}(O^p(L_\beta))$ either:*

(i) *V is a natural $\mathrm{Sz}(2)$-module for $L_\beta/R_\beta \cong \mathrm{Dih}(10)$ or $\mathrm{Sz}(2)$; or*

(ii) *V is a 2F-module for $L_\beta/R_\beta \cong (3 \times 3) : 2$ or $(3 \times 3) : 4$.*

Proof. By Lemma 5.4.7, V is not an FF-module and so, as V is a quadratic 2F-module and $m_p(S/Q_\beta) = 1$, the structure of V and L_β/R_β follows from Lemma 5.4.9. Since $Z_\alpha C_{V_\beta}(O^p(L_\beta))/C_{V_\beta}(O^p(L_\beta))$ is of order p and $G_{\alpha,\beta}$-invariant and $V_\beta = \langle Z_\alpha^{L_\beta}\rangle$, by Lemma 5.4.9, we conclude that $L_\beta/R_\beta \cong \mathrm{Sz}(2), \mathrm{Dih}(10)$, $(3 \times 3) : 2$ or $(3 \times 3) : 4$, as required. $\square$

Lemma 5.4.13. *Suppose that $C_{V_\beta}(V_{\alpha'}) < V_\beta \cap Q_{\alpha'}$, Z_α is not a natural $\mathrm{SL}_2(p)$-module and $V_\alpha^{(2)}/Z_\alpha$ contains a unique non-central chief factor U/V for L_α. Then U/V is not an FF-module for $\overline{L_\alpha}$.*

Proof. Suppose that U/V is an FF-module for $\overline{L_\alpha}$. By Lemma 5.2.13,

$V_\alpha^{(2)}/[V_\alpha^{(2)}, Q_\alpha]$ contains a non-central chief factor for L_α. Set C to be the preimage in $V_\alpha^{(2)}$ of $C_{V_\alpha^{(2)}/Z_\alpha}(O^p(L_\alpha))$. Then $[V_\alpha^{(2)}, Q_\alpha] \leq C$ and since U/V is an FF-module and $|S/Q_\alpha| = p$, by Lemma 2.3.10, $V_\alpha^{(2)}/C$ is isomorphic to a natural $\mathrm{SL}_2(p)$-module. In particular, as $V_\beta \not\leq C$, $V_\beta C/C$ is of order p for otherwise Q_β centralizes $V_\alpha^{(2)}/C$. But now, $V_\beta \cap C$ has index p in V_β and is normalized by L_α. By conjugacy, an index p subgroup of V_β is normalized by $L_{\alpha+2}$, and by transitivity, this subgroup is contained in $V_{\alpha+3}$ so that $V_\beta \cap V_{\alpha+3}$ is of index p in V_β. But then, as $V_\beta \not\leq Q_{\alpha'}$, $V_\beta \cap Q_{\alpha'} = V_\beta \cap V_{\alpha+3} = V_\beta \cap C_{\alpha'}$ and $[V_\beta \cap Q_{\alpha'}, V_{\alpha'}] = \{1\}$, contradicting the initial assumption. □

Lemma 5.4.14. *Suppose that $C_{V_\beta}(V_{\alpha'}) < V_\beta \cap Q_{\alpha'}$. Then $b = 3$.*

Proof. Suppose that $b > 3$ and Z_α is not a natural $\mathrm{SL}_2(p)$-module. Then Z_α is as described in Lemma 5.4.11. If $V_\alpha^{(2)} \leq Q_{\alpha'-2}$ or $V_\alpha^{(2)} \cap Q_{\alpha'-2} \leq Q_{\alpha'-1}$, then as $|S/Q_\alpha| = p$, $m_p(S/Q_\beta) = 1$ and $V_\alpha^{(2)}$ is elementary abelian, $Z_\alpha(V_\alpha^{(2)} \cap Q_{\alpha'})$ has index at most p in $V_\alpha^{(2)}$. Moreover, since Z_α is not the natural module, $V_\alpha^{(2)} \cap Q_{\alpha'} = (Z_\alpha \cap Q_{\alpha'})(V_\alpha^{(2)} \cap Q_{\alpha'+1})$ and it follows that there is a unique non-central chief factor in $V_\alpha^{(2)}/Z_\alpha$ for L_α, and that it is an FF-module for $\overline{L_\alpha}$, a contradiction by Lemma 5.4.13. Thus, $V_\alpha^{(2)} \not\leq Q_{\alpha'-2}$ and $V_\alpha^{(2)} \cap Q_{\alpha'-2} \not\leq Q_{\alpha'-1}$.

Suppose first that $L_\alpha/R_\alpha \cong \mathrm{SU}_3(2)'$. Then $H = [V_\beta \cap Q_{\alpha'}, V_{\alpha'}] = [V_{\alpha'}, Q_{\alpha'}]$ is of order 4 and strictly contained in $Z_{\alpha'}$. Moreover, since $b > 3$, H is centralized by $X_{\alpha'-1} := \langle V_\alpha^{(2)} \cap Q_{\alpha'-2}, R_{\alpha'-1}, Q_{\alpha'} \rangle$ and so either $Q_{\alpha'}Q_{\alpha'-1}$ is conjugate to $Q_{\alpha'}Q_{\alpha'-2}$ by an element of $R_{\alpha'-1}$; or $X_{\alpha'-1}/C_{X_{\alpha'-1}}(Z_{\alpha'-1}) \cong \mathrm{Sym}(3)$. In the latter case, it follows that H is invariant under the action of a subgroup of index 3 in $L_{\alpha'-1}$, a contradiction to structure of $Z_{\alpha'-1}$. In the former case, it follows that $[V_{\alpha'}, Q_{\alpha'}] = [V_{\alpha'-2}, Q_{\alpha'-2}]$ and since $V_\alpha^{(2)} \not\leq Q_{\alpha'-2}$, we may iterate backwards through critical pairs $(\alpha - 2k, \alpha' - 2k)$ for $k \geqslant 0$ so that $H = [V_{\alpha'}, Q_{\alpha'}] = [V_\beta, Q_\beta] \leq Z_\beta$ and so an index p subgroup of V_β/Z_β is centralized by $V_{\alpha'}$. We have a contradiction by Lemma 5.4.7.

Now, $Z_{\alpha'} = H \leq V_\beta$, so that $V_\alpha^{(2)} \cap Q_{\alpha'-2}$ is not contained in $Q_{\alpha'-1}$ and centralizes $Z_{\alpha'}Z_{\alpha'-2}$. It follows that $Z_{\alpha'} = Z_{\alpha'-2}$. Moreover, since $V_\alpha^{(2)} \not\leq Q_{\alpha'-2}$ there is some $\alpha-2$, with $(\alpha-2, \alpha'-2)$ a critical pair. By Lemma 5.4.6, we may assume that $(\alpha-2,\alpha'-2)$ satisfies the same hypothesis as (α,α'). Iterating through critical pairs, we conclude that $Z_{\alpha'} = \cdots = Z_\beta$. But then $H = [V_\beta \cap Q_{\alpha'}, V_{\alpha'}] = Z_\beta$ and $V_\beta/C_{V_\beta}(O^p(L_\beta))$ is a natural $\mathrm{SL}_2(p)$-module for L_β/R_β, a contradiction by Lemma 5.4.7. Hence, whenever $b > 3$, Z_α is a natural $\mathrm{SL}_2(p)$-module for $L_\alpha/R_\alpha \cong \mathrm{SL}_2(p)$.

By Lemma 5.4.12, $L_\beta/R_\beta \cong (3\times 3):4$ or $\mathrm{Sz}(2)$ and so $|S/Q_\beta| \neq p$. Moreover, since V_β is centralized by $V_\beta^{(3)}$ we deduce that $[V_{\alpha'}, V_\beta, V_\beta^{(3)}] = \{1\}$, $V_\beta^{(3)} \cap Q_{\alpha'-2} \cap Q_{\alpha'-1} = V_\beta(V_\beta^{(3)} \cap \cdots \cap Q_{\alpha'})$ and $[V_\beta^{(3)} \cap \cdots \cap Q_{\alpha'}, V_{\alpha'}] = [V_{\alpha'}, Q_{\alpha'}] = Z_{\alpha'} = H \leq V_\beta$ by Lemma 5.4.11. Since $|S/Q_\beta| \neq p$, any non-central chief factor within $V_\beta^{(3)}/V_\beta$ is not an FF-module for $\overline{L_\beta}$ and so $V_\beta^{(3)} \not\leq Q_{\alpha'-2}$ and $V_\beta^{(3)} \cap Q_{\alpha'-2} \not\leq Q_{\alpha'-1}$. But $V_\beta^{(3)} \cap Q_{\alpha'-2}$ centralizes $Z_{\alpha'-2}$ and $Z_{\alpha'} \leq V_\beta$ and since $V_\beta^{(3)} \cap Q_{\alpha'-2} \not\leq Q_{\alpha'-1}$, we deduce that $Z_{\alpha'} = Z_{\alpha'-2}$. Since $V_\beta^{(3)} \not\leq Q_{\alpha'-2}$ there is a critical pair $(\beta-3, \alpha'-2)$ satisfying the same hypothesis as (α,α') by Lemma 5.4.6, and iterating back through critical pairs, we conclude that $[V_\beta \cap Q_{\alpha'}, V_{\alpha'}] = H = Z_{\alpha'} = Z_\beta$ and $V_\beta/C_{V_\beta}(O^2(L_\beta))$ is a natural $\mathrm{SL}_2(p)$-module, a contradiction by Lemma 5.4.7. $\square$

Lemma 5.4.15. *Suppose that $C_{V_\beta}(V_{\alpha'}) < V_\beta \cap Q_{\alpha'}$ and $b = 3$. Then $L_\alpha/R_\alpha \cong \mathrm{Sym}(3)$, Z_α is natural $\mathrm{SL}_2(2)$-module, $O^2(L_\beta)$ centralizes C_β/V_β and one of the following holds:*

(i) *$\overline{L_\beta} \cong \mathrm{Sz}(2)$ and V_β/Z_β is a natural module $\mathrm{Sz}(2)$-module; or*

(ii) $\overline{L_\beta} \cong (3 \times 3) : 4$ *and* V_β/Z_β *is an irreducible* $2F$*-module.*

In particular, L_β is 2-minimal and $L_\beta \cap G_{\alpha,\beta} = S$ in either case.

Proof. Suppose that $L_\alpha/R_\alpha \cong \mathrm{SU}_3(2)'$ and Z_α is the restriction of a natural $\mathrm{SU}_3(2)$-module. Since Q_α is non-abelian, by the irreducibility of Z_α, $Z_\alpha \leq \langle (Z_\beta \cap \Phi(Q_\alpha))^{G_\alpha} \rangle \leq \Phi(Q_\alpha)$.

If $|S/Q_\beta| = 2$, then $Q_\alpha \cap Q_\beta \cap Q_{\alpha'-1} = Z_\alpha(Q_\alpha \cap \cdots \cap Q_{\alpha'+1})$ and since $Q_\alpha/\Phi(Q_\alpha)$ is not an FF-module, $\overline{L_\alpha} \cong \mathrm{SU}_3(2)'$ and $Q_\alpha/\Phi(Q_\alpha)$ contains a unique non-central chief factor, U/V say. Moreover, U/V is isomorphic to Z_α and $U \not\leq Q_\beta$. But $U \cap Q_\beta$ is $G_{\alpha,\beta}$-invariant subgroup of index 2 in U, a contradiction.

Applying Lemma 5.4.9, we see that $L_\beta/R_\beta \cong \mathrm{Sz}(2)$, $(3\times 3) : 4$, $\mathrm{SU}_3(2)'.2$ or $\mathrm{SU}_3(2)$. Now $V_\beta(Q_\beta \cap Q_{\alpha'-1} \cap Q_{\alpha'})$ has index at most 8 in Q_β and since $|S/Q_\beta| \neq 2$, no non-central chief factor is an FF-module for $\overline{L_\beta}$ and so Q_β/V_β contains a unique non-central chief factor for $\overline{L_\beta}$, and this chief factor lies in Q_β/C_β. Then, an application of the three subgroup lemma implies that $R_\beta = Q_\beta$. Suppose that $\overline{L_\beta} \cong \mathrm{SU}_3(2)'.2$ or $\mathrm{SU}_3(2)$. Since $V_\beta(Q_\beta \cap Q_{\alpha'})$ has index at most 8 in Q_β, one can compute that the non-central chief factor for L_β within Q_β/C_β is not an irreducible 8-dimensional $\mathrm{GF}(2)$-module for $\overline{L_\beta}$, and so it must be a natural $\mathrm{SU}_3(2)$-module. But $Q_\alpha \cap Q_\beta$ is a $G_{\alpha,\beta}$ subgroup of index 2, and we have a contradiction, as before. Thus, $\overline{L_\beta} \cong \mathrm{Sz}(2)$ or $(3 \times 3) : 4$. However, from the structure of Z_α, we conclude that $Z_\beta = Z_\alpha \cap C_{V_\beta}(O^p(L_\beta))$ has index 4 in Z_α so that a subgroup of order 4 of $V_\beta/C_{V_\beta}(O^2(L_\beta))$ is centralized by $S = Q_\alpha Q_\beta$, contradicting the structure of the 2F-modules associated to $\mathrm{Sz}(2)$ and $(3 \times 3) : 4$. Hence, $L_\alpha/R_\alpha \not\cong \mathrm{SU}_3(2)'$.

By Lemma 5.4.11, we may now assume that $Z_\alpha = Z_\beta \times Z_{\alpha-1}$ for some $\alpha-1 \in \Delta(\alpha)$.

Then $[V_\beta \cap Q_{\alpha'}, V_{\alpha'} \cap Q_\beta] \leq Z_{\alpha'} \cap Z_\beta$. But $Z_{\alpha'} Z_\beta \leq Z_{\alpha'-1}$ and by Lemma 5.4.11, either $Z_{\alpha'} = Z_\beta$, or $Z_{\alpha'} \cap Z_\beta = \{1\}$. If $Z_{\alpha'} = Z_\beta$, then $H = [V_\beta \cap Q_{\alpha'}, V_{\alpha'}] = Z_\beta$ and so $V_\beta / C_{V_\beta}(O^p(L_\beta))$ is a natural $\mathrm{SL}_2(p)$-module, a contradiction by Lemma 5.4.7. Hence, $[Z_\alpha \cap Q_{\alpha'}, Z_{\alpha'+1} \cap Q_\beta] \leq [V_\beta \cap Q_{\alpha'}, V_{\alpha'} \cap Q_\beta] \leq Z_{\alpha'} \cap Z_\beta = \{1\}$ and by Lemma 2.3.10, Z_α is an FF-module. Then, as $Z(L_\alpha) = \{1\}$, we have that Z_α is a natural $\mathrm{SL}_2(p)$-module and L_β / R_β is determined by Lemma 5.4.12.

Suppose that $|S/Q_\beta| = 2$ so that $L_\beta / R_\beta \cong \mathrm{Dih}(10)$ or $(3 \times 3) : 2$. Then $C_\beta \leq Q_{\alpha'-1}$ and $C_\beta = V_\beta(C_\beta \cap Q_{\alpha'})$. Since $[V_{\alpha'}, Q_{\alpha'}] = Z_{\alpha'} \leq V_\beta$, we deduce that $O^2(L_\beta)$ centralizes C_β / V_β. Then for $r \in R_\beta$ of odd order, if $[r, Q_\beta, V_\beta] = \{1\}$ then $[r, V_\beta, Q_\beta] = \{1\}$ by the three subgroup lemma, and so r centralizes Q_β. But now, $Q_\beta \cap Q_{\alpha'-1} = V_\beta(Q_\beta \cap Q_{\alpha'})$, and so Q_β / V_β contains a unique non-central chief factor for L_β, which is a faithful FF-module for $\overline{L_\beta}$, and $\overline{L_\beta} \cong \mathrm{Sym}(3)$ by Lemma 2.3.10 by Lemma 2.3.10.

Thus, $|S/Q_\beta| = 4$ and by Lemma 2.3.10, no non-central chief factor within Q_β is an FF-module for $\overline{L_\beta}$. Since $C_\beta \leq Q_{\alpha'-1}$, $V_\beta(C_\beta \cap Q_{\alpha'})$ has index at most 2 in C_β and since $[Q_{\alpha'}, V_{\alpha'}] = Z_{\alpha'} \leq V_\beta$, $V_{\alpha'}$ centralizes C_β / V_β so that $O^2(L_\beta)$ centralizes C_β / V_β. Now, applying the three subgroup lemma, any p'-element of R_β centralizes Q_β / C_β and V_β so centralizes Q_β, and we deduce that $R_\beta = Q_\beta$. By Lemma 5.4.12, $\overline{L_\beta} \cong \mathrm{Sz}(2)$ or $(3 \times 3) : 4$ and $V_\beta / C_{V_\beta}(O^2(L_\beta))$ is as described in Lemma 5.4.9. Since L_β is solvable, applying coprime action, we have that $V_\beta / Z_\beta = [V_\beta / Z_\beta, O^2(L_\beta)] \times C_{V_\beta / Z_\beta}(O^2(L_\beta))$ where $[V_\beta / Z_\beta, O^2(L_\beta)]$ is irreducible. Letting V^β be the preimage in V_β of $[V_\beta / Z_\beta, O^2(L_\beta)]$, we must have that $[V^\beta \cap Q_{\alpha'}, V_{\alpha'}] = Z_{\alpha'} \leq V^\beta$ so that $Z_{\alpha'-1} = Z_{\alpha'} \times Z_\beta \leq V^\beta$. But then, by definition, $V^\beta = V_\beta$ and V_β / Z_β is irreducible, as required. $\square$

Proposition 5.4.16. *Suppose that $C_{V_\beta}(V_{\alpha'}) < V_\beta \cap Q_{\alpha'}$ and $b > 1$. Then G is* *locally isomorphic to* ${}^2\mathrm{F}_4(2)$ *or* ${}^2\mathrm{F}_4(2)'$.

Proof. By Lemma 5.4.14 and Lemma 5.4.15, we have that $b = 3$, $L_\alpha/R_\alpha \cong \mathrm{Sym}(3)$, Z_α is natural $\mathrm{SL}_2(2)$-module and either $\overline{L_\beta} \cong \mathrm{Sz}(2)$ or $\overline{L_\beta} \cong (3 \times 3) : 4$. Suppose first that L_α is also 2-minimal group. Then the amalgam is determined in [Hay92], G has a weak BN-pair of rank 2 and the result follows by [DS85] and [Fan86]. Hence, to complete the proof, we assume that L_α is not 2-minimal and derive a contradiction. We may choose $P_\alpha < L_\alpha$ such that P_α is 2-minimal. Better, by McBride's lemma (Lemma 2.1.19), we may choose P_α such that $P_\alpha \not\leq R_\alpha$ and $L_\alpha = P_\alpha R_\alpha$. Moreover, we may assume that G is a minimal counterexample to Theorem 5.2.2. Form $X := \langle P_\alpha, L_\beta(G_{\alpha,\beta} \cap P_\alpha)\rangle$ and let Q be the largest subgroup of S which is normal in X.

If $Q = \{1\}$, then it follows that any non-trivial normal subgroup of X which is contained in $G_{\alpha,\beta} \cap P_\alpha$ is a $2'$-group, a contradiction for then Q_λ is not self centralizing in G_λ, where $\lambda \in \{\alpha, \beta\}$. Thus, no non-trivial normal subgroup of $G_{\alpha,\beta} \cap P_\alpha$ is normal in X and the triple $(P_\alpha, L_\beta(G_{\alpha,\beta} \cap P_\alpha), G_{\alpha,\beta} \cap P_\alpha)$ satisfies Hypothesis 5.2.1. Then, by minimality and comparing with the list of amalgams in Theorem 5.2.2, it follows that X is locally isomorphic to ${}^2\mathrm{F}_4(2)$ or ${}^2\mathrm{F}_4(2)'$. In particular, $P_\alpha/Q_\alpha \cong \mathrm{Sym}(3)$, $G_\beta/Q_\beta \cong \mathrm{Sz}(2)$ and S is isomorphic to a Sylow 2-subgroup of ${}^2\mathrm{F}_4(2)$ or ${}^2\mathrm{F}_4(2)'$. But then $2^2 \leqslant |Q_\alpha/\Phi(Q_\alpha)| \leqslant 2^3$ and so, $\overline{L_\alpha}$ is isomorphic to a subgroup of $\mathrm{GL}_3(2)$ which has a strongly 2-embedded subgroup. An elementary calculation, that may be performed in MAGMA, yields $\overline{L_\alpha} \cong \overline{P_\alpha} \cong \mathrm{Sym}(3)$ and L_α is 2-minimal, a contradiction.

Thus, $Q \neq \{1\}$ and since P_α does not centralize Z_β and $Q \trianglelefteq S$, we deduce that $Z_\alpha \leq Q$ and so $V_\beta \leq Q$. Moreover, since $Q \leq Q_\alpha \cap Q_\beta$ and $Q \trianglelefteq L_\beta$, $Q \leq C_\beta$.

If $\Phi(Q) \neq \{1\}$ then $Z_\beta \leq \Phi(Q)$ and arguing as above, $V_\beta \leq \Phi(Q)$. But then $O^2(L_\beta)$ centralizes $Q/\Phi(Q)$, a contradiction. Thus, Q is elementary abelian and since $C_S(Q) \leq C_\beta$, $C_S(Q) = C_{Q_\alpha}(Q) = C_{Q_\beta}(Q) \trianglelefteq X$ and $C_S(Q) = Q$.

Suppose that there is $r \in P_\alpha$ such that $[r, Q_\alpha] \leq Q$. If r centralizes $C_Q(Q_\alpha)$, then by the A$\times$B-lemma, r centralizes Q. But then r centralizes Q_α, and so r is trivial. Now, since Q_α is self centralizing in S, $C_Q(Q_\alpha) \leq Z(Q_\alpha)$. But $V_{\alpha'} \cap Q_\alpha$ is of index 4 in $V_{\alpha'}$, contains $Z_{\alpha'-1}$ and is centralized by $Z(Q_\alpha)$ from which it follows that $Z(Q_\alpha) = Z_\alpha(Z(Q_\alpha) \cap Q_{\alpha'})$. Since $Z_{\alpha'} \not\leq Z(Q_\alpha)$, otherwise $Z_{\alpha'-1} = Z_{\alpha'} \times Z_\beta$ would be normalized by $L_\beta = \langle Q_\beta, Q_\alpha, Q_{\alpha'-1}\rangle$, it follows that $V_{\alpha'} \cap Q_\beta$ centralizes $Z(Q_\alpha)/Z_\alpha$ and so $O^2(L_\alpha)$ centralizes $Z(Q_\alpha)/Z_\alpha$. Since $Z_\beta \leq Z_\alpha = [Z(Q_\alpha), O^2(L_\alpha)]$, it follows from coprime action that $Z(Q_\alpha) = Z_\alpha$. Hence, for r of odd order such that $[r, Q_\alpha] \leq Q$, we have that $r \not\leq R_\alpha$ and it follows that r is of order 3 and $\langle r\rangle Q/Q = O_{2'}(P_\alpha/Q)$. Then, by coprime action and as r acts non-trivially on Z_α, we have that $Q = [Q, r]$. But now, Q is elementary abelian and contains V_β, it follows that $Q \cap Q_{\alpha'} \cap Q_{\alpha'+1}$ is has index p^2 in Q and is centralized by $Z_{\alpha'+1} \cap Q_\beta \not\leq Q_\alpha$. In particular, Q contains at most two non-central chief factors for P_α and Q is acted upon quadratically $V_{\alpha'} \cap Q_\beta$. Note that $Q/[Q, Q_\alpha]$ is not centralized by r, and neither is $[Q, Q_\alpha]$. But then $[Q, Q_\alpha] \leq Z(Q_\alpha) = Z_\alpha$ and $Q/[Q, Q_\alpha]$ is an FF-module, absurd for then the action of r implies that $2^5 = |V_\beta| < |Q| = 2^4$. Thus, P_α/Q is of characteristic 2.

Suppose that there is $s \in L_\beta(P_\alpha \cap G_{\alpha,\beta})$ such that $[s, Q_\beta] \leq Q$. Since $L_\beta/Q_\beta \cong \mathrm{Sz}(2)$ it follows that $L_\beta(P_\alpha \cap G_{\alpha,\beta})/Q_\beta = L_\beta/Q_\beta \times (P_\alpha \cap G_{\alpha,\beta})/Q_\beta$. Since $Q \leq C_\beta$ and Q_β/C_β is an irreducible module for $\overline{L_\beta}$, $s \not\leq L_\beta$. Hence, s centralizes S/Q_β and so centralizes S/Q. Then $s \in P_\alpha$ and centralizes Q_α/Q, and by the previous paragraph, $s = 1$. Thus, $L_\beta(P_\alpha \cap G_{\alpha,\beta})/Q$ is of characteristic 2.

Moreover, no subgroup of S properly containing Q is normal in X and since P_α/Q is of characteristic 2, it follows that no non-trivial subgroup of $(G_{\alpha,\beta} \cap P_\alpha)/Q$ is normal in X/Q. Then the triple $(P_\alpha/Q, (L_\beta(G_{\alpha,\beta} \cap P_\alpha))/Q, (G_{\alpha,\beta} \cap P_\alpha)/Q)$ satisfies Hypothesis 5.2.1. By minimality and since $L_\beta/Q_\beta \cong \mathrm{Sz}(2)$, X/Q is locally isomorphic to ${}^2\mathrm{F}_4(2)$ or ${}^2\mathrm{F}_4(2)'$. But there is only one non-central chief factor in Q_β/Q for L_β, and we have a contradiction. $\square$

5.4.2 $C_{V_\beta}(V_{\alpha'}) = V_\beta \cap Q_{\alpha'}$

We continue with the analysis of the case $[Z_\alpha, Z_{\alpha'}] = \{1\}$, this time with the additional assumptions that $b > 1$ and $[V_\beta \cap Q_{\alpha'}, V_{\alpha'}] = \{1\}$. Recall from Lemma 5.4.4 and Lemma 5.4.5 that this hypothesis implies that $L_\alpha/R_\alpha \cong L_\beta/R_\beta \cong \mathrm{SL}_2(p)$ and Z_α and $V_\beta/C_{V_\beta}(O^p(L_\beta))$ are natural $\mathrm{SL}_2(p)$-modules.

Throughout this section, we fix the notation $V^\lambda := \langle (C_{V_\mu}(O^p(L_\mu)))^{G_\lambda} \rangle$ whenever $\lambda \in \alpha^G$, $\mu \in \Delta(\lambda)$ and $|V_\beta| \neq p^3$, and we remark that when $|V_\beta| \neq p^3$ and $b > 5$, for $\gamma \in \beta^G$ and some fixed $\delta \in \Delta(\gamma)$, the subgroup $\langle V^\mu \mid Z_\mu = Z_\delta, \mu \in \Delta(\gamma) \rangle$ is normal in $R_\gamma Q_\delta$ by essential the same argument as Lemma 5.2.19. Throughout, we set $R := [V_{\alpha'}, V_\beta]$ so that $R \leq Z_{\alpha+2} C_{V_\beta}(O^p(L_\beta)) \cap Z_{\alpha'-1} C_{V_{\alpha'}}(O^p(L_{\alpha'})) \leq V_\beta \cap V_{\alpha'}$ and, in particular, if $|V_\beta| = p^3$, then $R \leq Z_{\alpha+2} \cap Z_{\alpha'-1}$. By the work done in Section 5.4.1, we may assume in this section that every critical pair (α, α') satisfies the condition $C_{V_\beta}(V_{\alpha'}) = V_\beta \cap Q_{\alpha'}$.

As in Section 5.3.2, we intend to control the action of $O^p(R_\alpha)$ and $O^p(R_\beta)$ using the methods in Lemma 5.2.31-Lemma 5.2.35 in the expectation of applying Lemma 5.2.18 to force contradictions. In the following lemmas, we demonstrate that we satisfy Hypothesis 5.2.30, required for the application of these lemmas. Also, as in Section 5.3.2, since $L_\alpha/R_\alpha \cong L_\beta/R_\beta \cong \mathrm{SL}_2(p)$, we will often make a generic appeal to coprime action, utilizing that L_λ is solvable when $p = 2$ for $\lambda \in \{\alpha, \beta\}$, and that there is a central involution $t_\lambda \in L_\lambda/R_\lambda$ which acts fixed point freely on natural modules.

Lemma 5.4.17. *Suppose that $C_{V_\beta}(V_{\alpha'}) = V_\beta \cap Q_{\alpha'}$ and $V_{\alpha'} \leq Q_\beta$. Then $Z_\alpha = Z(Q_\alpha)$ and $Z_\beta = Z(Q_\beta)$.*

Proof. Suppose that $V_{\alpha'} \leq Q_\beta$. We aim to show that if the conclusion of the lemma fails to hold then $R = Z_\beta = Z_{\alpha'}$ for then, as $V_\beta \not\leq Q_{\alpha'}$, $O^p(L_{\alpha'})$ centralizes $V_{\alpha'}$, a contradiction.

Suppose that $V_{\alpha'} \leq Q_\beta$ and $Z_\alpha \neq Z(Q_\alpha)$. By minimality of b, and using that b is odd, we have that $V_\lambda \leq Q_\alpha$ and $Z(Q_\alpha) \leq Q_\lambda$ for all $\lambda \in \Delta^{(b-1)}(\alpha)$. In particular, $Z(Q_\alpha) \leq Q_{\alpha'-1}$ and $Z(Q_\alpha) = Z_\alpha(Z(Q_\alpha) \cap Q_{\alpha'})$. If $[V_{\alpha'}, Z(Q_\alpha) \cap Q_{\alpha'}] = \{1\}$, it follows that $O^p(L_\alpha)$ centralizes $Z(Q_\alpha)/Z_\alpha$ and an application of coprime action, observing that $Z_\beta \leq Z_\alpha = [Z(Q_\alpha), O^p(L_\alpha)]$, gives a contradiction. If $[V_{\alpha'}, Z(Q_\alpha) \cap Q_{\alpha'}] \neq \{1\}$, then $Z_{\alpha'} = [V_{\alpha'}, Z(Q_\alpha) \cap Q_{\alpha'}] \leq Z(Q_\alpha)$ and so $Z_{\alpha'}$ is centralized by $V_{\alpha'}Q_\alpha \in \mathrm{Syl}_p(L_\alpha)$ from which it follows that $Z_{\alpha'} = Z_\beta$, a contradiction. Thus, $Z_\alpha = Z(Q_\alpha)$. Since $Z(S) \leq Z(Q_\alpha)$ we conclude that $Z(S) = \Omega(Z(S)) = Z_\beta$ is of exponent p.

Since $V_\lambda \leq Q_{\alpha'}$ for all $\lambda \in \Delta^{(b-2)}(\alpha')$, again using the minimality of b and that b is odd, we argue that $Z(Q_{\alpha'}) \leq Q_{\alpha+2}$. If $Z(Q_{\alpha'}) \not\leq Q_\beta$ then, as $Z(S) = Z_\beta$, $\{1\} \neq [Z(Q_{\alpha'}), Z(Q_\beta)] \leq Z(Q_{\alpha'}) \cap Z(Q_\beta)$, for otherwise $Z(Q_\beta)$ is centralized by $Z(Q_{\alpha'})Q_\beta \in \mathrm{Syl}_p(L_\beta)$ and the result holds. Then, $[Z(Q_{\alpha'}), Z(Q_\beta)]$ is centralized by $Z(Q_{\alpha'})Q_\beta \in \mathrm{Syl}_p(L_\beta)$ and since $Z(S) = Z_\beta$, $[Z(Q_{\alpha'}), Z(Q_\beta)] = Z_\beta$. Moreover, since $[Z(Q_{\alpha'}), Z(Q_\beta)] \neq \{1\}$, $Z(Q_\beta) \not\leq Q_{\alpha'}$, and by a similar reasoning, $[Z(Q_{\alpha'}), Z(Q_\beta)] = Z_{\alpha'}$. But then $Z_\beta = Z_{\alpha'}$, a contradiction. Hence, $Z(Q_{\alpha'}) \leq Q_\beta$.

Observe that $Z(Q_{\alpha'}) \not\leq Q_\alpha$, else $Z(Q_{\alpha'})$ is centralized by $Z_\alpha Q_{\alpha'} \in \mathrm{Syl}_p(L_{\alpha'})$ and $Z(Q_{\alpha'}) = Z_{\alpha'}$, as desired. Then $Z_\beta = [Z(Q_{\alpha'}), Z_\alpha] \leq \Omega(Z(Q_{\alpha'}))$ so that Z_β is centralized by $Z_\alpha Q_{\alpha'} \in \mathrm{Syl}_p(L_{\alpha'})$ and $Z_\beta = Z_{\alpha'}$, again a contradiction. Therefore, if $V_{\alpha'} \leq Q_\beta$, we have shown that $Z(Q_\beta) = Z_\beta$. $\square$

Lemma 5.4.18. *Suppose that $C_{V_\beta}(V_{\alpha'}) = V_\beta \cap Q_{\alpha'}$ and $V_{\alpha'} \not\leq Q_\beta$. Then $Z_\alpha = Z(Q_\alpha)$ and $Z_\beta = Z(Q_\beta)$.*

Proof. Suppose that $V_{\alpha'} \not\leq Q_\beta$ and $Z(Q_{\alpha'}) \leq Q_\beta$. In addition, assume first that $Z(Q_{\alpha'}) \leq Q_\alpha$ so that $Z(Q_{\alpha'})$ is centralized by $Z_\alpha Q_{\alpha'} \in \mathrm{Syl}_p(L_{\alpha'})$. Set $Y^\beta := \langle Z(Q_\lambda) \mid Z_\lambda = Z_\alpha, \lambda \in \Delta(\beta)\rangle$ and let $r \in R_\beta Q_\alpha$. Since r is a graph automorphism, for $\lambda \in \Delta(\beta)$ such that $Z_\lambda = Z_\alpha$, $Z(Q_\lambda)^r = Z(Q_{\lambda\cdot r})$. But now, $Z_{\lambda\cdot r} = Z_\lambda^r = Z_\alpha^r = Z_\alpha$ and so $Z(Q_\lambda)^r \leq Y^\beta$. Thus, $Y^\beta \trianglelefteq R_\beta Q_\alpha$. Now, observe that by minimality of b, and using that b is odd, $V_\delta \leq Q_\lambda$ and $Z(Q_\lambda) \leq Q_\delta$ for all $\lambda \in \Delta(\beta)$ with $Z_\lambda = Z_\alpha$ and $\delta \in \Delta^{(b-1)}(\lambda)$ by Lemma 5.2.16. In particular, $Z(Q_\alpha) \leq Y^\beta \leq Q_{\alpha'-1}$. Thus, $Z(Q_\alpha) = Z_\alpha(Z(Q_\alpha) \cap Q_{\alpha'})$ and $Y^\beta = Z_\alpha(Y^\beta \cap Q_{\alpha'})$.

Since $Z(Q_\alpha) \cap Q_{\alpha'}$ is a maximal subgroup of $Z(Q_\alpha)$ not containing Z_α, we must have that $Z_\alpha \not\leq \Phi(Z(Q_\alpha))$. But then, by the irreducibility of Z_α under the action of G_α, $Z_\beta \cap \Phi(Z(Q_\alpha)) = \Omega(Z(S)) \cap \Phi(Z(Q_\alpha)) = \{1\}$ so that $\Phi(Z(Q_\alpha)) = \{1\}$ and $Z(Q_\alpha) = \Omega(Z(Q_\alpha))$ is elementary abelian.

Assume first that $[Y^\beta \cap Q_{\alpha'}, V_{\alpha'}] = Z_{\alpha'}$ so that $Y^\beta \not\leq V_\beta$ and there is some $\alpha'+1 \in \Delta(\alpha')$ with $Y^\beta \cap Q_{\alpha'} \not\leq Q_{\alpha'+1}$. Again, using the minimality of b and that b is odd, we deduce that $Z(Q_{\alpha'+1}) \leq Q_{\alpha+2}$. Write $Y_\beta = \langle Z(Q_\alpha)^{G_\beta}\rangle$ so that $Y^\beta \leq Y_\beta \trianglelefteq G_\beta$ and, as $b > 2$, Y_β is abelian. Then $Z(Q_{\alpha'+1})$ normalizes Y_β, $[Z(Q_{\alpha'+1}), Y^\beta \cap Q_{\alpha'}, Y^\beta \cap Q_{\alpha'}] \leq [Z(Q_{\alpha'+1}), Y_\beta, Y_\beta] = \{1\}$ and $Z(Q_{\alpha'+1})$ is quadratic module for $\overline{L_{\alpha'+1}}$. Moreover, by coprime action, $Z(Q_{\alpha'+1}) = [Z(Q_{\alpha'+1}), R_{\alpha'+1}] \times C_{Z(Q_{\alpha'+1})}(R_{\alpha'+1})$

is invariant under $T \in \mathrm{Syl}_p(G_{\alpha',\alpha'+1})$ and as $Z_{\alpha'} \leq Z_{\alpha'+1} \leq C_{Z(Q_{\alpha'+1})}(R_{\alpha'+1})$, we infer that $Z(Q_{\alpha'+1}) = C_{Z(Q_{\alpha'+1})}(R_{\alpha'+1})$ and $Z(Q_{\alpha'+1})$ is a faithful module for $L_{\alpha'+1}/R_{\alpha'+1} \cong \mathrm{SL}_2(p)$. But then by Lemma 2.3.11, $Z(Q_{\alpha'+1})$ is a direct sum of natural $\mathrm{SL}_2(p)$-modules. Now, since $[Z(Q_{\alpha'+1}), Y^\beta \cap Q_{\alpha'}]$ is of exponent p and centralized by $(Y^\beta \cap Q_{\alpha'})Q_{\alpha'+1} \in \mathrm{Syl}_p(G_{\alpha',\alpha'+1})$, we have that $[Z(Q_{\alpha'+1}), Y^\beta \cap Q_{\alpha'}] = Z_{\alpha'}$ is of order p from which it follows that $Z(Q_{\alpha'+1})$ contains a unique summand. Hence, $Z(Q_{\alpha'+1}) = Z_{\alpha'+1}$ and by conjugacy, $Z_\alpha = Z(Q_\alpha)$. But then $Y^\beta \leq V_\beta$, and we have a contradiction.

Suppose now that $[Y^\beta \cap Q_{\alpha'}, V_{\alpha'}] = \{1\}$. Then $[V_{\alpha'}, Y^\beta] \leq V_\beta$ and, as $Z_\alpha \neq Z_{\alpha+2}$, we conclude that $Y^\beta V_\beta \trianglelefteq L_\beta = \langle V_{\alpha'}, R_\beta, Q_\alpha \rangle$. But $V_{\alpha'}$ centralizes $Y^\beta V_\beta / V_\beta$ so that $O^p(L_\beta)$ centralizes $Y^\beta V_\beta / V_\beta$ and it follows that $Y^\beta V_\beta = Z(Q_\alpha) V_\beta \trianglelefteq L_\beta$. Then $[Z(Q_\alpha), Q_\beta] \trianglelefteq L_\beta$ and since $Q_\alpha \cap Q_\beta$ centralizes $[Z(Q_\alpha), Q_\beta]$ and $Q_\alpha \cap Q_\beta \ntrianglelefteq L_\beta$ by Proposition 5.2.25, we must have that $[Z(Q_\alpha), Q_\beta] \leq Z(S)$ and $[Z(Q_\alpha), Q_\beta, L_\beta] = \{1\}$. Now, $[O^p(L_\beta), Z(Q_\alpha), Q_\beta] \leq [V_\beta, Q_\beta] = Z_\beta$ and by the three subgroup lemma $[Q_\beta, O^p(L_\beta), Z(Q_\alpha)] \leq Z_\beta \leq Z_\alpha$. Since $[Q_\beta, O^p(L_\beta)] \not\leq Q_\alpha$, it follows that $O^p(L_\alpha)$ centralizes $Z(Q_\alpha)/Z_\alpha$ and coprime action yields $Z(Q_\alpha) = [Z(Q_\alpha), O^p(L_\alpha)] \times C_{Z(Q_\alpha)}(O^p(L_\alpha))$. But $Z_\beta \leq Z_\alpha = [Z(Q_\alpha), O^p(L_\alpha)]$ and $Z(Q_\alpha) = Z_\alpha$. Since $Z(Q_{\alpha'}) \leq Z(T)$, for $T \in \mathrm{Syl}_p(L_{\alpha'} \cap L_{\alpha'-1})$, we have that $Z(Q_{\alpha'}) = Z_{\alpha'}$ and $Z(Q_\alpha) = Z_\alpha$, as required. $\square$

Thus, throughout this subsection, whenever we assume the necessary values of b, we are able to apply Lemma 5.2.31 through Lemma 5.2.35. That the hypotheses of these lemmas are satisfied will often be left implicit in proofs.

The first goal in the analysis of the case $C_{V_\beta}(V_{\alpha'}) = V_\beta \cap Q_{\alpha'}$ will be to show that $b \leqslant 5$. Then the methods for $b = 5$ differ slightly from the techniques employed for lager values of b and so, for the most part, we treat the case when $b = 5$ independently from the the other cases. The case when $b = 3$ is different again and so this case is also treated separately.

The following lemma is also valid whenever $b = 3$ but, as mentioned above, since the techniques we apply when $b = 3$ are somewhat disparate from the rest of this subsection, we only prove it here whenever $b > 3$.

Lemma 5.4.19. *Suppose that $C_{V_\beta}(V_{\alpha'}) = V_\beta \cap Q_{\alpha'}$ and $b > 3$. If $V_\alpha^{(2)} \leq Q_{\alpha'-2}$ and $V_{\alpha'} \leq Q_\beta$ then $R = Z_\beta \leq Z_{\alpha'-1}$, $|V_\beta| = p^3$, $V_\alpha^{(2)}/Z_\alpha$ is an FF-module for $\overline{L_\alpha}$ and one of the following holds:*

(i) *$V_\alpha^{(2)} \leq Q_{\alpha'-1}$ and $[V_\alpha^{(2)} \cap Q_{\alpha'}, V_{\alpha'}] = Z_{\alpha'} \leq V_\alpha^{(2)}$; or*

(ii) *$V_\alpha^{(2)} \not\leq Q_{\alpha'-1}$ and $[V_\alpha^{(2)} \cap Q_{\alpha'}, V_{\alpha'}] = \{1\}$.*

Proof. Suppose first that $V_\alpha^{(2)} \leq Q_{\alpha'-1}$. Then $V_\alpha^{(2)} = Z_\alpha(V_\alpha^{(2)} \cap Q_{\alpha'})$ and since $V_\alpha^{(2)}/Z_\alpha$ contains a non-central chief factor for L_α, $[V_\alpha^{(2)} \cap Q_{\alpha'}, V_{\alpha'}] = Z_{\alpha'} \not\leq Z_\alpha$. Then, for $\alpha'+1 \in \Delta(\alpha')$ with $Z_{\alpha'+1} \not\leq Q_\alpha$ it follows that $[Z_{\alpha'+1}, V_\alpha^{(2)} \cap Q_{\alpha'} \cap Q_{\alpha'+1}] = \{1\}$ and $V_\alpha^{(2)}/Z_\alpha$ contains a unique non-central chief factor which is an FF-module for $\overline{L_\alpha}$. Then by Lemma 5.2.31, $|V_\beta| = p^3$, $[V_\alpha^{(2)}, Q_\alpha] = Z_\alpha$ and $Z_\beta = R \leq Z_{\alpha'-1} \cap Z_{\alpha+2}$.

Suppose now that $V_\alpha^{(2)} \not\leq Q_{\alpha'-1}$ and $C_{V_\beta}(O^p(L_\beta)) \neq Z_\beta$ so that by Lemma 5.2.31, $|V_\beta| = p^4$. Then, again by Lemma 5.2.31, both V^α/Z_α and $V_\alpha^{(2)}/V^\alpha$ contain a non-central chief factor for L_α. If $V^\alpha \not\leq Q_{\alpha'-1}$, then $V_\alpha^{(2)} = V^\alpha(V_\alpha^{(2)} \cap Q_{\alpha'})$ and so $Z_{\alpha'} \leq V_\alpha^{(2)}$ but $Z_{\alpha'} \not\leq V^\alpha$. Then, since $b > 3$, $V_\alpha^{(2)}$ is elementary abelian and $V_\alpha^{(2)} \not\leq Q_{\alpha'-1}$, $Z_{\alpha'} = Z_{\alpha'-2} = [V^\alpha, Z_{\alpha'-1}] \leq V^\alpha$, a contradiction. Thus, $V^\alpha \leq Q_{\alpha'-1}$ and since V^α/Z_α contains a non-central chief factor, it follows that $[V^\alpha \cap Q_{\alpha'}, V_{\alpha'}] =$

$Z_{\alpha'} \leq V^\alpha$ and V^α/Z_α is an FF-module for $\overline{L_\alpha}$. Since $V_\alpha^{(2)} \not\leq Q_{\alpha'-1}$ and $V_\alpha^{(2)}$ is abelian, $Z_{\alpha'} = Z_{\alpha'-2}$, $(V_\alpha^{(2)} \cap Q_{\alpha'-1})/V^\alpha$ is centralized by $V_{\alpha'}$ and $V_\alpha^{(2)}/V^\alpha$ is also an FF-module for $\overline{L_\alpha}$. Then, applying Lemma 5.2.32 and Lemma 5.2.18 to $Z_{\alpha'} = Z_{\alpha'-2}$, we conclude that $V_{\alpha'} = V_{\alpha'-2} \leq Q_\alpha$, a contradiction.

Thus, we assume now that $C_{V_\beta}(O^p(L_\beta)) = Z_\beta$, $|V_\beta| = p^3$ and $Z_\beta = R \leq Z_{\alpha'-1} \cap Z_{\alpha+2}$. If $[V_\alpha^{(2)} \cap Q_{\alpha'}, V_{\alpha'}] = Z_{\alpha'} \leq V_\alpha^{(2)}$, then $Z_{\alpha'-1} = Z_\beta \times Z_{\alpha'}$ is centralized by $V_\alpha^{(2)}$ and $V_\alpha^{(2)} \leq Q_{\alpha'-1}$, a contradiction. Thus, $[V_\alpha^{(2)} \cap Q_{\alpha'}, V_{\alpha'}] = \{1\}$, $(V_\alpha^{(2)} \cap Q_{\alpha'-1})/Z_\alpha$ is centralized by $V_{\alpha'}$ and $V_\alpha^{(2)}/Z_\alpha$ is an FF-module for $\overline{L_\alpha}$. □

Lemma 5.4.20. *Suppose that $C_{V_\beta}(V_{\alpha'}) = V_\beta \cap Q_{\alpha'}$ and $b > 5$. If $V_{\alpha'} \not\leq Q_\beta$ and $V_\alpha^{(2)} \leq Q_{\alpha'-2}$, then $|V_\beta| = p^3$.*

Proof. Suppose that $|V_\beta| \neq p^3$ so that both V^α/Z_α and $V_\alpha^{(2)}/V^\alpha$ contain a non-central chief factor for L_α. Choose $\alpha' + 1 \in \Delta(\alpha')$ with $Z_{\alpha'+1} \not\leq Q_\beta$. In particular, $(\alpha'+1, \beta)$ is a critical pair and we may assume that $C_{V_{\alpha'}}(V_\beta) = V_{\alpha'} \cap Q_\beta$. Set $U^\beta := \langle V^\lambda \mid \lambda \in \Delta(\beta), Z_\lambda = Z_\alpha \rangle$ so that $R_\beta Q_\alpha$ normalizes U^β by Lemma 5.2.19. Setting $U^{\alpha'} := \langle V^\mu \mid \mu \in \Delta(\alpha'), Z_\mu = Z_{\alpha'+1} \rangle$, it follows similarly that $U^{\alpha'} \trianglelefteq R_{\alpha'} Q_{\alpha'+1}$. Throughout, for $\mu \in \beta^G$, we set $U_\mu := \langle (V^{\mu+1})^{L_\mu} \rangle$ where $\mu + 1 \in \Delta(\mu)$. In particular, $U^\beta \leq U_\beta \trianglelefteq L_\beta$.

Notice throughout that if $R \leq Z_{\alpha'-1}$, then $Z_{\alpha'-1} Z_{\alpha'-1}^g$ is normalized by $L_{\alpha'} = \langle V_\beta, (V_\beta)^g, R_{\alpha'} \rangle$ for some suitable $g \in L_{\alpha'}$. Then, from the definition of $V_{\alpha'}$, we conclude that $V_{\alpha'} = Z_{\alpha'-1} Z_{\alpha'-1}^g$ is of order p^3, as required. A similar conclusion follows if $R \leq Z_{\alpha+2}$.

Suppose first that $U^\beta \not\leq Q_{\alpha'-2}$ and so there is some $\lambda \in \Delta(\beta)$ with $V^\lambda \not\leq Q_{\alpha'-2}$ and $Z_\lambda = Z_\alpha$. In particular, since $V_{\alpha'-2} \leq Q_\lambda$ and $Z_\alpha \not\leq V_{\alpha'-2}$, we deduce that $[V_{\alpha'-2}, V^\lambda] = Z_\beta \leq V_{\alpha'-2}$ and $Z_{\alpha'-2} \neq Z_\beta$. If, in addition, $U_{\alpha'-2} \not\leq Q_\beta$, then there is $\delta \in \Delta(\alpha'-2)$ with $[V^\delta, V_\beta] \leq Z_\delta$. In particular, it follows that $R \leq [V^\delta, V_\beta] \leq Z_\delta$ and since $R \not\leq Z_{\alpha'-1}$, otherwise $|V_{\alpha'}| = p^3$, it follows that $Z_\delta = R \times Z_{\alpha'-2}$ centralizes V^λ. But $V^\lambda \not\leq Q_{\alpha'-2}$ and since V^λ centralizes $Z_{\alpha'-3}C_{V_{\alpha'-2}}(O^p(L_{\alpha'-2}))$ we have that $Z_\delta = Z_{\alpha'-3}$ by Lemma 5.2.31. But now, $Z_{\alpha'-3} \leq V_{\alpha'-2} \cap V_{\alpha'}$ and again by Lemma 5.2.31, we conclude that $R \leq Z_{\alpha'-3} = Z_{\alpha'-1}$, a contradiction.

If $U_{\alpha'-2} \leq Q_\beta$, then for any $\delta \in \Delta(\alpha'-2)$, $[V^\delta, V_\beta] \leq Z_\beta \cap Z_\delta$ by Lemma 5.2.31. If $Z_\beta \leq Z_\delta$ for some δ, then $[V^\lambda, V_{\alpha'-2}] \leq Z_\beta \leq Z_\delta$ and $|V_{\alpha'-2}| = p^3$, a contradiction. Thus, $[U_{\alpha'-2}, V_\beta] = \{1\}$ and $U_{\alpha'-2} \leq Q_\lambda$ so that $[U_{\alpha'-2}, V^\lambda] = Z_\lambda \cap U_{\alpha'-2} = Z_\alpha \cap U_{\alpha'-2} \leq Z_\beta \leq V_{\alpha'-2}$ by Lemma 5.2.31, and V^λ centralizes $U_{\alpha'-2}/V_{\alpha'-2}$. But then $O^p(L_{\alpha'-2})$ centralizes $U_{\alpha'-2}/V_{\alpha'-2}$, a contradiction by Lemma 5.2.31, for then $V^{\alpha'-1}V_{\alpha'-2} \trianglelefteq L_{\alpha'-2}$. Thus, $U^\beta \leq Q_{\alpha'-2}$. Notice that $V_\alpha^{(2)}$ is not involved in the above arguments and so we may repeat the above arguments to conclude that $U^{\alpha'} \leq Q_{\alpha+3}$.

Assume now that $U^\beta \leq Q_{\alpha'-2}$ but $U^\beta \not\leq Q_{\alpha'-1}$. Then, as $Z_{\alpha'-1} \leq Q_\alpha$, it follows by Lemma 5.2.31 that $Z_{\alpha'-2} = [U^\beta, Z_{\alpha'-1}] \leq Z_\alpha$ and $Z_{\alpha'-2} = Z_\beta$ since $Z_\alpha \not\leq Q_{\alpha'}$. Then $[V^{\alpha'-1}, V_\beta] \leq Z_{\alpha'-1} \cap V_\beta$ and since $V_\beta U^\beta \leq V_\beta^{(3)}$ is abelian, it follows that $[V^{\alpha'-1}, V_\beta] \leq Z_{\alpha'-2} = Z_\beta$ and $V^{\alpha'-1} \leq Q_\beta$. If $V^{\alpha'-1} \leq Q_\alpha$, then $[V^{\alpha'-1}, V^\lambda] \leq Z_\alpha$ for $\lambda \in \Delta(\beta)$ with $Z_\lambda = Z_\alpha$ and $V^\lambda \not\leq Q_{\alpha'-1}$. Since $Z_\alpha \not\leq V_{\alpha'-1}^{(2)} \leq Q_{\alpha'}$, $[V^{\alpha'-1}, V^\lambda] \leq Z_\lambda \cap Q_{\alpha'} = Z_\alpha \cap Q_{\alpha'} = Z_\beta = Z_{\alpha'-2} \leq Z_\lambda$, a contradiction since $V^\lambda \not\leq Q_{\alpha'-1}$. Therefore $V^{\alpha'-1} \not\leq Q_\lambda$ and as

$$[V^\lambda \cap Q_{\alpha'-1}, V^{\alpha'-1}] \leq Z_{\alpha'-1} \cap V^\lambda = C_{Z_{\alpha'-1}}(U^\beta) = Z_{\alpha'-2} = Z_\beta \leq Z_\alpha = Z_\lambda,$$

V^λ/Z_λ is an FF-module for $\overline{L_\lambda}$. Moreover, $V_\lambda^{(2)} \cap Q_{\alpha'-2} = V^\lambda(V_\lambda^{(2)} \cap Q_{\alpha'-1})$ and $V_\lambda^{(2)}/V^\lambda$ is also an FF-module for $\overline{L_\lambda}$. Then Lemma 5.2.32 implies that $O^p(R_\lambda)$ centralizes $V_\lambda^{(2)}$. By Lemma 5.2.18, $Z_{\alpha+3} \neq Z_\beta = Z_{\alpha'-2}$ and so $V_{\alpha'}^{(3)} \cap Q_{\alpha+3}$

centralizes $Z_{\alpha+2}$ and $V_{\alpha'}^{(3)} \cap Q_{\alpha+3} = V_{\alpha'}(V_{\alpha'}^{(3)} \cap Q_\beta)$. Since $Z_\beta \leq V_{\alpha'}$, have that $V_{\alpha'}^{(3)}/V_{\alpha'}$ contains a unique non-central chief factor and by Lemma 5.2.34, $O^p(R_\beta)$ centralizes $V_\beta^{(3)}$.

By Lemma 5.2.18, $Z_\alpha = Z_\lambda$ implies that $V^\alpha = V^\lambda = U^\beta$ and $V_\alpha^{(2)} = V_\lambda^{(2)}$. Thus, $V^\alpha \not\leq Q_{\alpha'-1}$ and since $V_\alpha^{(2)} \leq Q_{\alpha'-2}$, we have that $V_\alpha^{(2)} = V^\alpha(V_\alpha^{(2)} \cap Q_{\alpha'-1})$. Since $Z_{\alpha'-1} \not\leq V_\alpha^{(2)}$, we conclude that $[V^{\alpha'-1}, V_\alpha^{(2)} \cap Q_{\alpha'-1}] = Z_{\alpha'-2} \leq V^\alpha$ so that $O^p(L_\alpha)$ centralizes $V_\alpha^{(2)}/V^\alpha$, a contradiction.

Thus, we may assume for the remainder of this proof that $U^\beta \leq Q_{\alpha'-1}$. If $[U^\beta \cap Q_{\alpha'}, V_{\alpha'}] \leq V_\beta U^\beta$, then $V_{\alpha'}$ normalizes $V_\beta U^\beta$ and so $U_\beta = V_\beta U^\beta \trianglelefteq L_\beta = \langle V_{\alpha'}, R_\beta, Q_\alpha \rangle$. But then $[Q_\alpha, V_\beta U^\beta] \leq Z_\alpha[Q_\alpha, V_\beta] \leq V_\beta$ and so, $O^p(L_\beta)$ centralizes U_β/V_β, $V^\alpha V_\beta \trianglelefteq L_\beta$ and a contradiction is provided by Lemma 5.2.31. Thus, $Z_{\alpha'} \leq U_\beta$, $Z_{\alpha'} \not\leq V_\beta U^\beta$ and $[U^\beta \cap Q_{\alpha'}, Z_{\alpha'+1}] = Z_{\alpha'}$. Furthermore, we have that $U^{\alpha'} \leq Q_{\alpha+3}$. If $U^{\alpha'} \not\leq Q_{\alpha+2}$, then as $Z_{\alpha+2} \leq C_{\alpha'}$, we deduce that $Z_{\alpha+3} = [Z_{\alpha+2}, U^{\alpha'}] \leq Z_{\alpha'+1} \cap Q_\beta = Z_{\alpha'}$, a contradiction for then $Z_{\alpha'} \leq V_\beta$. Thus, $U^{\alpha'} \leq Q_{\alpha+2}$.

If $V^{\alpha'+1} \cap Q_\beta \leq Q_\alpha$, $[V^{\alpha'+1} \cap Q_\beta, U^\beta \cap Q_{\alpha'}] \leq Z_\alpha \cap V^{\alpha'+1}$ and since $Z_\alpha \not\leq Q_{\alpha'}$ and $V^{\alpha'+1}/Z_{\alpha'+1}$ contains a non-central chief factor, we have that $[V^{\alpha'+1} \cap Q_\beta, U^\beta \cap Q_{\alpha'}] = Z_\beta \leq U^{\alpha'}$. But $U^{\alpha'} = Z_{\alpha'+1}(U^{\alpha'} \cap Q_\beta)$ and V_β normalizes $U^{\alpha'}V_{\alpha'}$ so that $U_{\alpha'} = U^{\alpha'}V_{\alpha'} \trianglelefteq L_{\alpha'} = \langle V_\beta, Q_{\alpha'+1}, R_{\alpha'} \rangle$. Thus, $[Q_{\alpha'+1}, U^{\alpha'}V_{\alpha'}] \leq V_{\alpha'}$ so that $O^p(L_{\alpha'})$ centralizes $U_{\alpha'}/V_{\alpha'}$ from which it follows that $V^{\alpha'+1}V_{\alpha'} \trianglelefteq L_{\alpha'}$, a contradiction by Lemma 5.2.31.

Suppose now that $V^{\alpha'+1} \cap Q_\beta \not\leq Q_\alpha$. Then $[U^\beta \cap Q_{\alpha'+1}, V^{\alpha'+1} \cap Q_\beta] \leq Z_{\alpha'+1} \cap U^\beta$ and since $(\alpha'+1, \beta)$ is critical, $[U^\beta \cap Q_{\alpha'+1}, V^{\alpha'+1} \cap Q_\beta] \leq Z_{\alpha'} \cap U^\beta$. Since $Z_{\alpha'} \not\leq U^\beta$, $[U^\beta \cap Q_{\alpha'+1}, V^{\alpha'+1} \cap Q_\beta] = \{1\}$. In particular, it follows that $[V^\alpha \cap Q_{\alpha'+1}, V^{\alpha'+1} \cap$

$Q_\beta] = \{1\}$ and since V^α/Z_α contains a non-central chief factor, $V^\alpha \cap Q_{\alpha'} \not\leq Q_{\alpha'+1}$ and V^α/Z_α is an FF-module for $\overline{L_\alpha}$. Since $Z_{\alpha'} \leq U_\beta$, we either have that $Z_{\alpha'} = Z_{\alpha'-2}$; or $Z_{\alpha'} \neq Z_{\alpha'-2}$ and $V_\beta^{(3)}$ centralizes $Z_{\alpha'-1}$.

Assume first that $Z_{\alpha'} = Z_{\alpha'-2}$. Since $Z_{\alpha'} \not\leq U^\beta$, $Z_{\alpha'-2} \not\leq V^\alpha$. In particular, since $V_\alpha^{(2)} \leq Q_{\alpha'-2}$, $[V_{\alpha'-2}, V_\alpha^{(2)}] \leq Z_{\alpha'-2} \cap V^\alpha = \{1\}$ and $V_\alpha^{(2)} = V^\alpha(V_\alpha^{(2)} \cap Q_{\alpha'+1})$. Since $V_\alpha^{(2)}/V^\alpha$ contains a non-central chief factor, we have that $[V_\alpha^{(2)} \cap Q_{\alpha'+1}, V^{\alpha'+1} \cap Q_\beta] = Z_{\alpha'} \leq V_\alpha^{(2)}$, $Z_{\alpha'} \not\leq V^\alpha$ and $V_\alpha^{(2)}/V^\alpha$ is an FF-module for $\overline{L_\alpha}$. But now, by Lemma 5.2.32, $O^p(R_\alpha)$ centralizes $V_\alpha^{(2)}$ and Lemma 5.2.18 applied to $Z_{\alpha'} = Z_{\alpha'-2}$ gives $V_{\alpha'} = V_{\alpha'-2} \leq Q_\beta$, a contradiction.

We assume that $Z_{\alpha'} \neq Z_{\alpha'-2}$ for the remainder of this proof. If $R \leq V_{\alpha'-2}$, then as $R \not\leq Z_{\alpha'-1}$, $|RZ_{\alpha'-1}| = p^3$ and $RZ_{\alpha'-1} = V_{\alpha'} \cap V_{\alpha'-2} = Z_{\alpha'-1}C_{V_{\alpha'}}(O^p(L_{\alpha'}))$. Since V^α/Z_α is an FF-module, the proof of Lemma 5.2.32 implies that $O^p(R_{\alpha'-1})$ centralizes R. Then $[R, Q_{\alpha'}] \leq Z_{\alpha'}$ and $[R, Q_{\alpha'-2}] \leq Z_{\alpha'-2}$ and so $Z_{\alpha'-1}C_{V_{\alpha'}}(O^p(L_{\alpha'})) = RZ_{\alpha'-1} \trianglelefteq L_{\alpha'-1} = \langle Q_{\alpha'}, Q_{\alpha'-2}, O^p(R_{\alpha'-1})\rangle$. But then, by definition, $V^{\alpha'-1} = RZ_{\alpha'-1}$ and $V^{\alpha'-1}/Z_{\alpha'-1}$ does not contain a non-central chief factor for $L_{\alpha'-1}$ and we have a contradiction by Lemma 5.2.31. Thus, $R \not\leq V_{\alpha'-2}$ and as $V_\beta \leq C_{\alpha'-2}$, we conclude that $R \not\leq [V_\beta, U_{\alpha'-2}] \leq V_{\alpha'-2}$.

If $U_\beta \not\leq Q_{\alpha'-2}$, then as $Z_{\alpha'} \leq U_\beta$ and $V_\beta^{(3)}$ centralizes $Z_{\alpha'-1} = Z_{\alpha'} \times Z_{\alpha'-2}$, $V_\beta^{(3)} = U_\beta(V_\beta^{(3)} \cap Q_{\alpha'})$ and $V_{\alpha'}$ centralizes $V_\beta^{(3)}/U_\beta$. Then, $O^p(L_\beta)$ centralizes $V_\beta^{(3)}/U_\beta$ and $V_\beta^{(3)} = V_\alpha^{(2)}U_\beta$. But then, by conjugacy $V_{\alpha'} \leq V_{\alpha'-2}^{(3)} = V_{\alpha'-3}^{(2)}U_{\alpha'-2}$ and since V_β centralizes $V_{\alpha'-3}^{(2)}$, $R = [V_\beta, V_{\alpha'}] \leq [V_\beta, U_{\alpha'-2}]$, a contradiction. Thus, $U_\beta \leq Q_{\alpha'-2}$ and as $Z_{\alpha'-1} = Z_{\alpha'} \times Z_{\alpha'-2}$ is centralized by U_β, $U_\beta \leq Q_{\alpha'-1}$. Then, as $V^\alpha V_\beta \ntrianglelefteq L_\beta$ by Lemma 5.2.31, U_β/V_β contains a unique non-central chief factor. Moreover, by a similar argument, $V_\beta^{(3)} \cap Q_{\alpha'-2} \leq Q_{\alpha'-1}$ and $V_\beta^{(3)}/U_\beta$ contains exactly one non-central chief factor too, otherwise $O^p(L_\beta)$ centralizes $V_\beta^{(3)}/U_\beta$ and we arrive at a contradiction as before. In both cases, the non-central chief factor is an FF-module for $\overline{L_\beta}$.

Set $R_1 := C_{L_\beta}(U_\beta/V_\beta)$ and $R_2 := C_{L_\beta}(V_\beta^{(3)}/U_\beta)$. Since the non-central chief factor within $V_\beta^{(3)}/U_\beta$ is an FF-module, it follows that either $R_2Q_\beta = R_\beta$; or $L_\beta = \langle R_2, R_\beta, S\rangle$ and $p \in \{2,3\}$ by Lemma 2.3.14 (iii) and Lemma 2.3.15 (ii), (iii). In the former case, since $V_\alpha^{(2)} \leq Q_{\alpha'-1}$, $V_\beta^{(3)} = V_\alpha^{(2)}U_\beta \trianglelefteq L_\beta = \langle V_{\alpha'}, R_\beta, Q_\alpha\rangle$. But $V_{\alpha'}$ centralizes $V_\alpha^{(2)}U_\beta/U_\beta$ so that $O^p(L_\beta)$ centralizes $V_\beta^{(3)}/U_\beta$, a contradiction. In the latter case, $V_\alpha^{(2)}U_\beta \trianglelefteq R_2S$ and if $[C_\beta, V_\alpha^{(2)}U_\beta] \leq V_\beta$, then $[C_\beta, V_\alpha^{(2)}U_\beta]$ is centralized by $O^p(R_\beta)$ and so $[C_\beta, V_\alpha^{(2)}U_\beta] \trianglelefteq L_\beta = \langle R_1, R_\beta, S\rangle$. Thus, $[C_\beta, V_\beta^{(3)}] = [C_\beta, V_\alpha^{(2)}U_\beta] \leq V_\beta$ and by conjugacy, $R \leq [V_{\alpha'-2}^{(3)}, V_\beta] \leq [V_{\alpha'-2}^{(3)}, C_{\alpha'-2}] \leq V_{\alpha'-2}$, a contradiction. Thus, $[C_\beta, V_\alpha^{(2)}] \leq V^\alpha$ but $[C_\beta, V_\alpha^{(2)}] \not\leq V_\beta$. If $R_1Q_\beta = R_2Q_\beta$ then, assuming that G is a minimal counterexample to Theorem 5.2.2, we may apply Lemma 5.2.29 with $\lambda = \beta$. Since $b > 5$, R_1Q_β normalizes $V_\alpha^{(2)}$ and $\lambda = \beta$, conclusion (d) holds. Then, $V_\alpha^{(4)} \leq V := \langle Z_\beta^X\rangle$ and the images of $Q_\alpha/C_{Q_\alpha}(V_\alpha^{(2)}$ and $C_{Q_\alpha}(V_\alpha^{(2)})/C_{Q_\alpha}(V_\alpha^{(4)})$ resp. Q_β/C_β and $C_\beta/C_{Q_\beta}(V_\beta^{(3)})$ contain a non-central chief factor for $\widetilde{L}_\alpha$ resp. $\widetilde{L}_\beta$, and we have a contradiction.

Thus, we may assume that $R_1Q_\beta \neq R_2Q_\beta$ and again by Lemma 2.3.14 (iii) and Lemma 2.3.15 (ii), (iii), we deduce that $L_\beta = \langle R_2, R_2, S\rangle$. Then $V_\alpha^{(2)}U_\beta \trianglelefteq R_2S$ so that $V^\alpha V_\beta \geq [C_\beta, V_\alpha^{(2)}U_\beta]V_\beta \trianglelefteq R_2S$. Furthermore, as $O^p(R_1)$ centralizes U_β/V_β, $[C_\beta, V_\alpha^{(2)}U_\beta]V_\beta \trianglelefteq R_1S$ so that $[C_\beta, V_\alpha^{(2)}U_\beta]V_\beta \trianglelefteq L_\beta$. Since $V^\alpha V_\beta \ntrianglelefteq L_\beta$, we may assume that $[C_\beta, V_\alpha^{(2)}]V_\beta < V^\alpha V_\beta$. Now, V^α/Z_α is an FF-module generated $C_{V_\beta}(O^p(L_\beta))/Z_\alpha$ of order p so that by Lemma 2.3.10, $p^2 \leqslant |V^\alpha/Z_\alpha| \leqslant p^3$ and $p^4 \leqslant |V^\alpha| \leqslant p^5$. Hence, $p^5 \leqslant |V^\alpha V_\beta| \leqslant p^6$, accordingly. But now, as $[C_\beta, V_\alpha^{(2)}U_\beta]V_\beta > V_\beta$, $|[C_\beta, V_\alpha^{(2)}U_\beta]V_\beta| \geqslant p^5$ and as $[C_\beta, V_\alpha^{(2)}]V_\beta < V^\alpha V_\beta$, we get that $|V^\alpha| = p^5$, $|V^\alpha V_\beta| = p^6$ and $[Q_\beta, V^\alpha] \not\leq Z_\alpha C_{V_\beta}(O^p(L_\beta))$.

Writing C^α for the preimage in V^α of $C_{V^\alpha/Z_\alpha}(O^p(L_\alpha))$, we have that $|C^\alpha| = p^3$, $C^\alpha \cap V_\beta = Z_\alpha$, $|Q_\alpha/C_{Q_\alpha}(C^\alpha)| \leqslant p^2$ and a calculation using the three subgroup lemma yields $[R_\alpha, Q_\alpha] \leq C_{Q_\alpha}(C^\alpha)$. Since $Z(Q_\alpha) = Z_\alpha$, calculating in $\mathrm{GL}_3(p)$, we infer that $Q_\alpha/C_{Q_\alpha}(C^\alpha)$ is a non-central chief factor of order p^2 for L_α. Hence, $Q_\alpha/C_{Q_\alpha}(C^\alpha)$ is a natural $\mathrm{SL}_2(p)$ module for L_α/R_α.

Now, by Lemma 5.2.13, $U_\beta/([U_\beta, Q_\beta]V_\beta)$ contains the unique non-central chief factor within U_β/V_β and so $O^p(L_\beta)$ centralizes $[U_\beta, Q_\beta]V_\beta/V_\beta$. Thus, $[V^\alpha, Q_\beta]V_\beta \trianglelefteq L_\beta$ from which it follows that $Z_\alpha \geq [V^\alpha, Q_\beta, Q_\beta] \trianglelefteq L_\beta$ and $[V^\alpha, Q_\beta, Q_\beta] = Z_\beta$. But $C^\alpha \leq Z_\alpha C_{V_\beta}(O^p(L_\beta))[V^\alpha, Q_\beta]$ so that $[Q_\beta, C^\alpha] = Z_\beta$. In particular, $C_{Q_\alpha}(C^\alpha) \leq Q_\beta$ for otherwise $Z_\beta = [C^\alpha, Q_\alpha \cap Q_\beta] = [C^\alpha, Q_\alpha] \trianglelefteq L_\alpha$, a contradiction.

If $V^{\alpha'-1} \not\leq Q_\beta$, then $RZ_\beta \leq [V^{\alpha'-1}, V_\beta]Z_\beta \leq Z_{\alpha'-1}Z_\beta$. Then, as $R \not\leq Z_{\alpha'-1}$, we get that $Z_\beta \leq RZ_{\alpha'-1} \leq V_{\alpha'}$. If $V^{\alpha'-1} \leq Q_\beta$ but $V_\beta \not\leq C_\beta$, we deduce that $Z_\beta = [V^{\alpha'-1}, V_\beta] \leq Z_{\alpha'-1}$. In either case, since $O^p(R_\alpha)$ centralizes $V_\alpha^{(2)}$, by Lemma 5.2.18, $Z_\beta \neq Z_{\alpha+3}$ and so $V_{\alpha'}^{(3)}$ centralizes $Z_{\alpha+2} = Z_\beta Z_{\alpha+3}$. But then $V_{\alpha'}^{(3)} \cap Q_{\alpha+3} = V_{\alpha'}(V_{\alpha'}^{(3)} \cap Q_\beta)$ and since $Z_\beta \leq Z_{\alpha'-1} \leq V_{\alpha'}$, $V_{\alpha'}^{(3)}/V_{\alpha'}$ contains a unique non-central chief factor, a contradiction. Thus, $[V_\beta, V^{\alpha'-1}] = \{1\}$ and $V_\beta \leq C_{Q_{\alpha'-1}}(C^{\alpha'-1}) \leq Q_{\alpha'}$, a final contradiction. $\square$

Lemma 5.4.21. *Suppose that $C_{V_\beta}(V_{\alpha'}) = V_\beta \cap Q_{\alpha'}$ and $b > 5$. If $V_{\alpha'} \not\leq Q_\beta$ and $V_\alpha^{(2)} \leq Q_{\alpha'-2}$, then $Z_{\alpha'-1} \leq V_\beta^{(3)} \leq Q_{\alpha'-1}$, $Z_{\alpha'} \not\leq V_\alpha^{(2)}$, $V_\beta^{(3)}/V_\beta$ contains a unique non-central chief factor for $\overline{L_\beta}$ which, as a $\overline{L_\beta}$-module, is an FF-module and $O^p(R_\beta)$ centralizes $V_\beta^{(3)}$.*

Proof. By Lemma 5.4.20, $|V_\beta| = p^3$ so that $R = [V_\beta, V_{\alpha'}] \leq Z_{\alpha'-1} \cap Z_{\alpha+2}$. Suppose first that $V_\alpha^{(2)} \not\leq Q_{\alpha'-1}$. Then $Z_{\alpha'-2} = [V_\alpha^{(2)}, V_{\alpha'-2}] \leq Z_\alpha$, so that $Z_\beta = Z_{\alpha'-2}$. But $Z_\beta \neq R \leq Z_{\alpha'-1}$ and so $Z_{\alpha'-1} = R \times Z_\beta \leq V_\beta$, a contradiction since $V_\alpha^{(2)}$ is abelian.

Thus, we may assume throughout that $V_\alpha^{(2)} \le Q_{\alpha'-1}$.

Suppose that $V_\beta^{(3)} \cap Q_{\alpha'-2} \le Q_{\alpha'-1}$. If $V_\beta^{(3)} \le Q_{\alpha'-2}$, then $V_\beta^{(3)} = V_\beta(V_\beta^{(3)} \cap Q_{\alpha'})$. Since $O^p(L_\beta)$ does not centralize $V_\beta^{(3)}/V_\beta$, $Z_{\alpha'} = [V_\beta^{(3)} \cap Q_{\alpha'}, V_{\alpha'}] \le V_\beta^{(3)}$. Even still, $V_\beta^{(3)}/V_\beta$ contains a unique non-central chief factor for $\overline{L_\beta}$ which is an FF-module and by Lemma 5.2.34, $O^p(R_\beta)$ centralizes $V_\beta^{(3)}$. If $Z_{\alpha'} \le V_\alpha^{(2)}$ or $[V_\alpha^{(2)} \cap Q_{\alpha'}, V_{\alpha'}] = \{1\}$, then $V_\alpha^{(2)} \trianglelefteq L_\beta = \langle V_{\alpha'}, Q_\alpha, R_\beta \rangle$, a contradiction. The lemma follows in this case so we may assume that $V_\beta^{(3)} \not\le Q_{\alpha'-2}$ and $Z_{\alpha'} = [V_\beta^{(3)} \cap Q_{\alpha'}, V_{\alpha'}] \le V_\beta^{(3)}$.

Continuing under the assumption that $V_\beta^{(3)} \not\le Q_{\alpha'-2}$ and $V_\beta^{(3)} \cap Q_{\alpha'-2} \le Q_{\alpha'-1}$, since $Z_{\alpha'-1} = Z_{\alpha'} \times R \le V_\beta^{(3)}$ and $b > 5$, we deduce that $Z_{\alpha'-1} = Z_{\alpha'-3}$, otherwise $V_\beta^{(3)}$ centralizes $V_{\alpha'-2}$. By Lemma 5.2.18, $O^p(R_\beta)$ does not centralize $V_\beta^{(3)}$ and so by Lemma 5.2.34, either $V_\beta^{(3)}/V_\beta$ contains more than one non-central chief factor, or a non-central chief factor within $V_\beta^{(3)}/V_\beta$ is not an FF-module. Hence, we infer that $Z_{\alpha'-1} = [V_\beta^{(3)} \cap Q_{\alpha'-2}, V_{\alpha'}] \not\le V_\beta$. Moreover, since $b > 5$, $[V_\beta^{(3)}, Z_{\alpha'+1}, Z_{\alpha'+1}] \le [V_\beta^{(3)}, V_{\alpha'-2}^{(3)}, V_{\alpha'-2}^{(3)}] = \{1\}$ and $V_\beta^{(3)}$ admits quadratic action. In particular, if $p \geqslant 5$ then the Hall–Higman theorem implies that $O^p(R_\beta)$ centralizes $V_\beta^{(3)}$ and so $p = 2$ or 3.

Notice that $Z_{\alpha'-1} = Z_{\alpha'-3} \le V_\beta^{(3)} \le Z(V_\beta^{b-4})$. Suppose that $b > 7$ and let $n \leqslant \frac{b-5}{2}$ be chosen minimally such that $V_\beta^{(2n+1)} \le Q_{\alpha'-2n}$. Since $V_\beta^{(3)} \not\le Q_{\alpha'-2}$, if such an n exists then $n \geqslant 2$. Notice $V_\beta^{(5)}$ centralizes $Z_{\alpha'-3} \le V_\beta^{(3)}$ so that either $Z_{\alpha'-3} = Z_{\alpha'-5} \le V_\beta^{(3)}$ or $V_\beta^{(5)} \le Q_{\alpha'-4}$ and $n = 2$. Extending through larger subgroups, it is clear that for a minimally chosen n, $Z_{\alpha'-1} = Z_{\alpha'-3} = \dots = Z_{\alpha'-2n+1} \le V_\beta^{(3)}$ is centralized by $V_\beta^{(2n+1)}$ so that $V_\beta^{(2n+1)} \le Q_{\alpha'-2n+1}$. Then $V_\beta^{(2n+1)} = V_\beta^{(2(n-1)+1)}(V_\beta^{(2n+1)} \cap Q_{\alpha'-2n+2})$. Moreover, $Z_{\alpha'-1} = \dots = Z_{\alpha'-2n+1}$, $V_\beta^{(2n+1)} \cap Q_{\alpha'-2a} \le Q_{\alpha'-2a+1}$ and $V_\beta^{(2n+1)} \cap Q_{\alpha'-2a} = V_\beta^{(2(a-2)+1)}(V_\beta^{(2n+1)} \cap Q_{\alpha'-2a+2})$ from which it follows that $V_\beta^{(2n+1)} = V_\beta^{(2(n-1)+1)}(V_\beta^{(2n+1)} \cap Q_{\alpha'})$ so that $O^p(L_\beta)$

centralizes $V_\beta^{(2n+1)}/V_\beta^{(2(n-1)+1)}$, a contradiction. Thus, no such n exists for $n \leqslant \frac{b-5}{2}$ and it follows that $V_\beta^{(b-4)} \not\leq Q_{\alpha'-b+5} = Q_{\alpha+5}$ and $Z_{\alpha'-1} = \cdots = Z_{\alpha+6} = Z_{\alpha+4}$. If $b = 7$, then $Z_{\alpha'-1} = Z_{\alpha'-3} = Z_{\alpha+4}$ by definition. Since $Z_{\alpha'-1} \not\leq V_\beta$, to obtain a contradiction, we need only show that $Z_{\alpha+2} = Z_{\alpha+4}$.

If Z_β is centralized by $V_{\alpha'}^{(3)}$, then $V_{\alpha'}^{(3)}$ centralizes $Z_{\alpha+2} = R \times Z_\beta$ and if $Z_{\alpha+2} \neq Z_{\alpha+4}$, then $V_{\alpha'}^{(3)}$ centralizes $V_{\alpha+3}$ and $V_{\alpha'}^{(3)} = V_{\alpha'}(V_{\alpha'}^{(3)} \cap Q_\beta)$ so that $V_{\alpha'}^{(3)}/V_{\alpha'}$ contains a unique non-central chief factor which is an FF-module and by Lemma 5.2.34, $O^p(R_{\alpha'})$ centralizes $V_{\alpha'}^{(3)}$. By conjugacy, $O^p(R_\beta)$ centralizes $V_\beta^{(3)}$, a contradiction. Thus, $V_{\alpha'}^{(3)}$ does not centralize Z_β. Since $V_{\alpha'}^{(3)}$ centralizes $Z_{\alpha+3} \times R \leq Z_{\alpha+2}$, we may assume that $R = Z_{\alpha+3}$. Furthermore, since $b > 5$ and $V_{\alpha'}^{(3)}$ is abelian, $V_{\alpha'}^{(3)} \cap Q_{\alpha+3} \cap Q_{\alpha+2} \cap Q_\beta \leq C_\beta$.

Now, $V_\beta \leq C_{\alpha'-2}$ and since $[Q_\lambda, V_\lambda^{(2)}] = Z_\lambda$ for all $\lambda \in \Delta(\alpha'-2)$, we have that $R \leq [V_\beta, V_{\alpha'-2}^{(3)}] \leq Z_{\alpha+2} \cap V_{\alpha'-2}$. If $Z_{\alpha+2} \leq V_{\alpha'-2}$, then $Z_{\alpha+2} = Z_{\alpha'-3} = Z_{\alpha'-1} \leq V_\beta$, a contradiction and so $[V_\beta, V_{\alpha'-2}^{(3)}] = R$ and $[V_\beta, V_{\alpha'-2}^{(3)} \cap Q_\beta] = R \cap Z_\beta = \{1\}$. Then $V_{\alpha'-2}^{(3)} \cap Q_\beta \leq C_\beta$ so that $[V_\beta^{(3)}, V_{\alpha'-2}^{(3)} \cap Q_\beta] \leq V_\beta \cap V_{\alpha'-2}^{(3)}$. Since $b > 5$, $V_\beta \not\leq V_{\alpha'-2}^{(3)}$ and since $R \leq V_{\alpha'-2}$, $Z_{\alpha+2} \leq V_{\alpha'-2}^{(3)}$ and $Z_\beta \leq V_{\alpha'-2}^{(3)}$ but $Z_\beta \not\leq V_{\alpha'-2}$. If $b > 7$, $V_{\alpha'}^{(3)}$ centralizes Z_β, a contradiction by the above.

Thus, we assume that $b = 7$, $V_\beta^{(3)} \not\leq Q_{\alpha'-2}$, $V_\beta^{(3)} \cap Q_{\alpha'-2} \leq Q_{\alpha'-1}$, $Z_{\alpha'-1} = Z_{\alpha'-3} \neq Z_{\alpha+2}$ and $[Z_\beta, V_{\alpha'}^{(3)}] \neq \{1\}$. Set $W^\beta = \langle V_\delta^{(2)} \mid Z_\delta = Z_\alpha, \delta \in \Delta(\beta) \rangle$ so that $[C_\beta, W^\beta] = [C_\beta, V_\alpha^{(2)}] \leq Z_\alpha$. Then $[W^\beta, V_{\alpha'-2}] \leq Z_{\alpha'-3} \cap Z_\alpha$ and by Lemma 5.2.19, $W^\beta \trianglelefteq R_\beta Q_\alpha$. If $Z_\beta \leq Z_{\alpha'-3} = Z_{\alpha'-1}$, then $Z_{\alpha'-1} = Z_\beta \times R = Z_{\alpha+2} \leq V_\beta$, a contradiction. Thus, $W^\beta = V_\beta(W^\beta \cap Q_{\alpha'})$. If $[W^\beta \cap Q_{\alpha'}, V_{\alpha'}] \leq W^\beta$, then $V_\alpha^{(2)} \leq W^\beta \trianglelefteq L_\beta = \langle V_{\alpha'}, Q_\alpha, R_\beta \rangle$ and $V_\beta^{(3)} = W^\beta \leq Q_{\alpha'-2}$, a contradiction. Thus, $W^\beta \cap Q_{\alpha'} \not\leq Q_{\alpha'+1}$ for some $\alpha'+1 \in \Delta(\alpha')$ and since $Z_{\alpha'+1} Z_{\alpha'-1} = V_{\alpha'} \not\leq Q_\beta$, $(\alpha'+1, \beta)$ is a critical pair.

Since $V_{\alpha+3} \leq Q_{\alpha'+1}$, $[V_{\alpha'+1}^{(2)} \cap Q_{\alpha+3}, V_{\alpha+3}] \leq Z_{\alpha'+1} \cap Z_{\alpha+3}$. If $Z_{\alpha+3} \leq Z_{\alpha'+1}$, then $Z_{\alpha+3} = Z_{\alpha'} \neq R$ so that $Z_{\alpha+2} = R \times Z_{\alpha'} = Z_{\alpha'-1} \leq V_\beta$, a contradiction. Thus, $[V_{\alpha'+1}^{(2)} \cap Q_{\alpha+3}, V_{\alpha+3}] = \{1\}$ and $V_{\alpha'+1}^{(2)} \cap Q_{\alpha+3} = Z_{\alpha'+1}(V_{\alpha'+1}^{(2)} \cap C_\beta)$. Furthermore, $[V_{\alpha'+1}^{(2)} \cap C_\beta, W^\beta \cap Q_{\alpha'}] \leq V_{\alpha'+1}^{(2)} \cap Z_\alpha$ and since $Z_\beta \not\leq V_{\alpha'}^{(3)}$, we have that $W^\beta \cap Q_{\alpha'}$ centralizes $(V_{\alpha'+1}^{(2)} \cap Q_{\alpha+3})/Z_{\alpha'+1}$. Thus, $V_{\alpha'+1}^{(2)} \not\leq Q_{\alpha+3}$ and $V_{\alpha'+1}^{(2)}/Z_{\alpha'+1}$ is an FF-module. By Lemma 5.2.32, $O^p(R_\alpha)$ centralizes $V_\alpha^{(2)}$ and since $V_\beta^{(3)}$ does not centralize $V_{\alpha'-2}$, it follows from Lemma 5.2.18 that $Z_{\alpha'-2} \neq Z_{\alpha'-4} = R$.

Suppose that $([V_\beta^{(3)}, Q_\beta]V_\beta)/V_\beta$ contains a non-central chief factor for L_β. In particular, $[Q_\beta, V_\alpha^{(2)}] \not\leq V_\beta$, and since $V_\alpha^{(2)}/Z_\alpha$ is an FF-module, $|V_\alpha^{(2)}| = p^5$. The non-central chief factor, U/V say, is an FF-module for $\overline{L_\beta}$ and $L_\beta/C_{L_\beta}(U/V) \cong \mathrm{SL}_2(p)$. Set $R_1 := C_{L_\beta}(U/V)$ and $R_2 := C_{L_\beta}(V_\beta^{(3)}/([V_\beta^{(3)}, Q_\beta]V_\beta))$, noticing that also $L_\beta/R_2 \cong \mathrm{SL}_2(p)$. If $R_1 \neq R_\beta$, and employing Lemma 2.3.15 (iii) when $p = 3$, we conclude that $L_\beta = \langle R_1, R_\beta, S\rangle$. Similarly, if $R_2 \neq R_\beta$ then $L_\beta = \langle R_2, R_\beta, S\rangle$.

Suppose that $R_1 \neq R_\beta$. Then $[V_\alpha^{(2)}, Q_\beta]V_\beta \trianglelefteq R_1$ and $[V_\alpha^{(2)}, Q_\beta, Q_\beta] \leq V_\beta$ so that $[V_\alpha^{(2)}, Q_\beta, Q_\beta] \trianglelefteq L_\beta = \langle R_1, R_\beta, S\rangle$. Since $[V_\alpha^{(2)}, Q_\beta, Q_\beta] \leq Z_\alpha$, we have that $[V_\alpha^{(2)}, Q_\beta, Q_\beta] = Z_\beta$. Setting C^α to be the preimage in $V_\alpha^{(2)}$ of $C_{V_\alpha^{(2)}/Z_\alpha}(O^p(L_\alpha))$, we have that $C^\alpha \leq V_\beta[V_\alpha^{(2)}, Q_\beta]$ and so $[C^\alpha, Q_\beta] = Z_\beta$. As in Lemma 5.4.20 (where C^α is defined slightly differently), we see that $|Q_\alpha/C_{Q_\alpha}(C^\alpha)| = p^2$ and $C_{Q_\alpha}(C^\alpha) \leq Q_\beta$. Now, $V_\beta \leq Q_{\alpha'-2}$ and so $[V_\beta, C^{\alpha'-1}] \leq Z_{\alpha'-2} \cap Z_{\alpha+2} = \{1\}$, for otherwise $Z_{\alpha+2} = Z_{\alpha'-1}$. But then, $V_\beta \leq C_{Q_{\alpha'-1}}(C^{\alpha'-1}) \leq Q_{\alpha'}$, a contradiction. Thus, $R_1 = R_\beta$.

Suppose that $R_2 \neq R_\beta$. Then $V_\alpha^{(2)}[V_\beta^{(3)}, Q_\beta] \trianglelefteq R_2$ and so $[V_\alpha^{(2)}, Q_\beta][V_\beta^{(3)}, Q_\beta, Q_\beta] \trianglelefteq L_\beta = \langle R_1, R_2, S\rangle$. Since $[V_\alpha^{(2)}, Q_\beta, Q_\beta] \leq Z_\alpha$, we have that $[V_\beta^{(3)}, Q_\beta, Q_\beta] \leq V_\beta$ and so $[V_\alpha^{(2)}, Q_\beta]V_\beta \trianglelefteq L_\beta$. But then $[V_\beta^{(3)}, Q_\beta]V_\beta = [V_\alpha^{(2)}, Q_\beta]V_\beta$ is centralized by Q_α, modulo V_β, and so $([V_\beta^{(3)}, Q_\beta]V_\beta)/V_\beta$ does not contain a non-central chief factor for

L_β. Thus, $R_2 = R_\beta$. But now $O^p(R_\beta)$ centralizes $V_\beta^{(3)}$ and Lemma 5.2.18 applied to $Z_{\alpha'-1} = Z_{\alpha'-3}$ gives $V_{\alpha'} \le V_{\alpha'-1}^{(2)} = V_{\alpha'-3}^{(2)} \le Q_\beta$, a contradiction.

Therefore, we may assume that $([V_\beta^{(3)}, Q_\beta]V_\beta)/V_\beta$ does not contain a non-central chief factor for L_β and $[V_\alpha^{(2)}, Q_\beta]V_\beta \trianglelefteq L_\beta$. As before, since $[V_\alpha^{(2)}, Q_\beta, Q_\beta] \le Z_\alpha$, we have that $[V_\alpha^{(2)}, Q_\beta, Q_\beta] = Z_\beta$ and either $|V_\alpha^{(2)}| = p^4$; or $[C^\alpha, Q_\beta] = Z_\beta$ for C^α as defined above. In the latter case, we again see that $V_\beta \le C_{Q_{\alpha'-1}}(C^{\alpha'-1}) \le Q_{\alpha'}$, a contradiction. Thus, $|V_\alpha^{(2)}| = p^4$, $[V_\alpha^{(2)}, Q_\beta] \le V_\beta$ and $[V_\beta^{(3)}, Q_\beta] = V_\beta$. Since $O^p(R_\beta)$ does not centralize $V_\beta^{(3)}$, by Lemma 5.2.34, $V_\beta^{(3)}/V_\beta$ is a quadratic $2F$-module for $\overline{L_\beta}$. Moreover, since $V_\alpha^{(2)}$ generates $V_\beta^{(3)}$, is $G_{\alpha,\beta}$-invariant and has order p modulo V_β, comparing with Lemma 2.3.22 and using that $|S/Q_\beta| = p$, it follows that $p = 2$ and $L_\beta/C_{L_\beta}(V_\beta^{(3)}/V_\beta) \cong \mathrm{Dih}(10)$ or $(3 \times 3) : 2$.

Now, $C_{L_\beta}(V_\beta^{(3)}/V_\beta)$ normalizes $V_\alpha^{(2)}$ so that $[V_\alpha^{(2)}, C_\beta] \le Z_\alpha$ is also normalized by $C_{L_\beta}(V_\beta^{(3)}/V_\beta)$. Since R_β normalizes Z_α, if $L_\beta = \langle S, R_\beta, C_{L_\beta}(V_\beta^{(3)}/V_\beta)\rangle$ then $[V_\alpha^{(2)}, C_\beta] = Z_\beta$ and $[V_\beta^{(3)}, C_\beta] = Z_\beta$. But then $R = [V_{\alpha'}, V_\beta] \le [V_{\alpha'-2}^{(3)}, V_\beta] = Z_{\alpha'-2}$, a contradiction. Thus $L_\beta/C_{L_\beta}(V_\beta^{(3)}/V_\beta) \cong (3 \times 3) : 2$ and $C_{L_\beta}(V_\beta^{(3)}/V_\beta) \le R_\beta$. Then $V_{\alpha'-1}^{(2)} < \langle (V_{\alpha'-3}^{(2)})^{R_\beta S}\rangle =: W$ and $|W/V_\beta| = 4$. But now, $[W, V_\beta^{(3)}] \le [W, Q_{\alpha'-3}] \le Z_{\alpha'-3}$ and $[V_{\alpha'-2}^{(3)} \cap Q_\beta, V_\beta] \le Z_\beta \cap V_{\alpha'-2} = \{1\}$ and $[V_{\alpha'-2}^{(3)} \cap Q_\beta, V_\beta^{(3)}] \le V_\beta \cap V_{\alpha'-2}^{(3)} = Z_{\alpha+2} \le V_{\alpha'-3}^{(2)}$. Therefore, $[V_\beta^{(3)}, V_{\alpha'-2}^{(3)}] \le V_{\alpha'-3}^{(2)}$, a contradiction since $V_\beta^{(3)}/V_\beta$ is not dual to an FF-module.

Hence, $V_\beta^{(3)} \cap Q_{\alpha'-2} \not\le Q_{\alpha'-1}$. Since $R \le Z_{\alpha'-1}$ and $R \ne Z_{\alpha'}$, it follows that $V_\beta^{(3)}$ does not centralize $Z_{\alpha'}$. Hence, as $b > 5$ and $V_\beta^{(3)}$ is abelian, we conclude that $[V_\beta^{(3)} \cap \dots \cap Q_{\alpha'}, V_{\alpha'}] = \{1\}$. In particular, $[V_\alpha^{(2)} \cap Q_{\alpha'}, V_{\alpha'}] = \{1\}$ and so $[V_{\alpha'}, V_\alpha^{(2)}] = R \le V_\alpha^{(2)}$. Additionally, since $V_\beta^{(3)}$ centralizes $Z_{\alpha'-2}$, we have that $R = Z_{\alpha'-2} \ne Z_\beta$.

Again, we set $W^\beta = \langle V_\delta^{(2)} \mid Z_\lambda = Z_\alpha, \lambda \in \Delta(\beta)\rangle$ noting that $W^\beta \trianglelefteq R_\beta Q_\alpha$ by Lemma 5.2.19. For such a $\lambda \in \Delta(\beta)$, (λ, α') is a critical pair. Suppose that $V_\lambda^{(2)} \not\leq Q_{\alpha'-2}$. Then $\{1\} \neq [V_{\lambda-1}, V_{\alpha'-2}] \leq Z_\lambda \cap Z_{\alpha'-3} = Z_\alpha \cap Z_{\alpha'-3}$ so that $Z_\beta \leq Z_{\alpha'-3}$ and so $Z_{\alpha'-3} = Z_{\alpha'-2} \times Z_\beta = Z_{\alpha+2}$. Now, there is $\alpha'+1 \in \Delta(\alpha')$ such that $(\alpha'+1, \beta)$ is a critical pair. As in the above steps, if $V_{\alpha'+1}^{(2)} \not\leq Q_{\alpha+3}$ then $Z_{\alpha'} = [V_{\alpha'+2}, V_{\alpha+3}] \leq V_{\alpha+3}$, a contradiction as $V_{\alpha+3}$ is centralized by $V_\beta^{(3)}$. Thus, $V_{\alpha'+1}^{(2)} \leq Q_{\alpha+2}$ and since $V_{\alpha'}^{(3)} \cap Q_{\alpha+3} \leq Q_{\alpha+2}$, applying the previous results in this proof, $O^p(R_{\alpha'})$ centralizes $V_{\alpha'}^{(3)}$. But then $V_\alpha^{(2)} \trianglelefteq L_\beta = \langle V_{\alpha'}, Q_\alpha, R_\beta\rangle$, a contradiction.

Thus, $W^\beta \leq Q_{\alpha'-1}$, $[W^\beta, Z_{\alpha'-1}] = Z_{\alpha'-2} \neq Z_\beta$ and $W^\beta = V_\beta(W^\beta \cap Q_{\alpha'})$. Then $V_{\alpha'}$ centralizes W^β/V_β so that $W^\beta \trianglelefteq L_\beta = \langle V_{\alpha'}, R_\beta, Q_\alpha\rangle$. Since $V_{\alpha'}$ centralizes W^β/V_β, it follows that $V_\alpha^{(2)} \trianglelefteq L_\beta$, a final contradiction. □

Lemma 5.4.22. *Suppose that $C_{V_\beta}(V_{\alpha'}) = V_\beta \cap Q_{\alpha'}$ and $b > 5$. If $V_\alpha^{(2)} \not\leq Q_{\alpha'-2}$ and $|V_\beta| \neq p^3$, then we may assume that $[V_\alpha^{(2)}, Z_{\alpha'-1}] \neq \{1\}$.*

Proof. Suppose that $|V_\beta| \neq p^3$. By Lemma 5.4.19 and Lemma 5.4.20, we may assume that for any critical pair $(\alpha^*, \alpha^{*\prime})$, $V_{\alpha^*}^{(2)} \not\leq Q_{\alpha^{*\prime}-2}$. In particular, there is an infinite path $(\alpha', \alpha'-1, \alpha'-2, \ldots, \beta, \alpha, \alpha-1, \alpha-2, \ldots)$ such that $(\alpha-2k, \alpha'-2k)$ is a critical pair for all $k \geqslant 0$. For $2k > b$, we have that $Z_{\alpha'-2k-1} \neq Z_{\alpha'-2k-3}$ and so we can arrange that for our chosen critical pair (α, α') we have that $Z_{\alpha'-1} \neq Z_{\alpha'-3}$. If $[V_\alpha^{(2)}, Z_{\alpha'-1}] = \{1\}$, then $V_\alpha^{(2)}$ centralizes $Z_{\alpha'-1}Z_{\alpha'-3}$ and since $V_\alpha^{(2)} \not\leq Q_{\alpha'-2}$, it follows that $Z_{\alpha'-1}Z_{\alpha'-3} = C_{V_{\alpha'-2}}(O^p(L_{\alpha'-2}))Z_{\alpha'-1} = C_{V_{\alpha'-2}}(O^p(L_{\alpha'-2}))Z_{\alpha'-3}$. But then, by Lemma 5.2.31 using that $|V_\beta| \neq p^3$, we conclude that $Z_{\alpha'-1} = Z_{\alpha'-3}$, a contradiction. □

Notice that by Lemma 5.4.19 and Lemma 5.4.20, whenever $|V_\beta| \neq p^3$ we have that $V_\lambda^{(2)} \not\leq Q_{\lambda+b-2}$ for any critical pair $(\lambda, \lambda+b)$ with $\lambda \in \Gamma$. Moreover, as demonstrated in Lemma 5.4.22, we may iterate backwards through critical pairs far enough that the conclusion of Lemma 5.4.22 holds for all critical pairs beyond a certain point. The net result of this that whenever $|V_\beta| \neq p^3$, we may assume that we have a critical pair (α, α') with $V_\alpha^{(2)} \not\leq Q_{\alpha'-2}$ and $[V_\alpha^{(2)}, Z_{\alpha'-1}] \neq \{1\}$, and for all $k \geqslant 0$ we also have that $(\alpha-2k, \alpha'-2k)$ is a critical pair with $V_{\alpha-2k}^{(2)} \not\leq Q_{\alpha'-2-2k}$ and $[V_{\alpha-2k}^{(2)}, Z_{\alpha'-1-2k}] \neq \{1\}$. We will use the fact in the following two lemmas.

Lemma 5.4.23. *Suppose that $C_{V_\beta}(V_{\alpha'}) = V_\beta \cap Q_{\alpha'}$ and $b = 7$. If $V_\alpha^{(2)} \not\leq Q_{\alpha'-2}$, then $|V_\beta| = p^3$.*

Proof. Suppose that $b = 7$. By Lemma 5.4.19 and Lemma 5.4.20, we may consider a critical pair (α, α') iterated backwards so that $(\alpha+2, \alpha'+2)$ is also a critical pair. Suppose first that $V^\alpha \not\leq Q_{\alpha'-2}$. Then $[V^\alpha, V_{\alpha'-2}] \leq Z_\alpha$ and so $[V^\alpha, V_{\alpha'-2}] = Z_\beta$. Since $Z_{\alpha+2} \not\leq Q_{\alpha'+2}$ and $b > 5$, we have that $Z_\beta = Z_{\alpha+3} \neq Z_{\alpha'-2}$. But now, $Z_{\alpha+3}Z_{\alpha+3}^g Z_{\alpha'-2} = Z_{\alpha'-3}Z_{\alpha'-3}^g$ is normalized by $L_{\alpha'-2} = \langle V^\alpha, (V^\alpha)^g, R_{\alpha'-2}\rangle$ for some appropriately chosen $g \in L_{\alpha'-2}$, so that $V_{\alpha'-2} = Z_{\alpha'-3}Z_{\alpha'-3}^g$ is of order p^3, a contradiction. Thus, we may assume that $V^\alpha \leq Q_{\alpha'-2}$.

If $V^\alpha \not\leq Q_{\alpha'-1}$, then $Z_{\alpha'-2} = [V^\alpha, V_{\alpha'-2}] \leq Z_\alpha$ and $Z_{\alpha'-2} = Z_\beta$. Moreover, for some $\alpha-2 \in \Delta^{(2)}(\alpha)$ with $(\alpha-2, \alpha'-2)$ a critical pair, $V_{\alpha-2}^{(2)}$ centralizes $Z_{\alpha'-2}$ and $Z_{\alpha'-2} = Z_{\alpha+3} = Z_\beta$. Now, $[V^{\alpha'-1}, V_\beta] \leq Z_{\alpha'-1}$ and since V^α does not centralize $Z_{\alpha'-1}$, $[V^{\alpha'-1}, V_\beta] \leq Z_\beta$ and $V^{\alpha'-1} \leq Q_\beta$. If $V^{\alpha'-1} \leq Q_\alpha$, then $[V^{\alpha'-1}, V^\alpha] \leq Z_\alpha$ so that $[V^{\alpha'-1}, V^\alpha] = Z_\beta = Z_{\alpha'-2}$ and V^α centralizes $V^{\alpha'-1}/Z_{\alpha'-1}$, a contradiction since $V^{\alpha'-1}/Z_{\alpha'-1}$ contains a non-central chief factor for $L_{\alpha'-1}$. Thus, $V^{\alpha'-1} \not\leq Q_\alpha$ and $V_{\alpha'-1}^{(2)} \cap Q_\beta = V^{\alpha'-1}(V_{\alpha'-1}^{(2)} \cap Q_\alpha)$. Since $Z_\alpha \not\leq V_{\alpha'-1}^{(2)}$, $[V_{\alpha'-1}^{(2)} \cap Q_\alpha, V^\alpha] = Z_\beta = Z_{\alpha'-2}$ and it follows that $V_{\alpha'-1}^{(2)}/V^{\alpha'-1}$ is an FF-module for $\overline{L_{\alpha'-1}}$. Similarly, $[V^{\alpha'-1} \cap Q_\alpha, V^\alpha] = Z_{\alpha'-2}$ and $V^{\alpha'-1}/Z_{\alpha'-1}$ is an FF-module for $\overline{L_{\alpha'-1}}$. Then

Lemma 5.2.32 and Lemma 5.2.18 applied to $Z_\beta = Z_{\alpha+3}$ implies that $V_\beta = V_{\alpha+3} \le Q_{\alpha'}$, a contradiction.

Thus, $V^\alpha = Z_\alpha(V^\alpha \cap Q_{\alpha'})$. Suppose that $V_{\alpha'} \le Q_\beta$ and again let $(\alpha-2, \alpha'-2)$ be a critical pair. Since V^α/Z_α contains a non-central chief factor, $Z_{\alpha'} \le V^\alpha$ and $Z_{\alpha'} \not\le Z_\alpha$. Then $Z_{\alpha'} = Z_{\alpha'-2}$, otherwise $[V_\alpha^{(2)}, Z_{\alpha'-1}] = \{1\}$. But now, since $b > 5$, $V_{\alpha-2}^{(2)}$ centralizes $Z_{\alpha'-2} \le V^\alpha$ and since $[V_{\alpha-2}^{(2)}, Z_{\alpha'-3}] \ne \{1\}$, it follows that $Z_{\alpha'-2} = Z_{\alpha'-4} = Z_{\alpha+3}$. Since $R = [V_{\alpha'}, V_\beta] = Z_\beta \le V_{\alpha'}$, as $Z_{\alpha+2} \not\le Q_{\alpha'+2}$, we must have that $Z_{\alpha+3} = Z_\beta$. But then $R = Z_\beta = Z_{\alpha'}$, a contradiction.

Finally, we have that $V^\alpha \le Q_{\alpha'-1}$ and $V_{\alpha'} \not\le Q_\beta$. Set $U^\beta = \langle V^\delta \mid Z_\delta = Z_\alpha, \delta \in \Delta(\beta)\rangle$. Then (δ, α') is a critical pair for all such $\delta \in \Delta(\beta)$ and so $V^\delta \le Q_{\alpha'-1}$ for all such δ. By Lemma 5.2.19, $R_\beta Q_\alpha$ normalizes U^β. Now, $U^\beta V_\beta = V_\beta(U^\beta \cap Q_{\alpha'})$ and either $Z_{\alpha'} \le V_\beta^{(3)}$; or $V_{\alpha'}$ centralizes $U^\beta V_\beta/V_\beta$. In the former case, since $V_\beta^{(3)}$ does not centralize $Z_{\alpha'-1}$, $Z_{\alpha'} = Z_{\alpha'-2}$. Iterating backwards through critical pairs, this eventually implies that $Z_{\alpha'} = Z_\beta$ and again, $V_{\alpha'}$ centralizes $U^\beta V_\beta/V_\beta$. Thus, in all cases, $U^\beta V_\beta \trianglelefteq L_\beta = \langle V_{\alpha'}, R_\beta, Q_\alpha\rangle$ and since $V_{\alpha'}$ centralizes $U^\beta V_\beta/V_\beta$, $O^p(L_\beta)$ centralizes $U^\beta V_\beta/V_\beta$. Then $V^\alpha V_\beta \trianglelefteq L_\beta$, a contradiction by Lemma 5.2.31. $\square$

Lemma 5.4.24. *Suppose that $C_{V_\beta}(V_{\alpha'}) = V_\beta \cap Q_{\alpha'}$ and $b > 5$. If $V_\alpha^{(2)} \not\le Q_{\alpha'-2}$, then $|V_\beta| = p^3$.*

Proof. By Lemma 5.4.23, we may assume that $b > 7$. In the following, the aim will be to prove that $Z_{\alpha'-2} = Z_{\alpha'-4}$ for then extending far enough backwards along the critical path, by Lemma 5.4.22, we can manufacture a situation in which (α, α') is a critical pair, $Z_{\alpha'-1-2k} \ne Z_{\alpha'-3-2k}$ for all $k \geqslant 0$ and $Z_{\alpha'} = Z_{\alpha'-2} = \cdots = Z_{\alpha+3} = Z_\beta$. Throughout we consider a critical pair (α, α') iterated backwards far enough so that $(\alpha+2, \alpha'+2)$ is also a critical pair.

Suppose first that $V_\beta^{(3)} \cap Q_{\alpha'-2} \not\leq Q_{\alpha'-1}$. Then $Z_{\alpha'-2} = [V_\beta^{(3)} \cap Q_{\alpha'-2}, Z_{\alpha'-1}] \leq V_\beta^{(3)}$ is centralized by $V_{\alpha-2}^{(2)}$ since $b > 7$. Since $V_{\alpha-2}^{(2)}$ does not centralizes $Z_{\alpha'-3}$, we have that $Z_{\alpha'-2} = Z_{\alpha'-4}$, as desired. Thus, $V_\beta^{(3)} \cap Q_{\alpha'-2} = V_\beta(V_\beta^{(3)} \cap Q_{\alpha'})$. If $Z_{\alpha'} = [V_\beta^{(3)} \cap Q_{\alpha'}, V_{\alpha'}] \leq V_\beta^{(3)}$ then, as $V_\beta^{(3)}$ does not centralize $Z_{\alpha'-1}$, we deduce that $Z_{\alpha'} = Z_{\alpha'-2} \leq V_\beta^{(3)}$. Similarly to the above, using $b > 7$, we have that $Z_{\alpha'-2} = Z_{\alpha'-4}$, as desired. Thus, $[V_\beta^{(3)} \cap Q_{\alpha'}, V_{\alpha'}] = \{1\}$.

Suppose that $V_{\alpha'} \leq Q_\beta$. Then, by the above, $V_\alpha^{(2)} \cap Q_{\alpha'-2} = Z_\alpha(V_\alpha^{(2)} \cap Q_{\alpha'})$ and $[V_\alpha^{(2)} \cap Q_{\alpha'}, V_{\alpha'}] = \{1\}$, a contradiction since both $V_\alpha^{(2)}/V^\alpha$ and V^α/Z_α contain a non-central chief factor. Thus, $V_{\alpha'} \not\leq Q_\beta$ and $V_\beta^{(3)}/V_\beta$ contains a unique non-central chief factor which is an FF-module for $\overline{L_\beta}$. By Lemma 5.2.34, $O^p(R_\beta)$ centralizes $V_\beta^{(3)}$. If $V^\alpha \leq Q_{\alpha'-2}$, then $V^\alpha V_\beta = V_\beta(V^\alpha V_\beta \cap Q_{\alpha'})$ and it follows that $V^\alpha V_\beta \trianglelefteq L_\beta = \langle V_{\alpha'}, Q_\alpha, R_\beta \rangle$, a contradiction by Lemma 5.2.31. Therefore, $V^\alpha \not\leq Q_{\alpha'-2}$ and since $V_{\alpha'-2} \leq Q_\alpha$, we have that $[V^\alpha, V_{\alpha'-2}] = Z_\beta \neq Z_{\alpha'-2}$.

Suppose that $b = 9$ and consider the critical pair $(\alpha - 2, \alpha' - 2)$. Then, as $V_{\alpha'-4} \leq Q_{\alpha-2}$, we have that $[V^{\alpha-2}, V_{\alpha'-4}] \leq Z_{\alpha-1}$. Suppose that $Z_{\alpha-1} = [V^{\alpha-2}, V_{\alpha'-4}] \leq V_{\alpha'-4}$. Since $Z_\alpha, Z_{\alpha+2} \not\leq V_{\alpha'-4}$, we must have that $Z_{\alpha-1} = Z_\beta = Z_{\alpha+3} = Z_{\alpha'-6}$. But then, $[V^{\alpha-2}, V_{\alpha'-4}] = Z_{\alpha'-6}$ and $Z_{\alpha'-5}Z_{\alpha'-5}^g \trianglelefteq L_{\alpha'-4} = \langle V^{\alpha-2}, (V^{\alpha-2})^g, R_{\alpha'-4} \rangle$ for some appropriately chosen $g \in L_{\alpha'-4}$. Then $V_{\alpha'-4} = Z_{\alpha'-5}Z_{\alpha'-5}^g$ is of order p^3, a contradiction. Thus, $[V^{\alpha-2}, V_{\alpha'-4}] = \{1\}$ so that $V^{\alpha-2}V_{\alpha-1} = V_{\alpha-1}(V^{\alpha-2}V_{\alpha-1} \cap Q_{\alpha'-2})$ and since $V^{\alpha-1}V_{\alpha-1} \not\trianglelefteq L_{\alpha-1}$, it follows that $Z_{\alpha'-2} \leq V_{\alpha-1}^{(3)}$. Then $V_{\alpha-2}^{(2)}$ centralizes $Z_{\alpha'-2}$ and so $Z_{\alpha'-2} = Z_{\alpha'-4}$, as desired.

Thus, we may assume that $b > 9$. Since $V_{\alpha'} \not\leq Q_\beta$, there is $\lambda \in \Delta(\alpha')$ such that (λ, β) is a critical pair with $V_\beta \not\leq Q_{\alpha'}$ and $V_\lambda^{(2)} \not\leq Q_{\alpha+3}$. In particular, since $b > 5$, $V_\lambda^{(2)}$ centralizes $Z_\beta \leq V_{\alpha'-2}$ and $Z_\beta = Z_{\alpha+3}$. Then $[V^\lambda, V_{\alpha+3}] \leq Z_{\alpha'}$ since $V_{\alpha+3} \leq Q_\lambda$. If $Z_{\alpha'} \leq V_{\alpha+3}$, since $b > 5$, $Z_{\alpha'}$ is centralized by $V_\alpha^{(2)}$, so that $Z_{\alpha'} = Z_{\alpha'-2}$. Since $b > 7$, $Z_{\alpha'-2} \leq V_{\alpha+3}$ is centralized by $V_{\alpha-2}^{(2)}$ and so $Z_{\alpha'-2} = Z_{\alpha'-4}$, as desired. Thus, $[V^\lambda, V_{\alpha+3}] = \{1\}$ and $V^\lambda V_{\alpha'} = V_{\alpha'}(V^\lambda V_{\alpha'} \cap Q_\beta)$. Since $V^\lambda V_{\alpha'} \ntrianglelefteq L_{\alpha'}$ by Lemma 5.2.31, we intend to force a contradiction by showing that $Z_\beta \leq V^\lambda V_{\alpha'}$.

By construction there is a critical pair $(\alpha + 2, \alpha' + 2)$ and we set $\alpha' + 1 \in \Delta(\alpha' + 2) \cap \Delta(\alpha')$ noting that $(\alpha' + 1, \beta)$ is not necessarily a critical pair. Since $V_{\alpha'} \leq Q_{\alpha+2}$, we infer that $[V_{\alpha'}, V^{\alpha+2}] \leq Z_{\alpha+3} \cap V_{\alpha'} = Z_\beta \cap V_{\alpha'}$. We may assume that $[V_{\alpha'}, V^{\alpha+2}] = \{1\}$. Then $V^{\alpha+2}V_{\alpha+3} = V_{\alpha+3}(V^{\alpha+2}V_{\alpha+3} \cap Q_{\alpha'+2})$ and either $[V^{\alpha+2}V_{\alpha+3} \cap Q_{\alpha'+2}, V_{\alpha'+2}] = \{1\}$, a contradiction for then $V^{\alpha+2}V_{\alpha+3} \trianglelefteq L_{\alpha+3}$; or $Z_{\alpha'+2} = [V^{\alpha+2}V_{\alpha+3} \cap Q_{\alpha'+2}, V_{\alpha'+2}] \leq V_{\alpha+3}^{(3)}$. If $Z_{\alpha'+1} \nleq Q_\beta$ then as $b > 7$, it follows that $Z_{\alpha'+2} = Z_{\alpha'}$. But, as $b > 7$, $Z_{\alpha'}$ is centralized by $V_\alpha^{(2)}$, so that $Z_{\alpha'} = Z_{\alpha'-2}$. Indeed, as $b > 9$, $Z_{\alpha'-2} \leq V_{\alpha+3}^{(3)}$ is centralized by $V_{\alpha-2}^{(2)}$ and so $Z_{\alpha'-2} = Z_{\alpha'-4}$, as desired. Thus, by Lemma 5.2.31, $Z_{\alpha'+1} = Z_{\alpha'-1}$. Since $Z_{\alpha'+2} \leq V_{\alpha+3}^{(3)}$ is centralized by $V_\alpha^{(2)}$, $Z_{\alpha'+2} = Z_{\alpha'-2}$, otherwise $Z_{\alpha'-1}$ is centralized by $V_\alpha^{(2)}$. Then as $b > 9$ and $Z_{\alpha'-2} \leq V_{\alpha+3}^{(3)}$ is centralized by $V_{\alpha-2}^{(2)}$, we get that $Z_{\alpha'-2} = Z_{\alpha'-4}$, as desired.

In all cases we have reduced to the case where $Z_{\alpha'-2} = Z_{\alpha'-4}$. By a previous observation we may now assume that (α, α') is a critical pair such that $Z_{\alpha'} = Z_{\alpha'-2} = \dots = Z_\beta = Z_{\alpha-1} = \dots$ and $Z_{\alpha'-1-2k} \neq Z_{\alpha'-3-2k}$ for any $k \geqslant 0$. Now, $[V_{\alpha'-2}, V^\alpha] \leq [Q_\alpha, V^\alpha] \leq Z_\alpha$ so that $[V_{\alpha'-2}, V^\alpha] = Z_\beta = Z_{\alpha'-2}$ and $V^\alpha \leq Q_{\alpha'-2}$. Moreover, $V_{\alpha'} \nleq Q_\beta$, otherwise $R = Z_\beta = Z_{\alpha'}$ and $O^p(L_{\alpha'})$ centralizes $V_{\alpha'}$.

Suppose that $V^\alpha \nleq Q_{\alpha'-1}$. Now, $V_\beta \leq Q_{\alpha'-1}$ and so $[V_\beta, V^{\alpha'-1}] \leq Z_{\alpha'-1}$ and since $V^\alpha \nleq Q_{\alpha'-1}$, $[V^{\alpha'-1}, V_\beta] = Z_{\alpha'-2} = Z_\beta$ and $V^{\alpha'-1} \leq Q_\beta$. Moreover, $V^{\alpha'-1} \nleq Q_\alpha$, else $[V^\alpha, V^{\alpha'-1}] = Z_\beta = Z_{\alpha'-2} \leq Z_{\alpha'-1}$ and $V^\alpha \leq Q_{\alpha'-1}$. Thus, $[V_\alpha^{(2)} \cap Q_{\alpha'-2}, V^{\alpha'-1}] = [V^\alpha(V_\alpha^{(2)} \cap Q_{\alpha'-1}), V^{\alpha'-1}] \leq V^\alpha Z_{\alpha'-2} = V^\alpha$. It follows that both $V_\alpha^{(2)}/V^\alpha$ and V^α/Z_α are FF-modules for $\overline{L_\alpha}$ and by Lemma 5.2.32 and

Lemma 5.2.18, we conclude that $Z_\beta = Z_{\alpha-3}$ implies that $V_\beta = V_{\alpha+3} \leq Q_{\alpha'}$, a contradiction.

Thus, $V^\alpha V_\beta = V_\beta(V^\alpha V_\beta \cap Q_{\alpha'})$. As in the $b = 7$ case, again set $U^\beta = \langle V^\delta \mid Z_\lambda = Z_\alpha, \lambda \in \Delta(\beta)\rangle \trianglelefteq R_\beta Q_\alpha$ so that (λ, α') is a critical pair for all such λ and, by the above, $V^\lambda \leq Q_{\alpha'-1}$. Then, $U^\beta V_\beta \trianglelefteq L_\beta = \langle V_{\alpha'}, R_\beta, Q_\alpha\rangle$ and since $V_{\alpha'}$ centralizes $U^\beta V_\beta/V_\beta$, $O^p(L_\beta)$ centralizes $U^\beta V_\beta/V_\beta$ and $V_\beta V^\alpha \trianglelefteq L_\beta$. A contradiction is provided by Lemma 5.2.31. $\square$

As a consequence of Lemma 5.4.24, we may assume that whenever $b > 5$, we have that $|V_\beta| = p^3$.

Lemma 5.4.25. *Suppose that $C_{V_\beta}(V_{\alpha'}) = V_\beta \cap Q_{\alpha'}$ and $b > 5$. If $V_\alpha^{(2)} \not\leq Q_{\alpha'-2}$ then either:*

(i) *$R = Z_{\alpha'-2} \leq Z_{\alpha+2} \cap Z_{\alpha'-1}$; or*

(ii) *$Z_{\alpha'-1} = Z_{\alpha'-3}$ and $V_{\alpha'} \leq Q_\beta$.*

Proof. By Lemma 5.4.24, we have that $|V_\beta| = p^3$, so that $R = [V_{\alpha'}, V_\beta] \leq Z_{\alpha'-1} \cap Z_{\alpha+2}$. Suppose that $R \neq Z_{\alpha'-2}$. Then $Z_{\alpha'-1} = R \times Z_{\alpha'-2}$ is centralized by $V_\alpha^{(2)}$ and since $V_\alpha^{(2)} \not\leq Q_{\alpha'-2}$, we deduce that $Z_{\alpha'-1} = Z_{\alpha'-3}$. Now, if $V_{\alpha'} \not\leq Q_\beta$, then $R \neq Z_\beta$ and since $[V_{\alpha'-2}, V_\alpha^{(2)}] \leq Z_\alpha$, we must have that $Z_\beta = [V_{\alpha'-2}, V_\alpha^{(2)}] \leq Z_{\alpha'-3} = Z_{\alpha'-1}$ and $Z_{\alpha'-1} = R \times Z_\beta \leq V_\beta$. Thus, $V_\beta^{(3)} \cap Q_{\alpha'-2} \leq Q_{\alpha'-1}$, $V_\beta^{(3)} \cap Q_{\alpha'-2} = V_\beta(V_\beta^{(3)} \cap Q_{\alpha'})$ and since $Z_{\alpha'} \leq Z_{\alpha'-1} \leq V_\beta$, $V_\beta^{(3)}/V_\beta$ contains a unique non-central chief factor for L_β which is an FF-module. Then, by Lemma 5.2.34 and Lemma 5.2.18, $Z_{\alpha'-1} = Z_{\alpha'-3}$ implies that $V_{\alpha'} \leq V_{\alpha'-1}^{(2)} = V_{\alpha'-3}^{(2)} \leq Q_\alpha$, a contradiction. $\square$

Lemma 5.4.26. *Suppose that $C_{V_\beta}(V_{\alpha'}) = V_\beta \cap Q_{\alpha'}$ and $b > 5$. Then there exists a critical pair $(\alpha^*, \alpha^{*\prime})$ such that $V_{\alpha^*}^{(2)} \leq Q_{\alpha^{*\prime}-2}$.*

Proof. Since $V_\alpha^{(2)} \not\leq Q_{\alpha'-2}$, there is another critical pair $(\alpha-2, \alpha'-2)$ and we may assume recursively, that there is a path $(\alpha', \alpha'-1, \dots, \alpha, \alpha-1, \alpha-2, \alpha-3, \dots)$ such that $(\alpha-2k, \alpha'-2k)$ is a critical pair satisfying $V_{\alpha-2k}^{(2)} \not\leq Q_{\alpha'-2k-2}$ for all $k \geqslant 0$. Set $R_k := [V_{\alpha-2k+1}, V_{\alpha'-2k}]$ for each critical pair $(\alpha-2k, \alpha'-2k)$. In particular, $R = R_0$.

Choose $k \geqslant (b-1)/2$ and suppose that $Z_{\alpha'-2k-1} = Z_{\alpha'-2k-3}$. Then as $k \geqslant (b-1)/2$, $2k+3 \geqslant b+2$ and so, by assumption, $(\alpha'-2k-3, \alpha'-2k-3+b)$ is a critical pair, a contradiction. Thus, for $k \geqslant (b-1)/2$, we may assume that for every critical pair $(\alpha-2k, \alpha'-2k)$, we have that $R_k = Z_{\alpha'-2k-2} \leq Z_{\alpha-2k+2}$. Now, if $R_k \neq Z_{\alpha-2k+3}$, then $Z_{\alpha-2k+2} = R_k \times Z_{\alpha-2k+3} \leq Q_{\alpha'-2k+2}$ a contradiction as $k \geqslant 1$ and $(\alpha-2k+2, \alpha'-2k+2)$ is a critical pair. Thus, we may assume that $Z_{\alpha'-2k-2} = Z_{\alpha-2k+3}$ for sufficiently large k. Then, $R_k = R_{k+1}$ for otherwise $Z_{\alpha-2k+2} = R_k \times R_{k+1} \leq Q_{\alpha'-2k+2}$ since $b > 5$. In particular, $Z_{\beta-2k} = Z_{\alpha-1-2k}$ and $(\alpha-(b-1)-2k, \beta-2k)$ is a critical pair with $R_{\frac{b-1}{2}-k} = Z_{\beta-2k-2} = Z_{\alpha-1-2k} = Z_{\beta-2k}$. But then $O^p(L_{\beta-2k})$ centralizes $V_{\beta-2k}/Z_{\beta-2k}$, a contradiction. $\square$

We aim to show that $b \leqslant 5$, and by Lemma 5.4.26, we can fix some pair (α, α') with $V_\alpha^{(2)} \leq Q_{\alpha'-2}$. We start with the case where $V_{\alpha'} \leq Q_\beta$.

Lemma 5.4.27. *Suppose that $C_{V_\beta}(V_{\alpha'}) = V_\beta \cap Q_{\alpha'}$ and $b > 5$. Assume that $V_{\alpha'} \leq Q_\beta$ and $V_\alpha^{(2)} \leq Q_{\alpha'-2}$. Then $V_\alpha^{(2)} \leq Q_{\alpha'-1}$.*

Proof. Suppose for a contradiction that $V_\alpha^{(2)} \not\leq Q_{\alpha'-1}$. Then, as $R \leq Z_{\alpha'-1}$, we conclude that $R = Z_{\alpha'-2}$. Let $\alpha-1 \in \Delta(\alpha)$ such that $V_{\alpha-1} \not\leq Q_{\alpha'-1}$. If $Z_{\alpha'-1} \leq Q_{\alpha-1}$, then $Z_{\alpha'-2} = [V_{\alpha-1}, Z_{\alpha'-1}] = Z_{\alpha-1}$ from which it follows that $Z_{\alpha-1} = Z_\beta$. Then, recalling that $O^p(R_\alpha)$ centralizes $V_\alpha^{(2)}$ by Lemma 5.4.19, by Lemma 5.2.18 we have that $V_{\alpha-1} = V_\beta \leq Q_{\alpha'-1}$, a contradiction. Thus, $(\alpha'-1, \alpha-1)$ is a critical pair and $Z_{\alpha-1} \neq Z_\beta$.

Note that if $Z_{\alpha'-2} = Z_{\alpha'-4}$, then $Z_{\alpha'-1} \leq V_{\alpha'-2} = V_{\alpha'-4}$ is centralized by $V_\alpha^{(2)}$, a contradiction. Thus, $Z_{\alpha'-3}$ is centralized by $V_\alpha^{(4)}$ and either $Z_{\alpha'-3} = Z_{\alpha'-5}$ or $Z_{\alpha'-3} \neq Z_{\alpha'-5}$ and $V_\alpha^{(4)} \leq Q_{\alpha'-3}$. Assume that $Z_{\alpha'-3} = Z_{\alpha'-5}$. Notice that $Z_{\alpha'-1} \leq V_{\alpha'-3}^{(2)}$ and $[V_\alpha^{(2)}, V_{\alpha'-5}^{(2)}] = \{1\}$, and so by Lemma 5.2.18 and Lemma 5.2.34, there is not a unique non-central chief factor within $V_{\alpha-1}^{(3)}/V_{\alpha-1}$ which is an FF-module. Suppose that $V_{\alpha-1}^{(3)} \leq Q_{\alpha'-4}$. Then $V_{\alpha-1}^{(3)} \cap Q_{\alpha'-2} = V_{\alpha-1}(V_{\alpha-1}^{(3)} \cap Q_{\alpha'-1})$, a contradiction. Thus, there is $\alpha - 4 \in \Delta^{(3)}(\alpha - 1)$ such that $(\alpha - 4, \alpha' - 4)$ is a critical pair. Then $\{1\} \neq [V_{\alpha'-4}, V_{\alpha-3}] \leq Z_{\alpha-2} \cap Z_{\alpha'-5}$. If $[V_{\alpha'-4}, V_{\alpha-3}] \neq Z_{\alpha-1}$ then, as $b > 5$, $Z_{\alpha-2} = Z_{\alpha-1} \times [V_{\alpha'-4}, V_{\alpha-3}] \leq Q_{\alpha'-1}$, a contradiction. Thus, again as $b > 5$, $Z_\alpha = [V_{\alpha'-4}, V_{\alpha-3}] \times Z_\beta \leq Q_{\alpha'}$, a contradiction.

Thus, $Z_{\alpha'-3} \neq Z_{\alpha'-5}$ and $V_\alpha^{(4)} \leq Q_{\alpha'-3}$. It follows that $Z_{\alpha'-2} \leq [V_\alpha^{(4)}, V_{\alpha'-2}] \leq Z_{\alpha'-3}$. If $Z_{\alpha'-2} = [V_\alpha^{(4)}, V_{\alpha'-2}]$, then $V_\alpha^{(4)} = V_\alpha^{(2)}(V_\alpha^{(4)} \cap Q_{\alpha'})$ and since $Z_{\alpha'} \not\leq V_\alpha^{(4)}$, otherwise $V_\alpha^{(2)}$ centralizes $Z_{\alpha'-1} = Z_{\alpha'} \times R$, it follows that $V_{\alpha'}$ centralizes $V_\alpha^{(4)}/V_\alpha^{(2)}$, a contradiction. Thus, $[V_\alpha^{(4)}, V_{\alpha'-2}] = Z_{\alpha'-3}$. Since $V_\alpha^{(4)} \cap Q_{\alpha'-2} = V_\alpha^{(2)}(V_\alpha^{(4)} \cap Q_{\alpha'})$, we have that $V_\alpha^{(4)}/V_\alpha^{(2)}$ contains a unique non-central chief factor and by Lemma 5.2.33, $O^p(R_\alpha)$ centralizes $V_\alpha^{(4)}$. Furthermore, since $V_{\alpha-1}^{(3)} \not\leq Q_{\alpha'-2}$, otherwise $Z_{\alpha'-1}$ centralizes $V_{\alpha-1}^{(3)}/V_{\alpha-1}$, we may suppose that $Z_{\alpha'-3} = [V_{\alpha-1}^{(3)}, V_{\alpha'-2}]$.

Suppose first that $b > 9$. Then, $V_\alpha^{(6)}$ centralizes $Z_{\alpha'-3} \leq V_{\alpha-1}^{(3)}$ and so centralizes $Z_{\alpha'-4}Z_{\alpha'-6}$. If $Z_{\alpha'-4} = Z_{\alpha'-6}$, then by Lemma 5.2.18 we have that $Z_{\alpha'-1} \leq V_{\alpha'-4}^{(3)} = V_{\alpha'-6}^{(3)}$ is centralized by $V_\alpha^{(2)}$, a contradiction. Thus, $V_\alpha^{(6)}$ centralizes $Z_{\alpha'-5}$ and so either $Z_{\alpha'-5} = Z_{\alpha'-7}$ or $V_\alpha^{(6)}$ centralizes $Z_{\alpha'-3}Z_{\alpha'-5}Z_{\alpha'-7} = V_{\alpha'-6}V_{\alpha'-4}$. In the latter case, $V_\alpha^{(6)} = V_\alpha^{(4)}(V_\alpha^{(6)} \cap Q_{\alpha'-2})$ and since $Z_{\alpha'} \not\leq V_\alpha^{(6)}$, we conclude that $O^p(L_\alpha)$ centralizes $V_\alpha^{(6)}/V_\alpha^{(4)}$, a contradiction. Thus, $Z_{\alpha'-5} = Z_{\alpha'-7}$ and as $Z_{\alpha'-1} \leq V_{\alpha'-5}^{(4)}$ and $V_\alpha^{(2)}$ centralizes $V_{\alpha'-7}^{(4)}$, by Lemma 5.2.18, Lemma 5.2.34 and Lemma 5.2.35, we need only show that both $V_\beta^{(5)}/V_\beta^{(3)}$ and $V_\beta^{(3)}/V_\beta$ contain a unique non-central chief factor which is an FF-module for $\overline{L_\beta}$. We may prove it for any $\lambda \in \beta^G$ and, following the steps in an earlier part of this proof, we infer that $V_\beta^{(3)}/V_\beta$ satisfies the required condition. By the steps above, $V_{\alpha-1}^{(3)} \not\leq Q_{\alpha'-2}$. Then, as $V_{\alpha'-4} = Z_{\alpha'-3}Z_{\alpha'-7}$ is centralized by $V_{\alpha-1}^{(5)}$, $V_{\alpha-1}^{(5)} \cap Q_{\alpha'-6} = V_{\alpha-1}^{(3)}(V_{\alpha-1}^{(3)} \cap Q_{\alpha'-2})$ and since $V_{\alpha'-2} \not\leq Q_{\alpha-1}$ and $Z_{\alpha'-2} \leq V_{\alpha-1}^{(3)}$, $V_{\alpha-1}^{(5)}/V_{\alpha-1}^{(3)}$ contains a unique non-central chief factor and satisfies the required conditions. This provides the contradiction.

Suppose that $b = 7$. Then $C_{Q_\alpha}(V_\alpha^{(4)}) \leq Q_{\alpha+4} = Q_{\alpha'-3}$. Thus, $V_\alpha^{(4)}C_{Q_\alpha}(V_\alpha^{(4)}) = V_\alpha^{(4)}(V_\alpha^{(4)}C_{Q_\alpha}(V_\alpha^{(4)}) \cap Q_{\alpha'})$ and since $Z_{\alpha'} \not\leq C_{Q_\alpha}(V_\alpha^{(2)}) \geq C_{Q_\alpha}(V_\alpha^{(4)})$, $O^p(L_\alpha)$ centralizes $V_\alpha^{(4)}C_{Q_\alpha}(V_\alpha^{(4)})/V_\alpha^{(4)}$. Then for $r \in O^p(R_\alpha)$ of order coprime to p, $[r, Q_\alpha, V_\alpha^{(4)}] = \{1\}$ by the three subgroup lemma and so $[Q_\alpha, r] = [Q_\alpha, r, r, r] \leq [C_{Q_\alpha}(V_\alpha^{(4)}), r, r] \leq [V_\alpha^{(4)}, r] = \{1\}$ so that $R_\alpha = Q_\alpha$ and $\overline{L_\alpha} \cong \mathrm{SL}_2(p)$. We may assume that $V_{\alpha-1}^{(3)} \leq Q_{\alpha'-4}$, $V_{\alpha-1}^{(3)} \not\leq Q_{\alpha'-2}$ and $O^p(R_{\alpha-1})$ centralizes $V_{\alpha-1}^{(3)}$. Moreover, $Z_{\alpha'-3} = [V_{\alpha'-2}, V_{\alpha-1}^{(3)}] \leq V_{\alpha-1}^{(3)}$ and so $Z_{\alpha'-3}$ is centralized by $C_{Q_{\alpha-1}}(V_{\alpha-1}^{(3)})$. Since $Z_{\alpha'-3} \neq Z_{\alpha+2}$, otherwise by Lemma 5.2.18, $Z_\alpha \leq V_{\alpha+2}^{(2)} = V_{\alpha'-3}^{(2)} \leq Q_{\alpha'}$, we have that $C_{Q_{\alpha-1}}(V_{\alpha-1}^{(3)})$ centralizes $V_{\alpha+3}$. It follows that $C_{Q_{\alpha-1}}(V_{\alpha-1}^{(3)}) = V_{\alpha-1}^{(3)}(C_{Q_{\alpha-1}}(V_{\alpha-1}^{(3)}) \cap Q_{\alpha'-2})$ and so $O^p(L_{\alpha-1})$ centralizes $C_{Q_{\alpha-1}}(V_{\alpha-1}^{(3)})/V_{\alpha-1}^{(3)}$. Now, letting $r \in O^p(R_{\alpha-1})$ of order coprime to p, $[r, Q_{\alpha-1}, V_{\alpha-1}^{(3)}] = \{1\}$ by the three subgroup lemma and $[Q_{\alpha-1}, r] = [Q_{\alpha-1}, r, r, r] = [C_{Q_{\alpha-1}}(V_{\alpha-1}^{(3)}), r, r] = [V_{\alpha-1}^{(3)}, r] = \{1\}$ so that $R_{\alpha-1} = Q_{\alpha-1}$ and $\overline{L_{\alpha-1}} \cong \mathrm{SL}_2(p)$. Thus, G has a weak BN-pair of rank 2 and by [DS85], no examples exist.

Suppose that $b = 9$. Then $C_{Q_\alpha}(V_\alpha^{(4)}) \leq Q_{\alpha+4} = Q_{\alpha'-5}$. Moreover, $Z_{\alpha'-5} \neq Z_{\alpha'-3} \leq V_\alpha^{(4)}$ so that $C_{Q_\alpha}(V_\alpha^{(4)}) \leq Q_{\alpha'-3}$ and $C_{Q_\alpha}(V_\alpha^{(4)}) = V_\alpha^{(4)}(C_{Q_\alpha}(V_\alpha^{(4)}) \cap Q_{\alpha'-2})$ and it follows that $O^p(L_\alpha)$ centralizes $C_{Q_\alpha}(V_\alpha^{(4)})/V_\alpha^{(4)}$. As in the $b = 7$ case, we get that $\overline{L_\alpha} \cong \mathrm{SL}_2(p)$. Since $Z_{\alpha'-3} \leq V_{\alpha-1}^{(3)}$, $Z_{\alpha'-4}$ is centralized by

$C_{Q_{\alpha-1}}(V_{\alpha-1}^{(3)})$ and $Z_{\alpha'-6} = Z_{\alpha+3}$ is centralized by $C_{Q_{\alpha-1}}(V_{\alpha-1}^{(3)})$ from which it follows that $C_{Q_{\alpha-1}}(V_{\alpha-1}^{(3)})$ centralizes $Z_{\alpha+4} = Z_{\alpha'-5}$. Continuing as above, we see that $C_{Q_{\alpha-1}}(V_{\alpha-1}^{(3)}) = V_{\alpha-1}^{(3)}(C_{Q_{\alpha-1}}(V_{\alpha-1}^{(3)}) \cap Q_{\alpha'-2})$ and $O^p(L_{\alpha-1})$ centralizes $C_{Q_{\alpha-1}}(V_{\alpha-1}^{(3)})/V_{\alpha-1}^{(3)}$ and an application of the three subgroup lemma and coprime action yields that $\overline{L_{\alpha-1}} \cong \mathrm{SL}_2(p)$ and G has a weak BN-pair of rank 2. By [DS85], no examples exist and the proof is complete. $\square$

Lemma 5.4.28. *Suppose that $C_{V_\beta}(V_{\alpha'}) = V_\beta \cap Q_{\alpha'}$ and $b > 5$. If $V_{\alpha'} \leq Q_\beta$ then $V_\alpha^{(4)} \not\leq Q_{\alpha'-4}$.*

Proof. By Lemma 5.4.27, we may suppose that $V_\alpha^{(2)} \leq Q_{\alpha'-1}$. Note that by Lemma 5.4.19, $Z_{\alpha'-1} = Z_{\alpha'} \times Z_\beta \leq V_\alpha^{(2)} \leq Z(V_\alpha^{(4)})$. Suppose that $V_\alpha^{(4)} \leq Q_{\alpha'-4}$ throughout. If $Z_{\alpha'-1} \neq Z_{\alpha'-3}$, then $V_\alpha^{(4)} \cap Q_{\alpha'-3} = V_\alpha^{(2)}(V_\alpha^{(4)} \cap Q_{\alpha'})$ and since $Z_{\alpha'} \leq V_\alpha^{(2)}$, $V_\alpha^{(4)}$ does not centralize $Z_{\alpha'-3}$. But $Z_{\alpha'-2} \leq Z_{\alpha'-1}$ so that $Z_{\alpha'-2}Z_{\alpha'-4}$ is centralized by $V_\alpha^{(4)}$ and $Z_{\alpha'-2} = Z_{\alpha'-4}$. Now, both $V_\alpha^{(4)}/V_\alpha^{(2)}$ and $V_\alpha^{(2)}/Z_\alpha$ contain unique non-central chief factors and by Lemma 5.2.32 and Lemma 5.2.33, we deduce that $O^p(R_\alpha)$ centralizes $V_\alpha^{(4)}$. Therefore, applying Lemma 5.2.18 to $Z_{\alpha'-2} = Z_{\alpha'-4}$, we conclude that $V_{\alpha'} \leq V_{\alpha'-2}^{(3)} = V_{\alpha'-4}^{(3)}$ is centralized by Z_α, a contradiction.

Thus, $Z_{\alpha'-1} = Z_{\alpha'-3}$ and $V_\alpha^{(4)} \not\leq Q_{\alpha'-2}$. In particular, it follows again by Lemma 5.2.33 that $O^p(R_\alpha)$ centralizes $V_\alpha^{(4)}$ and so, similarly to the above, $Z_{\alpha'-2} \neq Z_{\alpha'-4}$. Moreover, by Lemma 5.2.18, since $V_{\alpha'} \leq V_{\alpha'-1}^{(2)}$ and $V_{\alpha'-3}^{(2)} \leq Q_\alpha$, $O^p(R_\beta)$ does not centralize $V_\beta^{(3)}$. In particular, $Z_{\alpha'-1} \neq Z_{\alpha+2}$ for otherwise $V_{\alpha'}^{(3)} \cap Q_{\alpha+3} \leq Q_{\alpha+2}$, $[V_{\alpha'-3}^{(3)} \cap Q_{\alpha+3}, V_\beta] \leq Z_{\alpha+2} = Z_{\alpha'-1} \leq V_{\alpha'}$ and $V_{\alpha'}^{(3)}/V_{\alpha'}$ contains a unique non-central chief factor which is an FF-module, and we would have a contradiction by Lemma 5.2.34.

Suppose first that $b = 7$. Then $Z_\beta Z_{\alpha+3} \le Z_{\alpha+2} \cap Z_{\alpha'-3}$ and so either $Z_\beta = Z_{\alpha+3}$ or $Z_{\alpha'-1} = Z_{\alpha'-3} = Z_{\alpha+2}$. The latter case yields an immediate contradiction, while in the former case, Lemma 5.2.18 implies that $V_\beta = V_{\alpha+3} \le Q_{\alpha'}$, another contradiction. Thus, we may assume $b > 7$ throughout.

Assume that for $\alpha - 4 \in \Delta^{(4)}(\alpha)$, whenever $Z_{\alpha-4} \not\le Q_{\alpha'-2}$ we conclude that $Z_\beta = Z_{\alpha-1}$. Choose $\delta \in \Delta(\alpha)$ such that $Z_\delta \ne Z_\beta$ so that $V_\delta^{(3)} \le Q_{\alpha'-2}$. Moreover, $V_\delta^{(3)}$ centralizes $Z_{\alpha'-1} \le V_\alpha^{(2)}$ and $[V_\delta^{(3)}, V_{\alpha'}] = [V_\alpha^{(2)}, V_{\alpha'}][V_\delta^{(3)} \cap Q_{\alpha'}, V_{\alpha'}] \le V_\alpha^{(2)}$. Thus, $V_\delta^{(3)} \trianglelefteq L_\alpha = \langle V_{\alpha'}, R_\alpha, Q_\delta \rangle$, a contradiction. Thus, we may assume that there exists $\alpha - 4 \in \Delta^{(4)}(\alpha)$ with $Z_{\alpha-4} \not\le Q_{\alpha'-2}$ and $Z_\beta \ne Z_{\alpha-1}$.

Suppose that $V_{\alpha'-2} \not\le Q_{\alpha-1}$. Since $V_\alpha^{(2)} \le Q_{\alpha'-2}$, it follows that $Z_{\alpha'-2} = [V_\alpha^{(2)}, V_{\alpha'-2}] = Z_\beta$. Moreover, there is $\lambda \in \Delta(\alpha' - 2)$ such that $(\lambda, \alpha - 1)$ is a critical pair with $V_{\alpha-1} \le Q_{\alpha'-2}$. If $V_\lambda^{(2)} \le Q_\beta$, then by Lemma 5.4.27 $V_\lambda^{(2)} \le Q_\alpha$ and $Z_\alpha \le V_\lambda^{(2)}$, a contradiction since $b > 5$. Thus, $V_\lambda^{(2)} \not\le Q_\beta$ and $(\lambda + 2, \beta)$ is also a critical pair. Moreover, $\{1\} \ne [V_\beta, V_{\lambda+1}] \le Z_{\alpha+2} \cap Z_\lambda$. Since $Z_\lambda \not\le Q_{\alpha-1}$ and $Z_{\alpha'-2} \le V_\alpha^{(2)}$, it follows that $[V_\beta, V_{\lambda+1}] = Z_{\alpha'-2} = Z_\beta$. But then $V_{\lambda+1} \le Q_\beta$, a contradiction. Thus, $V_{\alpha'-2} \le Q_{\alpha-1}$ and $[V_{\alpha'-2}, V_{\alpha-1}] = \{1\}$, otherwise $Z_{\alpha-1} = [V_{\alpha'-2}, V_{\alpha-1}] = Z_{\alpha'-2}$ and since $Z_\alpha \not\le V_{\alpha'-2}$, $Z_{\alpha-1} = Z_\beta$, a contradiction. Therefore, $V_{\alpha'-2} \le Q_{\alpha-2}$.

Suppose that $[V_{\alpha'-2}, V_{\alpha-3}] = Z_\beta$ so that $Z_{\alpha'-2} \ne Z_\beta$. As $Z_\beta \le Z_{\alpha-2}$ and $Z_\beta \ne Z_{\alpha-1}$, $Z_\alpha = Z_{\alpha-2}$. Immediately, we have that $[V_\alpha^{(2)}, V_{\alpha'-2}] \le Z_{\alpha'-2} \cap Z_\alpha = \{1\}$ so that $V_\alpha^{(2)} \le C_{\alpha'-2}$.

Choose $\lambda \in \Delta(\alpha' - 2)$ such that $Z_\lambda \ne Z_{\alpha'-1}$ and set $W^{\alpha'-2} := \langle V_\delta^{(2)} \mid Z_\delta = Z_\lambda, \delta \in \Delta(\alpha' - 2) \rangle$. Then, for $\delta \in \Delta(\alpha' - 2)$ with $Z_\delta = Z_\lambda$, since $V_\alpha^{(2)} \le C_{\alpha'-2}$, we have that $[V_\beta, V_\delta^{(2)}] \le Z_\delta \cap Z_{\alpha+2}$. Since $Z_{\alpha+2} \le Z(V_\alpha^{(4)})$, $Z_\delta \cap Z_{\alpha+2} \le Z_{\alpha'-2}$, otherwise $V_\alpha^{(4)}$ centralizes $V_{\alpha'-2} = Z_\delta Z_{\alpha'-1}$. But now $[V_\beta, V_\delta^{(2)}] = \{1\}$, otherwise $Z_{\alpha+2} = Z_{\alpha'-2} \times Z_\beta = Z_{\alpha'-1}$, and we have a contradiction. Now, $[V_\alpha^{(2)}, V_\lambda^{(2)}] \leq Z_\lambda \cap Z_\alpha$ and for a similar reason as before, $[V_\alpha^{(2)}, V_\lambda^{(2)}] = \{1\}$. It follows that $W^{\alpha'-2} \leq Q_{\alpha-2}$ and $Z_\beta \leq [W^{\alpha'-2}, V_{\alpha-3}] \leq Z_{\alpha-2} = Z_\alpha$. Since $Z_\alpha \not\leq V_{\alpha'-2}^{(3)}$, we have that $[W^{\alpha'-2}, V_{\alpha-3}] = Z_\beta \leq V_{\alpha'-2}$ and $V_{\alpha-3}$ centralizes $W^{\alpha'-2}/V_{\alpha'-2}$. But now, by Lemma 5.2.19, $W^{\alpha'-2} \trianglelefteq L_{\alpha'-2} = \langle V_{\alpha-3}, R_{\alpha'-2}, Q_\lambda \rangle$. Since $V_{\alpha-3}$ centralizes $W^{\alpha'-2}/V_{\alpha'-2}$, it follows that $V_\lambda^{(2)} \trianglelefteq L_{\alpha'-2}$, a contradiction.

Suppose now that $Z_\beta \neq [V_{\alpha'-2}, V_{\alpha-3}] \leq Z_{\alpha-2} \cap Z_{\alpha'-3}$. Then $Z_\alpha \neq Z_{\alpha-2}$, else $Z_\alpha = Z_\beta \times [V_{\alpha'-2}, V_{\alpha-3}] \leq Z_{\alpha'-3}$, an obvious contradiction. Still, $Z_{\alpha'-3} = Z_{\alpha'-1} = Z_\beta[V_{\alpha'-2}, V_{\alpha-3}]$ so that $V_{\alpha-1} = Z_\alpha Z_{\alpha'-1}$. As $V_\beta \leq C_{\alpha'-2}$, it follows that $Z_\beta \leq [V_\beta, V_{\alpha'-2}^{(3)}] \leq Z_{\alpha+2} \cap V_{\alpha'-2}$. Since $Z_{\alpha+2} \neq Z_{\alpha'-1}$, $Z_{\alpha+2} \not\leq V_{\alpha'-2}$, otherwise $V_{\alpha'-2} = Z_{\alpha'-1}Z_{\alpha+2} \leq V_\alpha^{(2)}$ would be centralized by $V_\alpha^{(4)}$. Thus, $[V_\beta, V_{\alpha'-2}^{(3)}] = Z_\beta$ and $V_{\alpha'-2}^{(3)} \leq Q_\beta$. Then $V_{\alpha'-2}^{(3)} \cap Q_\alpha$ centralizes $V_{\alpha-1} = Z_\alpha Z_{\alpha'-1}$ and so $V_{\alpha'-2}^{(3)} \cap Q_\alpha \leq Q_{\alpha-2}$. Then $[V_{\alpha'-2}, V_{\alpha-3}] \leq [V_{\alpha'-2}^{(3)} \cap Q_\alpha, V_{\alpha-3}] \leq Z_{\alpha-2}$. If $[V_{\alpha'-2}^{(3)} \cap Q_\alpha, V_{\alpha-3}] = [V_{\alpha'-2}, V_{\alpha-3}]$, then $V_{\alpha'-2}^{(3)}/V_{\alpha'-2}$ contains a unique non-central chief factor which is an FF-module. By Lemma 5.2.34, $O^p(R_{\alpha'-2})$ centralizes $V_{\alpha'-2}^{(3)}$ and Lemma 5.2.18 applied to $Z_{\alpha'-1} = Z_{\alpha'-3}$ implies that $V_{\alpha'} \leq V_{\alpha'-1}^{(2)} = V_{\alpha'-3}^{(2)} \leq Q_\alpha$, a contradiction. Thus, $Z_{\alpha-1} \leq Z_{\alpha-2} \leq V_{\alpha'-2}^{(3)}$ and since $b > 5$, we have that $Z_\beta = Z_{\alpha-1}$, a final contradiction by the choice of $\alpha - 4$. $\square$

By Lemma 5.4.28, whenever $b > 5$ and $V_{\alpha'} \leq Q_\beta$, we may assume that there is a critical pair $(\alpha - 4, \alpha' - 4)$. In the following lemma, we let $(\alpha - 4, \alpha' - 4)$ be such a pair and and investigate the action of $V_{\alpha'-4}$ on $V_{\alpha-3}$ and vice versa.

Lemma 5.4.29. *Suppose that $C_{V_\beta}(V_{\alpha'}) = V_\beta \cap Q_{\alpha'}$ and $b > 5$. If $V_{\alpha'} \leq Q_\beta$ then $b > 7$, $Z_\alpha \neq Z_{\alpha-2}$, $O^p(R_\beta)$ centralizes $V_\beta^{(3)}$ and setting $R^\dagger = [V_{\alpha'-4}, V_{\alpha-3}]$, either:*

(i) $R^\dagger = Z_{\alpha-1} = Z_\beta$; *or*

(ii) $R^\dagger \neq Z_{\alpha-1}$.

Proof. By Lemma 5.4.27, $V_\alpha^{(2)} \leq Q_{\alpha'-1}$, $Z_{\alpha'-1} = Z_{\alpha'} \times Z_\beta \leq V_\alpha^{(2)} \leq Z(V_\alpha^{(4)})$, $V_\alpha^{(4)} \not\leq Q_{\alpha'-4}$ and there is a critical pair $(\alpha-4, \alpha'-4)$. Set $R^\dagger := [V_{\alpha'-4}, V_{\alpha-3}] \leq Z_{\alpha'-5} \cap Z_{\alpha-2}$. By assumption $R^\dagger \neq Z_{\alpha'-4}$.

Suppose first that $R^\dagger = Z_{\alpha-1} \leq Z_{\alpha'-5}$. Then, as $b > 5$, $Z_{\alpha-1} = Z_\beta$ so that by Lemma 5.2.18, $V_{\alpha-1} = V_\beta$. Then $[V_{\alpha'-4}^{(3)}, V_{\alpha-1}] = [V_{\alpha'-4}^{(3)}, V_\beta] = \{1\}$ and so $V_{\alpha'-4}^{(3)} \leq Q_{\alpha-2}$. Moreover, $V_{\alpha'-4} \not\leq Q_{\alpha-3}$, else $Z_{\alpha-3} = R^\dagger = Z_{\alpha-1}$ and by Lemma 5.2.18, $V_{\alpha-3} = V_{\alpha-1} \leq Q_{\alpha'-4}$, a contradiction as $(\alpha-4, \alpha'-4)$ is a critical pair. Then $V_{\alpha'-4}(V_{\alpha'-4}^{(3)} \cap Q_{\alpha'-3} \cap Q_{\alpha'-4})$ is an index p subgroup of $V_{\alpha'-4}^{(3)}$ which is centralized, modulo $V_{\alpha'-4}$, by $Z_{\alpha-4}$ and so, $V_{\alpha'-4}^{(3)}/V_{\alpha'-4}$ contains a unique non-central chief factor and by Lemma 5.2.34, and conjugacy, $O^p(R_\beta)$ centralizes $V_\beta^{(3)}$ and subject to proving $Z_\alpha \neq Z_{\alpha-2}$, (i) holds.

Assume now that $R^\dagger \neq Z_{\alpha-1}$ so that $Z_{\alpha-2} = Z_{\alpha-1} \times R^\dagger$ is centralized by $V_{\alpha'-4}^{(3)}$. If $Z_\alpha \neq Z_{\alpha-2}$ then it follows that $V_{\alpha'-4}^{(3)}$ centralizes $V_{\alpha-1}$ and $V_{\alpha'-4}^{(3)} \cap Q_{\alpha-3} \cap Q_{\alpha-4}$ is an index p^2 subgroup of $V_{\alpha'-4}^{(3)}$ centralized by $Z_{\alpha-4}$. Hence, $V_{\alpha'-4}^{(3)}$ contains only two non-central chief factors for $L_{\alpha'-4}$, one in $V_{\alpha'-4}$ and one in $V_{\alpha'-4}^{(3)}/V_{\alpha'-4}$. Moreover, both non-central chief factors are FF-modules for $\overline{L_{\alpha'-4}}$ and by Lemma 5.2.34, and conjugacy, we have that $O^p(R_\beta)$ centralizes $V_\beta^{(3)}$ and again, subject to proving $Z_\alpha \neq Z_{\alpha-2}$, (ii) holds.

It remains to prove that $b > 7$ and $Z_\alpha \neq Z_{\alpha-2}$. Observe that if $Z_\alpha = Z_{\alpha-2}$ and $O^p(R_\beta)$ centralizes $V_\beta^{(3)}$ then by Lemma 5.2.18, $Z_{\alpha-4} \leq V_{\alpha-2}^{(2)} = V_\alpha^{(2)} \leq Q_{\alpha'-4}$, a contradiction since $(\alpha-4, \alpha'-4)$ is a critical pair. Now, if $b > 7$ and $Z_\alpha = Z_{\alpha-2}$, then $V_{\alpha'-4}^{(3)} \cap Q_{\alpha-1}$ centralizes $Z_{\alpha-2}$ and $[V_{\alpha'-4}^{(3)} \cap Q_{\alpha-1}, V_{\alpha-3}] \leq Z_{\alpha-2} \cap V_{\alpha'-4}^{(3)} \leq Z_\alpha \cap$

$Q_{\alpha'} = Z_\beta$. Thus, there is a p-element in $V_{\alpha-3} \setminus Q_{\alpha'-4}$ which commutates a maximal subgroup of $V_{\alpha'-4}^{(3)}$ to a subgroup of order p. But then such an element centralizes an index p^2 subgroup of $V_{\alpha'-4}^{(3)}$ and as before, $V_{\alpha'-4}^{(3)}$ contains only two non-central chief factors for $L_{\alpha'-4}$, both being FF-modules for $\overline{L_{\alpha'-4}}$ and by Lemma 5.2.34, and conjugacy, we have that $O^p(R_\beta)$ centralizes $V_\beta^{(3)}$, a contradiction. We may assume that $b = 7$ for the remainder of the proof.

Suppose first that $R = Z_\beta = Z_{\alpha'-2}$. Since $Z_\beta \neq Z_{\alpha+3} = Z_{\alpha'-4}$, for otherwise by Lemma 5.2.18, $V_\beta = V_{\alpha+3} \leq Q_{\alpha'}$, we may assume that $Z_{\alpha+2} = Z_\beta \times Z_{\alpha+3} = Z_{\alpha'-2} \times Z_{\alpha'-4} = Z_{\alpha'-3}$. If $O^p(R_\beta)$ centralizes $V_\beta^{(3)}$ then Lemma 5.2.18 applied to $Z_{\alpha+2} = Z_{\alpha'-3}$ implies that $Z_\alpha \leq V_{\alpha+2}^{(2)} = V_{\alpha'-3}^{(2)} \leq Q_{\alpha'}$, a contradiction. But now, $V_{\alpha'}^{(3)} \cap Q_{\alpha+3}$ centralizes $Z_{\alpha+2} = Z_{\alpha'-3}$ and $[V_{\alpha'}^{(3)} \cap Q_{\alpha+3}, V_\beta] \leq Z_{\alpha+2} = Z_{\alpha'-3}$. In particular, we deduce that $Z_{\alpha'-3} \neq Z_{\alpha'-1}$ for otherwise $V_{\alpha'}^{(3)}/V_{\alpha'}$ contains a unique non-central chief factor for $L_{\alpha'}$ and by Lemma 5.2.34, $O^p(R_{\alpha'})$ centralizes $V_{\alpha'}^{(3)}$. But then, recalling from Lemma 5.4.19 that $Z_{\alpha'-1} \leq V_\alpha^{(2)}$, we have that $V_{\alpha'-2} = Z_{\alpha'-1}Z_{\alpha'-3} = Z_{\alpha'-1}Z_{\alpha+2} \leq V_\alpha^{(2)}$. Since $V_{\alpha'-2} \leq Q_{\alpha'}$, $Z_\alpha \not\leq V_{\alpha'-2}$ and so $Z_\alpha V_{\alpha'-2}$ is a subgroup of $V_\alpha^{(2)}$ of order p^4. Now, $V_\alpha^{(2)}/Z_\alpha$ is a FF-module for $\overline{L_\alpha}$ and V_β/Z_α has order p and generates $V_\alpha^{(2)}/Z_\alpha$, we infer that $p^4 \leqslant |V_\alpha^{(2)}| \leqslant p^5$. If $|V_\alpha^{(2)}| = p^4$, then $[V_\alpha^{(2)}, V_{\alpha'}] = [V_{\alpha'-2}Z_\alpha, V_{\alpha'}] = Z_\beta$, a contradiction by Lemma 5.4.19. Thus, $|V_\alpha^{(2)}| = p^5$ and the preimage of $C_{V_\alpha^{(2)}/Z_\alpha}(O^p(L_\alpha))$ in $V_\alpha^{(2)}$, which we write as C^α, has order p^3. By the action of Q_β on $V_\alpha^{(2)}$, we must have that $C^\alpha V_\beta \leq [V_\alpha^{(2)}, Q_\beta]V_\beta$. Moreover, since $Z_\alpha = Z(Q_\alpha)$, we must have that $[Q_\alpha, C^\alpha] = Z_\alpha$

If $[V_\beta^{(3)}, Q_\beta]V_\beta/V_\beta$ is centralized by $O^p(L_\beta)$ then we have that $C^\alpha V_\beta \trianglelefteq L_\beta$. But then $Z_\beta \leq [C^\alpha V_\beta, Q_\beta] \leq Z_\alpha$ so that $[C^\alpha V_\beta, Q_\beta] = Z_\beta$. Then, we deduce that $C_{Q_\alpha}(C^\alpha) \leq Q_\beta$ for otherwise $Z_\alpha = [Q_\alpha, C^\alpha] = [Q_\alpha \cap Q_\beta, C^\alpha] \leq Z_\beta$, a contradiction. But now, as $C^{\alpha'-1}V_{\alpha'-2} \trianglelefteq L_{\alpha'-2}$, V_β centralizes $C^{\alpha'-1} \leq C^{\alpha'-3}V_{\alpha'-2}$ so that $V_\beta \leq$

$C_{Q_{\alpha'-1}}(C^{\alpha'-1}) \leq Q_{\alpha'}$, a contradiction.

Thus, $[V_\beta^{(3)}, Q_\beta]V_\beta/V_\beta$ contains a non-central chief factor for L_β. Moreover, since $V_{\alpha'}^{(3)} \cap Q_{\alpha+3} \leq Q_{\alpha+2}$, an index p^2 subgroup of $V_{\alpha'}^{(3)}/V_{\alpha'}$ is centralized by Z_α and we conclude that $V_\beta^{(3)}/V_\beta$ contains two non-central chief factors for L_β, one in $V_\beta^{(3)}/[V_\beta^{(3)}, Q_\beta]V_\beta$ by Lemma 5.2.13 and one in $[V_\beta^{(3)}, Q_\beta]V_\beta/V_\beta$, and both are FF-modules for $\overline{L_\beta}$. Notice that $[V_\alpha^{(2)}, Q_\beta, Q_\beta] \leq Z_\alpha$ so that $[V_\beta^{(3)}, Q_\beta, Q_\beta] \leq V_\beta$ and write $R_1 := C_{L_\beta}([V_\beta^{(3)}, Q_\beta]V_\beta/V_\beta)$ and $R_2 := C_{L_\beta}(V_\beta^{(3)}/[V_\beta^{(3)}, Q_\beta]V_\beta)$ so that $L_\beta/R_1 \cong L_\beta/R_2 \cong L_\beta/R_\beta \cong \mathrm{SL}_2(p)$. Indeed, either $p \in \{2,3\}$ and $L_\beta = \langle R_1, R_2, S\rangle$ by Lemma 2.3.15 (ii) or $R_1 = R_2$. In the former case, we have that $V_\alpha^{(2)}[V_\beta^{(3)}, Q_\beta]V_\beta \trianglelefteq R_2S$ so that $[V_\alpha^{(2)}[V_\beta^{(3)}, Q_\beta]V_\beta, Q_\beta]V_\beta = [V_\alpha^{(2)}, Q_\beta]V_\beta \trianglelefteq R_2S$. But $[V_\alpha^{(2)}, Q_\beta]V_\beta \trianglelefteq R_1S$ so that $[V_\beta^{(3)}, Q_\beta]V_\beta = [V_\alpha^{(2)}, Q_\beta]V_\beta \trianglelefteq L_\beta$, impossible as then $[V_\beta^{(3)}, Q_\beta]V_\beta/V_\beta$ is centralized by Q_α, and so centralized by $O^p(L_\beta)$. Thus, $R_1 = R_2$ and as $O^p(R_\beta)$ does not centralize $V_\beta^{(3)}$ and R_β normalizes $Q_\alpha \cap Q_\beta$, we satisfy the hypothesis of Lemma 5.2.29 with $\lambda = \beta$. Since $b > 7$, outcome of Lemma 5.2.29 holds and we have that $V_\alpha^{(4)} \leq \langle Z_\beta^X\rangle \leq Z(O_p(X))$. IN particular, $V_\alpha^{(4)}$ is abelian, and by conjugacy $V_{\alpha'}, Z_\alpha \leq V_{\alpha'-3}^{(4)}$, impossible since $[Z_\alpha, V_{\alpha'}] \neq \{1\}$.

Thus, we have that $Z_{\alpha'-2} \neq Z_\beta$ so that $Z_{\alpha'-1} = Z_{\alpha'-2} \times Z_\beta$. If $Z_{\alpha'-2} \not\leq Z_{\alpha+2}$, then $V_{\alpha+3} = V_{\alpha'-4} = Z_{\alpha+2}Z_{\alpha'-2} \leq V_\alpha^{(2)}$ is centralized by $V_\alpha^{(4)}$, a contradiction by Lemma 5.4.28. Thus, $Z_{\alpha+2} = Z_{\alpha'-2} \times Z_\beta = Z_{\alpha'-1}$. Now, $[V_{\alpha'}^{(3)} \cap Q_{\alpha+3}, V_\beta] \leq Z_{\alpha+2} \leq V_{\alpha'}$ and by Lemma 5.2.34, $O^p(R_{\alpha'})$ centralizes $V_{\alpha'}^{(3)}$. In particular, $Z_\alpha \neq Z_{\alpha-2}$ and $Z_{\alpha'-1} \neq Z_{\alpha'-3}$, else by Lemma 5.2.18, $Z_{\alpha-4} \leq V_{\alpha-2}^{(2)} = V_\alpha^{(2)} \leq Q_{\alpha'-4}$ and $V_{\alpha'} \leq V_{\alpha'-1}^{(2)} = V_{\alpha'-3}^{(2)} \leq Q_\alpha$ respectively. Since $Z_{\alpha'-2}Z_{\alpha'-4} \leq Z_{\alpha+2} \cap Z_{\alpha'-3}$, we get that $Z_{\alpha'-2} = Z_{\alpha'-4}$.

We will show that whenever $(\alpha-4, \alpha'-4)$ is a critical pair, we have that $Z_\beta = Z_{\alpha-1}$. Choose $\alpha - 4$ such that $Z_{\alpha-4} \not\leq Q_{\alpha'-4}$. By the above, since $Z_\alpha \neq Z_{\alpha-2}$, assuming

$Z_\beta \neq Z_{\alpha-1}$, we deduce that (ii) holds and $R^\dagger := [V_{\alpha-3}, V_{\alpha'-4}] \neq Z_{\alpha-1}$. Then $Z_{\alpha-2} = R^\dagger \times Z_{\alpha-1}$. But $R^\dagger \leq Z_{\alpha+2} \leq V_\beta$ and $V_\beta = Z_\alpha Z_{\alpha-2} = V_{\alpha-1}$. Then, if $Z_\beta \neq Z_{\alpha-1}$, $V_\beta \trianglelefteq L_\alpha = \langle Q_\beta, Q_{\alpha-1}, R_\alpha \rangle$, a contradiction. Therefore, we have shown that whenever $Z_{\alpha-4} \not\leq Q_{\alpha'-4}$, $Z_\beta = Z_{\alpha-1}$.

Choose $\delta \in \Delta(\alpha)$ such that $Z_\delta \neq Z_\beta$ so that $V_\delta^{(3)} \leq Q_{\alpha'-4}$. Suppose that $V_\delta^{(3)} \not\leq Q_{\alpha'-3}$. There is $\delta - 2 \in \Delta^{(2)}(\delta)$ such that $Z_{\alpha'-4} = [V_{\delta-2}, Z_{\alpha'-3}] \leq Z_{\delta-1}$ and since $Z_{\alpha'-2} = Z_{\alpha'-4} = Z_{\alpha+3}$, $Z_{\alpha'-2} \leq V_\beta \cap V_\delta$. If $Z_{\alpha'-2} \leq Z_\alpha$, then $Z_\alpha = Z_\beta \times Z_{\alpha'-2} = Z_{\alpha'-1}$, a clear contradiction. Thus, $V_\beta = Z_{\alpha'-2} Z_\alpha = V_\delta$. But $Z_\beta \neq Z_\delta$ so that $V_\beta \trianglelefteq L_\alpha = \langle Q_\beta, Q_\delta, R_\alpha \rangle$, a contradiction.

Hence, $V_\delta^{(3)} \leq Q_{\alpha'-3}$ and since $Z_{\alpha'-3} \neq Z_{\alpha'-1} = Z_{\alpha+2}$, $V_\delta^{(3)}$ centralizes $V_{\alpha'-2}$ and $V_\delta^{(3)} \leq Q_{\alpha'-1}$. Setting $W^\alpha := \langle V_\lambda^{(3)} \mid Z_\lambda = Z_\delta, \lambda \in \Delta(\alpha) \rangle$, we have that $W^\alpha = V_\alpha^{(2)}(W^\alpha \cap Q_{\alpha'})$ and as $Z_{\alpha'} \leq V_\alpha^{(2)}$, $V_{\alpha'}$ centralizes $W^\alpha / V_\alpha^{(2)}$. Moreover, since $R_\alpha Q_\delta$ normalizes W^α by Lemma 5.2.19, $W^\alpha \trianglelefteq L_\alpha = \langle V_{\alpha'}, Q_\delta, R_\alpha \rangle$. Since $V_{\alpha'}$ centralizes $W^\alpha / V_\alpha^{(2)}$, $O^p(L_\alpha)$ centralizes $W^\alpha / V_\alpha^{(2)}$ and $V_\delta^{(3)} \trianglelefteq L_\alpha$, a final contradiction. □

Lemma 5.4.30. *Suppose that $C_{V_\beta}(V_{\alpha'}) = V_\beta \cap Q_{\alpha'}$ and $b > 5$. Then $V_{\alpha'} \not\leq Q_\beta$.*

Proof. Since $V_{\alpha'} \leq Q_\beta$, by Lemma 5.4.29, we may assume that $b > 7$ throughout. Recall from Lemma 5.4.19 that $Z_{\alpha'-1} \leq V_\alpha^{(2)} \leq Z(V_\alpha^{(4)}$. Notice that by Lemma 5.4.29, we have that $O^p(R_\beta)$ centralizes $V_\beta^{(3)}$ and by Lemma 5.2.18, if $Z_{\alpha'-1} = Z_{\alpha'-3}$ then $V_{\alpha'} \leq V_{\alpha'-1}^{(2)} = V_{\alpha'-3}^{(2)} \leq Q_\alpha$, a contradiction. Hence, we may assume that $Z_{\alpha'-1} \neq Z_{\alpha'-3}$ throughout the remainder of the proof. We fix $\alpha - 4 \in \Delta^{(4)}(\alpha)$ with $(\alpha - 4, \alpha' - 4)$ a critical pair.

Suppose first that $Z_{\alpha'-2} \neq Z_{\alpha'-4}$ so that $Z_{\alpha'-3} = Z_{\alpha'-2} \times Z_{\alpha'-4}$ is centralized by $V_\alpha^{(4)}$. Then, $V_{\alpha'-2} = Z_{\alpha'-1} Z_{\alpha'-3}$ is centralized by $V_\alpha^{(4)}$ so $V_\alpha^{(4)} \cap Q_{\alpha'-4} = V_\alpha^{(2)}(V_\alpha^{(4)} \cap Q_{\alpha'})$ and since $Z_{\alpha'} \leq V_\alpha^{(2)}$, it follows from Lemma 5.2.33 that $O^p(R_\alpha)$ centralizes

$V_\alpha^{(4)}$. In particular, we deduce that $Z_\beta \neq Z_{\alpha-1}$, otherwise by Lemma 5.2.18 we have that $V_{\alpha-3} \leq V_{\alpha-1}^{(3)} = V_\beta^{(3)} \leq Q_{\alpha'-4}$, a contradiction. Furthermore, as $V_\alpha^{(4)} \not\leq Q_{\alpha'-4}$ we have that $Z_{\alpha'-3} = Z_{\alpha'-5}$.

By Lemma 5.4.29, $Z_\alpha \neq Z_{\alpha-2}$, $O^p(R_\beta)$ centralizes $V_\beta^{(3)}$ and as $Z_{\alpha-1} \neq Z_\beta$, and again setting $R^\dagger := [V_{\alpha'-4}, V_{\alpha-3}]$, we have that $Z_{\alpha-1} < R^\dagger Z_{\alpha-1} \leq Z_{\alpha-2}$ and $R^\dagger Z_{\alpha-1}$ is centralized by $V_{\alpha'-4}^{(3)}$. Thus, $V_{\alpha'-4}^{(3)} \leq Q_{\alpha-2}$. Notice that, as $b > 7$, if $Z_{\alpha-2} \leq V_{\alpha'-4}^{(3)}$ then $Z_{\alpha-1} \leq V_{\alpha'-4}^{(3)} \leq Q_{\alpha'}$ and we conclude that $Z_{\alpha-1} = Z_\beta$, a contradiction. Thus, $Z_{\alpha-2} \not\leq V_{\alpha'-4}^{(3)}$.

If $V_{\alpha'-4} \not\leq Q_{\alpha-3}$ then $R^\dagger \neq Z_{\alpha-3}$ and $V_{\alpha'-4}^{(3)} = V_{\alpha'-4}(V_{\alpha'-4}^{(3)} \cap Q_{\alpha-3})$. Then $Z_{\alpha-3} = [V_{\alpha-3}, (V_{\alpha'-4}^{(3)} \cap Q_{\alpha-3})]$ for otherwise, $O^p(L_{\alpha'-4})$ centralizes $V_{\alpha'-4}^{(3)}/V_{\alpha'-4}$. But then $Z_{\alpha-2} = R^\dagger \times Z_{\alpha-3} \leq V_{\alpha'-4}^{(3)}$, a contradiction. Thus, $V_{\alpha'-4} \leq Q_{\alpha-3}$, $R^\dagger = Z_{\alpha-3}$ and $Z_{\alpha-3} \leq [V_{\alpha'-4}^{(3)}, V_{\alpha-3}] \leq Z_{\alpha-2} \cap V_{\alpha'-4}^{(3)} = Z_{\alpha-3}$ so that $[V_{\alpha'-4}^{(3)}, V_{\alpha-3}] = Z_{\alpha-3}$ and $V_{\alpha'-4}^{(3)} = V_{\alpha'-4}(V_{\alpha'-4}^{(3)} \cap Q_{\alpha-4})$. But then $O^p(L_{\alpha'-4})$ centralizes $V_{\alpha'-4}^{(3)}/V_{\alpha'-4}$, another contradiction.

Therefore, $Z_{\alpha'-2} = Z_{\alpha'-4}$ and by Lemma 5.2.18, $V_{\alpha'-2} = V_{\alpha'-4}$ so that $V_\alpha^{(4)} \cap Q_{\alpha'-4} \cap Q_{\alpha'-3} \leq Q_{\alpha'-2}$. Since $Z_{\alpha'-1}$ is centralized by $V_\alpha^{(4)}$, $V_\alpha^{(4)} \cap Q_{\alpha'-4} \cap Q_{\alpha'-3} = V_\alpha^{(2)}(V_\alpha^{(4)} \cap Q_{\alpha'})$. If $V_\alpha^{(4)}/V_\alpha^{(2)}$ contains a unique non-central chief factor which is an FF-module for $\overline{L_\alpha}$, then by Lemma 5.2.18, $V_{\alpha'} \leq V_{\alpha'-2}^{(3)} = V_{\alpha'-4}^{(3)} \leq Q_\alpha$, a contradiction. Thus, $V_\alpha^{(4)} \not\leq Q_{\alpha'-4}$ and $V_\alpha^{(4)} \cap Q_{\alpha'-4} \not\leq Q_{\alpha'-3}$.

Since $b > 7$, $Z_{\alpha'-4} = Z_{\alpha'-2} \leq Z_{\alpha'-1} \leq V_\alpha^{(2)} \leq Z(V_\alpha^{(6)})$. If $Z_{\alpha'-4} = Z_{\alpha'-6}$, then by Lemma 5.2.18, $V_{\alpha'-4} = V_{\alpha'-6}$ is centralized by $V_\alpha^{(4)}$, a contradiction. Thus, $Z_{\alpha'-5}Z_{\alpha'-7}$ is centralized by $V_\alpha^{(6)}$. If $Z_{\alpha'-5} \neq Z_{\alpha'-7}$ then $V_\alpha^{(6)} \leq Q_{\alpha'-5}$ and $V_\alpha^{(6)} = V_\alpha^{(4)}(V_\alpha^{(6)} \cap Q_{\alpha'})$. But then $O^p(L_\alpha)$ centralizes $V_\alpha^{(6)}/V_\alpha^{(4)}$, and we have a contradiction. Thus, $Z_{\alpha'-5} = Z_{\alpha'-7}$. But now, as $O^p(R_\beta)$ centralizes $V_\beta^{(3)}$ by

Lemma 5.4.29, by Lemma 5.2.18 we have that $V_{\alpha'-4} \le V_{\alpha'-5}^{(2)} = V_{\alpha'-7}^{(2)}$ is centralized by $V_\alpha^{(4)}$, a final contradiction. $\square$

Lemma 5.4.31. *Suppose that $C_{V_\beta}(V_{\alpha'}) = V_\beta \cap Q_{\alpha'}$. Then $b \leqslant 7$.*

Proof. By Lemma 5.4.30, $V_{\alpha'} \not\le Q_\beta$ and $V_\beta^{(3)} \le Q_{\alpha'-1}$. By Lemma 5.4.21, we have that $Z_{\alpha'-1} \le V_\beta^{(3)}$ and $O^p(R_\beta)$ centralizes $V_\beta^{(3)}$. In particular, if $Z_{\alpha'-1} = Z_{\alpha'-3}$, then $V_{\alpha'} \le V_{\alpha'-1}^{(2)} = V_{\alpha'-3}^{(2)}$ is centralized by Z_α, a contradiction. Hence, $V_{\alpha'-2} = Z_{\alpha'-1}Z_{\alpha'-3}$. Suppose throughout that $b > 7$.

Suppose first that $V_\beta^{(5)} \le Q_{\alpha'-4}$. Then, $V_\beta^{(5)} \cap Q_{\alpha'-3}$ centralizes $V_{\alpha'-2}$ and so $V_\beta^{(5)} \cap Q_{\alpha'-3} = V_\beta^{(3)}(V_\beta^{(5)} \cap Q_{\alpha'})$. Since $Z_{\alpha'} \le V_\beta^{(3)}$, $V_\beta^{(5)} \not\le Q_{\alpha'-3}$. Moreover by Lemma 5.2.35, we have that $O^p(R_\beta)$ centralizes $V_\beta^{(5)}$ and so $V_\alpha^{(4)} \not\le Q_{\alpha'-3}$, else $V_\alpha^{(4)} \trianglelefteq L_\beta = \langle V_{\alpha'}, Q_\alpha, R_\beta \rangle$. Thus, there is $\alpha - 4 \in \Delta^{(4)}(\alpha)$ such that $Z_{\alpha'-4} = [Z_{\alpha-4}, Z_{\alpha'-3}]$ and since $Z_{\alpha'-2} \le Z_{\alpha'-1} \le V_\beta^{(3)}$, we deduce that $Z_{\alpha'-2} = Z_{\alpha'-4}$.

Suppose that $Z_{\alpha'-3} \not\le Q_{\alpha-3}$. Then $(\alpha'-3, \alpha-3)$ is a critical pair with $V_{\alpha-3} \le Q_{\alpha'-4}$. By Lemma 5.4.30, $V_{\alpha'-3}^{(2)} \not\le Q_{\alpha-1}$ and either $Z_\alpha = Z_{\alpha-2}$ or $Z_{\alpha-1} = [V_{\alpha'-4}, V_{\alpha-3}] = Z_{\alpha'-4}$. In the former case it follows from Lemma 5.2.18 that $Z_{\alpha-4} \le V_{\alpha-2}^{(2)} = V_\alpha^{(2)} \le Q_{\alpha'-3}$, a contradiction. In the latter case, we have that $Z_\beta = Z_{\alpha-1} = Z_{\alpha'-4} = Z_{\alpha'-2}$. Then $R \ne Z_{\alpha'-2}$, so that $Z_{\alpha'} \le Z_{\alpha'-1} = R \times Z_{\alpha'-2} \le V_\beta$ and $V_{\alpha'}$ centralizes $V_\beta^{(3)}/V_\beta$, a contradiction.

Thus, $Z_{\alpha'-3} \le Q_{\alpha-3}$ and $Z_{\alpha'-4} = Z_{\alpha-3}$. If $Z_{\alpha-3} \le Z_\alpha$, then $Z_{\alpha-3} = Z_\beta = Z_{\alpha'-4} = Z_{\alpha'-2}$. But then $R \ne Z_{\alpha'-2}$ and $Z_{\alpha'-1} = R \times Z_\beta$ so that $Z_{\alpha'-1} \le V_\beta$ and $V_{\alpha'}$ centralizes $V_\beta^{(3)}/V_\beta$, a contradiction. Thus, $V_{\alpha-1} = Z_\alpha Z_{\alpha-3}$ is centralized by $V_{\alpha'-3}^{(2)}$ so that $V_{\alpha'-3}^{(2)} \le Q_{\alpha-2}$. Then, $Z_{\alpha-3} \le [V_{\alpha'-3}^{(2)}, V_{\alpha-3}] \le Z_{\alpha-2}$ and since $V_{\alpha-3}$ does not centralize $V_{\alpha'-3}^{(2)}/Z_{\alpha'-3}$, we may assume that $Z_{\alpha-2} \le V_{\alpha'-3}^{(2)}$. Still, $[V_{\alpha'-3}^{(2)} \cap Q_{\alpha-3}, V_{\alpha-3}] \le Z_{\alpha'-3}$ and it follows from Lemma 5.2.32 then $O^p(R_\alpha)$ centralizes

$V_\alpha^{(2)}$. Since $Z_{\alpha'-2} = Z_{\alpha'-4}$, Lemma 5.2.18 implies that $V_{\alpha'-2} = V_{\alpha'-4}$. Moreover, since $V_{\alpha'-4}$ is not centralized by $V_\beta^{(5)}$, but $Z_{\alpha'-1}Z_{\alpha'-5} \leq V_{\alpha'-4}$ is centralized, it follows that $Z_{\alpha'-1} = Z_{\alpha'-5}$.

Now, if $Z_{\alpha'-4} = Z_{\alpha'-6}$ then Lemma 5.2.18 implies that $Z_{\alpha'-3} \leq V_{\alpha'-4} = V_{\alpha'-6}$ is centralized by $V_\alpha^{(4)}$, a contradiction. Thus $Z_{\alpha'-5} = Z_{\alpha'-4} \times Z_{\alpha'-6}$ is centralized by $V_{\alpha-4}^{(2)}$ since $Z_{\alpha'-4} = Z_{\alpha-3}$. Moreover, $Z_{\alpha'-5} \neq Z_{\alpha'-7}$, otherwise Lemma 5.2.18 implies that $Z_{\alpha'-3} \leq V_{\alpha'-5}^{(2)} = V_{\alpha'-7}^{(2)}$ is centralized by $V_\alpha^{(4)}$, so that $V_{\alpha-4}^{(2)}$ centralizes $V_{\alpha'-6}$ and $V_{\alpha-4}^{(2)} \leq Q_{\alpha'-5}$. If $V_{\alpha-4}^{(2)} \leq Q_{\alpha'-4}$, then $V_{\alpha-4}^{(2)} = Z_{\alpha-4}(V_{\alpha-4}^{(2)} \cap Q_{\alpha'-3})$ is centralized, modulo $Z_{\alpha-4}$, by $Z_{\alpha'-3}$ so that $O^p(L_{\alpha-4})$ centralizes $V_{\alpha-4}^{(2)}/Z_{\alpha-4}$, a contradiction. Then $V_{\alpha-4}^{(2)} \not\leq Q_{\alpha'-4}$ and $[V_{\alpha-4}^{(2)}, V_{\alpha'-4}] \not\leq Z_{\alpha'-4}$. Since $Z_{\alpha'-4} = Z_{\alpha-3} \leq V_{\alpha-4}^{(2)}$, we assume that $Z_{\alpha'-5} \leq V_{\alpha-4}^{(2)}$.

Now, $V_{\alpha-4}^{(4)}$ centralizes $Z_{\alpha'-6} \leq Z_{\alpha'-5}$ and either $Z_{\alpha'-6} = Z_{\alpha'-8}$; or $V_{\alpha-4}^{(4)}$ centralizes $Z_{\alpha'-5}Z_{\alpha'-7}$. In the latter case, we may assume that $Z_{\alpha'-5} \neq Z_{\alpha'-7}$ for the same reason as above, and so either $V_{\alpha-4}^{(4)} \leq Q_{\alpha'-5}$ and $O^p(L_{\alpha-4})$ centralizes $V_{\alpha-4}^{(4)}/V_{\alpha-4}^{(2)}$, a contradiction; or $Z_{\alpha'-7} = Z_{\alpha'-9}$, $O^p(R_\beta)$ centralizes $V_\beta^{(5)}$ and $Z_{\alpha'-3} \leq V_{\alpha'-7}^{(4)} = V_{\alpha'-9}^{(4)}$ is centralized by $V_\alpha^{(4)}$, another contradiction. Thus, $Z_{\alpha'-6} = Z_{\alpha'-8}$ so that $V_{\alpha'-6} = V_{\alpha'-8}$. Suppose that $V_{\alpha-4}^{(4)} \leq Q_{\alpha'-8}$. Then $[V_{\alpha-4}^{(4)} \cap Q_{\alpha'-7}, V_{\alpha'-6}] = [V_{\alpha-4}^{(4)} \cap Q_{\alpha'-7}, V_{\alpha'-8}] = Z_{\alpha'-8} = Z_{\alpha'-6}$ and $V_{\alpha-4}^{(4)} \cap Q_{\alpha'-7} \leq Q_{\alpha'-6}$. But $V_{\alpha-4}^{(4)} \cap Q_{\alpha'-7}$ centralizes $Z_{\alpha'-5}$ so that $V_{\alpha-4}^{(4)} \cap Q_{\alpha'-7} = V_{\alpha-4}^{(2)}(V_{\alpha-4}^{(4)} \cap Q_{\alpha'-4})$ and by Lemma 5.2.33, $O^p(R_{\alpha-4})$ centralizes $V_{\alpha-4}^{(4)}$. But now, Lemma 5.2.18 applied to $Z_{\alpha'-2} = Z_{\alpha'-4}$ implies that $V_{\alpha'} \leq V_{\alpha'-2}^{(3)} = V_{\alpha'-4}^{(3)} \leq Q_\alpha$, a contradiction.

Thus, we have shown that there is a critical pair $(\alpha - 8, \alpha' - 8)$, $Z_{\alpha'-2} = Z_{\alpha'-4}$, $Z_{\alpha'-6} = Z_{\alpha'-8}$ and $V_{\alpha'-6} = V_{\alpha'-8}$. Since $Z_{\alpha'-5}Z_{\alpha'-9} \leq V_{\alpha'-8}$ is centralized by $V_{\alpha-4}^{(4)}$, we get that $Z_{\alpha'-1} = Z_{\alpha'-5} = Z_{\alpha'-9}$. We claim that the pair $(\alpha - 8, \alpha' - 8)$ satisfies the same initial hypothesis as (α, α'). By Lemma 5.4.30, $V_{\alpha-8}^{(2)} \not\leq Q_{\alpha'-10}$. But

$Z_{\alpha'-9} = Z_{\alpha'-5} \leq V_{\alpha-4}^{(2)}$ is centralized by $V_{\alpha-8}^{(2)}$ since $b > 7$, so that $Z_{\alpha'-9} = Z_{\alpha'-11}$. Then applying Lemma 5.2.18 gives $V_{\alpha'-8} \leq V_{\alpha'-9}^{(2)} = V_{\alpha'-11}^{(2)}$ is centralized by $V_{\alpha-4}^{(4)}$, a contradiction.

Suppose now that $V_\beta^{(5)} \not\leq Q_{\alpha'-4}$. Since $Z_{\alpha'-2} \leq Z_{\alpha'-1}$ is centralized by $V_\beta^{(5)}$, it follows that either $Z_{\alpha'-2} = Z_{\alpha'-4}$; or $Z_{\alpha'-3} = Z_{\alpha'-5}$. In the latter case, we have that $V_\beta^{(5)} \cap Q_{\alpha'-4}$ centralizes $V_{\alpha'-2}$ so that $V_\beta^{(5)} \cap Q_{\alpha'-4} = V_\beta^{(3)}(V_\beta^{(5)} \cap \dots \cap Q_{\alpha'})$ and Lemma 5.2.35 implies that $O^p(R_\beta)$ centralizes $V_\beta^{(5)}$. But then Lemma 5.2.18 applied to $Z_{\alpha'-3} = Z_{\alpha'-5}$ gives $V_{\alpha'} \leq V_{\alpha'-3}^{(4)} = V_{\alpha'-5}^{(4)} \leq Q_\alpha$, a contradiction. Thus, $Z_{\alpha'-2} = Z_{\alpha'-4}$. If $O^p(R_\alpha)$ centralizes $V_\alpha^{(2)}$, then using Lemma 5.2.18 and $Z_{\alpha'-2} = Z_{\alpha'-4}$, we have that $Z_{\alpha'-1}Z_{\alpha'-5} \leq V_{\alpha'-4}$ is centralized by $V_\beta^{(5)}$ and we conclude that $Z_{\alpha'-1} = Z_{\alpha'-5}$.

We have demonstrated, regardless of the hypothesis on $V_\beta^{(5)}$, that $Z_{\alpha'-2-4k} = Z_{\alpha'-4-4k}$ for $k \geqslant 0$, and there are suitable critical pairs to iterate upon. Suppose that $b = 9$. Applying the above, we infer that $Z_{\alpha'-2} = Z_{\alpha'-4}$ and $Z_{\alpha'-6} = Z_{\alpha+3} = Z_\beta$. Since $V_{\alpha'} \not\leq Q_\beta$, there is a critical pair $(\alpha'+1, \beta)$ with $V_\beta \not\leq Q_{\alpha'}$. Moreover, $V_{\alpha'+1}^{(2)} \leq Q_{\alpha+3}$, else by Lemma 5.4.25, $R = Z_{\alpha+3} = Z_\beta$, a clear contradiction. Thus, $(\alpha'+1, \beta)$ satisfies the same hypothesis as (α, α'). But then $Z_{\alpha'-6} = Z_{\alpha+3} = Z_{\alpha+5} = Z_{\alpha'-4}$ so that $Z_{\alpha'-2} = \dots = Z_\beta$. But then $R \neq Z_{\alpha'-2}$, $Z_{\alpha'-1} = Z_{\alpha'-2} \times R = Z_{\alpha+2}$ and $[V_{\alpha'}, V_\beta^{(3)}] = Z_{\alpha'-1} \leq V_\beta$, a contradiction for then $O^p(L_\beta)$ centralizes $V_\beta^{(3)}/V_\beta$. In fact, this applies whenever $b = 4k+1$ for $k \geqslant 2$ but we will only require this when $b = 9$.

Suppose that $V_\beta^{(5)} \not\leq Q_{\alpha'-4}$ and $b > 7$. Then $b \geqslant 11$ and $V_\beta^{(7)}$ centralizes $Z_{\alpha'-4} \leq Z_{\alpha'-1} \leq V_\beta^{(3)}$ and so, unless $Z_{\alpha'-4} = Z_{\alpha'-6}$, $[V_\beta^{(7)}, Z_{\alpha'-5}] = \{1\}$. Notice that if $Z_{\alpha'-5} = Z_{\alpha'-7}$, then Lemma 5.2.18 implies that $V_{\alpha'-4} \leq V_{\alpha'-5}^{(2)} = V_{\alpha'-7}^{(2)}$ is centralized by $V_\beta^{(5)}$, a contradiction. Thus, $V_\beta^{(7)}$ centralizes $V_{\alpha'-6} = Z_{\alpha'-5}Z_{\alpha'-7}$. But then

$V_\beta^{(7)} = V_\beta^{(5)}(V_\beta^{(7)} \cap Q_{\alpha'-4})$ and $V_\beta^{(5)} \cap Q_{\alpha'-4} \leq Q_{\alpha'-3}$, otherwise $V_\beta^{(7)} = V_\beta^{(5)}(V_\beta^{(7)} \cap Q_{\alpha'})$ so that $O^p(L_\beta)$ centralizes $V_\beta^{(7)}/V_\beta^{(5)}$, another contradiction. Then, $V_\beta^{(5)} \cap Q_{\alpha'-4} = V_\beta^{(3)}(V_\beta^{(5)} \cap \dots \cap Q_{\alpha'})$ and Lemma 5.2.35 implies that $O^p(R_\beta)$ centralizes $V_\beta^{(5)}$. In particular, $V_\alpha^{(4)} \not\leq Q_{\alpha'-4}$ for otherwise $V_\alpha^{(4)} \trianglelefteq L_\beta = \langle V_{\alpha'}, Q_\alpha, R_\beta \rangle$, a contradiction.

We have shown that, if $b > 7$ and $O^p(R_\alpha)$ centralizes $V_\alpha^{(2)}$ then $Z_{\alpha'-1} = Z_{\alpha'-1-4k}$ for all $k \geqslant 0$. Moreover, we can arrange that α lies along the infinite path $(\alpha', \alpha'-1, \dots, \alpha'-5, \dots)$; or for some critical pair $(\alpha^*, \alpha^{*\prime})$ we have that $Z_{\alpha^{*\prime}-2} = Z_{\alpha^{*\prime}-4} = Z_{\alpha^{*\prime}-6}$ and $V_{\beta^*}^{(5)} \not\leq Q_{\alpha^{*\prime}-4}$. In this latter case, Lemma 5.2.18 implies that $V_{\alpha^{*\prime}-4} = V_{\alpha^{*\prime}-6}$ and $V_{\beta^*}^{(5)}$ centralizes $V_{\alpha^{*\prime}-4}$, a clear contradiction. Now, since $Z_\alpha \neq Z_{\alpha'-1}$, $Z_{\alpha'-1} = Z_{\alpha+2} = Z_{\alpha-2}$. But then $[V_\beta^{(3)}, V_{\alpha'}] = Z_{\alpha'-1} \leq V_\beta$ and $O^p(L_\beta)$ centralizes $V_\beta^{(3)}/V_\beta$, a contradiction. In particular, if we ever arrive at a critical pair $(\alpha^*, \alpha^{*\prime})$ such that $V_{\alpha^*}^{(4)} \leq Q_{\alpha^{*\prime}-4}$, then $O^p(R_\alpha)$ centralizes $V_\alpha^{(2)}$ and we have a contradiction. Thus, whenever $b > 7$, we may assume that for every critical pair $(\alpha^*, \alpha^{*\prime})$, we have that $V_{\alpha^{*\prime}} \not\leq Q_{\beta^*}$, $V_{\beta^*}^{(3)} \leq Q_{\alpha^{*\prime}-1}$, $V_{\beta^*}^{(5)} \not\leq Q_{\alpha^{*\prime}-4}$ and $Z_{\alpha^{*\prime}-2} = Z_{\alpha^{*\prime}-4}$. Also, whenever $Z_{\alpha^{*\prime}-4} \neq Z_{\alpha^{*\prime}-6}$, $V_{\alpha^*}^{(4)} \not\leq Q_{\alpha^{*\prime}-4}$ and $V_{\alpha^*-4}^{(2)} \leq Q_{\alpha^{*\prime}-6}$

Suppose that $Z_{\alpha'-4} \neq Z_{\alpha'-6}$ so that there is a critical pair $(\alpha-4, \alpha'-4)$ and $O^p(R_\beta)$ centralizes $V_\beta^{(5)}$. We may also assume that $O^p(R_\alpha)$ does not centralize $V_\alpha^{(2)}$. Since $V_{\alpha'} \not\leq Q_\beta$, there is $\alpha'+1 \in \Delta(\alpha')$ such that $(\alpha'+1, \beta)$ is a critical pair. Suppose that $V_{\alpha'+1}^{(2)}$ centralizes Z_β. Since $Z_{\alpha+2} = Z_\beta \times R \neq Z_{\alpha+4}$, we have that $V_{\alpha'+1}^{(2)}$ centralizes $V_{\alpha+3}$ and $V_{\alpha'+1}^{(2)} = Z_{\alpha'+1}(V_{\alpha'+1}^{(2)} \cap Q_\beta)$. In particular, $V_{\alpha'+1}^{(2)} \cap Q_\beta \not\leq Q_\alpha$, otherwise $V_{\alpha'+1}^{(2)}$ is normalized by $L_{\alpha'} = \langle V_\beta, Q_{\alpha'+1}, R_{\alpha'} \rangle$. But now, $[V_\alpha^{(2)} \cap Q_{\alpha'} \cap Q_{\alpha'+1}, V_{\alpha'+1}^{(2)} \cap Q_\beta] \leq Z_{\alpha'+1} \cap V_\alpha^{(2)}$ and since $Z_{\alpha'} \not\leq V_\alpha^{(2)}$ by Lemma 5.4.21, $[V_\alpha^{(2)} \cap Q_{\alpha'} \cap Q_{\alpha'+1}, V_{\alpha'+1}^{(2)} \cap Q_\beta] = \{1\}$ and $V_\alpha^{(2)}/Z_\alpha$ is an FF-module for $\overline{L_\alpha}$, a contradiction by Lemma 5.2.32. Thus, it suffices to prove that Z_β is centralized by $V_{\alpha'}^{(3)}$. Since $V_\alpha^{(4)} \not\leq Q_{\alpha'-4}$, $\{1\} \neq [V_{\alpha-3}, V_{\alpha'-4}] \leq Z_{\alpha-2} \cap V_{\alpha'-4}$. If $[V_{\alpha-3}, V_{\alpha'-4}] = Z_{\alpha-1}$, then $Z_{\alpha-1} = Z_\beta \leq V_{\alpha'-4}$, for otherwise $Z_\alpha \leq Q_{\alpha'}$. Since $b > 7$, this leads to a contradiction. Thus, $Z_{\alpha-2} = [V_{\alpha-3}, V_{\alpha'-4}] \times Z_{\alpha-1}$ and $V_{\alpha'-4}^{(3)}$ centralizes $V_{\alpha-1} = Z_{\alpha-2}Z_\alpha$. Thus, since $V_{\alpha'-4}^{(3)}/V_{\alpha'-4}$ contains a non-central chief factor, $[V_{\alpha-3}, V_{\alpha'-4}] < [V_{\alpha-3}, V_{\alpha'-4}^{(3)}] \leq Z_{\alpha-2}$ so that $Z_{\alpha-2} \leq V_{\alpha'-4}^{(3}$. In particular, $Z_{\alpha-1} \leq V_{\alpha'-4}^{(3)}$ and since $b > 7$, we have that $Z_{\alpha-1} = Z_\beta \leq V_{\alpha'-4}^{(3)}$. Since $b > 9$, $V_{\alpha'}^{(3)}$ centralizes $V_{\alpha'-4}^{(3)}$ so that $V_{\alpha'+1}^{(2)}$ centralizes Z_β, as required.

Thus, we have shown that whenever $b > 7$, $Z_{\alpha'-2} = Z_{\alpha'-4} = Z_{\alpha'-6}$ and there is a critical pair $(\beta - 5, \alpha' - 4)$. Then, as $[V_{\beta-4}, V_{\alpha'-4}] \neq Z_{\alpha'-6}$ and $Z_{\alpha'-5} \neq Z_{\alpha'-7}$, $V_{\beta-5}^{(2)} \leq Q_{\alpha'-6}$ and by Lemma 5.4.30, we have that $V_{\alpha'-4} \not\leq Q_{\beta-4}$. In particular, $(\beta - 5, \alpha' - 4)$ satisfies the same hypothesis as (α, α') and applying the same methodology as above, we infer that $Z_{\alpha'-6} = Z_{\alpha'-8} = Z_{\alpha'-10}$. Applying this iteratively, we deduce that $Z_{\alpha'-2} = \cdots = Z_\beta$. In particular, $Z_{\alpha'-2} = Z_\beta \neq R \leq Z_{\alpha'-1} \cap Z_{\alpha+2}$ so that $Z_{\alpha'-1} = Z_{\alpha+2}$. But then $[V_\beta^{(3)}, V_{\alpha'}] = Z_{\alpha'-1} = Z_{\alpha+2} \leq V_\beta$ and $O^p(L_\beta)$ centralizes $V_\beta^{(3)}/V_\beta$, a final contradiction. □

Lemma 5.4.32. *Suppose that $C_{V_\beta}(V_{\alpha'}) = V_\beta \cap Q_{\alpha'}$. Then $b \neq 7$.*

Proof. By Lemma 5.4.30 and Lemma 5.4.31, we have that $V_{\alpha'} \not\leq Q_\beta$ and $b = 7$. Since $V_{\alpha'-2} = Z_{\alpha'-1}Z_{\alpha'-3} \leq V_\beta^{(3)}$ and $V_\beta^{(3)}$ is abelian, we have that $C_{Q_\beta}(V_\beta^{(3)}) = V_\beta^{(3)}(C_{Q_\beta}(V_\beta^{(3)}) \cap Q_{\alpha'})$ and since $Z_{\alpha'} \leq V_\beta^{(3)}$, $O^p(L_\beta)$ centralizes $C_{Q_\beta}(V_\beta^{(3)})/V_\beta^{(3)}$. In particular, $O^p(R_\beta)$ centralizes $C_{Q_\beta}(V_\beta^{(3)})$. But now, by the three subgroup lemma, for $r \in O^p(R_\beta)$ of order coprime to p, $[r, Q_\beta, V_\beta^{(3)}] = \{1\}$ and r centralizes Q_β. Thus, $R_\beta = Q_\beta$ and $\overline{L_\beta} \cong \mathrm{SL}_2(p)$.

Let $\alpha' + 1 \in \Delta(\alpha')$ such that $Z_{\alpha'+1} \not\leq Q_\beta$. Then, $V_\alpha^{(2)} \cap Q_{\alpha'} \not\leq Q_{\alpha'+1}$, for otherwise $V_{\alpha'}$ normalizes $V_\alpha^{(2)}$, a contradiction for then $L_\beta = \langle V_{\alpha'}, Q_\alpha, Q_\beta \rangle$ normalizes $V_\alpha^{(2)}$.

Notice that $[V_{\alpha'+1}^{(2)}, V_{\alpha+3}] \leq Z_{\alpha+4} \cap Z_{\alpha'+1}$. Since $(\alpha'+1, \beta)$ is a critical pair, we have that $Z_{\alpha+4} \cap Z_{\alpha'+1} = Z_{\alpha'-3} \cap Z_{\alpha'+1} \leq Z_{\alpha'}$. But if $Z_{\alpha'} \leq Z_{\alpha'-3}$, since $Z_{\alpha'-1} \neq Z_{\alpha'-3}$, we deduce that $Z_{\alpha'} = Z_{\alpha'-2} \neq R$. Then $R \neq Z_{\alpha+3}$ for otherwise $Z_{\alpha'-1} = Z_{\alpha'-2}R = Z_{\alpha'-2}Z_{\alpha'-4} = Z_{\alpha'-3}$, and so $V_{\alpha'+1}^{(2)}$ centralizes $Z_{\alpha+2} = Z_{\alpha+3}R$ and since $Z_{\alpha+2} \neq Z_{\alpha+4}$, we have that $[V_{\alpha'+1}^{(2)}, V_{\alpha+3}] = \{1\}$. Thus, whether $Z_{\alpha'} \leq Z_{\alpha+4}$ or not, $V_{\alpha'+1}^{(2)} \leq Q_{\alpha+2}$ and $V_{\alpha'+1}^{(2)} = Z_{\alpha'+1}(V_{\alpha'+1}^{(2)} \cap Q_\beta)$ and since $V_{\alpha'+1}^{(2)} \ntrianglelefteq L_{\alpha'} = \langle V_\beta, Q_{\alpha'+1}, Q_{\alpha'}\rangle$, we may assume that $V_{\alpha'+1}^{(2)} \cap Q_\beta \not\leq Q_\alpha$ and $Z_\beta \not\leq V_{\alpha'+1}^{(2)}$. But now, $[V_{\alpha'+1}^{(2)} \cap Q_\beta, V_\alpha^{(2)} \cap Q_{\alpha'} \cap Q_{\alpha'+1}] \leq Z_{\alpha'+1} \cap V_\alpha^{(2)}$ and since $Z_{\alpha'} \not\leq V_\alpha^{(2)}$, $[V_{\alpha'+1}^{(2)} \cap Q_\beta, V_\alpha^{(2)} \cap Q_{\alpha'} \cap Q_{\alpha'+1}] = \{1\}$ and $V_\alpha^{(2)}/Z_\alpha$ is an FF-module for $\overline{L_\alpha}$. Then by Lemma 5.2.32, $O^p(R_\alpha)$ centralizes $V_\alpha^{(2)}$.

It follows from the arguments above, that if $Z_{\alpha+3} = R \neq Z_{\alpha'-2}$, then $Z_{\alpha'-1} = Z_{\alpha'-3}$ and we have a contradiction. Similarly, $Z_{\alpha'-2} = R \neq Z_{\alpha+3}$ yields $Z_{\alpha+2} = Z_{\alpha+4}$, another contradiction. Suppose that $Z_{\alpha+3} \neq R \neq Z_{\alpha'-2}$. In particular, $R \not\leq Z_{\alpha'-3}$. But now, $V_{\alpha'-2} = RZ_{\alpha'-3} = V_{\alpha'-4}$. If $Z_{\alpha'-2} \neq Z_{\alpha'-4}$ then $L_{\alpha'-3} = \langle R_{\alpha'-3}, Q_{\alpha'-2}, Q_{\alpha'-4}\rangle$ normalizes $V_{\alpha'-2}$, a contradiction. Thus, $Z_{\alpha'-2} = Z_{\alpha'-4} = Z_{\alpha+3}$ so that $Z_{\alpha'-1} = RZ_{\alpha'-2} = RZ_{\alpha+3} = Z_{\alpha+2} \leq V_\beta$ from which it follows that $V_{\alpha'}$ centralizes $V_\beta^{(3)}/V_\beta$, a contradiction. Thus, $R = Z_{\alpha'-2} = Z_{\alpha'-4} = Z_{\alpha+3}$ and by Lemma 5.2.18, we conclude that $V_{\alpha'-2} = V_{\alpha'-4}$.

We may assume that $V_\alpha^{(4)}$ does not centralize $Z_{\alpha'-3}$, for otherwise $V_\alpha^{(4)}$ centralizes $V_{\alpha'-2} = V_{\alpha'-4} = Z_{\alpha'-3}Z_{\alpha+2}$, $V_\alpha^{(4)} = V_\beta^{(3)}(V_\alpha^{(4)} \cap Q_{\alpha'})$ and $V_\alpha^{(4)} \trianglelefteq L_\beta = \langle V_{\alpha'}, Q_\alpha\rangle$. Choose $\alpha - 4 \in \Delta^{(4)}(\alpha)$ such that $[Z_{\alpha-4}, Z_{\alpha'-3}] \neq \{1\}$. If $Z_{\alpha'-3} \leq Q_{\alpha-3}$, then $Z_{\alpha-3} = [Z_{\alpha-4}, Z_{\alpha'-3}] \leq Z_{\alpha+2}$. Then, if $Z_{\alpha-3} = Z_\beta$, either $Z_\alpha = Z_{\alpha-2}$, a contradiction for then Lemma 5.2.18 implies that $Z_{\alpha-4} \leq V_{\alpha-2}^{(2)} = V_\alpha^{(3)} \leq Q_{\alpha'-3}$; or $Z_\beta = Z_{\alpha-1} = Z_{\alpha-3}$ and by Lemma 5.2.18, $Z_{\alpha-4} \leq V_{\alpha-3} = V_\beta \leq Q_{\alpha'-3}$, another contradiction. Still, $Z_{\alpha-3} \leq V_{\alpha-1} \cap V_\beta$ and since $Z_{\alpha+2} = Z_\beta \times Z_{\alpha-3} \neq Z_\alpha$, we have that $V_\beta = V_{\alpha-1}$. Since $O^p(R_\alpha)$ centralizes $V_\alpha^{(2)}$, $Z_{\alpha-1} = Z_\beta$, for otherwise $V_\beta \trianglelefteq L_\alpha = \langle R_\alpha, Q_{\alpha-1}, Q_\beta\rangle$.

Suppose that $Z_{\alpha'-3} \not\leq Q_{\alpha-3}$ so that $(\alpha'-3, \alpha-3)$ is a critical pair. By Lemma 5.4.30, we may assume that $(\alpha'-3, \alpha-3)$ satisfies the same hypothesis as (α, α'), in which case $Z_{\alpha-1} = Z_\beta$; or $V_{\alpha'-3}^{(2)} \not\leq Q_{\alpha-1}$ and by Lemma 5.4.25, either $[V_{\alpha'-4}, V_{\alpha-3}] = Z_{\alpha-1} \leq Z_{\alpha+2}$, and again $Z_{\alpha-1} = Z_\beta$, or $Z_{\alpha-2} = Z_\alpha$, and by Lemma 5.2.18, we have a contradiction.

Thus, whenever there is $Z_{\alpha-4}$ such that $Z_{\alpha-4}$ does not centralizes $Z_{\alpha'-3}$, we have $Z_{\alpha-1} = Z_\beta$. Choose $\lambda \in \Delta(\alpha)$ such that $Z_\lambda \neq Z_\beta$ so that $V_\lambda^{(3)}$ centralizes $Z_{\alpha'-3}$. Then $V_\lambda^{(3)}$ centralizes $V_{\alpha'-4} = V_{\alpha'-2}$ so that $V_\lambda^{(3)} = V_\beta(V_\lambda^{(3)} \cap Q_{\alpha'})$. Then, $V_\lambda^{(3)}V_\beta^{(3)} \trianglelefteq L_\beta = \langle Q_\alpha, V_{\alpha'}\rangle$. In particular, $[C_\beta, V_\lambda^{(3)}V_\beta^{(3)}]$ is a normal subgroup L_β contained in $[C_\beta, V_\beta^{(3)}][Q_\alpha, V_\lambda^{(3)}]$. Noticing that $[V_{\alpha'+1} \cap Q_\beta, V_\alpha^{(2)}] = [V_{\alpha'+1} \cap Q_\beta, V_\beta(V_\alpha^{(2)} \cap Q_{\alpha'})] = Z_\beta R = Z_{\alpha+2}$, we have that $[S, V_\alpha^{(2)}] \leq V_\beta$ and $|V_\alpha^{(2)}| = p^4$. But then $[Q_\beta, V_\beta^{(3)}] = V_\beta$ and since $[V_{\alpha'}, V_\beta^{(3)}] = Z_{\alpha'-1} \leq V_{\alpha+3} \leq V_{\alpha+2}^{(2)}$, we must have that $|V_\beta^{(3)}| = p^5$ and $[Q_\alpha, V_\beta^{(3)}] = V_\alpha^{(2)}$. Thus, $V_\beta \not\geq [C_\beta, V_\lambda^{(3)}V_\beta^{(3)}] \leq V_\alpha^{(2)}$ and it follows that $V_\alpha^{(2)} = V_\beta[C_\beta, V_\lambda^{(3)}V_\beta^{(3)}] \trianglelefteq L_\beta$, a contradiction. □

Combining all the results in this subsection thus far, we have the following result.

Proposition 5.4.33. *Suppose that $C_{V_\beta}(V_{\alpha'}) = V_\beta \cap Q_{\alpha'}$. Then $b \leqslant 5$.*

In conjunction with the results proved in earlier sections, we have now proved that Hypothesis 5.2.1 implies that $b \leqslant 5$. In the next lemmas and proposition, we show this bound is tight by witnessing an example with $b = 5$. In [DS85] and [Del88], this configuration is shown to be parabolic isomorphic to F_3. In our case, we have demonstrated in Section 3.3 that this leads to an exotic fusion system.

The presence of this fusion system may go some way to explaining why it is so difficult to uniquely determine F_3 from a purely 3-local perspective.

Lemma 5.4.34. *Suppose that $C_{V_\beta}(V_{\alpha'}) = V_\beta \cap Q_{\alpha'}$ and $b = 5$. Then $V_{\alpha'} \not\leq Q_\beta$.*

Proof. Assume that $V_{\alpha'} \leq Q_\beta$. If $V_\alpha^{(2)} \leq Q_{\alpha'-2}$, then it follows from Lemma 5.4.19 that $|V_\beta| = p^3$ and $O^p(R_\alpha)$ centralizes $V_\alpha^{(2)}$. Now, $Z_\beta = R \leq Z_{\alpha'-1}$ and since $V_\beta \neq V_{\alpha'-2}$, by Lemma 5.2.18, we may assume that $Z_\beta \neq Z_{\alpha'-2}$ so that $Z_{\alpha'-1} = Z_{\alpha+2}$. But now, $V_{\alpha'}^{(3)} \cap Q_{\alpha'-2} \leq Q_{\alpha+2}$ so that $[V_{\alpha'}^{(3)} \cap Q_{\alpha'-2}, V_\beta] \leq Z_{\alpha+2} = Z_{\alpha'-1} \leq V_{\alpha'}$. By Lemma 5.2.34, $O^p(R_{\alpha'})$ centralizes $V_{\alpha'}^{(3)}$ and Lemma 5.2.18 applied to $Z_{\alpha'-1} = Z_{\alpha+2}$ implies that $V_\beta \leq V_{\alpha+2}^{(2)} = V_{\alpha'-1}^{(2)} \leq Q_{\alpha'}$, a contradiction.

Suppose now that $V_{\alpha'} \leq Q_\beta$, $|V_\beta| = p^3$ and $V_\alpha^{(2)} \not\leq Q_{\alpha'-2}$. If $Z_\beta = R \neq Z_{\alpha'-2}$ then, as above, $Z_{\alpha'-1} = Z_{\alpha+2}$ and $[V_{\alpha'}^{(3)} \cap Q_{\alpha'-2}, V_\beta] \leq Z_{\alpha+2} = Z_{\alpha'-1} \leq V_{\alpha'}$. Then $O^p(R_{\alpha'})$ centralizes $V_{\alpha'}^{(3)}$ and Lemma 5.2.18 provides a contradiction. Thus, $Z_\beta = Z_{\alpha'-2} \neq Z_{\alpha'}$. But now, $[V_{\alpha'-2}, V_\alpha^{(2)}] \leq Z_\alpha \cap Z_{\alpha+2} = Z_\beta = Z_{\alpha'-2}$ and $V_\alpha^{(2)} \leq Q_{\alpha'-2}$, a contradiction.

Thus, if $V_{\alpha'} \leq Q_\beta$ then $|V_\beta| \neq p^3$. Notice that if $Z_{\alpha'-2} = Z_\beta$, then $Z_\beta Z_\beta^g Z_{\alpha'} = Z_{\alpha'-1} Z_{\alpha'-1}^g$ is of order p^3 and normalized by $L_{\alpha'} = \langle V_\beta, V_\beta^g, R_{\alpha'} \rangle$, for some appropriately chosen $g \in L_{\alpha'}$, a contradiction. Now, if $Z_{\alpha'-2} \leq V^\alpha$, then $V_\beta = Z_{\alpha+2} Z_\alpha C_{V_\beta}(O^p(L_\beta)) \leq V^\alpha$. But then $V^\alpha = V_\alpha^{(2)}$ and we have a contradiction. Since $[Q_\alpha, V_\alpha^{(2)}] \leq V^\alpha$ and $V_{\alpha'-2} \leq Q_\alpha$, it follows that $V_\alpha^{(2)} \cap Q_{\alpha'-2}$ centralizes $V_{\alpha'-2}$ and $V_\alpha^{(2)} \cap Q_{\alpha'-2} \leq Q_{\alpha'-1}$. Since both $V_\alpha^{(2)}/V^\alpha$ and V^α/Z_α have non-central chief factors, $[V_\alpha^{(2)} \cap Q_{\alpha'}, V_{\alpha'}] = Z_{\alpha'} \leq V_\alpha^{(2)}$ and both $V_\alpha^{(2)}/V^\alpha$ and V^α/Z_α are FF-modules for $\overline{L_\alpha}$. Then by Lemma 5.2.32, we have that $O^p(R_\alpha)$ centralizes $V_\alpha^{(2)}$ and by Lemma 5.2.18, $Z_{\alpha'} \neq Z_{\alpha'-2}$ and $Z_{\alpha'-1} \leq V_\alpha^{(2)}$. Since $V_\alpha^{(2)} \not\leq Q_{\alpha'-2}$, and $V_\alpha^{(2)}$ centralizes $Z_{\alpha'-1} Z_{\alpha+2}$. By Lemma 5.2.31, we may assume that $Z_{\alpha'-1} = Z_{\alpha+2}$.

But now $[V_\beta, V_{\alpha'}] = Z_\beta \leq Z_{\alpha'-1}$ and $Z_{\alpha'-1}Z_{\alpha'-1}^g$ is of order p^3 and normalized by $L_{\alpha'} = \langle V_\beta, V_\beta^g, R_{\alpha'}\rangle$, a contradiction. $\square$

Lemma 5.4.35. *Suppose that $C_{V_\beta}(V_{\alpha'}) = V_\beta \cap Q_{\alpha'}$ and $b = 5$. Then $|V_\beta| = p^3$, $R = Z_{\alpha'-2} \neq Z_\beta \neq Z_{\alpha'} \neq R$ and $Z_{\alpha'-1} \neq Z_{\alpha+2}$.*

Proof. By Lemma 5.4.34, we have that $V_{\alpha'} \not\leq Q_\beta$ for all critical pairs (α, α'). Suppose that $|V_\beta| \neq p^3$ and fix $\alpha' + 1 \in \Delta(\alpha')$ such that $Z_{\alpha'+1} \not\leq Q_\beta$. In particular, $(\alpha' + 1, \beta)$ is a critical pair satisfying the same hypothesis as (α, α').

We suppose first that $Z_\beta \neq Z_{\alpha'-2}$. As in Lemma 5.4.34, this implies that $Z_{\alpha'-2} \not\leq V^\alpha$ so that $V_\alpha^{(2)} \cap Q_{\alpha'-2} \leq Q_{\alpha'-1}$. Moreover, $[V^\alpha, V_{\alpha'-2}] \leq Z_\alpha \cap V_{\alpha'-2} = Z_\beta$ and if $V^\alpha \not\leq Q_{\alpha'-2}$, then $Z_\beta Z_\beta^g Z_{\alpha'-2} = Z_{\alpha+2}Z_{\alpha+2}^g$ is of order p^3 and normalized by $L_{\alpha'-2} = \langle V^\alpha, (V^\alpha)^g, R_{\alpha'-2}\rangle$ for some appropriately chosen $g \in L_{\alpha'-2}$, a contradiction. Thus, $V^\alpha = Z_\alpha(V^\alpha \cap Q_{\alpha'})$.

Set $U_\beta := \langle (V^\alpha)^{G_\beta}\rangle$. Then $[U_\beta, V_{\alpha'-2}] \leq [U_\beta, C_\beta] \cap V_{\alpha'-2} \leq V_{\alpha'-2} \cap V_\beta$. Notice that if $V_{\alpha'-2} \cap V_\beta > Z_{\alpha+2}$ then, as $V_{\alpha'-2} \cap V_\beta$ is centralized by $V_{\alpha'}$, $V_{\alpha'-2} \cap V_\beta = Z_{\alpha+2}C_{V_\beta}(O^p(L_\beta)) = Z_{\alpha+2}C_{V_{\alpha'-2}}(O^p(L_{\alpha'-2}))$ and $Z_\beta = [Q_{\alpha+2}, V_{\alpha'-2} \cap V_\beta] = Z_{\alpha'-2}$, a contradiction. Thus, $U_\beta \leq Q_{\alpha'-2}$ for otherwise $Z_{\alpha+2}Z_{\alpha+2}^g$ is of order p^3 and normalized by $L_{\alpha'-2} = \langle U_\beta, U_\beta^g, R_{\alpha'-2}\rangle$, for some appropriate $g \in L_{\alpha'-2}$, another contradiction. Since $Z_{\alpha'-2} \neq Z_\beta$, it follows from a similar argument to above that $V^{\alpha'-1} \leq C_\beta$. Suppose that $V^\mu \not\leq Q_{\alpha'-1}$ for some $\mu \in \Delta(\beta)$. Then $\{1\} \neq [V^\mu, V^{\alpha'-1}] \leq Z_\mu \cap V^{\alpha'-1}$. Notice that $Z_\beta \not\leq V^{\alpha'-1}$ for otherwise $V_{\alpha'-2} = Z_{\alpha'-1}Z_{\alpha+2}C_{V_{\alpha'}}(O^p(L_{\alpha'-2})) \leq V^{\alpha'-1}$. Thus, $Z_\mu = [V^{\alpha'-1}, V^\mu] \times Z_\beta$ centralizes $V_{\alpha'}$ and since $R \neq \{1\}$, it follows that $Z_\mu = Z_{\alpha+2}$. Since $Z_{\alpha'-2} \leq V^{\alpha'-1}$ and $Z_\beta \not\leq V^{\alpha'-1}$, we have that $[V^\mu, V^{\alpha'-1}] = Z_{\alpha'-2} \leq Z_{\alpha'-1}$, a contradiction since $V^\mu \not\leq Q_{\alpha'-1}$. Thus, $U_\beta \leq Q_{\alpha'-1}$. Since $V^\alpha V_\beta \not\trianglelefteq L_\beta$, we conclude that $[U_\beta \cap Q_{\alpha'}, V_{\alpha'}] = Z_{\alpha'} \leq U_\beta$ and $Z_{\alpha'} \not\leq V_\beta$. But now $Z_{\alpha'-2} \neq Z_{\alpha'}$ and $V_{\alpha'-2} = Z_{\alpha'-1}Z_{\alpha+2}C_{V_{\alpha'-2}}(O^p(L_{\alpha'-2})) \leq U_\beta$.

Suppose that $[V_\alpha^{(2)}, Z_{\alpha'-1}] \neq \{1\}$. Then there is $\alpha - 1 \in \Delta(\alpha)$ such that $[V_{\alpha-1}, Z_{\alpha'-1}] \neq \{1\}$. If $Z_{\alpha'-1} \leq Q_{\alpha-1}$, then $Z_{\alpha-1} = [Z_{\alpha'-1}, V_{\alpha-1}] \leq [V_{\alpha'-2}, V_\alpha^{(2)}]$. Since $Z_\alpha \not\leq V_{\alpha'-2}$, it follows that $Z_{\alpha-1} = Z_\beta$. But then $[V_{\alpha'-2}, V_\alpha^{(2)}] \leq Z_\beta Z_{\alpha'-2}$ and if $V_\alpha^{(2)} \not\leq Q_{\alpha'-2}$, then $Z_{\alpha+2}Z_{\alpha+2}^g$ is of order p^3 and normalized by $L_{\alpha'-2} = \langle V_\alpha^{(2)}, (V_\alpha^{(2)})^g, R_{\alpha'-2}\rangle$ for some appropriately chosen $g \in L_{\alpha'-2}$, a contradiction. Thus, $Z_{\alpha-1} = [Z_{\alpha'-1}, V_{\alpha-1}] = Z_{\alpha'-2}$ and $Z_\alpha = Z_{\alpha'-2} \times Z_\beta = Z_{\alpha+2}$, a contradiction. Thus, $Z_{\alpha'-1} \not\leq Q_{\alpha-1}$ and $(\alpha'-1, \alpha-1)$ is a critical pair. Since $Z_{\alpha'-2} \neq Z_\beta$, $(\alpha'-1, \alpha-1)$ satisfies the same hypothesis as (α, α') and so we see that $V_\beta \leq U_{\alpha'-2}$. But then $R = [V_\beta, V_{\alpha'}] \leq [U_{\alpha'-2}, C_{\alpha'-2}] \leq V_{\alpha'-2}$ and $R \leq V_\beta \cap V_{\alpha'-2} \leq Z_{\alpha+2}$. Similarly to before, this implies that $|V_\beta| = p^3$, and we have a contradiction. Thus, $[V_\alpha^{(2)}, Z_{\alpha'-1}] = \{1\}$ and since $Z_{\alpha'-1} \neq Z_{\alpha'-3}$, it follows that $V_\alpha^{(2)}$ centralizes $V_{\alpha'-2}$ and $V_\alpha^{(2)} \leq Q_{\alpha'-1}$. In particular, this holds for any $\lambda \in \Delta(\beta)$ with $Z_\lambda = Z_\alpha$. Forming $W^\beta := \langle V_\lambda^{(2)} \mid Z_\lambda = Z_\alpha, \lambda \in \Delta(\beta)\rangle$, we have that $W^\beta U_\beta/U_\beta$ is centralized by $V_{\alpha'}$, and by Lemma 5.2.19, normalized by $R_\beta Q_\alpha$. But then $W^\beta U_\beta \trianglelefteq L_\beta = \langle V_{\alpha'}, R_\beta, Q_\alpha\rangle$ and since $V_{\alpha'}$ centralizes $W^\beta U_\beta/U_\beta$ we deduce that $V_\beta^{(3)} = V_\alpha^{(2)} U_\beta \trianglelefteq L_\beta$. Now, $R = [V_\beta, V_{\alpha'}] \leq [V_\beta, V_{\alpha'-1}^{(2)} U_{\alpha'-2}] = [V_\beta, V_{\alpha+2}^{(2)} U_{\alpha'-2}] = [V_\beta, U_{\alpha'-2}]$ and since $V_\beta \leq C_{\alpha'-2}$, it follows that $R \leq V_\beta \cap V_{\alpha'-2} = Z_{\alpha+2}$, which again implies that $|V_\beta| = p^3$, a contradiction.

Suppose now that $V_{\alpha'} \not\leq Q_\beta$, $|V_\beta| \neq p^3$ and $Z_\beta = Z_{\alpha'-2}$. Since $(\alpha'+1, \beta)$ is also a critical pair, by the above, we may assume that $Z_{\alpha'} = Z_{\alpha'-2}$ and $Z_{\alpha'} = Z_\beta$. Set $U^\beta := \langle V^\lambda \mid Z_\lambda = Z_\alpha, \lambda \in \Delta(\beta)\rangle$ so that (λ, α') is a critical pair for every such λ. For such a λ, $[V^\lambda, V_{\alpha'-2}] \leq Z_\alpha \cap V_{\alpha'-2}$ and since $Z_\beta = Z_{\alpha'-2}$, it follows that $V^\lambda \leq Q_{\alpha'-2}$. If $V^\lambda \not\leq Q_{\alpha'-1}$, then using that $[V^{\alpha'-1}, V_\beta] \leq Z_{\alpha'-1}$ and V_β

is centralized by V^λ, it follows that $[V^{\alpha'-1}, V_\beta] = Z_{\alpha'-2} = Z_\beta$ so that $V^{\alpha'-1} \leq Q_\beta$. Then $[V^{\alpha'-1} \cap Q_\lambda, V^\lambda] \leq Z_\lambda = Z_\alpha$ and since $Z_\alpha \not\leq V_{\alpha'-1}^{(2)}$, we have that $[V^{\alpha'-1} \cap Q_\lambda, V^\lambda] = Z_\beta \leq Z_{\alpha'-1}$. Since $V^{\alpha'-1}/Z_{\alpha'-1}$ contains a non-central chief factor, $V^{\alpha'-1} \not\leq Q_\lambda$ and $V^{\alpha'-1}/Z_{\alpha'-1}$ is an FF-module for $\overline{L_{\alpha'-1}}$. Then $V_{\alpha'-1}^{(2)} \cap Q_\beta = V^{\alpha'-1}(V_{\alpha'-1}^{(2)} \cap Q_\lambda)$ and since $Z_\lambda = Z_\alpha \not\leq V_{\alpha'-1}^{(2)}$, $V_{\alpha'-1}^{(2)}/V^{\alpha'-1}$ is also an FF-module for $\overline{L_{\alpha'-1}}$. Then Lemma 5.2.32 and Lemma 5.2.18 applied to $Z_\beta = Z_{\alpha'-2} = Z_{\alpha'}$ gives $V_{\alpha'} = V_\beta$, a contradiction. Thus $U^\beta \leq Q_{\alpha'-1}$ and $U^\beta V_\beta/V_\beta$ is centralized by $V_{\alpha'}$. Since $U^\beta \trianglelefteq R_\beta Q_\alpha$ by Lemma 5.2.19, $U^\beta \trianglelefteq L_\beta = \langle V_{\alpha'}, R_\beta, Q_\alpha \rangle$ and since $V_{\alpha'}$ centralizes U^β/V_β we have that $V^\alpha V_\beta \trianglelefteq L_\beta$, a contradiction by Lemma 5.2.31.

Thus, we have shown that $|V_\beta| = p^3$, $R \leq Z_{\alpha'-1} \cap Z_{\alpha+2}$ and $V_{\alpha'} \not\leq Q_\beta$. Suppose that $Z_{\alpha'-1} = Z_{\alpha+2}$. Then $V_\beta^{(3)} \cap Q_{\alpha'-2} \leq Q_{\alpha'-1}$ and $[V_\beta^{(3)} \cap Q_{\alpha'-2}, V_{\alpha'}] \leq Z_{\alpha'-1} \leq V_\beta$ and it follows that $V_\beta^{(3)}/V_\beta$ contains a unique non-central chief factor for $\overline{L_\beta}$ which is an FF-module. Then, Lemma 5.2.34 and Lemma 5.2.18 applied to $Z_{\alpha'-1} = Z_{\alpha+2}$ gives $V_{\alpha'} \leq V_{\alpha'-1}^{(2)} = V_{\alpha+2}^{(2)} \leq Q_\beta$, a contradiction. Now, $R \leq Z_{\alpha'-1} \cap Z_{\alpha+2}$ and so $R = Z_{\alpha'-2}$, otherwise $Z_{\alpha'-1} = Z_{\alpha+2}$. This completes the proof. $\square$

Lemma 5.4.36. *Suppose that $C_{V_\beta}(V_{\alpha'}) = V_\beta \cap Q_{\alpha'}$ and $b = 5$. If $[V_\beta^{(3)}, Q_\beta]V_\beta/V_\beta$ contains no non-central chief factor for L_β then $[V_\beta^{(3)}, Q_\beta]V_\beta \leq Z(V_\beta^{(3)})$ and $V_{\alpha'}$ acts quadratically on $V_\beta^{(3)}/V_\beta$. If, in addition, $V_\alpha^{(2)}/Z_\alpha$ is an FF-module for $\overline{L_\alpha}$, then $|V_\alpha^{(2)}| = p^4$ and $[V_\beta^{(3)}, Q_\beta] = V_\beta$.*

Proof. Suppose that $[V_\beta^{(3)}, Q_\beta]V_\beta/V_\beta$ contains no non-central chief factor for L_β. Then $O^p(L_\beta)$ centralizes $[V_\beta^{(3)}, Q_\beta]V_\beta/V_\beta$ and so $[V_\lambda^{(2)}, Q_\beta]V_\beta \trianglelefteq L_\beta$ for any $\lambda \in \Delta(\beta)$. It follows that $[V_\beta^{(3)}, Q_\beta]V_\beta = [V_\lambda^{(2)}, Q_\beta]V_\beta$ for any $\lambda \in \Delta(\beta)$. But $V_\lambda^{(2)}$ is elementary abelian and so $[V_\beta^{(3)}, Q_\beta]V_\beta \leq Z(V_\beta^{(3)})$. Moreover, $[V_\beta^{(3)}, Q_\beta, V_\beta^{(3)}] = \{1\}$ and it follows from the three subgroup lemma that $[V_\beta^{(3)}, V_\beta^{(3)}, Q_\beta] = \{1\}$ so that $[V_\beta^{(3)}, V_\beta^{(3)}] \leq Z(Q_\beta) = Z_\beta$. Since $V_\beta, V_{\alpha'} \leq V_{\alpha'-2}^{(3)}$, it follows by conjugacy that $V_\beta^{(3)}$

is non-abelian and so $[V_\beta^{(3)}, V_\beta^{(3)}] = Z_\beta$. Then $[V_\beta^{(3)}, V_{\alpha'}, V_{\alpha'}] \leq [V_\beta^{(3)}, V_{\alpha'-2}^{(3)}, V_{\alpha'-2}^{(3)}] \leq Z_{\alpha'-2} \leq V_\beta$, as required.

Suppose now, in addition, that $V_\alpha^{(2)}/Z_\alpha$ is an FF-module for $\overline{L_\alpha}$. Set C^α to be the preimage of $C_{V_\alpha^{(2)}/Z_\alpha}(O^p(L_\alpha))$. Then by Lemma 2.3.10, $V_\alpha^{(2)}/C^\alpha$ is a natural $\mathrm{SL}_2(p)$-module and since $|V_\beta| = p^3$, we may assume that $|C^\alpha| = p^3$, $|V_\alpha^{(2)}| = p^5$ and $C^\alpha \cap V_\beta = Z_\alpha$. In particular, if $O^p(L_\beta)$ centralizes $[V_\beta^{(3)}, Q_\beta]V_\beta/V_\beta$ then $C^\alpha V_\beta = [V_\alpha^{(2)}, Q_\beta]V_\beta \trianglelefteq L_\beta$ and $C^\alpha \leq Z(V_\beta^{(3)})$. Furthermore, as $Z_\alpha = Z(Q_\alpha)$, we must have that $[C^\alpha, Q_\alpha] = Z_\alpha$ and calculating in $\mathrm{GL}_3(p)$ and applying the three subgroup lemma, we infer that $|Q_\alpha/C_{Q_\alpha}(C^\alpha)| = p^2$ and $Q_\alpha/C_{Q_\alpha}(C^\alpha)$ is a natural $\mathrm{SL}_2(p)$-module for $L_\alpha/R_\alpha \cong \mathrm{SL}_2(p)$.

Now, as $C^\alpha V_\beta \trianglelefteq L_\beta$, $[C_\beta, C^\alpha]$ is normal in L_β and contained in Z_α. Note that C_β has index p^2 in Q_α and so $[C_\beta, C^\alpha] = \{1\}$ implies that $C_\beta = C_{Q_\alpha}(C^\alpha) \trianglelefteq \langle G_\alpha, G_\beta \rangle$, a contradiction. Thus, $[C_\beta, C^\alpha] = [Q_\beta, C^\alpha] = Z_\beta$ so that $C_{Q_\alpha}(C^\alpha) \leq Q_\beta$, for otherwise $Q_\alpha = (Q_\alpha \cap Q_\beta)(C_{Q_\alpha}(C^\alpha))$ and $Z_\beta = [Q_\alpha, C^\alpha] \trianglelefteq L_\alpha$. But now, since $C^{\alpha+2} \leq Z(V_{\alpha'-2}^{(3)})$, $V_{\alpha'} \leq C_{Q_{\alpha+2}}(C^{\alpha+2}) \leq Q_\beta$, a contradiction. Thus, $C^\alpha = Z_\alpha$, $|V_\alpha^{(2)}| = p^4$ and $[V_\beta^{(3)}, Q_\beta] = V_\beta$, as required. □

Lemma 5.4.37. *Suppose that $C_{V_\beta}(V_{\alpha'}) = V_\beta \cap Q_{\alpha'}$ and $b = 5$. Then we may assume that $V_\alpha^{(2)} \not\leq Q_{\alpha'-2}$.*

Proof. Since $L_\beta/R_\beta = \mathrm{SL}_2(p)$, we can arrange that there is $g \in L_\beta$ such that $g \not\leq G_{\beta,\alpha+2}R_\beta$ but $g^2 \leq G_{\beta,\alpha+2}R_\beta$. Then $Z_{\alpha+2}^g \neq Z_{\alpha+2}$ but $Z_{\alpha+2}^{g^2} = Z_{\alpha+2}$ and so we label $\alpha = (\alpha+2)^g$ so that (α, α') is still a critical pair. It then follows that $R_\beta Q_{\alpha+2}^g = R_\beta Q_\alpha$ and $R_\beta Q_\alpha^g = R_\beta Q_{\alpha+2}$. Moreover, as $V_{\alpha'} \not\leq Q_\beta$, there is $\alpha'+1 \in \Delta(\alpha')$ such that $Z_{\alpha'+1} \not\leq Q_\beta$ and $(\alpha'+1, \beta)$ is a critical pair. We arrange also that there is $h \in L_{\alpha'}$ with $h \not\leq G_{\alpha',\alpha'-1}R_{\alpha'}$ but $h^2 \in G_{\alpha',\alpha'-1}R_{\alpha'}$ such that

$(\alpha'+1)^h = \alpha'-1$, $R_{\alpha'}Q_{\alpha'+1}^h = R_{\alpha'}Q_{\alpha'-1}$ and $R_{\alpha'}Q_{\alpha'-1}^h = R_{\alpha'}Q_{\alpha'+1}$.

Set $W^\beta := \langle V_\lambda^{(2)} \mid \lambda \in \Delta(\beta), Z_\lambda = Z_\alpha\rangle$ so that by Lemma 5.2.19, $W^\beta \trianglelefteq R_\beta Q_\alpha$. Set $W^{\alpha'} = \langle V_\mu^{(2)} \mid \mu \in \Delta(\alpha'), Z_\mu = Z_{\alpha'+1}\rangle \trianglelefteq R_{\alpha'}Q_{\alpha'+1}$. Finally, we set $U^\beta := \langle V_\delta^{(2)} \mid \mu \in \Delta(\beta), Z_\delta = Z_{\alpha+2}\rangle \trianglelefteq R_\beta Q_{\alpha+2}$. In particular, $U^\beta W^\beta \trianglelefteq R_\beta$ and for $g \in L_\beta$ such that $g \not\le G_{\beta,\alpha+2}R_\beta$, $g^2 \le G_{\beta,\alpha+2}R_\beta$ and $\alpha = (\alpha+2)^g$, we have that $(U^\beta)^g = W^\beta$ and $(W^\beta)^g = U^\beta$.

For $\lambda \in \Delta(\beta)$ with $Z_\lambda = Z_\alpha$, $Z_{\alpha'-1} \le Q_\lambda$ and so $[Z_{\alpha'-1}, V_\lambda^{(2)}] \le Z_\lambda$. Thus, $[W^\beta \cap Q_{\alpha'-2}, Z_{\alpha'-1}] \le Z_\lambda \cap Z_{\alpha'-2} = \{1\}$ since $Z_\beta \ne Z_{\alpha'-2}$. Therefore, $W^\beta \cap Q_{\alpha'-2} \le Q_{\alpha'-1}$. Similarly, $W^{\alpha'} \cap Q_{\alpha'-2} \le Q_{\alpha+2}$.

Suppose that $W^\beta \le Q_{\alpha'-2}$ so that $V_\alpha^{(2)} \le Q_{\alpha'-2}$ and $W^\beta = V_\beta(W^\beta \cap Q_{\alpha'})$. If $[W^\beta \cap Q_{\alpha'}, V_{\alpha'}] \le W^\beta$, then $V_\beta^{(3)} = W^\beta \trianglelefteq L_\beta = \langle R_\beta, V_{\alpha'}, Q_\alpha\rangle$, and $V_\beta^{(3)} \le Q_{\alpha'-1}$. Then $V_\beta(V_\beta^{(3)} \cap Q_{\alpha'+1})$ is an index p subgroup of $V_\beta^{(3)}$ centralized, modulo V_β, by $Z_{\alpha'+1}$ and $V_\beta^{(3)}/V_\beta$ contains a unique non-central chief factor which is an FF-module for $\overline{L_\beta}$. By Lemma 5.2.34, $O^p(R_\beta)$ centralizes $V_\beta^{(3)}$. But then, for $\lambda \in \Delta(\beta)$ with $Z_\lambda = Z_\alpha$, it follows by Lemma 5.2.18 that $V_\lambda^{(2)} = V_\alpha^{(2)}$ and $V_\beta^{(3)} = W^\beta = V_\alpha^{(2)}$, a clear contradiction. Thus, if $W^\beta \le Q_{\alpha'-2}$, then $[W^\beta \cap Q_{\alpha'}, V_{\alpha'}] = Z_{\alpha'} \le V_\beta^{(3)}$ but $Z_{\alpha'} \not\le W^\beta$.

Now, still assuming that $W^\beta \le Q_{\alpha'-2}$, $[W^\beta \cap Q_{\alpha'} \cap Q_{\alpha'+1}, W^{\alpha'} \cap Q_\beta] \le Z_{\alpha'+1} \cap W^\beta = \{1\}$. In particular, $[V_\alpha^{(2)} \cap Q_{\alpha'} \cap Q_{\alpha'+1}, W^{\alpha'} \cap Q_\beta] = \{1\}$ and if $W^{\alpha'} \cap Q_\beta \not\le Q_\alpha$ then as $V_\alpha^{(2)} = Z_\alpha(V_\alpha^{(2)} \cap Q_{\alpha'})$, it follows that $V_\alpha^{(2)}/Z_\alpha$ is an FF-module for $\overline{L_\alpha}$ and $V_\alpha^{(2)} \cap Q_{\alpha'} \not\le Q_{\alpha'+1}$. If $W^{\alpha'} \cap Q_\beta \le Q_\alpha$, then $W^{\alpha'} \not\le Q_{\alpha'-2}$ otherwise we obtain the contradiction $V_{\alpha'}^{(3)} = V_{\alpha'+1}^{(2)}$ in the same manner as the case where $W^\beta \le Q_{\alpha'-2}$. We may as well assume that $V_{\alpha'+1}^{(2)} \not\le Q_{\alpha'-2}$ so that $V_{\alpha'+1}^{(2)}$ does not centralize $Z_{\alpha+2}$ and since $V_{\alpha'+1}^{(2)}$ is abelian, $Z_\beta \not\le V_{\alpha'+1}^{(2)}$. Then $V_{\alpha'+1}^{(2)} \cap Q_{\alpha'-2} = Z_{\alpha'+1}(V_{\alpha'+1}^{(2)} \cap Q_\alpha)$

and $[V^{(2)}_{\alpha'+1} \cap Q_\alpha, W^\beta \cap Q_{\alpha'}] \leq V^{(2)}_{\alpha'+1} \cap Z_\beta = \{1\}$ and $V^{(2)}_{\alpha'+1}/Z_{\alpha'+1}$ is an FF-module for $\overline{L_{\alpha'+1}}$.

Therefore, if $W^\beta \leq Q_{\alpha'-2}$ then $V^{(2)}_\alpha/Z_\alpha$ is an FF-module for $\overline{L_\alpha}$. Moreover, $[W^\beta, V_{\alpha'}] \leq Z_{\alpha'-1} \leq V^{(2)}_{\alpha+2} \leq U^\beta$ and so $V_{\alpha'}R_\beta = Q_{\alpha+2}R_\beta$ normalizes $W^\beta U^\beta$. But then $(Q_{\alpha+2}R_\beta)^g = Q_\alpha R_\beta$ normalizes $(W^\beta U^\beta)^g = W^\beta U^\beta$ and $V^{(3)}_\beta = W^\beta U^\beta \trianglelefteq L_\beta = \langle Q_{\alpha+2}, R_\beta, Q_\alpha\rangle$. If $U^\beta \not\leq Q_{\alpha'-2}$, then there is a critical pair with $(\beta-3, \alpha'-2)$ such that $Z_{\alpha+2} = Z_{\beta-1}$, a contradiction by Lemma 5.4.35; and so we conclude that $V^{(3)}_\beta \leq Q_{\alpha'-2}$.

Suppose that $V^{(3)}_\beta \not\leq Q_{\alpha'-1}$. Since $Z_{\alpha'-1} \leq V^{(3)}_\beta$, we have that $Z_{\alpha'-2} \leq [V^{(3)}_\beta, V^{(3)}_\beta] = V_\beta$ and it follows from Lemma 5.4.36 that $[V^{(3)}_\beta, Q_\beta]/V_\beta$ contains a non-central chief factor L_β. Moreover, by Lemma 5.2.13, $V^{(3)}_\beta/[V^{(3)}_\beta, Q_\beta]$ contains a non-central chief factor for L_β. Notice that if $Z_{\alpha'} \leq [V^{(3)}_\beta, Q_\beta]$, then $[V^{(3)}_\beta, Q_\beta] \leq Q_{\alpha'-1}$, for otherwise $V_{\alpha'}$ would centralize $V^{(3)}_\beta/[V^{(3)}_\beta, Q_\beta]$. But then $V^{(3)}_\beta = W^\beta[V^{(3)}_\beta, Q_\beta] \trianglelefteq L_\beta$ and $V^{(3)}_\beta \leq Q_{\alpha'-1}$, a contradiction. Thus, $Z_{\alpha'} \not\leq [V^{(3)}_\beta, Q_\beta]$ and since $[V^{(3)}_\beta, Q_\beta]/V_\beta$ contains a non-central chief factor, we infer that $[V^{(3)}_\beta, Q_\beta] \not\leq Q_{\alpha'-1}$. Now, since $W^\beta \leq Q_{\alpha'-1}$, we have that $[W^\beta, Q_\beta] \leq Q_{\alpha'-1}$ and as $Z_{\alpha'} \not\leq [V^{(3)}_\beta, Q_\beta]$, $[[W^\beta, Q_\beta], V_{\alpha'}] \leq V_\beta$ so that $[W^\beta, Q_\beta]V_\beta \trianglelefteq L_\beta = \langle V_{\alpha'}, R_\beta, Q_\alpha\rangle$. But then $[V^{(3)}_\beta, Q_\beta] = [W^\beta, Q_\beta]V_\beta \leq Q_{\alpha'-1}$, a contradiction.

Thus, $V^{(3)}_\beta \leq Q_{\alpha'-1}$ and it follows that $V_\beta(V^{(3)}_\beta \cap Q_{\alpha'})$ has index at most p in $V^{(3)}_\beta$. Then by Lemma 5.2.34, $O^p(R_\beta)$ centralizes $V^{(3)}_\beta$ and by Lemma 5.2.18, $W^\beta = V^{(2)}_\alpha$ and $U^\beta = V^{(2)}_{\alpha+2}$. Furthermore, by Lemma 5.4.36, $|V^{(2)}_\alpha| = p^4$. But then $V^{(3)}_\beta = V^{(2)}_\alpha V^{(2)}_{\alpha+2}$ and since $V^{(2)}_\alpha$ centralizes $V_{\alpha'-2}V_\beta = V^{(2)}_{\alpha+2}$, $V^{(3)}_\beta$ is abelian. Upon conjugating, $V^{(3)}_{\alpha'-2}$ is abelian, impossible since $[V_\beta, V_{\alpha'}] \neq \{1\}$. Thus, $W^\beta \not\leq Q_{\alpha'-2}$.

Using the symmetry in the critical pairs (α, α') and $(\alpha'+1, \beta)$, we may assume that $W^{\alpha'} \not\leq Q_{\alpha'-2}$. We may as well arrange that for the critical pairs (α, α') and $(\alpha'+1, \beta)$ we have that $V_\alpha^{(2)} \not\leq Q_{\alpha'-2}$ and $V_{\alpha'+1}^{(2)} \not\leq Q_{\alpha'-2}$, and the result holds. $\square$

Throughout the next lemmas and propositions, by the above work, we assume that $V_{\alpha'} \not\leq Q_\beta$, $Z_{\alpha'-1} \neq Z_{\alpha+2}$, $R = Z_{\alpha'-2} \neq Z_\beta \neq Z_{\alpha'} \neq R$, $V_\alpha^{(2)} \not\leq Q_{\alpha'-2}$ and for $\alpha'+1 \in \Delta(\alpha')$ with $(\alpha'+1, \beta)$ a critical pair, $V_{\alpha'+1}^{(2)} \not\leq Q_{\alpha'-2}$. In particular, $V_\alpha^{(2)}$ does not centralize $Z_{\alpha'-1}$ and so $Z_{\alpha'} \not\leq C_{Q_\alpha}(V_\alpha^{(2)})$. Similarly, $Z_\beta \not\leq C_{Q_{\alpha'+1}}(V_{\alpha'+1}^{(2)})$. We set $W^\beta := \langle V_\lambda^{(2)} \mid \lambda \in \Delta(\beta), Z_\lambda = Z_\alpha \rangle \trianglelefteq R_\beta Q_\alpha$ throughout.

Lemma 5.4.38. *Suppose that $C_{V_\beta}(V_{\alpha'}) = V_\beta \cap Q_{\alpha'}$ and $b = 5$. Then $O^p(L_\beta)$ centralizes $[V_\beta^{(3)}, Q_\beta]V_\beta/V_\beta$.*

Proof. Suppose that $[V_\beta^{(3)}, Q_\beta]V_\beta/V_\beta$ contains a non-central chief factor for L_β. In addition, suppose that $Z_{\alpha'} \not\leq [V_\beta^{(3)}, Q_\beta]V_\beta$. Notice that $[W^\beta, Q_\beta] = [W^\beta, (Q_\alpha \cap Q_\beta)][W^\beta, (Q_\alpha \cap Q_{\alpha+2})] \leq Z_\alpha[Q_{\alpha+2}, Q_{\alpha+2}] \leq Q_{\alpha'-2}$. Now, $[W^\beta \cap Q_{\alpha'-2}, Z_{\alpha'-1}] \leq Z_{\alpha'-2} \cap [W^\beta, Z_{\alpha'-1}] \leq Z_{\alpha'-2} \cap Z_\alpha = \{1\}$ and so $[W^\beta, Q_\beta, V_{\alpha'}] \leq Z_{\alpha'-1} \cap [V_\beta^{(3)}, Q_\beta] \leq Z_{\alpha'-2} \leq V_\beta$. In particular, it follows that $[W^\beta, Q_\beta]V_\beta \trianglelefteq L_\beta = \langle V_{\alpha'}, Q_\alpha, R_\beta \rangle$ and $[W^\beta, Q_\beta]V_\beta = [V_\beta^{(3)}, Q_\beta]V_\beta$. But then, $[[V_\beta^{(3)}, Q_\beta]V_\beta, Q_\alpha] \leq V_\beta$, a contradiction since $[V_\beta^{(3)}, Q_\beta]V_\beta/V_\beta$ contains a non-central chief factor.

Thus, $Z_{\alpha'} \leq [V_\beta^{(3)}, Q_\beta]V_\beta$. But then $V_{\alpha'-2} = Z_{\alpha'}Z_{\alpha+2} \leq [V_\beta^{(3)}, Q_\beta]V_\beta$. Now, since $V_\alpha^{(2)} \not\leq Q_{\alpha'-2}$, then is some $\alpha-2 \in \Delta^{(2)}(\alpha)$ with $Z_{\alpha-2} \not\leq Q_{\alpha'-2}$ and $(\alpha-2, \alpha'-2)$ a critical pair. But $[V_\beta^{(3)}, Q_\beta]V_\beta \leq [Q_\beta, Q_\beta]V_\beta \leq Q_{\alpha-1}$ since $Q_\beta \cap Q_{\alpha-1}$ has index p^2 in Q_β. Therefore, $V_{\alpha'-2} \leq Q_{\alpha-1}$, a contradiction by Lemma 5.4.37. $\square$

Lemma 5.4.39. *Suppose that $C_{V_\beta}(V_{\alpha'}) = V_\beta \cap Q_{\alpha'}$ and $b = 5$. Then $p \in \{2, 3\}$ and for $V := V_\alpha^{(2)}/Z_\alpha$ either:*

(i) *V is a quadratic module determined by Proposition 2.3.19;*

(ii) $V = [V, R_\alpha]$; *or*

(iii) $V = C_V(R_\alpha)$.

Moreover, $\overline{L_\beta} \cong \mathrm{SL}_2(p)$.

Proof. By Lemma 5.4.38, $[V_\beta^{(3)}, Q_\beta]V_\beta/V_\beta$ is centralized by $O^p(L_\beta)$. Then $[V_\alpha^{(2)}, Q_\beta]V_\beta \leq V_{\alpha+2}^{(2)}$. Now, $[V_\beta^{(3)}, Q_\beta]V_\beta \leq Z(V_\beta^{(3)})$ so that $[V_\beta^{(3)}, V_\beta^{(3)}] \leq \Omega(Z(Q_\beta)) = Z_\beta$ by the three subgroup lemma. Moreover, $C_\beta = V_\beta^{(3)}(C_\beta \cap Q_{\alpha'-2})$ and $[C_\beta \cap Q_{\alpha'-2}, V_{\alpha'-2}^{(3)}] \leq [V_{\alpha'-2}^{(3)}, Q_{\alpha'-2}] \leq V_{\alpha+2}^{(2)} \leq V_\beta^{(3)}$ so that $O^p(L_\beta)$ centralizes $C_\beta/V_\beta^{(3)}$. But then $O^p(R_\beta)$ centralizes $Q_\beta/V_\beta^{(3)}$. Indeed, $V_\beta^{(3)}/Z_\beta = [V_\beta^{(3)}/Z_\beta, O^p(R_\beta)] \times C_{V_\beta^{(3)}/Z_\beta}(O^p(R_\beta))$. Now, $[O^p(R_\beta), V_\beta^{(3)}, Q_\beta] \leq Z_\beta$ by the three subgroup lemma, and $[V_\beta, O^p(R_\beta), C_{V_\beta}(O^p(R_\beta))] = \{1\}$. If $[V_\beta, O^p(R_\beta)] \not\leq Q_{\alpha'-2}$, then $[V_{\alpha'-2}^{(3)} \cap Q_\beta, [V_\beta, O^p(R_\beta)]] \leq Z_\beta \leq V_{\alpha'-2}$ and we deduce that $V_{\alpha'-2}^{(3)}/V_{\alpha'-2}$ contains a unique non-central chief factor. Then Lemma 5.2.34 implies that $O^p(R_\beta)$ centralizes $V_\beta^{(3)}$. It is straightforward to show that $C_{Q_\beta}(V_\beta^{(3)})/V_\beta$ is centralized by $O^p(L_\beta)$ and a final application of the three subgroup lemma yields that $O^p(R_\beta)$ centralizes Q_β and $\overline{L_\beta} \cong \mathrm{SL}_2(p)$. Thus, $[V_\beta, O^p(R_\beta)] \leq Q_{\alpha'-2}$. Moreover, $Z_{\alpha'-2} \leq V_\beta \leq C_{V_\beta^{(3)}}(O^p(R_\beta))$ so that $[V_\beta, O^p(R_\beta)] \leq C_{\alpha'-2}$. If $Z_{\alpha'} \leq [V_\beta, O^p(R_\beta)]$, then $C_{V_\beta^{(3)}}(O^p(R_\beta))$ centralizes $V_{\alpha'-2} = Z_{\alpha+2}Z_{\alpha'}$ and $V_\beta^{(3)} \leq C_{\alpha'-2}$, a contradiction. Thus, $[V_\beta, O^p(R_\beta), V_{\alpha'-2}^{(3)}] \leq V_{\alpha'-2} \cap [V_\beta, O^p(R_\beta)] = Z_\beta$ so that $O^p(L_\beta)$ centralizes $[V_\beta, O^p(R_\beta)]$. Hence, $O^p(R_\beta)$ centralizes $V_\beta^{(3)}$ and the three subgroup lemma yields that $R_\beta = Q_\beta$ and $\overline{L_\beta} \cong \mathrm{SL}_2(p)$.

Now, writing $Q := Q_\beta \cap O^p(L_\beta)$, we have that $[V_\alpha^{(2)}, Q, Q] \leq [V_\beta^{(3)}, Q, Q] \leq V_\beta$. By coprime action, and setting $V := V_\alpha^{(2)}/Z_\alpha$, we have that $V = [V, R_\alpha] \times C_V(R_\alpha)$ and either $V_\beta/Z_\alpha \leq [V, R_\alpha]$ or Q acts quadratically on $[V, R_\alpha]$. Similarly, either $V_\beta/Z_\alpha \leq C_V(R_\alpha)$ or Q acts quadratically on $C_V(R_\alpha)$. Since both $[V, R_\alpha]$ and

$C_V(R_\alpha)$ are normalized by L_α, and V_β/Z_α generates V, we have shown that either Q acts quadratically on V, $V = [V, R_\alpha]$ or $V = C_V(R_\alpha)$.

In all cases, Q acts cubically on V and so if $p \geqslant 5$ the Hall-Higman theorem yields that $O^p(R_\alpha)$ centralizes $V_\alpha^{(2)}$. Since Q centralizes $C_\beta/V_\beta^{(3)}$, $[C_{Q_\alpha}(V_\alpha^{(2)}, Q, Q] \leq [C_\beta, Q, Q] \leq [V_\beta^{(3)}, Q] \leq V_\alpha^{(2)}$ and a standard argument implies that $O^p(R_\alpha)$ centralizes $C_{Q_\alpha}(V_\alpha^{(2)}$ and a final application of the three subgroup lemma yields that $O^p(R_\alpha)$ centralizes Q_α, G has a weak BN-pair of rank 2 and [DS85] provides a contradiction. Hence, $p \in \{2, 3\}$. □

Proposition 5.4.40. *Suppose that $C_{V_\beta}(V_{\alpha'}) = V_\beta \cap Q_{\alpha'}$ and $b = 5$. Then $p = 3$ and G is parabolic isomorphic to F_3.*

Proof. Let P_α be a $G_{\alpha,\beta}$-invariant subgroup of L_α such that $S \leq P_\alpha$ and $L_\alpha = P_\alpha C_{L_\alpha}(V)$, and form $X := \langle G_\beta, P_\alpha \rangle$. Let T be the largest subgroup of S which is normalized by X. Suppose that $T \neq \{1\}$. Then $\langle Z_\beta^X \rangle \leq Z(T)$ and by construction, $Z_\alpha \not\leq T$, otherwise $V_\beta^{(3)} \leq \langle Z_\alpha^X \rangle$ is abelian, a contradiction. Even still, $[T, Z_\alpha] = \{1\}$ and taking normal closures under X, we deduce that $T \leq C_{Q_\beta}(V_\beta^{(3)})$. But $O^p(L_\beta)$ centralizes $C_{Q_\beta}(V_\beta^{(3)})/V_\beta$ and so G_β/T is of characteristic p. Assume that P_α/T is not of characteristic p so that $O^p(P_\alpha)$ acts non-trivially on T. Since $Z_\alpha \not\leq T$, T is not self-centralizing and we may assume that $C_S(T) \leq Q_\alpha$ and $C_S(T) \not\leq Q_\beta$. If $C_S(T)^x \cap Q_\beta \not\leq Q_\alpha$ for some $x \in L_\beta$, then $[C_S(T)^x \cap Q_\beta, T] = \{1\}$ so that $[O^p(P_\alpha), T] \leq [\langle (C_S(T)^x \cap Q_\beta)^{P_\alpha} \rangle, T] = \{1\}$, a contradiction. Thus, $\langle (C_S(T) \cap Q_\beta)^{L_\beta} \rangle \leq Q_\alpha$ and so $[O^2(L_\beta), Q_\beta] \leq [\langle C_S(T)^{L_\beta} \rangle, Q_\beta] \leq \langle (C_S(T) \cap Q_\beta)^{L_\beta} \rangle \leq Q_\alpha$ and $Q_\alpha \cap Q_\beta \trianglelefteq L_\beta$, a contradiction by Proposition 5.2.25. Thus, the triple $(G_\beta/T, P_\alpha/T, G_{\alpha,\beta}/T)$ satisfies Hypothesis 5.2.1 and assuming that G is a minimal counterexample to Theorem 5.2.2, we conclude that $P_\alpha/Q_\alpha \cong (3 \times 3) : 2$ and $|S/T| = 2^6$. But Q_β contains three non-central chief factors for $\overline{L_\beta}$ and we have a contradiction. Hence, for every subgroup of P of L_α which contains S and is normalized by $G_{\alpha,\beta}$, $L_\alpha = PC_{L_\alpha}(V)$ implies that $L_\alpha = P$. In particular, applying Lemma 5.2.32 and Lemma 2.3.15 (iii) when $p=3$, we deduce that if V is an FF-module then $R_\alpha \le C_{L_\alpha}(C_{Q_\alpha}(V_\alpha^{(2)}))S$ and the three subgroup lemma yields that $R_\alpha = Q_\alpha$ and G has a weak BN-pair of rank 2. Then [DS85] gives that V is not an FF-module, and we have a contradiction.

Note that $V_\beta^{(3)}/[V_\beta^{(3)}, Q_\beta]$ is a quadratic 2F-module for $\overline{L_\beta} \cong \mathrm{SL}_2(p)$ by Lemma 5.4.39. Hence, applying Lemma 2.3.11, we have that $[V_\beta^{(3)}/[V_\beta^{(3)}, Q_\beta], O^p(L_\beta)]$ is a direct sum of at most two natural modules for $\overline{L_\beta}$. Assume that $[V_\beta^{(3)}/[V_\beta^{(3)}, Q_\beta], O^p(L_\beta)]$ contains two natural modules. Then $V_{\alpha'-2}$ projects as a subgroup of order p in $[V_\beta^{(3)}/[V_\beta^{(3)}, Q_\beta], O^p(L_\beta)]$. Indeed, we have that $V_\beta^{(3)}/[V_\beta^{(3)}, Q_\beta] = [V_\beta^{(3)}/[V_\beta^{(3)}, Q_\beta], O^p(L_\beta)]$. Since $C_\beta/V_\beta^{(3)}$ is centralized by $O^p(L_\beta)$, $[C_{Q_\alpha}(V_\alpha^{(2)}), Q] \le V_\alpha^{(2)}$ and so $O^p(L_\alpha)$ centralizes $C_{Q_\alpha}(V_\alpha^{(2)}$. Then the three subgroup lemma yields that $R_\alpha \cap C_{L_\alpha}(V) = Q_\alpha$. By Lemma 2.2.7, for $W := \langle V_{\alpha'-2}^{L_\beta}\rangle[V_\beta^{(3)}, Q_\beta]$, $V_\beta^{(3)}/W$ is a natural module for $\overline{L_\beta} \cong \mathrm{SL}_2(p)$.Then $W \le C_{\alpha'-2}$ for otherwise $W \not\le Q_{\alpha'-2}$, $V_\beta^{(3)} = W(V_\beta^{(3)} \cap C_{\alpha'-2})$ so that $[V_\beta^{(3)}, V_{\alpha'-2}^{(3)}] \le W$, a contradiction since $V_\beta^{(3)}/W$ contains a non-central chief factor. Hence, $[W, V_{\alpha'-2}] = \{1\}$ so that W is abelian. Then $W = V_\beta(W \cap Q_{\alpha'})$ and since W/V_β contains a non-central chief factor for L_β, $W \cap Q_{\alpha'} \not\le Q_{\alpha'+1}$ for some $\alpha'+1 \in \Delta(\alpha')$. Since W is abelian, $W \cap Q_{\alpha'}$ acts quadratically on $V_{\alpha'+1}^{(2)}$. Hence, V is also a quadratic module. Since V_β/Z_α has order p and generates V, by Lemma 2.3.22, $p=2$ and $L_\alpha/C_{L_\alpha}(V) \cong \mathrm{Dih}(10)$ or $(3\times 3):2$. Then $R_\alpha S$ is a maximal subgroup of L_α containing S which is normalized by $G_{\alpha,\beta}$ and we deduce that $L_\alpha/Q_\alpha \cong (3\times 3):2$. Let $P_\alpha^i \le L_\alpha$ with $S \le P_\alpha^i$, $L_\alpha = P_\alpha^i R_\alpha$, $P_\alpha^i/Q_\alpha \cong \mathrm{Sym}(3)$ and $Q_\alpha \cap Q_\beta \ntrianglelefteq P_\alpha^i$ for $i \in \{1,2\}$. Then the triple $(L_\beta, P_\alpha^1, P_\alpha^2)$ satisfies the hypothesis of [Che86, Theorem B] and as $\mathrm{Sym}(4)$ is not a homomorphic image of

$(3 \times 3) : 2$, we have a contradiction.

Hence, $[V_\beta^{(3)}/[V_\beta^{(3)}, Q_\beta], O^p(L_\beta)]$ contains a unique non-central chief factor for L_β. Moreover, $V_\alpha^{(2)}/V_\beta \cap C_{V_\beta^{(3)}/V_\beta}(O^p(L_\beta))$ has index p in $V_\alpha^{(2)}$. Setting $Q := Q_\beta \cap O^p(L_\beta)$ there is an index p subgroup U of $V_\alpha^{(2)}$ such that $[U, Q] \leq V_\beta$. It follows that there is an index p^2 subgroup U^* of $V_\alpha^{(2)}$ with $[U^*, Q] \leq Z_\alpha$ so that V is a 2F-module for $L_\alpha/C_{L_\alpha}(V)$.

Suppose now that V is a quadratic module for $\overline{L_\alpha}$. Then, since V_β/Z_α has order p and generates V, by Lemma 2.3.22, $p = 2$ and $L_\alpha/C_{L_\alpha}(V) \cong \mathrm{Dih}(10)$ or $(3\times 3) : 2$. Since $R_\alpha S$ is a maximal subgroup of L_α containing S which is normalized by $G_{\alpha,\beta}$, we deduce that $L_\alpha/Q_\alpha \cong (3 \times 3) : 2$. Let $P_\alpha \leq L_\alpha$ with $L_\alpha = P_\alpha R_\alpha$, $S \leq P_\alpha$, $P_\alpha Q_\alpha \cong \mathrm{Sym}(3)$ and $O_3(P_\alpha/Q_\alpha) \trianglelefteq L_\alpha/Q_\alpha$. Let T be the largest normal subgroup of S which is normalized by both L_β and P_α, and assume that $T \neq \{1\}$. Then $\langle V_\beta^{P_\alpha} \rangle \leq Z(T)$ and $\langle V_\beta^{P_\alpha} \rangle \not\leq Z(V_\beta^{(3)})$ so that $V_\beta^{(3)} \leq (Z(T) \cap V_\beta^{(3)})Z(V_\beta^{(3)})$. But then $V_\beta^{(3)}$ is abelian, a contradiction. Thus, the triple $(P_\alpha G_{\alpha,\beta}, G_\beta, G_{\alpha,\beta}))$ satisfies Hypothesis 5.2.1. Assuming that G is a minimal counterexample to Theorem 5.2.2, we deduce that $|S| \leqslant 2^7$. But $V_\beta^{(3)} \geqslant 2^7$ and we have a contradiction. Thus, V is not a quadratic module.

Since whenever $p = 2$, $|S/Q_\alpha| = 2$ and there is always an element $x \in S/ \setminus Q_\alpha$ which acts quadratically on V. Thus, for the remainder of this proof, we may assume that $p = 3$ and $V = [V, R_\alpha]$. Moreover, $V_\alpha^{(2)}$ projects with order p in $[V_\beta^{(3)}/[V_\beta^{(3)}, Q_\beta], O^3(L_\beta)]$. Let $Z_\alpha < U < V_\alpha^{(2)}$ with $U \trianglelefteq L_\alpha$. Then U/Z_α contains a non-central chief factor for L_α and as $U < V_\alpha^{(2)}$, $U \cap V_\beta = Z_\alpha$. Then $V_\alpha^{(2)} = U(V_\alpha^{(2)} \cap [V_\beta^{(3)}, Q_\beta])$ for otherwise, $[Q, U] \leq U \cap V_\beta = Z_\alpha$, a contradiction since U/Z_α contains a non-central chief factor. But now, $[V_\alpha^{(2)}, Q, Q] = [U, Q, Q] \leq V_\beta \cap U = Z_\alpha$, a contradiction since V is not quadratic. Hence, we conclude that V is an irreducible

2F-module for $L_\alpha/C_{L_\alpha}(V)$.

Note that by Lemma 5.2.17, R_α does not normalize S so that for $L := O^{3'}(R_\alpha S)$, L/Q_α has a strongly 3-embedded subgroup and $O^3(L/Q_\alpha) = O_{3'}(L/Q_\alpha)$. By coprime action, $V = [V, O_{3'}(L/Q_\alpha)] \times C_V(O_{3'}(L/Q_\alpha))$ is an S-invariant decomposition. Using that $[V, Q, Q] = V_\beta/Z_\alpha$, in a similar manner to Lemma 5.4.39, either $V = [V, O_{3'}(L/Q_\alpha)]$ or $V = C_V(O_{3'}(L/Q_\alpha))$. In the latter case, we have that $O_{3'}(L/Q_\alpha) \le R_\alpha/Q_\alpha \cap C_{L_\alpha}(V)/Q_\alpha = \{1\}$, a contradiction. Hence, $V = [V, O_{3'}(L/Q_\alpha)]$.

Suppose that there is $Z_\alpha < U < V_\alpha^{(2)}$ with $U \trianglelefteq L$. Since $C_V(O_{3'}(L/Q_\alpha)) = \{1\}$, U contains a non-central chief factor for L. If $U \le Z(V_\beta^{(3)})$ then $[U, Q] = V_\beta$ so that U is dual to an FF-module for $L/C_L(U/Z_\alpha) \cong \mathrm{SL}_2(3)$ by Lemma 2.3.10. But then an index 3 subgroup of $V_\alpha^{(2)}/U$ is centralized by Q so that by Lemma 2.3.10 $L/C_L(V_\alpha^{(2)}/U) \cong \mathrm{SL}_2(3)$ and $V_\alpha^{(2)}/U$ is an FF-module. Since $C_V(O_{3'}(L/Q_\alpha)) = \{1\}$, we conclude that $|V| = 3^4$. Similarly, if $U \not\le Z(V_\beta^{(3)})$, then we may assume that $V_\beta \not\le U$, otherwise $V_\alpha^{(2)}/U$ is centralized by Q, a contradiction since $C_V(O_{3'}(L/Q_\alpha)) = \{1\}$. Hence, an index 3 subgroup of U is centralized modulo Z_α by Q and U is an FF-module for $L/C_L(U/Z_\alpha) \cong \mathrm{SL}_2(3)$ by Lemma 2.3.10. Moreover, $[V_\alpha^{(2)}, Q] = [U, Q]V_\beta$ and $V_\alpha^{(2)}/U$ is dual to an FF-module for $L/C_L(V_\alpha^{(2)}/U) \cong \mathrm{SL}_2(3)$ and again we deduce that $|V| = 3^4$. In either case, by Lemma 2.3.15 (ii), $L/Q_\alpha \cong \mathrm{SL}_2(3)$ or $(Q_8 \times Q_8) : 3$. In the latter case, for two distinct central involutions t_1, t_2 in L/Q_α, we have that $V = [V, t_1] \times [V, t_2]$ and so V is a quadratic module, a contradiction. Thus, $L/Q_\alpha \cong \mathrm{SL}_2(3)$. Now, V is an irreducible module of dimension 4 for $L_\alpha/C_{L_\alpha}(V)$ and $L_\alpha/C_{L_\alpha}(V)$ contains a subgroup of $3'$-index isomorphic to $\mathrm{SL}_2(3)$. Considering irreducible subgroups of $\mathrm{SL}_4(3)$ which have strongly 3-embedded subgroups and which do not have a 3-element which acts quadratically, we calculate (e.g. using MAGMA) that $L_\alpha/C_{L_\alpha}(V)$ is of order $2^5.3$, V is the unique irreducible module of dimension 4 for $L_\alpha/C_{L_\alpha}(V)$ and $L = R_\alpha S$. Hence, $|R_\alpha C_{L_\alpha}(V)/R_\alpha| = Z(L_\alpha/R_\alpha)$ so that $|C_{L_\alpha}(V)/Q_\alpha| = 2 = |Z(R_\alpha/Q_\alpha)|$, $|\overline{L_\alpha}| = 2^6$, $Z(\overline{L_\alpha}) = C_{L_\alpha}(V)/Q_\alpha \times Z(R_\alpha/Q_\alpha)$. Once again, calculating in MAGMA we conclude that $\overline{L_\alpha} \cong Q_8 \times Q_8) : 3$. But then, since $\overline{L_\beta} \cong \mathrm{SL}_2(3)$, there is $t \leq L_\alpha$ such that $T \leq Z(\overline{L_\alpha})$ and t centralizes $\overline{L_\beta}$, a contradiction by Proposition 5.2.6.

Thus, V is irreducible for L/Q_α so that $\langle V_\beta^{R_\alpha S} \langle = V_\alpha^{(2)}$. Let T be the largest subgroup of S normalized by both L_β and $R_\alpha S$. Suppose that $T \neq \{1\}$. Then $Z_\alpha \not\leq T$, for otherwise, $Z_\alpha \leq Z(T)$ and taking respective normal closures yields $V_\beta^{(3)} \leq Z(T)$ is abelian, a contradiction. Since Z_α centralizes T, we infer that $[T, V_\beta^{(3)}] = \{1\}$ and $T \leq C_{Q_\beta}(V_\beta^{(3)})$. Since coprime elements of $O^3(L_\beta)$ act faithfully Q_β/C_β, we conclude that L_β/T is of characteristic 3. Assume that $R_\alpha S/T$ is not of characteristic 3 so that $O^3(L)$ acts non-trivially on T. Since $Z_\alpha \not\leq T$, T is not self-centralizing and we may assume that $C_S(T) \leq Q_\alpha$ and $C_S(T) \not\leq Q_\beta$. If $C_S(T)^x \cap Q_\beta \not\leq Q_\alpha$ for some $x \in L_\beta$, then $[C_S(T)^x \cap Q_\beta, T] = \{1\}$ so that $[O^3(L), T] \leq [\langle (C_S(T)^x \cap Q_\beta)^{R_\alpha S}\rangle, T] = \{1\}$, a contradiction. Thus, $\langle (C_S(T) \cap Q_\beta)^{L_\beta}\rangle \leq Q_\alpha$ and so $[O^3(L_\beta), Q_\beta] \leq [\langle C_S(T)^{L_\beta}\rangle, Q_\beta] \leq \langle (C_S(T) \cap Q_\beta)^{L_\beta}\rangle \leq Q_\alpha$ and $Q_\alpha \cap Q_\beta \trianglelefteq L_\beta$, a contradiction by Proposition 5.2.25. Thus, the triple $(G_\beta/T, R_\alpha G_{\alpha\beta}/T, G_{\alpha,\beta}/T)$ satisfies Hypothesis 5.2.1 and assuming that G is a minimal counterexample to Theorem 5.2.2, we conclude that $L/Q_\alpha \cong \mathrm{SL}_2(3)$. Since V is an irreducible non-quadratic 2F-module for $\mathrm{SL}_2(3)$, $C_L(V) \neq \{1\}$ a contradiction. Thus, $T = \{1\}$ and the triple $(G_\beta, R_\alpha G_{\alpha\beta}, G_{\alpha,\beta})$ satisfies Hypothesis 5.2.1. As before, this implies that $L/Q_\alpha \cong \mathrm{SL}_2(3)$ and since V is an irreducible module for L/Q_α, we deduce that $C_L(V) \neq \{1\}$, a final contradiction. $\square$

Now, we may assume that $b = 3$. Unfortunately, most of the techniques introduced earlier in this section are not applicable in this setting and so the methodology for this case is different from the rest of this subsection. The aim throughout will be to show that $R_\beta = Q_\beta$ and $R_\alpha = Q_\alpha$ for then an appeal to [DS85] yields $p = 2$ and G is parabolic isomorphic to M_{12} or $\mathrm{Aut}(\mathrm{M}_{12})$.

Lemma 5.4.41. *Suppose that $C_{V_\beta}(V_{\alpha'}) = V_\beta \cap Q_{\alpha'}$ and $b = 3$. Then $R_\beta = Q_\beta$, $\overline{L_\beta} \cong \mathrm{SL}_2(p)$ and $O^p(L_\beta)$ centralizes C_β/V_β.*

Proof. Notice that $RZ_{\alpha+2} \leq V_\beta \cap V_{\alpha'}$. If $Z_{\alpha+2} = V_{\alpha'} \cap V_\beta \geq R$, then $Z_{\alpha+2}Z_{\alpha+2}^g \trianglelefteq L_{\alpha'} = \langle V_\beta, V_\beta^g, R_{\alpha'} \rangle$ for some appropriately chosen $g \in L_{\alpha'}$, and $|V_\beta| = p^3$. Otherwise, $RZ_{\alpha+2} = V_{\alpha'} \cap V_\beta$ is of order p^3 and $|V_\beta| = p^4$. Indeed, it follows that $Z_{\alpha+2}C_{V_{\alpha'}}(O^p(L_{\alpha'})) = RZ_{\alpha+2} = Z_{\alpha+2}C_{V_\beta}(O^p(L_\beta))$ so that $Z_\beta = [RZ_{\alpha+2}, Q_{\alpha+2}] = Z_{\alpha'}$.

Now, if $V_{\alpha'} \leq Q_\beta$, then $R = Z_\beta \leq Z_{\alpha+2}$ and $|V_\beta| = p^3$. Then $[C_{\alpha'}, V_\beta] \leq Z_{\alpha+2} \leq V_{\alpha'}$ and $O^p(L_{\alpha'})$ centralizes $C_{\alpha'}/V_{\alpha'}$. By conjugation, $O^p(L_\beta)$ centralizes C_β/V_β. If $V_{\alpha'} \not\leq Q_\beta$, then $[C_{\alpha'}, V_\beta] = [V_{\alpha'}(C_{\alpha'} \cap Q_\beta), V_\beta] \leq RZ_\beta \leq V_{\alpha'}$ and again, by conjugation, $O^p(L_\beta)$ centralizes C_β/V_β. Thus, in all cases $O^p(L_\beta)$ centralizes C_β/V_β. In particular, for $r \in R_\beta$ of order coprime to p, the three subgroup lemma implies that $[r, Q_\beta] \leq C_\beta$ so that $[r, Q_\beta] = \{1\}$ and $r = 1$. Thus, $R_\beta = Q_\beta$ and $\overline{L_\beta} \cong \mathrm{SL}_2(p)$. □

Lemma 5.4.42. *Suppose that $C_{V_\beta}(V_{\alpha'}) = V_\beta \cap Q_{\alpha'}$ and $b = 3$. Then $Z_{\alpha'} \neq Z_\beta$, $|V_\beta| = p^3$ and $\Phi(V_\alpha^{(2)}) = Z_\alpha$. Moreover, if $[C_\beta, C_\beta] \leq V_\beta$ and $R_\alpha \neq Q_\alpha$, then $[C_\beta, C_\beta] \leq Z_\beta$ and $p \in \{2, 3\}$.*

Proof. Assume now that whenever (α, α') is a critical pair we have that $Z_\beta = Z_{\alpha'}$. In particular, $V_{\alpha'} \not\leq Q_\beta$ for any critical pair. Since $Z_\beta \ntrianglelefteq L_{\alpha+2}$, there is $\lambda \in \Delta(\alpha+2)$

such that $Z_\lambda \neq Z_\beta$. Moreover, by assumption, $V_\lambda \leq Q_\beta$ and $V_\beta \leq Q_\lambda$ so that $[V_\lambda, V_\beta] \leq Z_\beta \cap Z_\lambda = \{1\}$. Then, $[C_\beta \cap Q_\lambda, V_\lambda] \leq [C_\beta, C_\beta] \cap Z_\lambda$. If $Z_\lambda \leq \Phi(C_\beta)$, then $Z_{\alpha+2} = Z_\lambda \times Z_\beta \leq \Phi(C_\beta)$ and $V_\beta \leq \Phi(C_\beta)$. But $O^p(L_\beta)$ centralizes C_β/V_β, a contradiction by coprime action. Therefore, $C_\beta \cap Q_\lambda = C_\beta \cap C_\lambda$ is of index at most p in C_β and C_λ. By the same reasoning, $C_{\alpha'} \cap Q_\lambda = C_{\alpha'} \cap C_\lambda$ and since $V_{\alpha'} \leq C_\lambda$ and $V_{\alpha'} \not\leq C_\beta$, $C_\beta \not\leq Q_\lambda$ and $C_\beta \cap C_\lambda$ is proper in C_β.

Since $V_\beta \leq C_\beta \cap C_\lambda$, $C_\beta \cap C_\lambda \neq C_{\alpha'} \cap C_\lambda$ so that $C_\lambda = (C_\beta \cap C_\lambda)(C_{\alpha'} \cap C_\lambda)$. Moreover, since $V_\beta V_\lambda \leq C_\beta \cap C_\lambda$, we have that $C_\beta \cap C_\lambda \trianglelefteq \langle Q_{\alpha+2}, O^p(L_\lambda), O^p(L_\beta)\rangle = \langle L_\beta, L_\lambda\rangle$ and C_λ is non-abelian. It follows that either $Z_{\alpha+2} = Z_\beta \times Z_\lambda \leq \Phi(C_\beta \cap C_\lambda) \leq \Phi(C_\beta)$ and $V_\beta \leq \Phi(C_\beta)$, a contradiction for then $O^p(L_\beta)$ centralizes C_β/V_β; or $C_\beta \cap C_\lambda$ is elementary abelian. Then $\Omega(Z(C_\lambda)) = C_\lambda \cap C_\beta \cap C_{\alpha'}$ and $C_\lambda = V_\beta V_{\alpha'} \Omega(Z(C_\lambda))$. But then $[C_\lambda, C_\lambda] = [V_\beta, V_{\alpha'}] = R$ so that $Z_{\alpha+2} = Z_\lambda Z_\beta \leq [C_\lambda, C_\lambda] Z_\beta \leq R Z_\beta$ and since $|RZ_\beta| = p^2$, we have that $R \leq Z_{\alpha+2}$ so that $R = Z_\lambda$. Now, there is $\mu \in \Delta(\alpha+2)$ such that $Z_\beta \neq Z_\mu \neq Z_\lambda$ and we may repeat the above arguments with μ in place of λ. But then $Z_\mu = R = Z_\lambda$, a contradiction.

Thus, there is a critical pair (α, α') with $Z_{\alpha'} \neq Z_\beta$ and by an argument in the proof of Lemma 5.4.41, we infer that $|V_\beta| = p^3$. Thus, $[V_\alpha^{(2)}, V_\beta] \leq Z_\alpha$ and since $V_\alpha^{(2)}$ is non-abelian, otherwise by conjugacy $V_{\alpha'} \leq V_{\alpha+2}^{(2)}$ centralizes V_β, we have that $[V_\alpha^{(2)}, V_\alpha^{(2)}] = Z_\alpha$. But now, $V_\alpha^{(2)}$ is generated by V_λ for $\lambda \in \Delta(\alpha)$ and since V_λ/Z_α is of order p, $V_\alpha^{(2)}/Z_\alpha$ is elementary abelian and $\Phi(V_\alpha^{(2)}) = Z_\alpha$.

Suppose now that $[C_\beta, C_\beta] \leq V_\beta$. We have that $[C_\beta, C_\beta] \neq V_\beta$, otherwise $O^p(L_\beta)$ centralizes $C_\beta/\Phi(C_\beta)$. Thus, $[C_\beta, C_\beta] \leq Z_\beta$. Notice that if $V_\alpha^{(2)} \cap Q_\beta \not\leq C_\beta$, then $[C_\beta, V_\alpha^{(2)} \cap Q_\beta] \leq [Q_\alpha, V_\alpha^{(2)}] \leq Z_\alpha$ and since $Q_\beta = C_\beta \langle (V_\alpha^{(2)} \cap Q_\beta)^{L_\beta}\rangle$ and $[C_\beta, C_\beta] = Z_\beta$, it follows that $[Q_\beta, Q_\beta] \leq V_\beta$ and Q_β acts cubically on Q_α/Z_α. If $V_\alpha^{(2)} \cap Q_\beta \leq C_\beta$, then as $[Q_\beta, O^p(L_\beta), C_\beta] \leq V_\beta$ by the three subgroup lemma, setting $Q := [Q_\beta, O^p(L_\beta)]$ and noticing that $Q \not\leq Q_\alpha$, we have that $[V_\alpha^{(2)}, Q, Q, Q] \leq [V_\alpha^{(2)} \cap Q_\beta, Q, Q] \leq [C_\beta, Q, Q] \leq Z_\beta$ and Q acts cubically on $V_\alpha^{(2)}/Z_\alpha$. Moreover, since $[Q, Q] \leq C_\beta$, $[Q, Q, Q] \leq V_\beta \leq V_\alpha^{(2)}$ and Q acts at most cubically on $Q_\alpha/V_\alpha^{(2)}$. Therefore, if $p \geqslant 5$, an application of the Hall–Higman theorem implies that $R_\alpha = Q_\alpha$, and G has a weak BN-pair of rank 2. Then [DS85] provides a contradiction. $\square$

Lemma 5.4.43. *Suppose that $C_{V_\beta}(V_{\alpha'}) = V_\beta \cap Q_{\alpha'}$ and $b = 3$. Then both $V_\beta(C_{\alpha'} \cap C_\beta)$ and $V_{\alpha'}(C_{\alpha'} \cap C_\beta)$ are elementary abelian, and $V_{\alpha'} \not\leq Q_\beta$.*

Proof. By Lemma 5.4.42, there is a critical pair (α, α') such that $Z_{\alpha'} \neq Z_\beta$. Moreover, by Lemma 5.4.41, $V_\beta(C_{\alpha'} \cap C_\beta) \trianglelefteq L_\beta = O^p(L_\beta)Q_{\alpha+2}$ from which it follows that $\Phi(C_{\alpha'} \cap C_\beta) = \Phi(V_\beta(C_{\alpha'} \cap C_\beta)) \trianglelefteq L_\beta$. If $C_{\alpha'} \cap C_\beta$ is not elementary abelian, then $Z_\beta \leq \Phi(C_{\alpha'} \cap C_\beta)$ and by a similar argument, $Z_{\alpha'} \leq \Phi(C_{\alpha'} \cap C_\beta)$ from which it follows that $Z_{\alpha+2} \leq \Phi(C_{\alpha'} \cap C_\beta) \leq \Phi(C_\beta)$. But then $V_\beta \leq \Phi(C_\beta)$, a contradiction since $O^p(L_\beta)$ centralizes C_β/V_β. Thus, $C_{\alpha'} \cap C_\beta$ is elementary abelian so that both $V_\beta(C_{\alpha'} \cap C_\beta)$ and $V_{\alpha'}(C_{\alpha'} \cap C_\beta)$ are elementary abelian.

Suppose that $V_{\alpha'} \leq Q_\beta$. Then, by Lemma 5.4.42, $|V_\beta| = p^3$, $C_\beta = Q_\alpha \cap Q_\beta \cap Q_{\alpha+2}$ has index p^2 in both Q_β and Q_α, and $V_\beta(C_\beta \cap C_{\alpha'})$ is elementary abelian and has index at most p in C_β. Similarly, $V_{\alpha'}(C_{\alpha'} \cap C_\beta)$ is elementary abelian of index at most p in $C_{\alpha'}$. Assume first that $C_{\alpha'}$ is elementary abelian so that by Lemma 5.4.42, $p \in \{2, 3\}$. If $C_{\alpha'} \cap C_\beta$ has index p^2 in C_β then $Q_{\alpha+2} = C_\beta C_{\alpha'}$ and $C_\beta \cap C_{\alpha'} \leq \Omega(Z(Q_{\alpha+2})) = Z_{\alpha+2}$. In particular, $|Q_{\alpha+2}/Z_{\alpha+2}| \leqslant p^4$. Let $\lambda \in \Delta(\alpha+2)$ with $Z_\beta \neq Z_\lambda \neq Z_{\alpha'}$. Then we again deduce that $|Q_{\alpha+2}/Z_{\alpha+2}| \leqslant p^4$ if $C_{\alpha'} \cap C_\lambda$ or $C_\beta \cap C_\lambda$ has index p^2 in C_λ; or $C_\lambda \cap C_{\alpha'}$ and $C_\lambda \cap C_\beta$ have index p in C_λ, $Q_{\alpha+2} = C_\beta C_{\alpha'} C_\lambda$ and, as before, we conclude that $|Q_{\alpha+2}/Z_{\alpha+2}| \leqslant p^4$. Checking p-solvable subgroups of $\mathrm{GL}_4(p)$ with an $\mathrm{SL}_2(p)$ quotient, we deduce that $R_\alpha = Q_\alpha$; or $\overline{L_\alpha} \cong (3 \times 3) : 2$ when $p = 2$ or $\overline{L_\alpha} \cong (Q_8 \times Q_8) : 3$ when $p = 3$. In the former case, since $V_{\alpha'} \leq Q_\beta$, [DS85, (9.6)] provides a contradiction.

Assume now that C_β is not elementary abelian so that $V_\beta(C_\beta \cap C_{\alpha'})$ has index p in C_β. Hence, $\Phi(C_\beta) \neq \{1\}$ and since V_β contains the unique non-central chief for L_β inside C_β, we have that $\Phi(C_\beta) \cap V_\beta = Z_\beta$. Note that $C_\beta \cap Q_{\alpha'}$ contains $C_{\alpha'} \cap C_\beta$ and is distinct from $V_\beta(C_\beta \cap C_{\alpha'})$ from which it follows that $C_\beta/(C_\beta \cap C_{\alpha'})$ is elementary abelian of order p^2. In particular, $\Phi(C_\beta) \leq C_{\alpha'}$ so that $L_{\alpha'} = O^p(L_{\alpha'})Q_{\alpha+2}$ normalizes $\Phi(C_\beta)V_{\alpha'}$. But then $\Phi(C_\beta) \geq [\Phi(C_\beta), C_{\alpha'}] = [\Phi(C_\beta)V_{\alpha'}, C_{\alpha'}] \trianglelefteq L_{\alpha'}$ and since $Z_{\alpha'} \not\leq \Phi(C_\beta)$, we deduce that $\Phi(C_\beta) \leq Z(C_{\alpha'})$. Now, as $C_{\alpha'} \cap C_\beta$ has index p^2 in C_β and C_β has index p^2 in $Q_{\alpha+2}$, we have that $Q_{\alpha+2} = C_\beta C_{\alpha'}$. Then, there is $x \in (L_{\alpha+2} \cap G_{\alpha',\alpha+2}) \setminus R_{\alpha+2}$ such that $Z_\beta^x \neq Z_\beta$. Applying a similar argument as for α', we see that $\Phi(C_\beta)$ is centralized by C_β^x and so $\Phi(C_\beta)$ is centralized by $Q_{\alpha+2} = C_{\alpha'}C_\beta^x$. Thus, $\Phi(C_\beta \leq Z_{\alpha+2}$ so that $\Phi(C_\beta) = [C_\beta, C_\beta] = Z_\beta$. Now, for any $x \in C_\beta \setminus V_\beta(C_\beta \cap C_{\alpha'})$, $C_{V_\beta(C_\beta \cap C_{\alpha'})}(x) = Z(C_\beta)$ so that $Z(C_\beta)$ is the kernel of the homomorphism $\theta : V_\beta(C_\beta \cap C_{\alpha'}) \to V_\beta(C_\beta \cap C_{\alpha'})$ such that $v\theta = [v, x]$ for $v \in V_\beta(C_\beta \cap C_{\alpha'})$. Then, the image of θ is $[C_\beta, C_\beta] = Z_\beta$ from which it follows that $|V_\beta(C_\beta \cap C_{\alpha'})/Z(C_\beta)| = p$ and $Z(C_\beta)$ is elementary abelian of index p^2 in C_β.

Since C_β is not elementary abelian, $\Omega(Z(C_\beta)) \cap Q_{\alpha'} \leq C_{\alpha'}$, for otherwise $C_\beta = \Omega(Z(C_\beta))(C_\beta \cap C_{\alpha'})$. Thus, $\Omega(Z(C_\beta)) \cap C_{\alpha'} \cap C_{\alpha'}^g$ has index at most p^4 in C_β and is centralized by $Q_\beta = C_\beta V_{\alpha'} V_{\alpha'}^g$ for some appropriately chosen $g \in L_\beta$. Hence $Z_\beta = \Omega(Z(C_\beta)) \cap C_{\alpha'} \cap C_{\alpha'}^g$ has index at most p^4 in C_β so that $|C_\beta| \leqslant p^5$. In particular, $[C_\beta, C_\beta] \leq V_\beta$, $|Q_\alpha/Z_\alpha| \leqslant p^5$ and we may assume that $p \in \{2, 3\}$ by Lemma 5.4.42.

For any $r \in O^p(L_\alpha)$ of p'-order, by the three subgroup lemma and coprime action, if r centralizes $V_\alpha^{(2)}/Z_\alpha$, then r centralizes $V_\alpha^{(2)}$ and $Q_\alpha/C_{Q_\alpha}(V_\alpha^{(2)})$. Notice that $C_{Q_\alpha}(V_\alpha^{(2)}) \leq C_\beta \leq Q_{\alpha+2}$ and so $[V_{\alpha'}, C_{Q_\alpha}(V_\alpha^{(2)})] \leq Z_{\alpha+2}$ and since $V_\beta \not\leq Z(V_\alpha^{(2)})$, we have that $V_{\alpha'}$ centralizes $C_{Q_\alpha}(V_\alpha^{(2)})/Z_\alpha$ and so $O^p(L_\alpha)$ centralizes $C_{Q_\alpha}(V_\alpha^{(2)})/Z_\alpha$. In particular, if r centralizes $V_\alpha^{(2)}/Z_\alpha$, then r centralizes Q_α and $r = 1$. Therefore, $V_\alpha^{(2)}/Z_\alpha$ is a faithful $\overline{L_\alpha}$-module and $|V_\alpha^{(2)}/Z_\alpha| \leqslant p^5$ with equality if and only if $Q_\alpha = V_\alpha^{(2)}$. For the remainder of this proof, set $V := V_\alpha^{(2)}/Z_\alpha$. Additionally, set $Q := \langle V_{\alpha'}^{L_\beta}\rangle$ so that $Q \cap C_\beta = V_\beta$, $Q_\beta = QC_\beta$ and $[Q, Q] \leq V_\beta$.

Let $U_\alpha \trianglelefteq L_\alpha$ chosen minimally such that $Z_\alpha < U_\alpha \leq V_\alpha^{(2)}$ and $\overline{L_\alpha}$ acts faithfully on $U^* := U_\alpha/Z_\alpha$. Set $U := U^*/C_{U^*}(\overline{L_\alpha})$. If U is irreducible for $\overline{L_\alpha}$ then $\overline{L_\alpha}$ is isomorphic to an irreducible subgroups of $\mathrm{GL}_r(p)$ for $r \leqslant 5$ which is p-solvable, contains a strongly p-embedded subgroup and has some quotient isomorphic to $\mathrm{SL}_2(p)$. We deduce, using MAGMA, that $R_\alpha = Q_\alpha$, and a contradiction is provided by [DS85] since $V_{\alpha'} \leq Q_\beta$. Thus, U contains two non-trivial composition factors and $|U| \geqslant p^4$. By the restrictions on $\overline{L_\alpha}$, $\overline{L_\alpha}$ acts as $\mathrm{SL}_2(p)$ on factors of order p^2, and as $\mathrm{PSL}_2(3)$ or $13 : 3$ on factors of order p^3 where necessarily $p = 3$. In the latter cases, we may choose a p'-element r such that U splits as a direct sum of two $\overline{L_\alpha}$-modules, one of order p^2 and one of order p^3. Then, Q does not act quadratically on U and for U^1 the factor of order p^3, $[U, Q, Q] = [U^1, Q, Q] = V_\beta/Z_\alpha$, a contradiction for then $U^1 = U$ is irreducible. Thus, U has two non-trivial factors, both of order p^2 and, assuming that $R_\alpha \neq Q_\alpha$, it follows from Lemma 2.3.14 (ii) and Lemma 2.3.15 (iii) that $\overline{L_\alpha} \cong (3 \times 3) : 2$ or $(Q_8 \times Q_8) : 3$. Thus, whether C_β is elementary abelian or not, we have deduced the isomorphism type of $\overline{L_\alpha}$.

If $p = 2$, then by Lemma 2.3.14 there is P_α with $L_\alpha = P_\alpha R_\alpha$, $P_\alpha/Q_\alpha \cong \mathrm{Sym}(3)$ and neither V_β nor C_β normal in P_α. Note that if there is $\{1\} \neq Q \leq S$ with $Q \trianglelefteq P_\alpha$ and $Q \trianglelefteq L_\beta$, then $V_\beta < Z(Q) < C_\beta$ and since $Q_{\alpha+2}/Z_{\alpha+2}| \leqslant p^4$, we have a contradiction. Hence, (P_α, L_β, S) satisfies Hypothesis 5.2.1. Since we could have chosen G minimally, and as $|S| = 2^7$, we deduce that (P_α, L_β, S) is parabolic isomorphic to $\mathrm{Aut}(\mathrm{M}_{12})$. But then one can calculate, e.g. using MAGMA, that $|\mathrm{Aut}(Q_\alpha)|_3 = 3$, a contradiction. If $p = 3$, then there is $t \in L_\alpha \cap G_{\alpha,\beta}$ an involution with $[t, L_\alpha] \leq Q_\alpha$ and, since $\overline{L_\beta} \cong \mathrm{SL}_2(3)$, $[t, L_\beta] \leq Q_\beta$, a contradiction by Proposition 5.2.6 (v). $\square$

Lemma 5.4.44. *Suppose that $C_{V_\beta}(V_{\alpha'}) = V_\beta \cap Q_{\alpha'}$ and $b = 3$. Then C_β is elementary abelian.*

Proof. Suppose throughout that C_β is not elementary abelian. Notice that if $C_\beta \cap Q_{\alpha'} \leq C_{\alpha'}$, then as $V_\beta \not\leq Q_{\alpha'}$ and $C_\beta \cap C_{\alpha'}$ is elementary abelian, $C_\beta = V_\beta(C_\beta \cap C_{\alpha'})$ is elementary abelian. Additionally, if $\Omega(Z(C_\beta)) \cap Q_{\alpha'} \not\leq C_{\alpha'}$, then as $V_\beta \leq \Omega(Z(C_\beta)) \not\leq Q_{\alpha'}$, $C_\beta = \Omega(Z(C_\beta))(C_\beta \cap C_{\alpha'})$ is elementary abelian.

Thus, we may suppose that $C_\beta \cap Q_{\alpha'} \not\leq C_{\alpha'}$ and $\Omega(Z(C_\beta)) \cap Q_{\alpha'} \leq C_{\alpha'}$. Since $V_\beta(C_\beta \cap C_{\alpha'})$ has index p in C_β, arguing as in Lemma 5.4.43 we have that $\Omega(Z(C_\beta))$ has index p^2 in C_β and $[C_\beta, C_\beta] = \Phi(C_\beta) = Z_\beta$. By Lemma 5.4.42, we may assume that $p \in \{2, 3\}$.

Since $\Omega(Z(C_\beta)) \cap Q_{\alpha'} \leq C_\beta \cap C_{\alpha'}$, it follows that $\Omega(Z(C_\beta)) \cap \Omega(Z(C_{\alpha'}))$ has index at most p^4 in C_β. But $\Omega(Z(C_\beta)) \cap \Omega(Z(C_{\alpha'}))$ is centralized by $Q_{\alpha+2} = C_\beta C_{\alpha'}$ and so $\Omega(Z(C_\beta)) \cap \Omega(Z(C_{\alpha'})) = Z_{\alpha+2}$ has index at most p^6 in $Q_{\alpha+2}$. Note that $[Q_\beta \cap O^p(L_\beta), C_\beta] \leq [O^p(L_\beta), C_\beta] = V_\beta$ and since $C_{Q_\alpha}(V_\alpha^{(2)}) \leq C_\beta$ and $V_\beta \not\leq Z(V_\alpha^{(2)})$, we have that $O^p(L_\alpha)$ centralizes $C_{Q_\alpha}(V_\alpha^{(2)})/Z_\alpha$. Moreover, since $Z_\alpha = \Phi(Q_\alpha)$, applying the three subgroup lemma, we see that $O^p(L_\alpha)$ acts faithfully on $V_\alpha^{(2)}\Phi(Q_\alpha)/\Phi(Q_\alpha)$. As in Lemma 5.4.43, we know the suitable subgroups of $\mathrm{GL}_4(p)$ which contain strongly p-embedded subgroups and obtain contradictions in much the same way. Thus, we may as well assume that $V_\alpha^{(2)}\Phi(Q_\alpha)/\Phi(Q_\alpha)$ has order at least p^5. Since Q_α/Z_α has order at most p^6 and $\Phi(V_\alpha^{(2)}) = Z_\alpha$, we conclude that

$\Phi(Q_\alpha) = Z_\alpha$.

Let W_α be chosen minimally such that $W_\alpha \trianglelefteq L_\alpha$, $Z_\alpha < W_\alpha \leq V_\alpha^{(2)}$ and $O^p(L_\alpha)$ acts non-trivially on W_α/Z_α. Set $V := V_\alpha^{(2)}/Z_\alpha$. Then, for $R := C_{L_\alpha}(W_\alpha)$, $V := C_V(R) \times [V, R]$ by coprime action. Moreover, $V_\beta/Z_\alpha \not\leq [V, R]$, for otherwise $C_V(R)$ is centralized by $O^p(L_\alpha)$, a contradiction since $W_\alpha/Z_\alpha \leq C_V(R)$.

Suppose that $[Q_\beta, Q_\beta] \leq V_\beta$. Then $[[V, R] \cap (Q_\beta/Z_\alpha), Q_\beta] = \{1\}$ and either $V_\beta/Z_\alpha \leq C_V(R)$ or $[C_V(R) \cap (Q_\beta/Z_\alpha), Q_\beta] = \{1\}$. If both $C_V(R)$ and $[V, R]$ are FF-modules for $\overline{L_\alpha}$ then applying Lemma 2.3.14 (ii) and Lemma 2.3.15 (ii), we get that $\overline{L_\alpha} \cong (3 \times 3) : 2$ or $(Q_8 \times Q_8) : 3$. As in Lemma 5.4.43, using generation properties of V when $p = 2$ and Proposition 5.2.6 (v) when $p = 3$ yield contradictions. Thus, we may assume that $V_\beta/Z_\alpha \leq C_V(R)$ and since $V_\alpha^{(2)} = \langle V_\beta^{L_\alpha} \rangle$, $C_V(R) = V$ admits $\overline{L_\alpha}$ faithfully and R is p-group. We may as well assume that $\overline{L_\alpha}$ acts irreducibly on W_α/Z_α. We appeal to MAGMA to see that if $\overline{L_\alpha}$ is isomorphic to some irreducible subgroups of $\mathrm{GL}_r(p)$ for $r \leqslant 5$ which is p-solvable, contains a strongly p-embedded subgroup and has some quotient isomorphic to $\mathrm{SL}_2(p)$, then $R_\alpha = Q_\alpha$, G has a weak BN-pair of rank 2 and [DS85] implies that C_β is elementary abelian, a contradiction. Thus, we may as well assume that $W_\alpha = Q_\alpha = V_\alpha^{(2)}$ and V is an irreducible module of order p^6.

Now, if $[Q_\beta, Q_\beta] \not\leq V_\beta$ then using that both $(Q_\alpha \cap Q_\beta)/V_\beta$ and $(Q_{\alpha+2} \cap Q_\beta)/V_\beta$ are elementary abelian of index p in Q_β/V_β, we deduce that $C_\beta/V_\beta = Z(Q_\beta/V_\beta)$. Now, for any $x \in Q_\beta/V_\beta \setminus (Q_\alpha \cap Q_\beta)/V_\beta$, we have that $Z(Q_\beta/V_\beta)$ is the kernel of the homomorphism $\theta : (Q_\alpha \cap Q_\beta)/V_\beta \to (Q_\alpha \cap Q_\beta)/V_\beta$ such that $v\theta = [v, x]$ for $v \in (Q_\alpha \cap Q_\beta)/V_\beta$. Then, $[Q_\beta, Q_\beta]V_\beta/V_\beta$ is the image of θ and has order p. Similarly, since $(Q_\alpha \cap Q_\beta)/Z_\alpha$ is an abelian subgroup of index p in Q_β/Z_α, we conclude that $[Q_\beta, Q_\beta]Z_\alpha/Z_\alpha \cong ((Q_\alpha \cap Q_\beta)/Z_\alpha)/Z(Q_\beta/Z_\alpha)$ has order at most p^2, and $Z(Q_\beta/Z_\alpha) \leq C_\beta/Z_\alpha$.

If $C_\beta = Z(Q_\beta/Z_\alpha)$, then $|[Q_\beta, Q_\beta]Z_\alpha/Z_\alpha| = p$ and observing that $[C_\beta, Q_\beta] \trianglelefteq L_\beta$, we have that $[C_\beta, Q_\beta] = Z_\beta$. By the three subgroup lemma, $[Q_\beta, Q_\beta] \leq Z(C_\beta) \leq C_{\alpha'}$ so that $[Q_\beta, Q_\beta]V_{\alpha'} \trianglelefteq L_\beta$. But then, either $[Q_\beta, Q_\beta]$ is centralized by $Q_{\alpha+2} = C_{\alpha'}C_\beta$ so that $[Q_\beta, Q_\beta] \leq Z_{\alpha+2}$, a contradiction; or $Z_{\alpha'} \leq [Q_\beta, Q_\beta]$ so that $V_\beta \leq [Q_\beta, Q_\beta]$. Since $|[Q_\beta, Q_\beta]Z_\alpha/Z_\alpha| = p$ and $[Q_\beta, Q_\beta] \not\leq V_\beta$, we have another contradiction.

Thus, $Z(Q_\beta/Z_\alpha) < C_\beta/Z_\alpha < (Q_\alpha \cap Q_\beta)/Z_\alpha$ and V_β index p in $[Q_\beta, Q_\beta]$. Now, $Q_\beta/[Q_\beta, Q_\beta]$ splits by coprime action and we may set $Q \leq (Q_\beta \cap O^2(L_\beta))[Q_\beta, Q_\beta]$ such that $Q/[Q_\beta, Q_\beta]$ is elementary abelian of order p^2 and $Q_\beta = QC_\beta$. Then Q/V_β is non-abelian of order p^3 and $(Q \cap Q_\alpha)/V_\beta$ is an elementary abelian subgroup of order p^2. Moreover, $\overline{L_\beta} \cong \mathrm{SL}_2(p)$ acts faithfully on Q/V_β and so we may assume that $p = 3$ and $Q/V_\beta \cong 3^{1+2}_+$. But now, $|Q/Z_\alpha| = 3^4$, $|[Q/Z_\alpha, Q/Z_\alpha]| = 9$, $Z(Q/Z_\alpha) = V_\beta/Z_\alpha$ is of order 3, $(Q/Z_\alpha)/Z(Q/Z_\alpha) \cong 3^{1+2}_+$ and $m_3(Q/Z_\alpha) = 3$. One can check that the only group satisfying these properties is $3 \wr 3$. But then, every normal subgroup of Q/Z_α contained in $[Q/Z_\alpha, Q/Z_\alpha]$ contains $Z(Q/Z_\alpha) = V_\beta/Z_\alpha$.

Now, $[[V, R], Q/Z_\alpha, Q/Z_\alpha] \trianglelefteq Q/Z_\alpha$ and since $V_\beta/Z_\alpha \not\leq [V, R]$ we have that $[[V, R], Q/Z_\alpha] \leq Z(Q/Z_\alpha) = V_\beta/Z_\alpha$. Finally, this implies that $[[V, R], Q] = \{1\}$ and since $Q \not\leq Q_\alpha$, it follows that $O^p(L_\alpha)$ centralizes $[V, R]$, R centralizes V and $\overline{L_\alpha}$ acts faithfully on W_α/Z_α. Again, we may as well assume that $W_\alpha = Q_\alpha = V_\alpha^{(2)}$ and V is an irreducible module of order p^6.

We appeal to MAGMA for a list of solvable irreducible subgroups of $\mathrm{GL}_6(p)$ for $p \in \{2, 3\}$. We investigate groups H such that for $P \in \mathrm{Syl}_p(H)$, $|P| = p$ and $H = \langle P^H \rangle$. Moreover, H contains a normal p'-subgroup N with $H/N \cong \mathrm{SL}_2(p)$. Notice that a Hall p'-subgroup of the preimage of $Z(\overline{L_\alpha})$ lies in $G_{\alpha,\beta}$ and so acts on

$\overline{L_\beta} \cong \mathrm{SL}_2(p)$. In particular, it follows by Proposition 5.2.6 (v) that $Z(\overline{L_\alpha}) = \{1\}$ if $p = 2$; and $|Z(\overline{L_\alpha})| \leqslant 2$ if $p = 3$. Imposing these conditions on the candidate subgroup $H \leq \mathrm{GL}_6(p)$, we reduce to three possibilities when $p = 2$, and four possibilities when $p = 3$. Suppose that $p = 2$. Then the candidates for H are $\{\mathrm{Dih}(18), 3^{1+2}_+ : \mathrm{Sym}(3), 7^2 : \mathrm{Sym}(3)\}$. If $\overline{L_\alpha} \cong \mathrm{Dih}(18)$, then we appeal to [Hay92] to obtain a contradiction. If $\overline{L_\alpha} \cong 3^{1+2}_+ : \mathrm{Sym}(3)$ then $\overline{S}$ is isomorphic to a Sylow 3-subgroup of $\mathrm{Sym}(9)$ and we identify a subgroup $P_\alpha \leq L_\alpha$ such that $\overline{P_\alpha}$ is isomorphic to $\mathrm{Dih}(18)$. Since $\mathrm{GL}_5(2)$ does not have any elements of order 9, this group acts irreducibly on Q_α/Z_α. If $\overline{L_\alpha} \cong 7^2 : \mathrm{Sym}(3)$, then we define P_α to be the preimage in L_α of $\overline{S}O_3(\overline{L_\alpha})$ so that $\overline{P_\alpha} \cong \mathrm{Sym}(3)$. Then, in either case, $\overline{P_\alpha}$ acts faithfully on Z_α. Forming $X := \langle L_\beta, P_\alpha \rangle$ and assuming that G is a minimal counterexample to Theorem 5.2.2, since $|S| = 2^9$ and all suitable examples in Theorem 5.2.2 have $|S| \leqslant 2^7$, some subgroup of $G_{\alpha,\beta}$ is normal in X. Indeed, since L_β is of characteristic p, some subgroup of S is normal in X. Call this subgroup Q and observe that as $Q \trianglelefteq S$, $Z_\beta \leq Q \leq Q_\alpha \cap Q_\beta$. Indeed, by the choice of P_α, $V_\beta \leq Q \leq C_\beta$. If $\Phi(Q) \neq \{1\}$, then as $\Phi(Q) \trianglelefteq S$, $Z_\beta \leq \Phi(Q)$ so that $V_\beta \leq \Phi(Q) \leq Q \leq C_\beta$, a contradiction for then $O^2(L_\beta)$ acts trivially on $V_\beta \leq Q$. Thus, $\Phi(Q) = \{1\}$ and Q is elementary abelian.

When $\overline{P_\alpha} \cong \mathrm{Dih}(18)$, taking consecutive closures of Z_β under P_α and G_β gives $Q_\alpha \leq Q$, a clear contradiction. Thus, we may assume that $\overline{L_\alpha} \cong 7^2 : \mathrm{Sym}(3)$ and $\overline{P_\alpha} \cong \mathrm{Sym}(3)$, and we have that $V_\beta < \langle Z_\beta^X \rangle$ and X/Q satisfies Hypothesis 5.2.1. Moreover, in this case the 3-element in P_α acts fixed point freely on Q_α/Z_α. Since $|S/Q| \leqslant 2^5$, Q_α/Q is elementary abelian and $J(S) \not\leq Q$, we have by Theorem 5.2.28 that $X/C_X(\langle Z_\beta^X \rangle)$ is locally isomorphic to $\mathrm{PSL}_3(2)$ or $\mathrm{Sp}_4(2)$ and $Q = C_S(\langle Z_\beta^X \rangle)$. If $X/C_X(\langle Z_\beta^X \rangle)$ is locally isomorphic to $\mathrm{PSL}_3(2)$, then $|S/Q| = 8$ and as $Q \leq C_\beta$, we have that $Q = C_\beta$, a contradiction since C_β is not elementary abelian. Thus,

$X/C_X(\langle Z_\beta^X \rangle)$ is locally isomorphic to $\mathrm{Sp}_4(2)$ and using that $C_\beta \neq \langle Z_\beta^X \rangle$, applying [CD91, Theorem A] we must have that $|S/Q| = 2^4$, $|Q| = 2^5$, $X/C_X(\langle Z_\beta^X \rangle) \cong \mathrm{Sp}_4(2)$ and $\langle Z_\beta^X \rangle$ is a natural $\mathrm{Sp}_4(2)$-module. But then $|Q/\langle Z_\beta^X \rangle| = 2$ so that X centralizes $Q/\langle Z_\beta^X \rangle$, a contradiction since a 3-element of P_α acts fixed point freely on Q_α/Z_α.

Suppose that $p = 3$. We briefly describe the four candidates. First, there is a group of shape $(Q_8 \times 2^2) : 3$ which occurs as a product of $\mathrm{SL}_2(3)$ and $\mathrm{PSL}_2(3)$ with their Sylow 3-subgroup identified, which we refer to as H_1. Next, there is a group of shape $2^2.\mathrm{SL}_2(3)$ where the extension is non-split, which we refer to as H_2. Then, there is a group of shape $(Q_8 \times 13) : 3$ which occurs as a product of $\mathrm{SL}_2(3)$ and the Frobenius group $13 : 3$ with their Sylow 3-subgroups identified, which we refer to as H_3. Finally, we have a group $2^{1+2+2}.\mathrm{SL}_2(3)$ where the extension is non-split. Indeed, the center of the Sylow 2-subgroup in this case has order 2^3 and the quotient by this center is isomorphic to H_2. We refer to this group as H_4.

Suppose that $\overline{L_\alpha}$ is isomorphic to H_1 or H_3. In the latter case, we have that $\overline{S}O_2(\overline{L_\alpha}) \cong \mathrm{SL}_2(3)$, while in the former case, while in the former case, there are four subgroups isomorphic to $\mathrm{SL}_2(3)$. Letting $\overline{L_\alpha}$ be isomorphic to H_1 and $t_\beta \in L_\beta \cap G_{\alpha,\beta}$ be an involution, we infer that t_β inverts S/Q_α and centralizes $Z(\overline{L_\alpha})$. Then $O_2(\overline{L_\alpha}/Z(\overline{L_\alpha}))$ splits as a direct sum of two non-trivial modules for $t_\beta\overline{S} \cong \mathrm{Sym}(3)$. Then by [Gor07, (I.3.5.6)], there are three submodules of $O_2(\overline{L_\alpha}/Z(\overline{L_\alpha}))$, one of which corresponds to the image of $\overline{R_\alpha}$, while the others correspond to $G_{\alpha,\beta}$-invariant subgroups of $\overline{L_\alpha}$ isomorphic to Q_8. Thus, whether $\overline{L_\alpha}$ is isomorphic to H_1 or H_3, we have a $G_{\alpha,\beta}$-invariant subgroup of L_α, call it P_α, such that $O^3(P_\alpha)$ acts non-trivially on Z_α and $\overline{P_\alpha} \cong \mathrm{SL}_2(3)$.

In either scenario, form $X := \langle L_\beta(G_{\alpha,\beta} \cap P_\alpha), P_\alpha(G_{\alpha,\beta} \cap L_\beta) \rangle$. Assuming that G is a minimal counterexample to Theorem 5.2.2, if no non-trivial subgroup of $G_{\alpha,\beta} \cap X$ is normal in X, then X is described in Theorem 5.2.2. Since no configurations described there have $|S| = 3^9$ and satisfy the requirements, we have a contradiction. Thus, some subgroup of $G_{\alpha,\beta}$ is normal in X. Indeed, we may as well suppose that a non-trivial subgroup of S is normal in X, calling this group Q. By the choice of P_α, we have that $V_\beta < \langle Z_\beta^X \rangle \leq Q \leq C_\beta$ and X/Q satisfies Hypothesis 5.2.1. Then, Theorem 5.2.28 implies that $X/C_X(\langle Z_\beta^X \rangle)$ is locally isomorphic to $\mathrm{SL}_3(3)$. By [CD91, Theorem A], and since $V_\beta < \langle Z_\beta^X \rangle$, it follows that $Q = \langle Z_\beta^X \rangle$ is a direct sum of two natural $\mathrm{SL}_3(3)$ modules and $Q = C_\beta$ is elementary abelian, a contradiction.

If $\overline{L_\alpha}$ is isomorphic to H_2 or H_4 then set $\overline{P_\alpha}$ to be the subgroup generated by the unique normal subgroup of $\overline{L_\alpha}$ of order 4 and $\overline{S}$. Then $\overline{P_\alpha} \cong \mathrm{PSL}_2(3)$ and P_α is normalized by $G_{\alpha,\beta}$. Moreover, $P_\alpha \leq R_\alpha S$. Setting $X := \langle P_\alpha(G_{\alpha,\beta} \cap L_\beta), L_\beta \rangle$, and writing Q for the largest subgroup of S which is normal in X, we have that $Q \leq C_\beta$ and both P_α/Q and L_β/Q are of characteristic 3. In particular, if G is a minimal counterexample to Theorem 5.2.2, then by minimality, X/Q is locally isomorphic to $\mathrm{PSp}_4(3)$ and $|Q| = 3^5$. If $Z_\alpha \leq Q$, then $\langle Z_\alpha^X \rangle$ is a non-trivial module for X/Q and since Z_β is centralized by X, $Q = \langle Z_\alpha^X \rangle$. But $Q < C_\beta$ and $[C_\beta, Q] \leq [C_\beta, C_\beta] = Z_\beta$ from which it follows that $O^3(L_\beta)$ centralizes Q, a contradiction. Thus, $Z_\alpha \not\leq Q$ and is follows that $Q \cap V_\beta = Z_\beta$. But then, Q_β/Q contains two non-central chief factors for L_β, a final contradiction. $\square$

Proposition 5.4.45. *Suppose that $C_{V_\beta}(V_{\alpha'}) = V_\beta \cap Q_{\alpha'}$ and $b = 3$. Then $p = 2$ and G is parabolic isomorphic to* M_{12} *or* $\mathrm{Aut}(\mathrm{M}_{12})$*.*

Proof. Suppose first that that G has a weak BN-pair of rank 2. Then by [DS85], $p = 2$ and G is parabolic isomorphic to M_{12} or $\mathrm{Aut}(\mathrm{M}_{12})$. Since $L_\alpha/R_\alpha \cong L_\beta/R_\beta \cong$

$\mathrm{SL}_2(p)$, to prove the proposition it suffices to prove that $R_\beta = Q_\beta$ and $R_\alpha = Q_\alpha$. We assume throughout that (α, α') is a critical pair with $V_{\alpha'} \not\leq Q_\beta$, $Z_{\alpha'} \neq Z_\beta$. Moreover, $R \leq Z_{\alpha+2}$, $|V_\beta| = p^3$, C_β is elementary abelian and has index p^2 in both $Q_{\alpha+2}$ and Q_β, and $O^p(L_\beta)$ centralizes C_β/V_β.

Suppose first that there is $\lambda, \mu \in \Delta(\alpha + 2)$ with $Q_{\alpha+2} = C_\lambda C_\mu$. Then $Z_{\alpha+2} = \Omega(Z(Q_{\alpha+2})) = C_\lambda \cap C_\mu$ has index p^4 in $Q_{\alpha+2}$.

Suppose now that $C_{\alpha'}C_\beta$ has index p in $Q_{\alpha+2}$. If $C_{\alpha'}C_\beta \trianglelefteq L_{\alpha+2}$ then $O^p(L_{\alpha+2}) \cap Q_{\alpha+2} \leq C_{\alpha'}C_\beta$. Moreover, $V_{\alpha+2}^{(2)} \leq C_{\alpha'}C_\beta$ so that $C_{\alpha'} \cap C_\beta = \Omega(Z(C_{\alpha'}C_\beta))$ is normal in $L_{\alpha+2}$ and centralizes $V_{\alpha+2}^{(2)}$. In particular, $C_{\alpha'}C_\beta = V_{\alpha+2}^{(2)}\Omega(Z(C_{\alpha'}C_\beta))$. By coprime action $\Omega(Z(C_{\alpha'}C_\beta)) = [\Omega(Z(C_{\alpha'}C_\beta)), O^p(R_{\alpha+2})] \times C_{\Omega(Z(C_{\alpha'}C_\beta))}(O^p(R_{\alpha+2}))$ and since $Z_\beta \leq C_{\Omega(Z(C_{\alpha'}C_\beta))}(O^p(R_{\alpha+2}))$, it follows that $[\Omega(Z(C_{\alpha'}C_\beta)), O^p(R_{\alpha+2})] = \{1\}$. Now, for any p'-element $r \in O^p(L_{\alpha+2})$, if $[r, V_{\alpha+2}^{(2)}] \leq \Omega(Z(C_{\alpha'}C_\beta))$, then $[r, V_{\alpha+2}^{(2)}, V_{\alpha+2}^{(2)}] = \{1\}$. By the three subgroup lemma, such an r centralizes $[V_{\alpha+2}^{(2)}, V_{\alpha+2}^{(2)}] = Z_\alpha$ so that $r \in O^p(R_{\alpha+2})$. But then r centralizes $C_{\alpha'}C_\beta = V_{\alpha+2}^{(2)}\Omega(Z(C_{\alpha'}C_\beta))$ and so $r = 1$. Thus, every p'-element acts faithfully on $V_\alpha^{(2)}/(V_\alpha^{(2)} \cap \Omega(Z(C_{\alpha'}C_\beta)))$ which has order p^2. Since $L_{\alpha+2}/R_{\alpha+2} \cong \mathrm{SL}_2(p)$ and by conjugacy, $R_\alpha = Q_\alpha$, as required.

Thus, we may assume that $C_{\alpha'}C_\beta \ntrianglelefteq L_{\alpha+2}$ and so there is $\mu \in \Delta(\alpha + 2)$ such that $Q_{\alpha+2} = C_{\alpha'}C_\beta C_\mu$. If $Q_{\alpha+2} = C_\mu C_\beta$, then $C_\mu \cap C_\beta = Z_{\alpha+2}$ has index p^4 in $Q_{\alpha+2}$ and $|C_\beta/V_\beta| = p$. We get a similar result if $Q_{\alpha+2} = C_\mu C_{\alpha'}$. Thus, we may assume that $C_{\alpha'} \cap C_\beta \cap C_\mu = Z_{\alpha+2}$ has index p^2 in C_β and so, again, $Z_{\alpha+2}$ has index p^4 in $Q_{\alpha+2}$.

Thus, we have reduced to the case where $|C_\beta/V_\beta| = p$, $|Q_\beta/Z_\beta| = p^5$ and $|Q_\alpha/Z_\alpha| = p^4$. By Lemma 5.4.42, we may assume that $p \in \{2, 3\}$ and we may as well assume that $\Phi(Q_\alpha) = Z_\alpha$. Then $\overline{L_\alpha}$ is isomorphic to a subgroup of $\mathrm{GL}_4(p)$ which has a strongly p-embedded subgroup and some quotient isomorphic to $\mathrm{SL}_2(p)$. It follows that $R_\alpha = Q_\alpha$, or $\overline{L_\alpha} \cong (3 \times 3) : 2$ or $(Q_8 \times Q_8) : 3$. If $p = 2$, then by Lemma 2.3.15, there is $P_\alpha \leq L_\alpha$ such that $P_\alpha/Q_\alpha \cong \mathrm{Sym}(3)$, $L_\alpha = P_\alpha R_\alpha$ and we may choose P_α such that neither V_β nor C_β are normal in P_α. It follows that no subgroup of S is normal in both P_α and L_β so that (P_α, L_β, S) satisfies Hypothesis 5.2.1. Since we could have chosen G minimally, and as $|S| = 2^7$, we deduce that (P_α, L_β, S) is parabolic isomorphic to $\mathrm{Aut}(\mathrm{M}_{12})$. But then one can calculate, e.g. using MAGMA, that $|\mathrm{Aut}(Q_\alpha)|_3 = 3$, a contradiction. If $p = 3$, then $Z(\overline{L_\alpha})$ is elementary abelian of order 4 and since $\overline{L_\beta} \cong \mathrm{SL}_2(3)$, it follows that there is $t \in G_{\alpha,\beta}$ such that $[t, L_\beta] \leq Q_\beta$ and $tQ_\alpha \leq Z(\overline{L_\alpha})$, a contradiction by Proposition 5.2.6 (v). □

5.4.3 $b = 1$

From this point on, restating Lemma 5.4.1, we may assume the following:

- $b = 1$ so that $Z_\alpha \not\leq Q_\beta$;
- $\Omega(Z(S)) = Z_\beta = \Omega(Z(L_\beta))$; and
- $Z(L_\alpha) = \{1\}$.

Proposition 5.4.46. *Suppose that $p \geqslant 5$. Then $\overline{L_\beta} \cong \mathrm{SL}_2(q)$ or $\mathrm{(P)SU}_3(q)$.*

Proof. Since $[Q_\beta, Z_\alpha, Z_\alpha] = \{1\}$ the result follows immediately from Lemma 2.3.5. □

Proposition 5.4.47. *Suppose that $p \geqslant 5$. Then G has a weak BN-pair of rank 2 and is locally isomorphic to H where $F^*(H) = \mathrm{PSp}_4(p^n)$, $\mathrm{PSU}_4(p^n)$ or $\mathrm{PSU}_5(p^n)$.*

Proof. Let K_β be a critical subgroup of Q_β. By Theorem 2.1.26, $O^p(L_\beta)$ acts faithfully on $K_\beta/\Phi(K_\beta)$. Assume that $K_\beta \leq Q_\alpha$. Since $\overline{L_\beta} \cong \mathrm{SL}_2(q)$ or $\mathrm{(P)SU}_3(q)$, we have that $[K_\beta, O^p(L_\beta)] \leq [K_\beta, \langle Z_\alpha^{L_\beta}\rangle] = \{1\}$, a contradiction. Hence, $K_\beta \not\leq Q_\alpha$, $[Q_\alpha, K_\beta, K_\beta, K_\beta] = \{1\}$ and K_β acts cubically on Q_α.

Since $Q_\alpha/\Phi(Q_\alpha$ is a faithful $\overline{L_\alpha}$-module which admits cubic action, we may apply Corollary 2.3.24 so that $\overline{L_\alpha} \cong \mathrm{(P)SL}_2(q)$ or $\mathrm{(P)SU}_3(q)$, or $p = 5$ and $\overline{L_\alpha} \cong 3\cdot\mathrm{Alt}(6)$ or $3\cdot\mathrm{Alt}(7)$ and for W some irreducible constituent of $Q_\alpha/\Phi(Q_\alpha)$, $|W| \geqslant 5^6$. If $\overline{L_\alpha} \cong \mathrm{(P)SL}_2(q)$ or $\mathrm{(P)SU}_3(q)$ then G has a weak BN-pair of rank 2 and is determined in [DS85]. Therefore, G is locally isomorphic to H where $F^*(H) = \mathrm{PSp}_4(p^{n+1})$, $\mathrm{PSU}_4(p^n)$ or $\mathrm{PSU}_5(p^n)$ for $n \geqslant 1$. Thus it remains to check that $\overline{L_\alpha} \not\cong 3\cdot\mathrm{Alt}(6)$ or $3\cdot\mathrm{Alt}(7)$ and so have that $p = 5$ and $|S/Q_\alpha| = 5$. Since Q_β is not centralized by Z_α, else $Z_\alpha \leq \Omega(Z(S))$, $\overline{L_\beta} \cong \mathrm{SL}_2(5)$ and Q_β contains exactly one non-central chief factor for L_β, which is isomorphic to a natural $\mathrm{SL}_2(5)$-module. Since $Z(L_\alpha) = \{1\}$, Z_α contains a non-central chief factor for L_α and admits cubic action, Z_α is also a faithful $\overline{L_\alpha}$-module and $|Z_\alpha| \geqslant 5^6$, so that $R_\alpha = Q_\alpha$.

Suppose that $Z_\alpha \cap Q_\beta \leq Q_\lambda$ for all $\lambda \in \Delta(\beta)$. Since $L_\beta = \langle Z_\lambda, Q_\beta \mid \lambda \in \Delta(\beta)\rangle$, it follows that $Z_\alpha \cap Q_\beta$ is centralized by $O^p(L_\beta)$. Since $Q_\alpha \cap Q_\beta \not\trianglelefteq L_\beta$, $O^p(L_\beta) \cap Q_\beta \not\leq Q_\alpha$ and so $[Z_\alpha, Q_\beta, Q_\beta \cap O^p(L_\beta)] = \{1\}$ and Z_α is a quadratic module, a contradiction to Lemma 2.3.5. Thus, $Z_\alpha \cap Q_\beta \not\leq Q_{\alpha+2}$ for some $\alpha+2 \in \Delta(\beta)$ and $Z_\alpha \cap Q_\beta \cap Q_{\alpha+2}$ has index at most 25 in Z_α. If $Z_{\alpha+2} \cap Q_\beta \leq Q_\alpha$ then $[Z_{\alpha+2}, Z_\alpha, Z_\alpha] = \{1\}$ and so, $Z_\alpha \cap Q_\beta$ acts quadratically on $Z_{\alpha+2}$ and since $\alpha+2$ is conjugate to α, we have a contradiction. Thus, $Z_{\alpha+2} \cap Q_\beta \not\leq Q_\alpha$. But now, $\overline{L_\alpha}$ is generated by two conjugates of $(Z_{\alpha+2} \cap Q_\beta)Q_\alpha/Q_\alpha$, and as an index 25 subgroup of Z_α is centralized by $Z_{\alpha+2} \cap Q_\beta$ and $Z(L_\alpha) = \{1\}$, we have that $|Z_\alpha| \leqslant 5^4$, a contradiction $\square$

Given the above proposition, we suppose that $p \in \{2,3\}$ for the remainder of this subsection. We introduce some notation specific to the case where $b = 1$.

Notation 5.4.48.
- F_β is a normal subgroup of G_β which satisfies $[F_\beta, O^p(L_\beta)] \neq \{1\}$ and is minimal by inclusion with respect to adhering to these conditions.
- $W_\beta := \langle (Z_\alpha \cap Q_\beta)^{G_\beta} \rangle$.
- $D_\beta := C_{Q_\beta}(O^p(L_\beta))$.

Lemma 5.4.49. *The following hold:*

(i) $F_\beta \not\leq Q_\alpha$;

(ii) $F_\beta = [F_\beta, O^p(L_\beta)] \leq O^p(L_\beta)$; *and*

(iii) *for any p-subgroup* $U \trianglelefteq L_\alpha$ *with* $U \not\leq Q_\beta$, $[F_\beta, Q_\beta] \leq U$.

Proof. We have that $[F_\beta, O^p(L_\beta)] \leq O^p(L_\beta)$ and by coprime action $[F_\beta, O^p(L_\beta), O^p(L_\beta)] = [F_\beta, O^p(L_\beta)]$. By minimality of F_β, $F_\beta = [F_\beta, O^p(L_\beta)]$.

If $F_\beta \leq Q_\alpha$, then $[F_\beta, S]$ is strictly contained in F_β and normalized by $L_\beta = \langle Z_\alpha^{L_\beta} \rangle (G_{\alpha,\beta} \cap L_\beta)$ and, by minimality, $[F_\beta, S] \leq D_\beta$. But then $[F_\beta, L_\beta] \leq D_\beta$, a contradiction.

Let $H_\beta := \langle (U \cap F_\beta)^{G_\beta} \rangle \trianglelefteq G_\beta$. By minimality of F_β, either $H_\beta = F_\beta$ or $H_\beta \leq D_\beta$. Suppose the latter. Then $[F_\beta, U] \leq F_\beta \cap U \leq H_\beta \leq D_\beta$ so that $[F_\beta, \langle U^{G_\beta} \rangle] \leq D_\beta$. Now, $F_\beta = [F_\beta, O^p(L_\beta)] \leq [F_\beta, \langle U^{G_\beta} \rangle (G_{\alpha,\beta} \cap L_\beta)] \leq D_\beta [F_\beta, G_{\alpha,\beta} \cap L_\beta]$. Then, by minimality of F_β, $F_\beta / F_\beta \cap D_\beta$ is an irreducible $\overline{G_{\alpha,\beta} \cap L_\beta}$ so that $[S, F_\beta] \leq D_\beta$. As above, this implies that $[F_\beta, L_\beta] \leq D_\beta$, a contradiction. Thus, $H_\beta = F_\beta$. Now,

$[U \cap F_\beta, Q_\beta] \leq [F_\beta, Q_\beta] \leq D_\beta$ and so $[U \cap F_\beta, Q_\beta] \trianglelefteq G_\beta$. But then $[U \cap F_\beta, Q_\beta] = [U \cap F_\beta, Q_\beta]^{G_\beta} = [\langle (U \cap F_\beta)^{G_\beta}, Q_\beta] = [F_\beta, Q_\beta]$ and $U \geq [U \cap F_\beta, Q_\beta] = [F_\beta, Q_\beta]$, completing the proof. $\square$

Lemma 5.4.50. *Suppose that $m_p(S/Q_\alpha) = 1$. Then $p = 3$, $\overline{L_\beta} \cong \mathrm{SL}_2(3)$, Z_α is an irreducible 2F-module for $\overline{L_\alpha}$ and Q_α is elementary abelian.*

Proof. Assume that $m_p(S/Q_\alpha) = 1$. Since W_β is generated by elements of order p and $m_p(S/Q_\alpha) = 1$, $|W_\beta Q_\alpha/Q_\alpha| = p$ and Z_α centralizes an index p subgroup of W_β. Since $[Z_\alpha, Q_\beta] \leq W_\beta$, W_β contains all non-central chief factors for L_β in Q_β and so, $W_\beta/C_{W_\beta}(O^p(L_\beta))$ is the unique non-central chief factor for L_β inside Q_β. Moreover, $W_\beta/C_{W_\beta}(O^p(L_\beta))$ is a natural $\mathrm{SL}_2(p)$-module for $\overline{L_\beta} \cong \mathrm{SL}_2(p)$ and $L_\beta = \langle Q_\alpha, Q_\beta, Z_{\alpha+2}\rangle$ for some $\alpha+2 \in \Delta(\beta)$. Then $Z_\alpha \cap Q_\beta \leq (Z_\alpha \cap W_\beta)(Z_{\alpha+2} \cap W_\beta) \trianglelefteq L_\beta$ and so $W_\beta = (Z_\alpha \cap W_\beta)(Z_{\alpha+2} \cap W_\beta)$.

Suppose first that W_β is abelian. Then, as $Z_\alpha \cap Q_\beta \leq W_\beta$, an index p subgroup of Z_α is centralized by W_β and Z_α is a natural $\mathrm{SL}_2(p)$-module. But then $Z_\alpha \cap Q_\beta = Z_\beta$ and $W_\beta = Z_\beta$, a contradiction.

Since W_β is non-abelian, and $W_\beta \cap Q_\alpha \cap Q_{\alpha+2}$ has index p^2 in W_β, $W_\beta \cap Q_\alpha \cap Q_{\alpha+2} = \Omega(Z(W_\beta)) = C_{W_\beta}(O^p(L_\beta))$. Notice that every element of W_β lies in $(Z_\lambda \cap W_\beta)\Omega(Z(W_\beta))$ for some $\lambda \in \Delta(\beta)$, and that $(Z_\lambda \cap W_\beta)\Omega(Z(W_\beta))$ is of exponent p, from which it follows that W_β is of exponent p. In particular, since W_β is not elementary abelian, $p \neq 2$. Therefore, $\Omega(Z(W_\beta))$ has index 9 in Z_α, Z_α is 2F-module and since $[Z_\alpha, W_\beta] \not\leq \Omega(Z(W_\beta))$ and S/Q_α has a unique element of order 3, Z_α does not admit quadratic action by any element $x \in S \setminus Q_\alpha$.

Now, by minimality of F_β, $\Phi(F_\beta) \leq Q_\alpha$ so that $F_\beta(Q_\alpha \cap Q_\beta) = W_\beta(Q_\alpha \cap Q_\beta)$ since S/Q_α has a unique subgroup of order p. Then $[F_\beta, Z_\alpha] = [W_\beta, Z_\alpha]$. Moreover,

$F_\beta = [F_\beta, O^p(L_\beta)] \leq [F_\beta, Z_\alpha]^{L_\beta} \leq W_\beta$ and since F_β contains a non-central chief factor, $W_\beta = F_\beta Z(W_\beta)$. Then, since $[F_\beta, Q_\alpha] = [F_\beta, Z_\alpha(Q_\alpha \cap Q_\beta)] \leq Z_\alpha$ by Lemma 5.4.49, it follows that $O^3(L_\alpha)$ centralizes Q_α/Z_α. In particular, every p'-element of L_α acts non-trivially on Z_α.

Let $U < Z_\alpha$ be a non-trivial subgroup of Z_α which is normal in L_α. If $C_S(U) \not\leq Q_\alpha$, then $O^3(L_\alpha)$ centralizes U and as $U \trianglelefteq S$, $U \cap Z_\beta \neq \{1\}$ and $Z(L_\alpha) \neq \{1\}$, a contradiction. If $U \not\leq Q_\beta$, then $Z_\alpha = U(Z_\alpha \cap Q_\beta)$ and by Lemma 5.4.49, it follows that $[F_\beta, Z_\alpha] \leq U$ so that $[O^3(L_\alpha), Z_\alpha] \leq U$ and $C_{Z_\alpha}(O^3(L_\alpha)) \neq \{1\}$ by Lemma 2.3.2. But then $Z(L_\alpha) \geq Z_\beta \cap C_{Z_\alpha}(O^3(L_\alpha)) \neq \{1\}$, a contradiction. Thus, $U \leq Q_\beta$ and as Z_α is 2F, we may assume that both Z_α/U and U are FF-modules for $\overline{L_\alpha}$ and by Lemma 2.3.15 (ii), either $\overline{L_\alpha} \cong \mathrm{SL}_2(3)$ or $(Q_8 \times Q_8) : 3$. If $\overline{L_\alpha} \cong \mathrm{SL}_2(3)$, then G has a weak BN-pair of rank 2 and by [DS85], we have a contradiction. If $\overline{L_\alpha} \cong (Q_8 \times Q_8) : 3$, since $|\mathrm{Out}(\overline{L_\beta})| = 2$ and a Hall $3'$-subgroup of $L_\alpha \cap G_{\alpha,\beta}$ is isomorphic to an elementary abelian group of order 4, it follows that that there is an involution $t \in G_{\alpha,\beta}$ such that $[L_\alpha, t] \leq Q_\alpha$ and $[L_\beta, t] \leq Q_\beta$, a contradiction by Proposition 5.2.6 (v).

Thus, we may now assume that Z_α is an irreducible 2F-module. Since Z_α is irreducible and $Z_\alpha \not\leq \Phi(Q_\alpha)$, we have that $Z_\alpha \cap \Phi(Q_\alpha) = Z_\beta \cap \Phi(Q_\alpha) = \{1\}$ so that $\Phi(Q_\alpha) = \{1\}$ and Q_α is elementary abelian. $\square$

Proposition 5.4.51. *Suppose that $m_p(S/Q_\alpha) = 1$ and $p \in \{2,3\}$. Then $p = 3$, $Z_\alpha = Q_\alpha$ is an irreducible $\mathrm{GF}(3)\overline{L_\alpha}$-module and one of the following holds:*

(i) *G has a weak BN-pair of rank 2 and G is locally isomorphic to H where $F^*(H) \cong \mathrm{PSp}_4(3)$;*

(ii) *$|S| = 3^5$, $\overline{L_\alpha} \cong \mathrm{Alt}(5)$, Z_α is the restriction of the permutation module,*

$\overline{L_\beta} \cong \mathrm{SL}_2(3)$ *and* $Q_\beta \cong 3 \times 3^{1+2}_+$;

(iii) $|S| = 3^5$, $\overline{L_\alpha} \cong O^{3'}(2 \wr \mathrm{Sym}(4))$, Z_α *is a reflection module,* $\overline{L_\beta} \cong \mathrm{SL}_2(3)$ *and* $Q_\beta \cong 3 \times 3^{1+2}_+$*; or*

(iv) $|S| = 3^6$, $\overline{L_\alpha} \cong O^{3'}(2 \wr \mathrm{Sym}(5))$, Z_α *is a reflection module,* $\overline{L_\beta} \cong \mathrm{SL}_2(3)$ *and* $Q_\beta \cong 3 \times 3 \times 3^{1+2}_+$.

Consequently, if G is a completion of an amalgam determined by a fusion system $\mathcal{F}$ satisfying Hypothesis 5.1.12, then $\mathcal{F} = \mathcal{F}_S(H)$ where $H \cong \mathrm{PSp}_4(3)$, $\mathrm{Aut}(\mathrm{PSp}_4(3))$, $\mathrm{PSU}_5(2)$, $\mathrm{Aut}(\mathrm{PSU}_5(2))$, $\Omega_8^+(2)$, $\mathrm{O}_8^+(2)$,, $\Omega_{10}^-(2)$ or $\mathrm{Sp}_{10}(2)$.

Proof. By Lemma 5.4.50, Z_α is the unique non-central chief factor for L_α in Q_α and Q_α is elementary abelian. Moreover, $W_\beta/C_{W_\beta}(O^p(L_\beta))$ is the unique non-central chief factor for L_β inside Q_β, and is a natural $\mathrm{SL}_2(3)$-module for $\overline{L_\beta} \cong \mathrm{SL}_2(3)$.

Suppose first that $|Z_\alpha| = 3^3$. Then $\overline{L_\alpha}$ is isomorphic to a subgroup X of $\mathrm{GL}_3(3)$ which has a strongly 3-embedded subgroup. One can check that the only groups which satisfy $X = O^{3'}(X)$ are $\mathrm{PSL}_2(3)$, $\mathrm{SL}_2(3)$ and $13 : 3$. In the first two cases, G has a weak BN-pair of rank 2 and comparing with [DS85], we have that $\overline{L_\alpha} \cong \mathrm{PSL}_2(3)$ and G to locally isomorphic to H, where $F^*(H) \cong \mathrm{PSp}_4(3)$. Suppose that $\overline{L_\alpha} \cong 13 : 3$ and let $t_\beta \in L_\beta \cap G_{\alpha,\beta}$ be an involution. Then $t_\beta \in G_\alpha$ and writing $\overline{t_\beta} := t_\beta Q_\alpha/Q_\alpha$, $\overline{t_\beta}$ acts on $\overline{L_\alpha}$ and inverts $\overline{S} = Q_\beta Q_\alpha/Q_\alpha$, a contradiction since any involutary automorphism of $13 : 3$ centralizes a Sylow 3-subgroup.

Thus, we may assume that $|Z_\alpha| > 3^3$. Again, let $t_\beta \leq G_{\alpha,\beta} \cap L_\beta$ be an involution. Then, using coprime action, $[t_\beta, Q_\alpha] \leq W_\beta$ and $[t_\beta, C_{W_\beta}(O^3(L_\beta))] = \{1\}$. In particular, it follows that t_β centralizes an index 3 subgroup of Q_α. Let $L^* := \langle t_\beta^{G_\alpha} \rangle$ and $\overline{L^*} = L^* Q_\alpha/Q_\alpha \leq \overline{G_\alpha}$. Since $\overline{L^*} \trianglelefteq \overline{G_\alpha}$, we have that $[\overline{L^*}, \overline{L_\alpha}] \leq \overline{L^*}$. Note that t_β inverts $W_\beta Q_\alpha/Q_\alpha \cong W_\beta/W_\beta \cap Q_\alpha$ and so $W_\beta Q_\alpha/Q_\alpha = [W_\beta Q_\alpha/Q_\alpha, t_\beta] \leq [\overline{L_\alpha}, \overline{L^*}] \leq \overline{L^*}$. If $\overline{G_\alpha}$ is not 3-solvable, then $\overline{L_\alpha}/O_{3'}(\overline{L_\alpha})$ is a non-abelian finite simple group and since $\overline{L^*} \trianglelefteq \overline{G_\alpha}$, we have that $\overline{L_\alpha} \leq \overline{L^*}$.

If $\overline{G_\alpha}$ is 3-solvable, let O_α be the preimage of $O_{3'}(\overline{L_\alpha})$ in L_α. By coprime action, we have that $Q_\alpha = [Q_\alpha, O_\alpha] \times C_{Q_\alpha}(O_\alpha)$ is an S-invariant decomposition. Since Z_α is irreducible, we infer that $[Q_\alpha, O_\alpha] = [Z_\alpha, O_\alpha] = Z_\alpha$ and as $Z_\beta \leq Z_\alpha$, it follows that $C_{Q_\alpha}(O_\alpha) = \{1\}$ and $Q_\alpha = Z_\alpha$. If $|S/Q_\alpha| > 3$, then $W_\beta \leq \Phi(Q_\beta)(Z_\alpha \cap Q_\beta)$ and it follows from the Dedekind modular law that $W_\beta = \Phi(Q_\beta)(Z_\alpha \cap Q_\beta) \cap W_\beta = (Z_\alpha \cap Q_\beta)(\Phi(Q_\beta) \cap W_\beta)$. Since W_β contains all non-central chief factors for L_β inside Q_β, $\Phi(Q_\beta) \cap W_\beta \leq Z(Q_\beta$ so that $W_\beta = (Z_\alpha \cap Q_\beta)Z(W_\beta)$, a contradiction. Thus, $|S/Q_\alpha| = 3$ and, again, $\overline{L_\alpha} \leq \overline{L^*}$.

Since S/Q_α does not act quadratically on Z_α, $\overline{L^*}$ is not generated by transvections and as $|Z_\alpha| \geqslant 3^4$, we may apply the main result of [ZS81]. Using that S/Q_α is cyclic, we have that $\overline{L^*}$ is isomorphic to the reduction modulo 3 of a finite irreducible reflection group of degree n in characteristic 0, and $3^4 \leqslant |Z_\alpha| \leqslant 3^5$.

Suppose that there is $t_\alpha \in L^* \cap G_{\alpha,\beta}$ an element of order 4 with $t_\alpha^2 Q_\alpha \in Z(\overline{L^*})$. Then $t_\alpha \in G_\beta$ and t_α acts on $\overline{L_\beta}$. We may assume that t_α^2 acts non-trivially on $\overline{L_\beta}$ for otherwise $t_\alpha^2 Q_\alpha$ is centralized by $\overline{L_\alpha}$ and $t_\alpha^2 Q_\beta$ is centralized by $\overline{L_\beta}$, a contradiction by Proposition 5.2.6 (v). But t_α normalizes S/Q_β and so either t_α inverts S/Q_β or centralizes S/Q_β. In either case, t_α^2 centralizes S/Q_β and by Lemma 2.2.1 (viii), t_α^2 acts trivially on $\overline{L_\beta}$, a contradiction.

Upon comparing the groups listed in [ZS81] and the orders of $\mathrm{GL}_4(3)$ and $\mathrm{GL}_5(3)$ we are left with the groups $G(1,1,5)$, $G(2,1,4)$, $G(2,2,4)$, $G(2,1,5)$ and $G(2,2,5)$ (in the Todd-Shepherd enumeration convention) as candidates for $\overline{L^*}$. In particular, $|S/Q_\alpha| = 3$. If $\overline{L^*} \cong G(1,1,5) \cong \mathrm{Sym}(5)$, then $\overline{L_\alpha} \cong \mathrm{Alt}(5)$. Then G is determined in [JJS89] and outcome (ii) follows in this case. Thus, $O_2(\overline{L^*}) \neq \{1\}$ and writing O_α for the preimage of $O_2(\overline{L^*})$ in G_α, we have by coprime action that $Q_\alpha = [Q_\alpha, O_\alpha] \times C_{Q_\alpha}(O_\alpha)$ and since Z_α is irreducible and is the unique non-central chief factor within Q_α, $Q_\alpha = [Q_\alpha, O_\alpha] = Z_\alpha$. In particular, $W_\beta = Q_\beta$, $|Q_\beta| \leqslant 3^5$ and Q_β/Z_β is a natural $\mathrm{SL}_2(3)$-module for $\overline{L_\beta}$.

Now, $G(2,1,4) \cong 2 \wr \mathrm{Sym}(4)$ and $G(2,2,4)$ is isomorphic to an index 2 subgroup of $G(2,1,4)$. Therefore, if $|Z_\alpha| = 3^4$, $\overline{L_\alpha} \cong O^{3'}(2 \wr \mathrm{Sym}(4))$ and the possible actions of $\overline{L_\alpha}$ are determined up to conjugacy in $\mathrm{GL}_4(3)$. Indeed, it follows in this case that S is isomorphic to a Sylow 3-subgroup of $\mathrm{Alt}(12)$. Furthermore, Q_β has exponent 3, is of order 3^4 and $Z(Q_\beta) = Z_\beta$ is elementary abelian of order 9. Indeed, $Q_\beta \cong 3_+^{1+2} \times 3$.

Finally, $G(2,1,5) \cong 2 \wr \mathrm{Sym}(5)$ and $G(2,2,5)$ is isomorphic to an index 2 subgroup of $G(2,1,5)$. Therefore, if $|Z_\alpha| = 3^5$, $\overline{L_\alpha} \cong O^{3'}(2 \wr \mathrm{Sym}(5))$ and the possible actions of $\overline{L_\alpha}$ are determined up to conjugacy in $\mathrm{GL}_5(3)$. Indeed, it follows in this case that S is isomorphic to a Sylow 3-subgroup of $\mathrm{Alt}(15)$. Furthermore, Q_β has exponent 3, is of order 3^5 and $Z(Q_\beta) = Z_\beta$ is elementary abelian of order 27. Indeed, $Q_\beta \cong 3_+^{1+2} \times 3 \times 3$.

If G is obtained from a fusion system $\mathcal{F}$ then as $|S| \leqslant 3^6$, and $S \in \mathrm{Syl}_3(O^3(L_\alpha))$ or G has a weak BN-pair of rank 2, we may assume that $O^3(\mathcal{F}) = \mathcal{F}$ and use the results in [PS21] to completely determine $\mathcal{F}$. □

Lemma 5.4.52. *Suppose that $m_p(S/Q_\alpha) \geqslant 2$ and $m_p(S/Q_\beta) = \{1\}$. Then there is $\alpha + 2 \in \Delta(\beta)$ such that $Z_\alpha \cap Q_\beta \not\leq Q_{\alpha+2}$ and $Z_{\alpha+2} \cap Q_\beta \not\leq Q_\alpha$. Moreover, $|(Z_\alpha \cap Q_\beta)Q_{\alpha+2}/Q_{\alpha+2}| = |(Z_{\alpha+2} \cap Q_\beta)Q_\alpha/Q_\alpha|$.*

Proof. Suppose that $Z_\alpha \cap Q_\beta \leq Q_\lambda$ for all $\lambda \in \Delta(\beta)$. Then $Z_\alpha \cap Q_\beta$ is centralized by $\langle Z_\alpha^{G_\beta}\rangle$. In particular, as $G_\beta = \langle Z_\alpha^{G_\beta}\rangle G_{\alpha,\beta}$ by Lemma 5.2.8 (iii), $Z_\alpha \cap Q_\beta \trianglelefteq G_\beta$. But then, Z_α centralizes $Q_\beta/(Q_\beta \cap Z_\alpha$ and $Q_\beta \cap Z_\alpha$, impossible as $Z_\alpha \not\leq Q_\beta$. Thus, we may choose $\alpha+2 \in \Delta(\beta)$ such that $Z_\alpha \cap Q_\beta \not\leq Q_{\alpha+2}$. If $Z_{\alpha+2} \cap Q_\beta \leq Q_\alpha$, then an index p subgroup of $Z_{\alpha+2}$ is centralized by $Z_\alpha \cap Q_\beta \not\leq Q_{\alpha+2}$ and as $\alpha+2$ is conjugate to α and $m_p(S/Q_\alpha) > 1$, by Lemma 2.3.10 we have a contradiction.

Observe that $Z_\alpha \cap Q_\beta \cap Q_{\alpha+2} \leq C_{Z_\alpha}(Z_{\alpha+2} \cap Q_\beta)$. Set $r_\alpha = |(Z_\alpha \cap Q_\beta)Q_{\alpha+2}/Q_{\alpha+2}|$ and define $r_{\alpha+2}$ similarly. If $r_{\alpha+2} > r_\alpha$, then

$$|Z_\alpha/C_{Z_\alpha}(Z_{\alpha+2} \cap Q_\beta)| \leqslant pr_\alpha \leqslant r_{\alpha+2} = (Z_{\alpha+2} \cap Q_\beta)/C_{Z_{\alpha+2}\cap Q_\beta}(Z_\alpha)$$

and $Z_{\alpha+2} \cap Q_\beta$ is an offender on Z_α. Then, by Lemma 2.3.10, Z_α is an FF-module for $L_\alpha/R_\alpha \cong \mathrm{SL}_2(p^n)$ and $L_\alpha = Q_\alpha O^p(L_\alpha)$. In particular, since $Z(L_\alpha) = \{1\}$, $C_{Z_\alpha}(O^p(L_\alpha)) = \{1\}$ and Z_α is irreducible of order p^{2n}. But then, we have that $[Z_\alpha, F_\beta] \leq Z_\beta$, a contradiction since F_β contains a non-central chief factor for L_β. Hence, $r_{\alpha+2} \leqslant r_\alpha$ and by a symmetric calculation, $r_\alpha \leqslant r_{\alpha+2}$ so that $r_\alpha = r_{\alpha+2}$ and the result holds. □

Lemma 5.4.53. *Suppose that $m_p(S/Q_\alpha) \geqslant 2$. Then S/Q_α is elementary abelian.*

Proof. Assume that $m_p(S/Q_\alpha) \geqslant 2$ and S/Q_α is not elementary abelian. In particular, $\overline{L_\alpha} \cong \mathrm{PSU}_3(p^n), \mathrm{SU}_3(p^n), \mathrm{Sz}(2^n)$ or $\mathrm{Ree}(3^n)$. If $m_p(S/Q_\beta) \geqslant 2$, since Z_α acts quadratically on Q_β, by Lemma 2.3.5 we have that $\overline{L_\beta}$ is isomorphic to a central extension of a simple group of Lie type by a p'-group. In particular, G has a weak BN-pair and is determined in [DS85]. No examples occur. Thus, we may assume that $m_p(S/Q_\beta) = 1$ throughout. By Lemma 5.4.52, there is $\alpha+2 \in \Delta(\beta)$ such that an index $r_\alpha p$ subgroup of Z_α is centralized by $(Z_{\alpha+2} \cap Q_\beta)Q_\alpha/Q_\alpha$, where

$r_\alpha = |(Z_{\alpha+2} \cap Q_\beta)Q_\alpha/Q_\alpha|$. Since $Z(L_\alpha) = \{1\}$, $C_{Z_\alpha}(O^p(L_\alpha))$ and so, if $O^p(\overline{L_\alpha})$ is generated by d conjugates of $(Z_{\alpha+2} \cap Q_\beta)Q_\alpha/Q_\alpha$, it follows that $|Z_\alpha| \leqslant (r_\alpha p)^d$.

Suppose that $S \neq Q_\alpha Q_\beta$. Since S/Q_α is not elementary abelian and $Q_\alpha Q_\beta$ is a $G_{\alpha,\beta}$-invariant, it follows that $S/Q_\alpha Q_\beta$ is elementary abelian of order strictly greater than 4, unless $\overline{L_\alpha} \cong \mathrm{Ree}(3)$. Since S/Q_β is cyclic or generalized quaternion, the largest elementary abelian quotient of S/Q_β has order at most 4 and we have a contradiction unless $\overline{L_\alpha} \cong \mathrm{Ree}(3)$ and $|S/Q_\alpha Q_\beta| = 3$.

If $\overline{L_\alpha} \cong \mathrm{Ree}(3)$ then $O^3(\overline{L_\alpha}) \cong \mathrm{PSL}_2(8)$ is generated by two conjugates of $(Z_\lambda \cap Q_\beta)Q_\alpha/Q_\alpha$. Since the minimal degree of a GF(3)-representation of $\mathrm{PSL}_2(8)$ is 7, and $O^3(L_\alpha)$ does not centralize Z_α, we have that $3^7 \leqslant |Z_\alpha| \leqslant r_\alpha^2 3^2 \leqslant 3^6$, a contradiction. Thus, $S = Q_\alpha Q_\beta$. Notice also that $(Z_{\alpha+2} \cap Q_\beta)Q_\alpha \trianglelefteq Q_\alpha Q_\beta = S$.

Suppose that $\overline{L_\alpha} \cong \mathrm{SU}_3(p^n)$ or $\mathrm{PSU}_3(p^n)$. If $(Z_{\alpha+2} \cap Q_\beta)Q_\alpha/Q_\alpha \not\leq Z(S/Q_\alpha)$ then it follows from Lemma 2.2.2 (viii) that two conjugates generate $\overline{L_\alpha}$ and $|Z_\alpha| \leqslant r_\alpha^2 p^2$. Since $|S/Q_\alpha| = p^{3n}$, $|Z(S/Q_\alpha)| = p^n$ and $Z_{\alpha+2}$ is abelian, we have that $(Z_{\alpha+2} \cap Q_\beta)Q_\alpha/Q_\alpha$ has index at least p^2 in S/Q_α and $|Z_\alpha| < p^{6n}$ unless perhaps $p = 3$ and $n = 1$ in which case $|Z_\alpha| \leqslant 3^6$ anyway. Since the minimal degree of a GF(p)-representation of $\overline{L_\alpha}$ is $6n$ it follows that $p = 3$, $n = 1$ and Z_α is the natural module. But now, $Z_\alpha \cap Q_\beta$ is a $G_{\alpha\beta}$-invariant subgroup of index 3 in Z_α, a contradiction by Lemma 2.2.13 (iii). Assume now that $(Z_{\alpha+2} \cap Q_\beta)Q_\alpha/Q_\alpha \leq Z(S/Q_\alpha)$ so that $r_\alpha \leqslant p^n$. By Lemma 2.2.2 (vi), (vii), $\overline{L_\alpha}$ is generated by at most 4 conjugates of $(Z_{\alpha+2} \cap Q_\beta)Q_\alpha/Q_\alpha \not\leq Z(S/Q_\alpha)$ and so $|Z_\alpha| \leqslant p^{4n+4}$. If $n > 2$, then $|Z_\alpha| < p^{6n}$ and since the minimal degree of a GF(p)-representation of $\overline{L_\alpha}$ is $6n$, we have a contradiction. Suppose that $n = 2$. If $r_\alpha = p^2$, then $(Z_{\alpha+2} \cap Q_\beta)Q_\alpha/Q_\alpha = Z(S/Q_\alpha)$ and by Lemma 2.2.2 (vi), three conjugates of $(Z_{\alpha+2} \cap Q_\beta)Q_\alpha/Q_\alpha$ generate $\overline{L_\alpha}$ and $|Z_\alpha| \leqslant p^9 < p^{12}$, a contradiction. If $r_\alpha = p$, then $|Z_\alpha| \leqslant p^8 < p^{12}$, another contradiction. Suppose finally that $n = 1$ so that $p = 3$. Then $(Z_{\alpha+2} \cap Q_\beta)Q_\alpha/Q_\alpha = Z(S/Q_\alpha)$ and $\overline{L_\alpha}$ is generated by three conjugates of $(Z_{\alpha+2} \cap Q_\beta)Q_\alpha/Q_\alpha$. Then $|Z_\alpha| \leqslant 3^6$ and the only possibility is that Z_α is the natural module. As above, $Z_\alpha \cap Q_\beta$ is a $G_{\alpha,\beta}$-invariant subgroup of index 3 in Z_α, and we have a contradiction.

Suppose that $\overline{L_\alpha} \cong \mathrm{Sz}(2^n)$ with $n \geqslant 3$. If $(Z_{\alpha+2} \cap Q_\beta)Q_\alpha/Q_\alpha \not\leq Z(S/Q_\alpha)$ then it follows from Lemma 2.2.3 (vii) that two conjugates generate and $|Z_\alpha| \leqslant r_\alpha^2 2^2$. Since $|S/Q_\alpha| = 2^{2n}$, $|Z(S/Q_\alpha)| = 2^n$, $n \geqslant 3$ and $Z_{\alpha+2}$ is abelian, we have that $(Z_{\alpha+2} \cap Q_\beta)Q_\alpha/Q_\alpha$ has index at least p^2 and $|Z_\alpha| < p^{4n}$. Since the minimal degree of a $\mathrm{GF}(p)$-representation of $\overline{L_\alpha}$ is $4n$, we have a contradiction. If $(Z_{\alpha+2} \cap Q_\beta)Q_\alpha/Q_\alpha \leq Z(S/Q_\alpha)$ then it follows from Lemma 2.2.3 (vi) that three conjugates generate and $|Z_\alpha| \leqslant r_\alpha^3 2^3 \leqslant 2^{3n+3}$. Since the minimal degree of a $\mathrm{GF}(p)$-representation of $\overline{L_\alpha}$ is $4n$ and $n \geqslant 3$, we have that $n = 3$ and $r_\alpha = 8$. But then $(Z_{\alpha+2} \cap Q_\beta)Q_\alpha/Q_\alpha = Z(S/Q_\alpha)$ and only two conjugates are required to generate $\overline{L_\alpha}$ from which it follows that $|Z_\alpha| \leqslant 2^8 < 2^{12}$, a contradiction.

Suppose that $\overline{L_\alpha} \cong \mathrm{Ree}(3^n)$. By the above, $n \geqslant 3$. By Lemma 2.2.4 (vi), we infer that $\overline{L_\alpha}$ is generated by three conjugates of $(Z_{\alpha+2} \cap Q_\beta)Q_\alpha/Q_\alpha$ and since the minimal degree of a $\mathrm{GF}(3)$-representation of $\overline{L_\alpha}$ is $7n$, we deduce that $3^{7n} \leqslant |Z_\alpha| \leqslant r_\alpha^3 p^3$. Since $Z_{\alpha+2}$ is elementary abelian and $|\Omega(S/Q_\alpha)| = 3^{2n}$, we have that $r_\alpha^3 p^3 \leqslant 3^{6n} 3^3$ and since $n \geqslant 3$, we conclude that $n = 3$, $\overline{L_\alpha} \cong \mathrm{Ree}(27)$ and $(Z_{\alpha+2} \cap Q_\beta)Q_\alpha/Q_\alpha = \Omega(S/Q_\alpha)$. But then, it may be checked that $\overline{L_\alpha}$ is generated by two conjugates of $(Z_{\alpha+2} \cap Q_\beta)Q_\alpha/Q_\alpha$ and $3^{21} \leqslant 3^{12} 3^3 = 3^{15}$, a clear contradiction. $\square$

Proposition 5.4.54. *Suppose that $m_p(S/Q_\alpha) \geqslant 2$, $m_p(S/Q_\beta) \geqslant 2$ and $p \in \{2, 3\}$. Then one of the following holds:*

(i) *G has a weak BN-pair of rank 2 and G is locally isomorphic to H where $F^*(H) \cong \mathrm{PSU}_4(p^{n+1}), \mathrm{PSU}_5(2^{n+1}), \mathrm{PSU}_5(3^n)$ or $\mathrm{PSp}_4(3^{n+1})$ for $n \geqslant 1$; or*

(ii) *$p = 3$, $|S| = 3^7$, $\overline{L_\alpha} \cong \mathrm{M}_{11}$, $Z_\alpha = Q_\alpha$ is the "code" module for $\overline{L_\alpha}$, $\overline{L_\beta} \cong \mathrm{SL}_2(9)$ and $Q_\beta \cong 3^{1+4}_+$.*

Moreover, if G is obtained from a fusion system $\mathcal{F}$ satisfying Hypothesis 5.1.12 then one of the following holds:

(i) *$p = 2$ and $\mathcal{F} = \mathcal{F}_S(H)$ where $F^*(H) \cong \mathrm{PSU}_4(2^n)$ or $\mathrm{PSU}_5(2^n)$ and $n \geqslant 2$; or*

(ii) *$p = 3$ and $\mathcal{F} = \mathcal{F}_S(H)$ where $F^*(H) \cong \mathrm{PSp}_4(3^{n+1})$, $\mathrm{PSU}_4(3^{n+1})$, $\mathrm{PSU}_5(3^n)$ for $n \geqslant 1$; or*

(iii) *$p = 3$ and $\mathcal{F} = \mathcal{F}_S(H)$ where $H \cong \mathrm{Co}_3$.*

Proof. Assume that $m_p(S/Q_\alpha) \geqslant 2$ so that S/Q_α is elementary abelian by Lemma 5.4.53. Then by Proposition 3.2.7, $\overline{L_\alpha} \cong \mathrm{SL}_2(p^n)$ or $\mathrm{PSL}_2(p^n)$ for $n \geqslant 2$ and $p \in \{2, 3\}$; or $\overline{L_\alpha} \cong \mathrm{M}_{11}$ or $3'$-central extension of $\mathrm{PSL}_3(4)$ and $p = 3$. In particular, $(G_{\alpha,\beta} \cap L_\alpha)/Q_\alpha$ acts irreducibly on S/Q_α and so $Q_\beta = F_\beta(Q_\beta \cap Q_\alpha)$ and F_β contains all non-central chief factors for L_β. Further, $D_\beta \leq Q_\alpha$ for otherwise $Q_\beta = D_\beta(Q_\alpha \cap Q_\beta)$ and $O^p(L_\beta)$ centralizes Q_β, a contradiction.

If both $\overline{L_\alpha}$ and $\overline{L_\beta}$ are isomorphic to central extensions of Lie type groups, then G has a weak BN-pair of rank 2 and G is determined up to local isomorphism in [DS85]. Comparing with the amalgams determined there, we have that G is locally isomorphic to H where $F^*(H) \cong \mathrm{PSU}_4(p^n), \mathrm{PSU}_5(p^n)$ or $\mathrm{PSp}_4(3^n)$ for $n \geqslant 2$, or $\mathrm{PSU}_5(3)$. Hence, $p = 3$. Since Q_β admits quadratic action, by Lemma 2.3.5,

$L_\beta \cong \mathrm{SL}_2(3^{a+1})$ or (P)$\mathrm{SU}_3(3^a)$ for $a \geqslant 1$; and $L_\alpha \cong \mathrm{M}_{11}$ or a central extension of $\mathrm{PSL}_3(4)$. Set $r_\alpha := |(Z_\alpha \cap Q_\beta)Q_{\alpha+2}/Q_{\alpha+2}|$ and d the number of conjugates of $(Z_\alpha \cap Q_\beta)Q_{\alpha+2}/Q_{\alpha+2}$ required to generate $\overline{L_{\alpha+2}}$. In a similar way to Lemma 5.4.52, we see that $r_\alpha = r_{\alpha+2}$ and the value of d is consistent for both $\overline{L_\alpha}$ and $\overline{L_{\alpha+2}}$.

If $\overline{L_\beta} \cong$ (P)$\mathrm{SU}_3(3^a)$, then since $F_\beta \cap Q_\alpha$ is index 9 in F_β and is centralized by Z_α, we have that $\overline{L_\beta} \cong \mathrm{SU}_3(3)$ and $F_\beta/F_\beta \cap D_\beta$ is a natural module. Then, $|Z_\alpha| \leqslant (r_\alpha 3)^d$. One can check that for $\overline{L_\alpha} \cong \mathrm{M}_{11}$ or a central extension of $\mathrm{PSL}_3(4)$, $\overline{L_\alpha}$ is generated by two conjugate Sylow 3-subgroups, or three conjugates 3-elements and so $|Z_\alpha| \leqslant 3^6$, $S = (Z_{\alpha+2} \cap Q_\beta)Q_\alpha$ and $Z_\beta = Z_\alpha \cap Q_\beta \cap Q_\lambda$ is index 3^3 in Z_α. Since the minimal degree of a GF(3)-representation of M_{11} is 5 and the minimal degree of a GF(3)-representation of a central extension of $\mathrm{PSL}_3(4)$ is 6, Z_α contains a unique non-trivial irreducible constituent and $r_\alpha = 9$. Since $C_{Z_\alpha}(O^p(L_\alpha)) = Z(L_\alpha) = \{1\}$, it follows from Lemma 2.3.2 that $Z_\alpha = [Z_\alpha, L_\alpha]$ is irreducible. Since $S = Q_\alpha Q_\beta$ and S/Q_β is non-abelian, it follows that $Z_\alpha \leq \langle (Z_\beta \cap \Phi(Q_\alpha))^{G_\alpha} \rangle \leq \Phi(Q_\alpha)$. But then $Z_\alpha(Q_\alpha \cap Q_\beta \cap Q_{\alpha+2})$ has index 3^2 in Q_α and there is an index 3^4 subgroup of $Q_\alpha/\Phi(Q_\alpha)$ which is centralized by $O^3(L_\alpha)$. A consideration of the minimal degrees of GF(3)-representations of $\overline{L_\alpha}$ yields that $O^3(L_\alpha)$ centralizes $Q_\alpha/\Phi(Q_\alpha)$, a contradiction.

If $\overline{L_\beta} \cong \mathrm{SL}_2(3^{a+1})$, then $L_\beta = \langle Q_\beta, Z_\alpha, Z_{\alpha+2} \rangle$ and $Q_\beta \cap Q_\alpha$ is an index 9 subgroup of Q_β which is centralized by Z_α. It follows that $\overline{L_\beta} \cong \mathrm{SL}_2(9)$ and Q_β contains one non-central chief factor, which is isomorphic to the natural module. Suppose that $\overline{L_\alpha} \cong \mathrm{M}_{11}$. Then the amalgam is described in [Pap97] and we have (v) as a conclusion in this case. If G is obtained from a fusion system $\mathcal{F}$ satisfying Hypothesis 5.1.12, then since $S \in \mathrm{Syl}_3(O^3(G_\alpha))$, it follows that $O^3(\mathcal{F}) = \mathcal{F}$ and we may apply the results in [PS21]. Indeed, $\mathcal{F}$ is isomorphic to the 3-fusion system of Co_3.

Let T_β be the preimage in L_β of $Z(\overline{L_\beta})$. Then, by coprime action, $Q_\beta/\Phi(Q_\beta) = [Q_\beta/\Phi(Q_\beta), T_\beta] \times C_{Q_\beta/\Phi(Q_\beta)}(T_\beta)$ where $|[Q_\beta/\Phi(Q_\beta), T_\beta]| = 3^4$. Since S/Q_α is elementary abelian, $\Phi(Q_\beta) \leq Q_\alpha$ and so $[Z_\alpha, \Phi(Q_\beta)] = \{1\}$ and $\Phi(Q_\beta) \leq D_\beta$. It follows that D_β is index 3^4 in Q_β and $Q_\beta = F_\beta D_\beta$.

We may assume that $\overline{L_\alpha}$ is isomorphic to a central extension of $\mathrm{PSL}_3(4)$ so that $|Z_\alpha| \leqslant (r_\alpha 3^2)^d \leqslant (3^4)^d = 3^8$. Thus, Z_α contains a unique irreducible constituent and, as above, Z_α is an irreducible module and $|Z_\alpha| = 3^6$ or 3^8. Since $Q_\alpha = Z_\alpha(Q_\alpha \cap Q_\beta)$ and $[Q_\beta, F_\beta] \leq Z_\alpha$ by Lemma 5.4.49, it follows that Z_α contains all non-central chief factors for L_α and the irreducibility of Z_α implies that $\Phi(Q_\alpha) = \{1\}$ and Q_α is elementary abelian. Since the minimal degree of a GF(3)-representation of $\mathrm{PSL}_3(4)$ is 15, $Z(\overline{L_\alpha})$ acts non-trivially on Z_α and since Z_α is irreducible, for T_α the preimage in L_α of $Z(\overline{L_\alpha})$, $Z_\alpha = [Z_\alpha, T_\alpha]$. Since Q_α is abelian, it follows from coprime action that $Q_\alpha = [Q_\alpha, T_\alpha] \times C_{Q_\alpha}(T_\alpha) = Z_\alpha \times C_{Q_\alpha}(T_\alpha)$ and since $C_{Q_\alpha}(T_\alpha)$ is normalized by S and intersects Z_β trivially, $C_{Q_\alpha}(T_\alpha) = \{1\}$ and $Q_\alpha = Z_\alpha$. Now, D_β is centralized by $S = Z_\alpha F_\beta$ and so $Z_\beta = D_\beta$ has index 3^4 in Q_β. If $|Z_\alpha| = 3^8$, then Z_β has order 9 and so $|S| = 3^2.|Q_\beta/Z_\beta||Z_\beta| = 3^8$, a contradiction. Thus, $|Z_\alpha| = 3^6$. Then, one can check that for either irreducible module of dimension 6, S splits over Z_α and since Z_α is self-centralizing, $|Z(\overline{L_\alpha})| = 2$. Moreover, S is of order 3^8 and is isomorphic to a Sylow 3-subgroup of Suz or $\mathrm{PSp}_4(9)$. In the former case, Z_β is of order 3, so that $|S| = 3^2.|F_\beta/Z_\beta||Z_\beta| = 3^7$, a clear contradiction.

When S is isomorphic to a Sylow 3-subgroup of $\mathrm{PSp}_4(9)$, we apply [HS19, Theorem 3.13] to see that $\overline{G_\alpha}$ embeds as a subgroup of $2 \cdot \mathrm{PSL}_3(4).2^2$ and for any element $x \in \overline{G_\alpha}$ of order 8, $[x^4, Z_\alpha] \not\leq [S, Z_\alpha] = Z_\alpha \cap Q_\beta$. Let $t_\beta \in L_\beta \cap G_{\alpha,\beta}$ be an element of order 8, so that $t_\beta^4 Q_\beta \leq Z(\overline{L_\beta})$. But then $[t_\beta^4, Z_\alpha] \leq Z_\alpha \cap Q_\beta$ and since $t_\beta \leq G_\alpha$, we have a contradiction. $\square$

Proposition 5.4.55. *Suppose that $m_p(S/Q_\alpha) \geqslant 2$, $m_p(S/Q_\beta) = 1$ and $p \in \{2, 3\}$. Then one of the following holds:*

(i) *G has a weak BN-pair of rank 2 and G is locally isomorphic to H where $F^*(H) \cong \mathrm{PSU}_4(p)$ or $\mathrm{PSU}_5(2)$;*

(ii) *$p = 3$, $|S| = 3^6$, $\overline{L_\alpha} \cong \mathrm{PSL}_2(9)$, $Z_\alpha = Q_\alpha$ is a natural $\Omega_4^-(3)$-module, $\overline{L_\beta} \cong (Q_8 \times Q_8) : 3$ and $Q_\beta \cong 3_+^{1+4}$;*

(iii) *$p = 3$, $|S| = 3^6$, $\overline{L_\alpha} \cong \mathrm{PSL}_2(9)$, $Z_\alpha = Q_\alpha$ is a natural $\Omega_4^-(3)$-module, $\overline{L_\beta} \cong 2 \cdot \mathrm{Alt}(5)$ and $Q_\beta \cong 3_+^{1+4}$;*

(iv) *$p = 3$, $|S| = 3^6$, $\overline{L_\alpha} \cong \mathrm{PSL}_2(9)$, $Z_\alpha = Q_\alpha$ is a natural $\Omega_4^-(3)$-module, $\overline{L_\beta} \cong 2_-^{1+4}.\mathrm{Alt}(5)$ and $Q_\beta \cong 3_+^{1+4}$;*

(v) *$p = 3$, $|S| = 3^7$, $\overline{L_\alpha} \cong \mathrm{M}_{11}$ and $Z_\alpha = Q_\alpha$ is the "cocode" module for $\overline{L_\alpha}$, $\overline{L_\beta} \cong \mathrm{SL}_2(3)$ and $Q_\beta \cong 3^{1+1+4} \cong T \in \mathrm{Syl}_3(\mathrm{SL}_3(9))$; or*

(vi) *$p = 3$, $|S| = 3^7$, $\overline{L_\alpha} \cong \mathrm{M}_{11}$ and $Z_\alpha = Q_\alpha$ is the "cocode" module for $\overline{L_\alpha}$, $\overline{L_\beta} \cong \mathrm{SL}_2(5)$ and $Q_\beta \cong 3^{1+1+4} \cong T \in \mathrm{Syl}_3(\mathrm{SL}_3(9))$.*

Moreover, if G is obtained from a fusion system $\mathcal{F}$ satisfying Hypothesis 5.1.12 then one of the following holds:

(i) *$p = 2$ and $\mathcal{F} = \mathcal{F}_S(H)$ where $H \cong \mathrm{PSU}_4(2)$, $\mathrm{Aut}(\mathrm{PSU}_4(2))$, $\mathrm{PSU}_5(2)$ or $\mathrm{Aut}(\mathrm{PSU}_5(2))$;*

(ii) *$p = 3$ and $\mathcal{F} = \mathcal{F}_S(H)$ where $F^*(H) \cong \mathrm{PSU}_4(3)$; or*

(iii) $p = 3$ *and* $\mathcal{F} = \mathcal{F}_S(H)$ *where* $H \cong \mathrm{McL}$, $\mathrm{Aut}(\mathrm{McL})$, Co_2, Ly, Suz, $\mathrm{Aut}(\mathrm{Suz})$, $\mathrm{PSU}_6(2)$ *or* $\mathrm{PSU}_6(2).2$.

Proof. Suppose that $m_p(S/Q_\alpha) \geqslant 2$ and $m_p(S/Q_\beta) = 1$. Then by Lemma 5.4.53, S/Q_α is elementary abelian and as in Proposition 5.4.54, we have that if $O^p(\overline{L_\alpha})$ is generated by d conjugates of $(Z_{\alpha+2} \cap Q_\beta)Q_\alpha/Q_\alpha$, then $|Z_\alpha| \leqslant (r_\alpha|Z_\alpha Q_\beta/Q_\beta|)^d$, where $r_\alpha = |(Z_{\alpha+2} \cap Q_\beta)Q_\alpha/Q_\alpha|$. In particular, since $m_p(S/Q_\beta) = 1$, $|Z_\alpha| \leqslant (r_\alpha p)^d$.

Suppose that $\overline{L_\alpha} \cong \mathrm{SL}_2(p^n)$ or $\mathrm{PSL}_2(p^n)$ for any $n > 1$. Applying Lemma 2.2.1 (iii),(iv), (v), unless $r_\alpha = p$ we have that $|Z_\alpha| \leqslant r_\alpha^2 p^2 \leqslant p^{2n+2}$ and if $r_\alpha = p$, then $|Z_\alpha| \leqslant p^6$. Since the minimal degree of a $\mathrm{GF}(p)$-representation of $\overline{L_\alpha}$ is $2n$ and $n \geqslant 2$, $r_\alpha \geqslant p^{n-1}$ and it follows that there is at most one non-trivial irreducible constituent within Z_α. Since $C_{Z_\alpha}(O^p(L_\alpha)) = Z(L_\alpha) = \{1\}$, by Lemma 2.3.2, $Z_\alpha = [Z_\alpha, L_\alpha]$ is irreducible. Setting K to be Hall p'-subgroup of $L_\alpha \cap G_{\alpha,\beta}$, it follows from Smith's theorem ([GLS98, Theorem 2.8.11]) that $Z_\beta = C_{Z_\alpha}(S)$ and $Z_\alpha/[Z_\alpha, S]$ are irreducible and 1-dimensional as $\overline{F}K$-modules, where $\overline{F}$ is an algebraically closed field of characteristic p. But $[Z_\alpha, S] = [Z_\alpha, Q_\beta] \leq Z_\alpha \cap Q_\beta$ and since $Z_\alpha \cap Q_\beta$ has index p in Z_α, $[Z_\alpha, S] = Z_\alpha \cap Q_\beta$ and $|Z_\beta| = |Z_\alpha/[Z_\alpha, S]| = p$. If $n > 2$, then $|Z_\alpha| \leqslant p^{2n+2} < p^{3n}$ and Lemma 2.3.12 implies that Z_α is a triality module for $\overline{L_\alpha} \cong \mathrm{SL}_2(p^3)$ and $|Z_\alpha| = p^8$. Since $|Z_\alpha| \leqslant r_\alpha^2 p^2$, we have that $r_\alpha = p^3$ and $S = (Z_{\alpha+2} \cap Q_\beta)Q_\alpha$ centralizes $Z_\alpha \cap Q_\beta \cap Q_{\alpha+2}$. But then $Z_\beta = Z_\alpha \cap Q_\beta \cap Q_{\alpha+2}$ is index p^4 in Z_α. Since $|Z_\beta| = p$, $p^5 = |Z_\alpha| = p^8$, a contradiction.

Thus, we may assume that $|S/Q_\alpha| = p^2$ for the remainder of the proof. Then $F_\beta/F_\beta \cap D_\beta$ is a quadratic $2F$-module and so, by Proposition 2.3.19, both $\overline{L_\beta}$ and $F_\beta/F_\beta \cap D_\beta$ are determined. If $\overline{L_\beta} \cong \mathrm{SU}_3(2)$, then since $p = 2$, $\overline{L_\alpha} \cong \mathrm{PSL}_2(4)$ and G has a weak BN-pair of rank 2 and by [DS85], G is locally isomorphic to H where

$F^*(H) \cong \mathrm{PSU}_5(2)$. Hence, we may assume that S/Q_β is abelian.

We have that $\overline{L_\alpha}$ is isomorphic to $\mathrm{PSL}_2(p^2), \mathrm{SL}_2(p^2), \mathrm{M}_{11}$ or a $3'$-central extension of $\mathrm{PSL}_3(4)$. Since M_{11} and central extensions of $\mathrm{PSL}_3(4)$ are generated by two conjugate Sylow 3-subgroups, or three conjugates elements of order 3, we see that $|Z_\alpha| \leqslant 3^6$ and by the above, in all cases we conclude that $|Z_\alpha| \leqslant 3^6$. Checking against the degrees of the minimal $\mathrm{GF}(p)$-representations of the candidates for $\overline{L_\alpha}$, we see that Z_α contains a unique irreducible constituent and since $C_{Z_\alpha}(O^p(L_\alpha)) = Z(L_\alpha) = \{1\}$, it follows from Lemma 2.3.2, that $Z_\alpha = [Z_\alpha, L_\alpha]$ is irreducible.

If $|S/Q_\beta| > p$, then $p = 2$ and $\Phi(Q_\alpha) \neq \{1\}$ and it follows from the irreducibility of Z_α, that $Z_\alpha \leq \Phi(Q_\alpha)$. But then $\Phi(Q_\alpha)(Q_\alpha \cap Q_\beta)$ is an index 2 subgroup of Q_α and $[\Phi(Q_\alpha)(Q_\alpha \cap Q_\beta), F_\beta] \leq \Phi(Q_\alpha)$ by Lemma 5.4.49. Since $m_p(S/Q_\alpha) \geqslant 2$, it follows that $O^2(L_\alpha)$ centralizes $Q_\alpha/\Phi(Q_\alpha)$, a contradiction. Thus, we have that $|S/Q_\beta| = p$. Then, $Q_\alpha = Z_\alpha(Q_\alpha \cap Q_\beta)$ and by Lemma 5.4.49, $[O^p(L_\alpha), Q_\alpha] \leq Z_\alpha$. Then the irreducibility of Z_α implies that $\Phi(Q_\alpha) = \{1\}$ and Q_α is elementary abelian.

Now, checking against the list of groups provided in Proposition 2.3.19, either $\overline{L_\beta}$ is p-solvable or has a non-trivial center, and for T_β the preimage in L_β of $O_{p'}(\overline{L_\beta})$, we have by coprime action $Q_\beta/\Phi(Q_\beta) = [Q_\beta/\Phi(Q_\beta), T_\beta] \times C_{Q_\beta/\Phi(Q_\beta)}(T_\beta)$ where $[Q_\beta/\Phi(Q_\beta), T_\beta]$ contains all non-central chief factors in $Q_\beta/\Phi(Q_\beta)$ and $C_{Q_\beta/\Phi(Q_\beta)}(T_\beta) = C_{Q_\beta/\Phi(Q_\beta)}(O^p(L_\beta))$. In particular, $F_\beta\Phi(Q_\beta)/\Phi(Q_\beta) = [Q_\beta/\Phi(Q_\beta), T_\beta]$. Since $\Phi(Q_\beta) \leq Q_\alpha$, $[\Phi(Q_\beta), Z_\alpha] = \{1\}$ and it follows that $\Phi(Q_\beta) \leq D_\beta$ so that $Q_\beta = F_\beta D_\beta$. Since $D_\beta \leq Q_\alpha$ is elementary abelian and $F_\beta \leq O^p(L_\beta)$, $S = F_\beta Q_\alpha$ centralizes D_β so that $D_\beta = Z_\beta$.

Suppose that $\overline{L_\alpha}$ is isomorphic to a central extension of $\mathrm{PSL}_3(4)$. Then $p = 3$

and comparing with the modules in Proposition 2.3.19, $|Q_\beta/Z_\beta| = 3^4$ so that $|S/Z_\beta| = 3^5$. Since $|Z_\alpha| = 3^6$, we have that $|S| \geqslant 3^8$ and $|Z_\beta| \geqslant 3^3$. Checking the relevant irreducible GF(3)-modules associated to $\overline{L_\alpha}$, we have that $|Z_\beta| \leqslant 3^2$, a contradiction.

Suppose that $\overline{L_\alpha} \cong \mathrm{M}_{11}$. Then $p = 3$, $|Z_\alpha| = 3^5$ and $\overline{L_\beta} \cong 2 \cdot \mathrm{Alt}(5)$, $2_-^{1+4}.\mathrm{Alt}(5)$, $\mathrm{SL}_2(3)$ or $(Q_8 \times Q_8) : 3$ by Proposition 2.3.19. In the first three cases, the structure of L_α and L_β is determined in [Pap97] and outcomes (vi) and (vii) follow in these cases. Suppose that $\overline{L_\beta} \cong (Q_8 \times Q_8) : 3$ with $|Q_\beta/Z_\beta| = p^4$ and let K_β be a Hall $2'$-subgroup of $G_{\alpha,\beta} \cap L_\beta$. Then $K_\beta \leq G_\alpha$ and so K_β acts on L_α/Q_α. Since M_{11} has no outer automorphisms, if $K_\beta \not\leq L_\alpha$, then there is an involution $t \in K_\beta$ such that $[t, L_\alpha] \leq Q_\alpha$ and $[t, L_\beta] \leq Q_\beta$, a contradiction by Proposition 5.2.6 (v). Thus, $K_\beta \leq L_\alpha$ so that $L_\alpha = G_\alpha$. Since $[K_\beta, Z_\alpha] \leq Z_\alpha \cap Q_\beta$ and K_β centralizes Z_β it follows that $|C_{Z_\alpha}(K_\beta)| = 3^3$, and one can check (e.g. using MAGMA) that this provides a contradiction. If G is obtained from a fusion system $\mathcal{F}$ satisfying Hypothesis 5.1.12, then since $S \in \mathrm{Syl}_3(O^3(G_\alpha))$, it follows that $O^3(\mathcal{F}) = \mathcal{F}$ and we may apply the results in [PS21]. Indeed, $\mathcal{F}$ is isomorphic to the 3-fusion system of Suz, Aut(Suz) or Ly.

Finally, suppose that $\overline{L_\alpha} \cong \mathrm{PSL}_2(p^2)$ or $\mathrm{SL}_2(p^2)$. Then, again by Smith's theorem, $|Z_\beta| = p$ so that $F_\beta = Q_\beta$. By the minimality of F_β, it follows that $Z(Q_\beta) = \Phi(Q_\beta) = Z_\beta$ is of order p and Q_β is extraspecial. Since $Q_\beta \cap Q_\alpha$ is an elementary abelian subgroup of index p^2 in Q_β, we have that $|Q_\beta| = p^5$. In particular, $|S| = p^6$ and $Z_\alpha = Q_\alpha$ is of order p^4.

If $p = 2$, then $\overline{L_\beta} \cong \mathrm{Dih}(10), \mathrm{Sym}(3)$ or $(3 \times 3) : 2$ since $\mathrm{SU}_3(2)'$ does not embed in $\mathrm{Aut}(Q_\beta\Phi(Q_\beta)) \cong \mathrm{GL}_4(2)$. In the first two cases, G has a weak BN-pair and so comparing with [DS85], we have that $\overline{L_\beta} \cong \mathrm{Sym}(3)$ and G is locally isomorphic to

H where $F^*(H) \cong \mathrm{PSU}_4(2)$. Since Q_β is extraspecial, comparing with [Win72], $\overline{L_\beta}$ is isomorphic to a subgroup of $O_4^+(2)$ if $Q_\beta \cong 2_+^{1+4}$; or $O_4^-(2)$ if $Q_\beta \cong 2_-^{1+4}$. Note that 9 does not divide $|O_4^-(2)|$ and so, we deduce that $Q_\beta \cong 2_+^{1+4}$. Let K be a Sylow 3-subgroup of $L_\alpha \cap G_{\alpha,\beta}$. Then K acts non-trivially on Q_β and so K also embeds into $O_4^+(2)$ while normalizing $\overline{L_\beta} \cong (3 \times 3) : 2$. But for $H \leq O_4^+(2)$ with $H \cong (3 \times 3) : 2$ we have that $|N_{O_4^+(2)}(H)/H| = 2$, a contradiction.

Thus, we may assume that $p = 3$ and $L_\beta \cong \mathrm{SL}_2(3), (Q_8 \times Q_8) : 3, 2 \cdot \mathrm{Alt}(5)$ or $2_-^{1+4}.\mathrm{Alt}(5)$. Since $|Z_\alpha| = 3^4$, Z_α is a faithful $\overline{L_\alpha}$-module and Z_α is not a quadratic module, we have that $\overline{L_\alpha} \cong \mathrm{PSL}_2(9)$ and Z_α is a natural $\Omega_4^-(3)$-module. If $\overline{L_\beta} \cong \mathrm{SL}_2(3)$ then G has a weak BN-pair and comparing with [DS85], G is locally isomorphic to H where $F^*(H) \cong \mathrm{PSU}_4(3)$. If $\overline{L_\beta} \cong 2 \cdot \mathrm{Alt}(5)$ or $2_-^{1+4}.\mathrm{Alt}(5)$ then the structure of L_α and L_β is determined in [Pap97] and we obtain conclusions (iii) and (iv). If G is obtained from a fusion system $\mathcal{F}$ satisfying Hypothesis 5.1.12, then applying the results in [PS21], $\mathcal{F}$ is isomorphic to the 3-fusion system of McL, Aut(McL) or Co_2. Finally, suppose that $\overline{L_\beta} \cong (Q_8 \times Q_8) : 3$. Since Q_β is extraspecial of order 3^5 and $\overline{L_\beta}$ embeds in the automorphism group of Q_β, it follows from [Win72] that $Q_\beta \cong 3_+^{1+4}$. If S acted quadratically on Z_α, then Z_α is a natural $\mathrm{SL}_2(9)$-module, a contradiction since $C_{Z_\alpha}(S) = Z_\beta$ is of order 3. It follows that Z_α is a natural $\Omega_4^-(3)$-module for $\overline{L_\alpha}$ and since Z_α is self-centralizing, $\overline{L_\alpha} \cong \mathrm{PSL}_2(9)$ and we have (ii) as a conclusion. If G is obtained from a fusion system $\mathcal{F}$ satisfying Hypothesis 5.1.12, then applying the results in [PS21], $\mathcal{F}$ is isomorphic to the 3-fusion system of $\mathrm{PSU}_6(2)$ or $\mathrm{PSU}_6(2).2$. $\square$

We conclude this section by summarizing what has been shown:

Theorem 5.4.56. *Suppose that $\mathcal{A} = \mathcal{A}(G_\alpha, G_\beta, G_{\alpha,\beta})$ is an amalgam satisfying Hypothesis 5.2.1. If $Z_{\alpha'} \leq Q_\alpha$, then one of the following holds:*

(i) *$\mathcal{A}$ is a weak BN-pair of rank 2; or*

(ii) *$p = 3$, $b = 1$, $|S| \leqslant 3^7$ and the shapes of L_α and L_β are known.*

Consequently, if $\mathcal{A}$ is obtained from a fusion system satisfying Hypothesis 5.1.12, then $\mathcal{F}$ is not a counterexample to the Main Theorem.

5.5 Some Further Classification Results

We first prove Corollary A. That is, we classify saturated fusion systems in which there are exactly two essentials.

Corollary 5.5.1. *Suppose that $\mathcal{F}$ is a saturated fusion system on a p-group S such that $O_p(\mathcal{F}) = \{1\}$. Assume that $\mathcal{F}$ has exactly two essential subgroups E_1 and E_2. Then $N_S(E_1) = N_S(E_2)$ and writing $\mathcal{F}_0 := \langle N_{\mathcal{F}}(E_1), N_{\mathcal{F}}(E_2)\rangle_{N_S(E_1)}$, $\mathcal{F}_0$ is a saturated normal subsystem of $\mathcal{F}$ and either*

(i) *$\mathcal{F} = \mathcal{F}_0$ is determined by the Main Theorem;*

(ii) *p is arbitrary, $\mathcal{F}_0$ is isomorphic to the p-fusion category of H, where $F^*(H) \cong \mathrm{PSL}_3(p^n)$, and $\mathcal{F}$ is isomorphic to the p-fusion category of G where G is the extension of H by a graph or graph-field automorphism;*

(iii) *$p = 2$, $\mathcal{F}_0$ is isomorphic to the 2-fusion category of H, where $F^*(H) \cong \mathrm{PSp}_4(2^n)$, and $\mathcal{F}$ is isomorphic to the 2-fusion category of G where G is the extension of H by a graph or graph-field automorphism; or*

(iv) *$p = 3$, $\mathcal{F}_0$ is isomorphic to the 3-fusion category of H, where $F^*(H) \cong \mathrm{G}_2(3^n)$, and $\mathcal{F}$ is isomorphic to the 3-fusion category of G where G is the extension of H by a graph or graph-field automorphism.*

Proof. Note that if both E_1 and E_2 are $\mathrm{Aut}_{\mathcal{F}}(S)$-invariant then, appealing to Proposition 5.2.9 to verify that E_1 and E_2 are maximally essential, $\mathcal{F} = \mathcal{F}_0$ is determined by the Main Theorem. Assume throughout that at least one of E_1 and E_2 is not $\mathrm{Aut}_{\mathcal{F}}(S)$-invariant, and without loss of generality, E_1 is not $\mathrm{Aut}_{\mathcal{F}}(S)$-invariant. Then $N_S(E_1)\alpha \leq N_S(E_1\alpha)$ and since E_1 is fully $\mathcal{F}$-normalized, it follows that $N_S(E_1)\alpha = N_S(E_1\alpha)$. Moreover, $E_1\alpha$ is also essential in $\mathcal{F}$ and so $E_1\alpha = E_2$. By a similar reasoning, $E_2\alpha = E_1$, $\alpha^2 \in N_{\mathcal{F}}(E_1) \cap N_{\mathcal{F}}(E_2)$ and both E_1 and E_2 are maximally essential. Suppose first that p is odd. Then $S = N_S(E_1) = N_S(E_2)$ and by [AKO11, Lemma I.7.6(b)] and the Alperin–Goldschmidt theorem, $\mathcal{F}_0$ is a saturated subsystem of $\mathcal{F}$ of index 2 and by [AKO11, Theorem I.7.7], $\mathcal{F}_0$ is normal in $\mathcal{F}$. Hence, $O_p(\mathcal{F}_0)$ is normalized by $\mathcal{F}$ and as $O_p(\mathcal{F}) = \{1\}$, $O_p(\mathcal{F}_0) = \{1\}$ and $\mathcal{F}_0$ is determined by the Main Theorem.

Since there is $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$ such that $E_1\alpha = E_2$, we must have that $E_1 \cong E_2$ as abstract p-groups. Thus, comparing with the Main Theorem, $\mathcal{F}_0$ is isomorphic to the p-fusion category of H where $F^*(H)$ is one of $\mathrm{PSL}_3(p^n)$ or $\mathrm{G}_2(3^n)$ (where $p > 2$ is arbitrary or $p = 3$ respectively). Indeed, since $\mathcal{F}_0 \trianglelefteq \mathcal{F}$, there is $\mathcal{F}^0 \trianglelefteq \mathcal{F}$ with $\mathcal{F}^0$ is isomorphic to the p-fusion category of $F^*(H)$ and supported on S. At this point, we can either apply [BMO19, Theorem A]; or recognize that the possible fusion systems correspond exactly to overgroups G of $F^*(H)$ such that $F^*(G) = F^*(H)$ by applying [AKO11, Theorem I.7.7].

Suppose now that $p = 2$. Then $N_S(E_1) = N_S(E_2) = E_1E_2$ has index 2 in S. Let G_i be a model for $N_{\mathcal{F}}(E_i)$ for $i \in \{1, 2\}$. Note that if there is $Q \leq N_S(E_1)$ with Q normal in both $N_{\mathcal{F}}(E_1)$ and $N_{\mathcal{F}}(E_2)$, then $Q\alpha = Q$ is normal in $\mathcal{F}$. Since $O_2(\mathcal{F}) = \{1\}$, we deduce that Q is trivial. Moreover, applying [Asc10, (2.2.4)], $N_{G_1}(E_2) = N_{G_1}(N_S(E_2))$ is isomorphic to $N_{G_2}(E_1) = N_{G_2}(N_S(E_2)$ by an isomorphism fixing

$N_S(E_1)$.

Hence, suppressing the necessary inclusion maps, we form the rank 2 amalgam $\mathcal{A} := \mathcal{A}(G_1, G_2, G_{12}^*)$ writing G_{12}^* for the group gained by identifying $N_{G_1}(N_S(E_1))$ with $N_{G_2}(N_S(E_2))$ in the previously described isomorphism. Then $\mathcal{F}_0 = \langle \mathcal{F}_{N_S(E_1)}(G_1), \mathcal{F}_{N_S(E_2)}(G_2)\rangle = \mathcal{F}_{N_S(E_1)}(G_1 *_{G_{12}^*} G_2)$ by Theorem 5.1.6, and $O_2(\mathcal{F}_0) = \{1\}$. Moreover, $\mathcal{A}$ satisfies Hypothesis 5.2.1 and since $E_2 = E_1\alpha$, E_1 and E_2 are isomorphic as abstract 2-groups. Then $G_1 *_{G_{12}^*} G_2$ is locally isomorphic to H where $H \in \bigwedge^0$ is as described after Definition 5.1.7, and $F^*(H) \cong \mathrm{PSL}_3(2^n)$ or $\mathrm{PSp}_4(2^n)$. Then by Corollary 5.1.9, $\mathcal{F}_0$ is isomorphic to the 2-fusion category of Y and so $\mathcal{F}_0$ is saturated. Moreover, applying [AKO11, Theorem I.7.4] and the Alperin–Goldschmidt theorem, $\mathcal{F}_0$ is a normal subsystem of index 2 in $\mathcal{F}$. Again, there is $\mathcal{F}^0 \trianglelefteq \mathcal{F}$ with $\mathcal{F}^0$ isomorphic to the p-fusion category of $F^*(H)$ and supported on $N_S(E_1)$ and we can either apply [BMO19, Theorem A]; or recognize that the possible fusion systems correspond exactly to overgroups G of $F^*(H)$ such that $F^*(G) = F^*(H)$ by applying [AKO11, Theorem I.7.4] and [AKO11, Theorem I.7.7]. $\square$

We now turn our attention to identifying some finite simple groups from a situation motivated by Hypothesis 5.2.1. In Theorem 5.2.2, when a group has a weak BN-pair and is determined up to local isomorphism, then almost all the groups occurring as appropriate Sylow completions are known (see [PR06]). Thus, we investigate the cases where the amalgam is described up to some weaker form of isomorphism. For this, we make use of several identification results already present in the literature, and often implicitly use MAGMA and the list of maximal subgroups in the Atlas [Con+85] for computations. Moreover, we assume all the details regarding the amalgams which were collected in earlier sections.

It is worth pointing out that a consistent theme in these identification results is that the centralizer of an element of the center of a Sylow p-subgroup, for some appropriate prime p, is of characteristic p. Recall that a finite group G is of *parabolic characteristic* p if the normalizers of p-subgroups which contain some Sylow p-subgroup of G are of characteristic p. One can prove, using some *balance* arguments, that it suffices to check that the centralizers of elements of order p which contain some Sylow p-subgroup of G are constrained. In most of our examples, for an appropriate $S \in \mathrm{Syl}_p(G)$, $|Z(S)| = p$ and so the condition G has a parabolic characteristic p is equivalent to demanding that $N_G(Z(S))$ is of characteristic p, which in the cases listed here is equivalent to $C_G(Z(S))$ being of characteristic p.

First, recall that an element $x \in S \in \mathrm{Syl}_p(G)$, where G is some finite group, is *weakly closed* in S with respect to G if $x^G \cap S = \{x\}$. Throughout, for $S \in \mathrm{Syl}_p(G)$ as specified, we let $Z := Z(S)$, $N := N_G(Z)$ and $C := C_G(Z)$.

Theorem 5.5.2. *Suppose that G is a finite group and $H, M \leq G$ such that*

(i) *there is $H_1, H_2 \trianglelefteq H$ with $H_1 \cong H_2 \cong \mathrm{SL}_2(3)$, $|H : H_1H_2| = 2$, $|H_1 \cap H_2| = 2$, and $H = C_G(H_1 \cap H_2)$; and*

(ii) *$H_1 \cap H_2 \leq V \trianglelefteq M$ with $V \cong 2^3$ and $M/V \cong \mathrm{PSL}_3(2)$.*

Then $G \cong \mathrm{G}_2(3)$.

Proof. This is the main theorem of [Asc02]. □

Corollary 5.5.3. *Suppose that G is a finite group such that C is of characteristic 2 and G is a Sylow completion of the amalgam described in Proposition 5.3.16 (ii). Then $G \cong \mathrm{G}_2(3)$.*

Proof. Since $G_\beta \leq C$ and $O_2(C)$ is self-centralizing in C, we have that $O_2(C) = Q_\beta$. But $\Phi(Q_\beta) = Z_\beta$ and by Lemma 2.1.8, C/Q_β embeds as a subgroup of $\mathrm{GL}_4(2)$. We search for subgroups Y of $\mathrm{GL}_4(2)$ such $|Y|_2 = 2$, $O_2(Y) = \{1\}$ and, as $G_\beta/Q_\beta \leq C/Q_\beta$, some subgroup of Y is isomorphic to $(3 \times 3) : 2$. One can check (e.g. using MAGMA) that this implies that $C = G_\beta$.

Let $H = G_\beta$ so that $Z = Z(H)$ and $H/O_2(H) \cong (3 \times 3) : 2$. Choose r_1, r_2 3-elements in H such that $|C_{O_2(H)/Z}(r_i)| = 4$. Then $H_i := O^2(S\langle r_i \rangle) \cong \mathrm{SL}_2(3)$, $H_i \trianglelefteq H$, $|H : H_1H_2| = 2$ and $H_1 \cap H_2 = Z$. Thus, G satisfies (i) of Theorem 5.5.2.

Set $V = \bigcap_{\lambda \in \Delta(\alpha)}(Q_\alpha \cap Q_\lambda)$ so that V is elementary abelian of order 8 and contains Z_α. Moreover, $|H : N_H(V)| = 3$ and $N_H(V)/O_2(H) \cong \mathrm{Sym}(3)$. Setting $M := \langle G_\alpha, N_H(V) \rangle$, we have that $V = O_2(M)$ and M/V has weak BN-pair locally isomorphic to $\mathrm{PSL}_3(2)$. Since $J(S) \not\leq V$ and $Z \ntrianglelefteq M$, we have that V is an FF-module for M. It follows from [CD91, Theorem A], that $M/V \cong \mathrm{PSL}_3(2)$ and so G satisfies (ii) of Theorem 5.5.2. Thus, $G \cong \mathrm{G}_2(3)$, as required. □

Theorem 5.5.4. *Let G be a finite group, z an involution in G, $H = C_G(z)$, $Q = O_2(H)$ and $X \in \mathrm{Syl}_3(H)$. Assume that*

(i) *Q is extraspecial of order 32;*

(ii) *$H/Q \cong \mathrm{Sym}(3)$ and $C_Q(X) = \langle z \rangle$; and*

(iii) *z is not weakly closed in Q w.r.t G.*

Then one of the following holds:

(i) *There is $V \trianglelefteq G$ such that V is elementary abelian of order 8 and $G/V \cong \mathrm{PSL}_3(2)$.*

(ii) $G \cong \mathrm{Alt}(8)$ *or* $\mathrm{Alt}(9)$ *and the two* Q_8*-subgroups of* Q *are not normal in* H*.*

(iii) $G \cong \mathrm{M}_{12}$ *and the two* Q_8*-subgroups of* Q *are normal in* H*.*

Proof. This is [Asc03]. $\square$

Corollary 5.5.5. *Suppose that* G *is a finite group such that* $C_G(Z)$ *is of characteristic* 2 *and* G *is a Sylow completion of an amalgam parabolic isomorphic to* M_{12}*. Then* $G \cong \mathrm{M}_{12}$ *or* $\mathrm{G}_2(3)$*.*

Proof. Note that $G_\beta \leq C$ and since C is of characteristic 2, we either have that $O_2(C) = Q_\beta$, or $O_2(C)$ is elementary abelian of order 8. In the latter case, it follows that $C/O_2(C)$ embeds into a subgroup of the automorphism group of $O_2(C)$ which fixes Z. But such a subgroup is isomorphic to $2^2 : \mathrm{Sym}(3)$ and so $C = G_\beta$ and $O_2(C) = Q_\beta$, a contradiction. Thus, we have that $O_2(C) = Q_\beta$ and $\Phi(Q_\beta) = Z_\beta$. Since $O_2(C/Q_\beta) = \{1\}$, by Lemma 2.1.8, C/Q_β embeds faithfully into $\mathrm{Aut}(Q_\beta/\Phi(Q_\beta)) \cong \mathrm{GL}_4(2)$. We search for subgroups Y of $\mathrm{GL}_4(2)$ such $|Y|_2 = 2$, $O_2(Y) = \{1\}$ and, as $G_\beta/Q_\beta \leq C/Q_\beta$, some subgroup of Y is isomorphic to $\mathrm{Sym}(3)$. Thus, $Y \in \{\mathrm{Sym}(3), (3 \times 3) : 2, \mathrm{Sym}(3) \times 3\}$.

If $C/Q_\beta \cong (3 \times 3) : 2$, then in a similar manner to Corollary 5.5.3, we have that G satisfies the hypothesis of Theorem 5.5.2 and $G \cong \mathrm{G}_2(3)$. If $C/Q_\beta \cong \mathrm{Sym}(3) \times 3$ then a Sylow 3-subgroup of $N_C(S)$ normalizes $Q_\alpha = C_S(\Omega(Z_2(S)))$. But $|Q_\alpha/\Phi(Q_\alpha)| = 2^3$ and by Lemma 2.1.8, $N_G(Q_\alpha)/Q_\alpha$ is isomorphic to a subgroup of $\mathrm{GL}_3(2)$ with Sylow 2-subgroup of order 2, no non-trivial normal 2-subgroups and contains a subgroup isomorphic to $\mathrm{Sym}(3)$, so that $N_G(Q_\alpha)/Q_\alpha \cong \mathrm{Sym}(3)$ and as $N_C(S) \leq N_G(Q_\alpha)$, we arrive at a contradiction.

If $C/Q \cong \mathrm{Sym}(3)$ then letting $H := C$ and $z \in Z_\beta$ so that $\langle z \rangle = Z = Z(H)$, G

satisfies (i) and (ii) of the hypothesis of Theorem 5.5.4. Moreover, since Z_β is not normalized by G_α, G also satisfies (iii). Since $O_2(G) = \{1\}$, it remains to show that outcome (ii) of Theorem 5.5.4 does not occur. Since the M_{12} amalgam is determined up to parabolic isomorphism, S is determined up to isomorphism. In particular, $m_2(S) = 3$. However, for $T \in \mathrm{Syl}_2(\mathrm{Alt}(8))$, $m_2(T) = 4$ and so outcome (ii) does not occur. □

We remark that, by work of Fan [Fan86], when G is parabolic isomorphic to M_{12}, then G is locally isomorphic to M_{12} and so this case is reasonably well understood without the need for Aschbacher's result.

Theorem 5.5.6. *Suppose that G is a finite group and $S \in \mathrm{Syl}_2(G)$. Further assume that G has an involution z such that*

(i) *$C_G(z)$ is of characteristic 2;*

(ii) *$O_2(C_G(z)) \cong 2_-^{1+4}$;*

(iii) *$C_G(z)/O_2(C_G(z)) \cong \mathrm{Alt}(5)$; and*

(iv) *Z is not weakly closed in S w.r.t G.*

Then either G has two classes of involutions and $G \cong \mathrm{J}_2$; or G has a unique class of involutions and $G \cong \mathrm{J}_3$.

Proof. See [Asc94, Section 47] for the uniqueness of J_2 and [Fro83] for the uniqueness of J_3. □

Corollary 5.5.7. *Suppose that G is a finite group such that C is of characteristic 2 and G is a Sylow completion of an amalgam parabolic isomorphic to J_2. Then $G \cong \mathrm{J}_2$ or J_3.*

Proof. Since $G_\beta \leq C$, G_β is irreducible on Q_β/Z_β and C is of characteristic 2, we deduce that $Q_\beta = O_2(C)$ and (ii) of Theorem 5.5.6 is satisfied. By Lemma 2.1.8, using that $O_2(C/Q_\beta) = \{1\}$, we have that C/Q_β embeds as a subgroup of $\mathrm{GL}_4(2)$ with Sylow 2-subgroup of order 4 and contains a subgroup isomorphic to $\mathrm{PSL}_2(4) \cong G_\beta/Q_\beta$. It transpires that either $C = G_\beta$ or $C/Q_\beta \cong \mathrm{PSL}_2(4) \times 3$.

In the latter case, for y the 3-element in $C_C(S/Q_\beta)$, we have that y normalizes S so normalizes $Q_\alpha = C_S(Z_2(S))$. But $Z_\alpha = \Phi(Q_\alpha)$ and $|Q_\alpha/Z_\alpha| = 2^4$ so that, again by Lemma 2.1.8, $N_G(Q_\alpha)/Q_\alpha$ embeds as a subgroup of $\mathrm{GL}_4(2)$ and as in Corollary 5.5.5, we have that $N_G(Q_\alpha)/Q_\alpha$ is isomorphic to one of $\mathrm{Sym}(3), (3 \times 3) : 2$ or $\mathrm{Sym}(3) \times 3)$. Moreover, since $y \in N_G(S) \leq N_G(Q_\alpha)$ we must have that $N_G(Q_\alpha)/Q_\alpha \cong \mathrm{Sym}(3) \times 3$. But then, the index of $C_{G_\lambda}(y)$ in G_λ is a 2-group for $\lambda \in \{\alpha, \beta\}$ and as $Z_\beta \leq C_G(y)$, the actions of G_λ/Q_λ implies that $S \leq C_G(y)$, impossible since y acts non-trivially on Q_β/Z_β.

Thus, $C = G_\beta$ and (iii) of Theorem 5.5.6 is satisfied. Moreover, since Z is not normalized by G_α, G also satisfies (iv) and the result follows. $\square$

For the next characterization, we define a $\mathcal{K}$-proper finite group to be a finite group in which every proper subgroup is a $\mathcal{K}$-group.

Theorem 5.5.8. *Let G be a finite $\mathcal{K}$-proper group with $S \in \mathrm{Syl}_3(G)$. Suppose that:*

(i) *Z has order 3 and $Z_2(S)$ has order 9;*

(ii) *$N_G(Z_2(S)) \sim 3^{2+3+2+2} : 2.\mathrm{Sym}(4)$ is of characteristic 3;*

(iii) *$N \sim 3^{1+2+1+2+1+2} : 2.\mathrm{Sym}(4)$ is of characteristic 3; and*

(iv) $G = \langle N, N_G(Z_2(S))\rangle$ *and* $O_3(G) = \{1\}$.

Then $G \cong \mathrm{F}_3$.

Proof. This is the main result of [Fow07]. $\square$

Corollary 5.5.9. *Suppose that G is a finite $\mathcal{K}$-proper group such that C is of characteristic* 3 *and G is a Sylow completion of an amalgam parabolic isomorphic to* F_3*-amalgam. Then* $G \cong \mathrm{F}_3$.

Proof. From the structure of the F_3-amalgam, in order to apply Theorem 5.5.8 it suffices to show, in the language of Section 5.4.2, that $N = G_\beta$ and $N_G(Z_2(S)) = G_\alpha$, remarking that $Z = Z_\beta$ and $Z_2(S) = Z_\alpha$. Notice that $G_\alpha/Q_\alpha \cong \mathrm{Aut}(Z_2(S))$ and so $N_G(Z_2(S)) = G_\alpha C_G(Z_2(S)) \leq G_\alpha C_{C_G(Z(S))}(Z_2(S))$. In particular, upon demonstrating that $N_G(Z(S)) = G_\beta$, we have that $N_G(Z_2(S)) \leq G_\alpha C_{G_\beta}(Z_2(S)) \leq G_\alpha$. We shall adopt the language of Section 5.4.2 throughout. We first aim to show that $Q_\beta = O_3(N)$ and as $O_3(C) \trianglelefteq N$, we may as well demonstrate that $Q_\beta = O_3(C)$.

Since G is of parabolic characteristic 3, we have that $O_3(C)$ is self-centralizing and properly contains Z_β. In particular, $O_3(C)$ is normal in S and so $(O_3(C)/Z_\beta) \cap Z(S/Z_\beta) \neq \{1\}$. Then, as $Z_\alpha = Z_2(S)$ and $L_\beta \leq C$, $V_\beta \leq O_3(C)$. Suppose first that $\Omega(Z(O_3(C)) = Z_\beta$. Then $O_3(C) \not\leq C_\beta$ and $Q_\beta = O_3(C)C_\beta$. Furthermore, $[V_\beta^{(3)}, O_3(C)]V_\beta = \Omega(Z(V_\beta^{(3)})) \leq O_3(C)$ and $[C_\beta, O_3(C)]\Omega(Z(V_\beta^{(3)})) = V_\beta^{(3)} \leq O_3(C)$. If $C_\beta \leq O_3(C)$, then $O_3(C) = Q_\beta$ and the result holds and so, we may assume that $O_3(C) \cap C_\beta = V_\beta^{(3)}$. Note that $\Omega(Z(V_\beta^{(3)})) = [O_3(C), V_\beta^{(3)}] \leq \Phi(O_3(C))$ and so $V_\beta^{(3)}$ is equal to one of the characteristic subgroups $\Phi(O_3(C))$ or $C_{O_3(C)}(\Phi(O_3(C)))$, and so $\Omega(Z(V_\beta^{(3)}))$ is also characteristic in $O_3(C)$. But then C_β

centralizes the chain $\{1\} \trianglelefteq Z_\beta \trianglelefteq \Omega(Z(V_\beta^{(3)})) \trianglelefteq V_\beta^{(3)} \trianglelefteq O_3(C)$ and by Lemma 2.1.9, $C_\beta \leq O_3(C)$, a contradiction.

If $Z_\beta < \Omega(Z(O_3(C)))$, $V_\beta \leq \Omega(Z(O_3(C)))$ then $O_3(C) \leq C_\beta$. Moreover, it follows that $V_\beta \leq \Omega(Z(O_3(C))) \leq \Omega(Z(V_\beta^{(3)}))$. Then $[O_3(C), \Omega(Z(V_\beta^{(3)}))] \leq Z_\beta \leq \Omega(Z(O_3(C)))$ and by Lemma 2.1.9, $\Omega(Z(V_\beta^{(3)})) \leq O_3(C)$. If $V_\beta = \Omega(Z(O_3(C)))$, then $V_\beta^{(3)}$ centralizes the chain $\{1\} \trianglelefteq \Omega(Z(O_3(C))) \trianglelefteq O_3(C)$ and by Lemma 2.1.9, $V_\beta^{(3)} \leq O_3(C)$ so that $C_\beta = O_3(C)$. Now, $\Phi(O_3(C)) = V_\beta$ and so, $C/O_3(C)$ acts faithfully on C_β/V_β and so embeds into $\mathrm{GL}_4(3)$. Moreover, $C_C(V_\beta/Z_\beta)$ is a normal subgroup of C which has Q_β as its Sylow 3-subgroup. Thus, we turn our attention to subgroups H of $\mathrm{GL}_4(3)$ such that $|H|_3 = 3^3$, $O_3(H) = \{1\}$ and H has a normal subgroup N such that $|N|_3 = 3^2$. One can calculate, using MAGMA, that no groups satisfy this property, providing a contradiction.

Finally, if $\Omega(Z(O_3(C))) = \Omega(Z(V_\beta^{(3)}))$, then $O_3(C) \leq V_\beta^{(3)}$ and since $O_3(C)$ is normalized by L_β and is self-centralizing, we have that $O_3(C) = V_\beta^{(3)}$. But then, C_β centralizes the chain $\{1\} \trianglelefteq Z_\beta \trianglelefteq \Omega(Z(O_3(C))) \trianglelefteq O_3(C)$, a contradiction by Lemma 2.1.9.

Thus, we have shown that $Q_\beta = O_3(C)$. Furthermore, one can compute that $\Phi(Q_\beta) = C_\beta$ has index 9 in Q_β, and by Lemma 2.1.8, N/Q_β is isomorphic to a subgroup of $\mathrm{GL}_2(3)$. Since $G_\beta/Q_\beta \cong \mathrm{GL}_2(3)$, we have that $N = G_\beta$, as required. This completes the proof. □

Theorem 5.5.10. *Suppose that G is a group and $S \in \mathrm{Syl}_3(G)$. Further assume that*

(i) $|N| = 2^7.3^6$;

(ii) *$O_3(N)$ is extraspecial of order 3^5;*

(iii) $O_2(N) = \{1\}$;

(iv) $O_2(N/O_3(N)) \cong Q_8 \times Q_8$;

(v) $|N/O^2(N)| = 2$; and

(vi) $O_3(N)/Z(O_3(N))$ is an N-chief factor.

Then either Z is weakly closed in S or $G \cong \mathrm{PSU}_6(2)$.

Proof. See [Par06, Theorem 1]. □

Corollary 5.5.11. *Suppose that G is a finite group such that C is of characteristic 3 and G is a Sylow completion of the amalgam described in Proposition 5.4.55 (ii). Then $G \cong \mathrm{PSU}_6(2)$.*

Proof. From the structure of the amalgam in Proposition 5.4.55 (ii), we may choose $t \in L_\alpha \cap G_{\alpha,\beta}$ of order 4, such that $t \in N_G(Z)$ and $t^2 \in C$. Moreover, since Z_α is isomorphic to an $\Omega_4^-(3)$-module, t acts irreducibly on $Z_\alpha \cap Q_\beta/Z$ and t inverts $S/Q_\beta \cong Z_\alpha/Z_\alpha \cap Q_\beta$. Then, one can calculate that $t^2 \in L_\beta$, Q_β/Z is irreducible as a $L_\beta\langle t\rangle$-module and $G_\beta = L_\beta\langle t\rangle$.

In order to apply Theorem 5.5.10, we need only show that $N = G_\beta$. Since C is of characteristic 3, we have that $Z_\beta < O_3(C) \leq O_3(N)$ and since $G_\beta \leq N$, $O_3(N) = Q_\beta$. Thus, N/Q_β embed into the automorphism group of Q_β and so by [Win72], N/Q_β is isomorphic to a subgroup of $\mathrm{Sp}_4(3) : 2$. Moreover, $|N/Q_\beta|_3 = 3$ and N/Q_β contains a subgroup isomorphic to G_β/Q_β which has order $2^7.3$ and a comparison with the maximal subgroups of $\mathrm{Sp}_4(3) : 2$ yields $N = G_\beta$, as required. □

Theorem 5.5.12. *Suppose that G is a finite group, $S \in \mathrm{Syl}_3(G)$ and J is an elementary abelian subgroup of S of order 3^4. Further assume that*

(i) $O^{3'}(N) \cong 3_+^{1+4}.2.\mathrm{Alt}(5)$;

(ii) $O^{3'}(N_G(J)) \cong 3^4.\mathrm{Alt}(6)$; *and*

(iii) C *is of characteristic* 3.

Then $G \cong \mathrm{McL}$ *or* $\mathrm{Aut}(\mathrm{McL})$.

Proof. See [PStr14, Theorem 1.1]. □

Corollary 5.5.13. *Suppose that G is a finite group such that C is of characteristic 3 and G is a Sylow completion of the amalgam described in Proposition 5.4.55 (iii). Then* $G \cong \mathrm{McL}$ *or* $\mathrm{Aut}(\mathrm{McL})$.

Proof. By Proposition 5.4.55 (iii), in order to apply Theorem 5.5.12, taking $J = Z_\alpha$, it suffices to show that $N = G_\beta$ and $O^{3'}(N_G(J)) = L_\alpha$. Since C is of characteristic 3, $O_3(C)$ is self-centralizing. Moreover, $L_\beta \le C$ and acts irreducibly on Q_β/Z from which it follows that $O_3(C) = Q_\beta$, and as $C \trianglelefteq N$, we have that $Q_\beta = O_3(N)$. Thus, N/Q_β embeds into the automorphism group of Q_β and so again by [Win72], N/Q_β is isomorphic to a subgroup of $\mathrm{Sp}_4(3) : 2$. Moreover, $|N/Q_\beta|_3 = 3$ and N/Q_β contains a subgroup isomorphic to G_β/Q_β, remarking that $|G_\beta| = 2|L_\beta|$ and $L_\beta/Q_\beta \cong \mathrm{SL}_2(5)$. Computing in $\mathrm{Sp}_4(3)$, we have that $N = G_\beta$, as desired. Now, $N_G(J)/Z_\alpha$ embeds as a subgroup of $\mathrm{GL}_4(3)$, $|N_G(J)/Z_\alpha| = 9$ and $N_G(J)/Z_\alpha$ contains a subgroup isomorphic to $L_\alpha/Z_\alpha \cong \mathrm{PSL}_2(9)$. But for all such subgroups, the normal closure of a Sylow 3-subgroup is isomorphic to $\mathrm{PSL}_2(9)$, as desired. □

Theorem 5.5.14. *Suppose that G is a finite group and $S \in \mathrm{Syl}_3(G)$. Further assume that*

(i) $O_3(C)$ *is extraspecial of order* 3^5*;*

(ii) $O_2(C/O_3(C))$ *is extraspecial of order* 2^5*; and*

(iii) $C/O_{3,2}(C) \cong \mathrm{Alt}(5)$.

Then either Z *is weakly closed in* S *or* $G \cong \mathrm{Co}_2$.

Proof. See [PR10, Theorem 1.1]. □

Corollary 5.5.15. *Suppose that* G *is a finite group such that* C *is of characteristic* 3 *and* G *is a Sylow completion of the amalgam described in Proposition 5.4.55 (iv). Then* $G \cong \mathrm{Co}_2$.

Proof. By Proposition 5.4.55 (iv), and since Z is not normalized by G_α, to apply Theorem 5.5.14, it suffices to show that $C = L_\beta$. Since $O_3(C)$ is self-centralizing and $L_\beta \leq C$ is irreducible on Q_β/Z, we have that $O_3(C) = Q_\beta$. Now, C/Q_β embeds into the automorphism group of Q_β and again by [Win72], C/Q_β is isomorphic to a subgroup of $\mathrm{Sp}_4(3)$. Moreover, $|C/Q_\beta|_3 = 3$, C/Q_β contains a subgroup isomorphic to $L_\beta/Q_\beta \cong 2_-^{1+4}.\mathrm{Alt}(5)$ and computing in $\mathrm{Sp}_4(3)$, we have that $C = L_\beta$, as required. □

Theorem 5.5.16. . *Suppose that* G *is a finite group,* $S \in \mathrm{Syl}_3(G)$ *and* $J \leq S$*. Further assume that*

(i) $N \sim 3^+_{1+4}.2.2.\mathrm{PSL}_2(9).2$*; and*

(ii) $N_G(J) \sim 3^5 : (2 \times \mathrm{M}_{11})$.

Then $G \cong \mathrm{Co}_3$.

Proof. This is [KPR07, Theorem 1]. □

Corollary 5.5.17. *Suppose that G is a finite group such that $C_G(Z)$ is of characteristic 3 and G is a Sylow completion of the amalgam described in Proposition 5.4.54 (iv). Then $G \cong \mathrm{Co}_3$.*

Proof. Comparing with proofs in [KPR07], to apply Theorem 5.5.16 it is enough in the context of Proposition 5.4.54 (iv) to show that $N = G_\beta$ and $N_G(J) = G_\alpha$. Since $O_3(C)$ is self-centralizing, and $L_\beta \leq C$ and acts irreducibly on Q_β/Z, we have that $O_3(C) = Q_\beta$. Since $C \trianglelefteq N$, we have that $Q_\beta = O_3(N)$. Thus, N/Q_β embeds into the automorphism group of Q_β and by [Win72], we have that N/Q_β is isomorphic to a subgroup of $\mathrm{Sp}_4(2):2$. Furthermore, $|N/Q_\beta|_3 = 3$ and N/Q_β contains a subgroup isomorphic to $L_\beta/Q_\beta \cong \mathrm{SL}_2(9)$. Computing in $\mathrm{Sp}_4(3)$, we infer that $N = G_\beta$, as required. Now, $N_G(J)/Z_\alpha$ embeds as a subgroup of $\mathrm{GL}_5(3)$, $|N_G(J)/Z_\alpha| = 9$ and $N_G(J)/Z_\alpha$ contains a subgroup isomorphic to $L_\alpha/Z_\alpha \cong \mathrm{M}_{11}$, remarking that $|G_\alpha| = 2|L_\alpha|$. Since M_{11} is a maximal subgroup of $\mathrm{SL}_5(3)$ and $|\mathrm{GL}_5(3)/\mathrm{SL}_5(3)| = 2$, we conclude that $N_G(J) = G_\alpha$, as required. □

Recall that a group is of local characteristic p if the normalizers of non-trivial p-subgroups are of characteristic p. Thus, groups of local characteristic p are of parabolic characteristic, but not necessarily the other way about. As in the case of parabolic characteristic p, it suffices to check the normalizers of elements of order p. Here, we set $\mathcal{L}$ to be the amalgam described in Proposition 5.3.15 (v) and define a $\mathcal{K}_2$-group to be a finite group in which the normalizer of every non-trivial 2-subgroup is a $\mathcal{K}$-group.

Theorem 5.5.18. *Suppose that G is a $\mathcal{K}_2$-group of local characteristic 5 which is a finite faithful completion of $\mathcal{L}$. If $L_\alpha \cap L_\beta \in \mathrm{Syl}_5(G)$, then there is an involution*

t in G such that $C_G(t) \cong 2.\mathrm{Alt}(11)$ and $G \cong \mathrm{Ly}$.

Proof. This is [PR04, Theorem 1.1]. □

Theorem 5.5.19. *Suppose that G is a $\mathcal{K}$-proper finite group, $S \in \mathrm{Syl}_7(G)$, $Z(S)$ has order 7, $Z_2(S)$ has order 49 and*

(i) *$N_G(Z_2(S)) \sim 7^{2+1+2}.\mathrm{GL}_2(7)$ is of characteristic 7;*

(ii) *$N_G(Z) \sim 7_+^{1+4}.2.\mathrm{Alt}(7).6$ is of characteristic 7; and*

(iii) *$G = \langle N_G(Z), N_G(Z_2(S))\rangle$ and $O_7(G) = \{1\}$.*

Then $G \cong \mathrm{M}$.

Proof. See [PW05, Theorem 1.1]. □

Corollary 5.5.20. *Suppose that G is a $\mathcal{K}$-proper finite group such that C is of characteristic 7 and G is a Sylow completion of the amalgam described in Proposition 5.3.15 (vi). Then $G \cong \mathrm{M}$.*

Proof. By Proposition 5.3.15 (vi), to apply Theorem 5.5.19, it suffices to prove that $N = G_\beta$ and $N_G(Z_2(S)) = G_\alpha$. Note that since C is of characteristic 7, $Z < O_7(C) \leq O_3(N)$ and since $L_\beta \leq C$ acts irreducibly on Q_β/Z, we have that $Q_\beta = O_7(C) = O_7(N)$ and N is of characteristic 7. Now $G_\alpha/Q_\alpha \cong \mathrm{GL}_2(7)\mathrm{Aut}(Z_2(S))$ and so $N_G(Z_2(S)) = G_\alpha C_G(Z_2(S)) = G_\alpha C_C(Z_2(S))$,and upon demonstrating that $C = L_\beta$, we have that $G_\alpha C_C(Z_2(S)) = G_\alpha C_{L_\beta}(Z_2(S)) = G_\alpha$. Hence, we need only show that $N = G_\beta$. Note that N/Q_β embeds into $\mathrm{Aut}(Q_\beta)$ so that by [Win72], N/Q_β is isomorphic to a subgroup of $\mathrm{Sp}_4(7) : 6$. Since $L_\beta/Q_\beta \cong 2.\mathrm{Alt}(7)$ and

$2.\mathrm{Alt}(7)$ is a maximal subgroup of $\mathrm{Sp}_4(7)$, and as $G_\beta = |L_\beta|6$, we have that $N = G_\beta$, as required. □

We are left with the amalgams coinciding with Proposition 5.3.17 (ii) when $p = 2$; Proposition 5.4.51 (ii), (iii) and (iv) when $p = 3$; Proposition 5.4.55 (v) and (vi) when $p = 3$ and Proposition 5.3.15 (iii),(iv) when $p = 5$. These have example completions $\mathrm{PSp}_6(3)$, $\mathrm{PSU}_5(2)$, $O_8^+(2)$, $\Omega_{10}^-(2)$, Suz, Ly, HN and B respectively.

In Proposition 5.3.17 (ii), taking $X := \langle R_\alpha G_{\alpha,\beta}, G_\beta \rangle$, we have that $C_\beta \trianglelefteq X$, $L_\beta/C_\beta \cong 2^4.\mathrm{PSL}_2(4)$ and $O^{2'}(R_\alpha S)/C_\beta \cong 2^{1+2+2}.\mathrm{Sym}(3)$. Thus, X/C_β is locally isomorphic to $\mathrm{PSU}_4(2) \cong \mathrm{PSp}_4(3)$. Indeed, it seems likely that in the finite groups which occur as suitable completions of the amalgam described in Proposition 5.3.17 (ii), there is a component in the centralizer of $\Omega(Z(S))$ which is isomorphic to a central extension of $\mathrm{PSU}_4(2)$ and so this type of configuration belongs in the analysis of groups or fusion systems which are of component type. Indeed, in the group $\mathrm{PSp}_6(3)$, the centralizer of $Z(S)$ for S a Sylow 2-subgroup is isomorphic to $2 \cdot (\mathrm{Alt}(4) \times \mathrm{PSU}_4(2))$. We will not say much more about this case.

In the situation of Proposition 5.4.51 (ii), and taking the stabilizer of a point in the action of $\mathrm{Alt}(5)$ on Z_α, we retrieve the group $\mathrm{Alt}(4) \cong \mathrm{PSL}_2(3)$. Indeed, one can choose the stabilized point, x say, to lie in Z_β. Then letting $L \leq L_\alpha$ such that $S \leq L$ and $L/Z_\alpha \cong \mathrm{Alt}(4)$, we get that for $X := \langle LG_{\alpha,\beta}, G_\beta \rangle$, we have that $Q := \langle x \rangle \trianglelefteq X$ and X/Q is locally isomorphic to $\mathrm{PSp}_4(3)$. As above, it seems likely that in the finite groups which occur as suitable completion of the amalgam described in Proposition 5.4.51 (ii), there is a component in the centralizer of some central element of a Sylow 3-subgroup which is isomorphic to $\mathrm{PSp}_4(3) \cong \mathrm{PSU}_4(2)$. This occurs in the group $\mathrm{PSU}_5(2)$.

In the situation of Proposition 5.4.51 (iii) or (iv), and taking the stabilizer of point in the action of L_α/Z_α on Z_α, we have a group L such that $O^{3'}(L)/Z_\alpha \cong \mathrm{PSL}_2(3)$ or $2^3.\mathrm{Alt}(4)$ respectively. In the latter case, the group coincides with L_α/Z_α in the former case. As above, one can choose the point x to lie in Z_β. Forming an appropriate X, we have that $Q := \langle x \rangle \trianglelefteq X$, and X/Q is locally isomorphic to $\mathrm{PSp}_4(3)$ or X/Q has the form of an amalgam satisfying Proposition 5.4.51 (iii) respectively. Again, it seems likely that finite groups occurring as good completions of these amalgams have some component in the centralizer of an element of order 3 which is central in a Sylow 3-subgroup, which is isomorphic to $\mathrm{PSp}_4(3)$ or $O_8^+(2)$ respectively. Indeed, $O_8^+(2)$ and $\Omega_{10}^-(2)$ have such a structure.

In Proposition 5.4.55 (v) and (vi), we again consider the stabilizer of a point in the action of L_α/Z_α on Z_α where this time $L_\alpha/Z_\alpha \cong \mathrm{M}_{11}$. We obtain a group L containing S such that $L/Q_\alpha \cong M_{10} \cong \mathrm{PSL}_2(9).2$. Choosing this point in $x \in Z(S)$ and making an appropriate X we get that $Q := \langle x \rangle \trianglelefteq X$ and X/Q is locally isomorphic to $\mathrm{PSU}_4(3)$ in Proposition 5.4.55 (v); or, in Proposition 5.4.55 (vi), is of the same type as in amalgam in Proposition 5.4.55 (iii) which had example completion McL. Again it seems likely that in any good finite group completion of these amalgams this subgroup corresponds to a component in the centralizer of some central element of a Sylow 3-subgroup. This is the case in the groups Suz and Ly.

It seems to it should be possible to characterize the finite groups occurring as parabolic characteristic 5 completions of the amalgams in Proposition 5.3.15 (iii) and (iv). It appears that the simple groups HN and B are the "unique" appropriate completions. This result is not available in the literature yet, but see [PW04, Theorem 2.1, Theorem 2.2].

Glossary of Notations

$\mathrm{GF}(q)$	The field of order q, where $q = p^n$ for some prime p.
$\Omega_i(P)$	If P is a p-group, the subgroup generated by all elements of order p^i in P, with convention $\Omega(P) = \Omega_1(P)$.
$\mho^i(P)$	If P is a p-group, the subgroup generated by the p^i-powers of all elements in P, with convention $\mho(P) = \mho^1(P)$.
$[A, B]$	For two subgroups $A, B \leq G$, the group generated by all elements of the form $a^{-1}b^{-1}ab$ for $a \in A$, $b \in B$.
$[A, B; i]$	For $A, B \leq G$, the group $[[A, \underbrace{B], B], \ldots, B}_{i \text{ times}}]$.
G'	$G' := [G, G]$, referred to as the commutator subgroup, or derived subgroup, of G.
$[V, G]$	The module generated by all elements of the form $x \cdot v - v$, $x \in G$, $v \in V$, where V is a module acted on by G.
$[V, G; i]$	For V a G-module, the submodule $[[V, \underbrace{G], G], \ldots, G}_{i \text{ times}}]$.
$G^{(i)}$	The subgroup of G such that $G^{(i)} = [G^{(i-1)}, G]$ chosen so that $G^{(1)} = G'$.
$C_A(B)$	All elements $a \in A$ such that $ab = ba$ for all $b \in B$, for subgroups $A, B \leq G$. We use the notation $C_A(b) := C_A(\langle b \rangle)$ where $b \in B$. This forms a subgroup of A.

$C_V(G)$	All elements $v \in V$ which are fixed under the action of G, where V is a module acted on by G. This forms a submodule of V.
$N_A(B)$	The largest subgroup of $A \leq G$ which normalizes $B \leq G$.
$\mathrm{Aut}(G)$	The automorphism group of G.
$\mathrm{Inn}(G)$	The inner automorphism group of G, that is, all automorphisms induced by the conjugation action of G on itself.
$\mathrm{Out}(G)$	The outer automorphism group of G, explicitly the quotient $\mathrm{Aut}(G)/\mathrm{Inn}(G)$.
$\mathrm{Hom}_G(A, B)$	The group of homomorphisms from a group A to a group B induced by conjugation by elements of G.
$\mathrm{Aut}_G(B)$	The group of automorphisms of B induced by conjugation by elements of G on B.
$\langle A^G \rangle$	The smallest subgroup containing A which is normal in G, referred to as the normal closure of A in G.
$Z(G)$	The center of G.
$Z_i(G)$	The subgroups of G satisfying $Z_i/Z_{i-1} = Z(G/Z_{i-1})$ chosen so that $Z_1(G) := Z(G)$. The ordered set $\{Z_1(G), Z_2(G), \dots\}$ is referred to as the upper central series of G.
$\Phi(G)$	The intersection of all maximal subgroups of G, known as the Frattini subgroup of G. If G is a p-group, then $\Phi(G) = [G, G]G^p$ is the smallest normal subgroup in which G has an elementary abelian quotient.

$O_\pi(G)$	The largest normal π-group of a group G, for π a set of primes. If $\pi = \{p\}$, then referred to as the p-core of G.
$O^\pi(G)$	The smallest normal subgroup of a group G such that the quotient is a π-group, for π a set of primes. Equivalently, $O^\pi(G)$ is the normal subgroup generated by all elements whose orders are coprime to all the primes in π.
$\lvert G\rvert_p$	The largest prime power p^n dividing the order of G.
$\mathrm{Syl}_p(G)$	For a prime p, the set of all Sylow p-subgroups of G. That is, all subgroups P of G such that $\lvert P\rvert = \lvert G\rvert_p$.
$m_p(G)$	For a prime p, the maximum rank of an elementary abelian p-subgroup of G.
$\mathcal{A}(P)$	For P a p-group, the collection of elementary abelian subgroups Q of P such that $\lvert Q\rvert = p^{m_p(P)}$.
$J(P)$	For P a p-group, the subgroup of P generated by all subgroups in $\mathcal{A}(P)$, referred to as the Thompson subgroup of P.
$F(G)$	The largest normal nilpotent subgroup of G, referred to as the Fitting subgroup of G
$E(G)$	The normal subgroup of G generated by all components of G, referred to as the layer of G.
$F^*(G)$	The normal subgroup generated by the Fitting subgroup and the layer, referred to as the generalized Fitting subgroup of G.

$A : B$	The semidirect product of A and B, where A is normalized by B.
$A.B$	An arbitrary extension of B by A. That is, A is a normal subgroup of $A.B$ such that the quotient of $A.B$ by A is isomorphic to B.
$A \cdot B$	A central extension of B by A.
$A * B$	The central product of A and B, where the intersection of A and B will be clear whenever this arises.

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