A GLOBAL APPROACH TO NONLINEAR BRASCAMP–LIEB INEQUALITIES AND RELATED TOPICS

by

JENNIFER H DUNCAN

A thesis submitted to
The University of Birmingham
for the degree of
Doctor of Philosophy

School of Mathematics
The University of Birmingham
February 14, 2022

Abstract

In this thesis, we investigate global nonlinear Brascamp–Lieb inequalities and some related problems in multilinear harmonic analysis. The body of this thesis is split into three parts, the first is concerning the near-monotonicity properties of nonlinear Brascamp–Lieb functionals under heat-flow. We establish a global nonlinear analogy to the heat-flow monotonicity property enjoyed by linear Brascamp–Lieb inequalities, which we use to prove a slight improvement of the local nonlinear Brascamp–Lieb inequality due to Bennett, Bez, Buschenhenke, Cowling, and Flock, as well as a global stability property of the finiteness of nonlinear Brascamp–Lieb inequalities. In the second part we prove a diffeomorphism-invariant weighted nonlinear Brascamp–Lieb inequality for maps that admit a certain structure that generalises the class of polynomial maps. Like polynomials, they have a well-defined notion of degree, and the best constant in this inequality depends explicitly on only the degree of these maps, as well as the underlying dimensions and exponents. Lastly, we refine an induction-on-scales method due to Bennett, Carbery, and Tao to prove a global multilinear L^2 estimate on oscillatory integral operators in general dimensions.

This thesis is dedicated to my cats Ellie, Charlie, and Betty, who every day teach me the value of persistence, be it in pursuit of a difficult L^p bound or a shoelace on a stick.

Acknowledgements

The author would firstly like to thank her supervisor Jonathan Bennett, whose patience, wisdom, and empathy were essential to the completion of this thesis, as well as her co-supervisor Alessio Martini for his many insightful suggestions that greatly improved the quality of this work. The author would also like to thank her parents for their endless love and support, and her many friends and colleagues at the University of Birmingham, thanks to whom the past few years as a postgraduate student have been the among the happiest in her life. This research was funded by a PhD scholarship from the Engineering and Physical Sciences Research Council.

Contents

1 Background and Context **1**

1.1 Linear Brascamp–Lieb Inequalities 2

1.2 Nonlinear Brascamp–Lieb Inequalities 8

1.3 Heat-flow Monotonicity 10

1.4 Nonlinear Multilinear Kakeya Inequalities 14

1.5 The Linear Theory of Oscillatory Integrals 21

1.6 The Wavepacket Decomposition for Oscillatory Integrals 23

1.7 The Multilinear Theory of Oscillatory Integrals 26

1.8 Guide to the Thesis 28

2 A Nonlinear Variant of Ball's Inequality: The Simple Case **29**

2.1 Preliminaries 30

2.2 Notation 32

2.3 Statement of the Theorem 33

2.4 Outline of the proof of Theorem 2.3.1 34

2.5 Gaussian Lemmas 37

2.6 Proof of Proposition 2.4.5 43

3 A Nonlinear Variant of Ball's Inequality: The General Case **47**

3.1 Setup and Notation 47

3.1.1 Statements of Results 48

3.1.2 Reduction of Theorem 3.1.2 52

3.1.3 A Regularised Effective Lieb's Theorem 54

3.1.4 Definition of $G_{x,\tau,j}$ 60

3.2 Gaussian Lemmas 64

3.3 Proof of Proposition 3.1.11 73
3.4 Proof of Corollaries 3.1.8 and 3.1.9 76

4 An Algebraic Brascamp–Lieb Inequality 79
4.1 Introduction 79
4.1.1 Preliminary Definitions and Notation 80
4.1.2 Main Results 82
4.2 Setup for the Proof of Theorem 4.1.3 86
4.2.1 Reductions 86
4.2.2 Central Constructions 90
4.2.3 Heuristic Explanation of Proof Strategy 92
4.3 Lemmas 94
4.4 Proof of Proposition 4.2.4 102
4.5 Two Kakeya–Brascamp–Lieb Versions 104
4.5.1 Setup 106
4.5.2 Reduction to a discrete inequality 106
4.5.3 Proof of Propositions 4.5.6 and 4.5.10 110

5 Global L^2-bounds for Multilinear Oscillatory Integrals 114
5.1 Statement of Results 115
5.2 Stationary Phase 117
5.3 Induction-on-Scales 124
5.4 The Remaining Proof 129

6 Further Research 133
6.1 Exact Heat-flow Monotonicity 133
6.2 Sharp Nonlinear Brascamp–Lieb Inequalities 135
6.3 Scale-Dependent Nonlinear Brascamp–Lieb Inequalities 137
6.4 Invariant Inequalities in Multilinear Harmonic Analysis 139

List of References 151

Chapter 1

Background and Context

In this chapter, we will introduce the ideas and results that we will refer to over the course of this thesis. Many central problems of interest to harmonic analysis, such as the restriction conjecture for example, involve operators whose functional-analytic properties depend on the geometric properties of some underlying manifold. Generally speaking, in multilinear settings, inequalities of interest are expected to hold provided that a collection of underlying manifolds are sufficiently 'transversal' in a suitable sense. As will become clear, one may view the Brascamp–Lieb inequalities as a fundamental manifestation of this type of transversality in multilinear analysis, that exist at the conceptual base of a hierarchy of related inequalities, including the celebrated multilinear Kakeya and Restriction inequalities. Indeed, the suitable notion of transversality required to access high-dimensional generalisations of Kakeya and restriction inequalities is itself formulated in terms of the optimal constant for an associated Brascamp–Lieb inequality; in this sense, Brascamp–Lieb inequalities serve to quantify a certain higher-order notion of transversality, their best constants acting as generalised wedge products. We begin with some background in their linear theory.

1.1 Linear Brascamp–Lieb Inequalities

For each $j \in \{1, ..., m\}$, let $L_j : \mathbb{R}^n \to \mathbb{R}^{n_j}$ be a linear surjection and $p_j \in [0, 1]$. The *Brascamp–Lieb inequality* associated with the pair $(\mathbf{L}, \mathbf{p}) := ((L_j)_{j=1}^m, (p_j)_{j=1}^m)$ is the following:

$$\int_{\mathbb{R}^n} \prod_{j=1}^m (f_j \circ L_j)^{p_j} \leqslant C \prod_{j=1}^m \left(\int_{\mathbb{R}^{n_j}} f_j \right)^{p_j} \quad \forall f_j \in L^1(\mathbb{R}^{n_j}), f_j \geqslant 0. \tag{1.1.1}$$

Using the notation of [14], we refer to the pair $(\mathbf{L}, \mathbf{p})$ as a *Brascamp–Lieb datum* (we shall often abuse this terminology and refer to $\mathbf{L}$ as a datum as well, given that $\mathbf{p}$ may often be regarded as fixed). We define the Brascamp–Lieb constant, $\mathrm{BL}(\mathbf{L},\mathbf{p})$, to be the infimum over all constants $C \in (0, \infty]$ for which the above inequality holds. For a given m-tuple of non–zero, non–negative functions $\mathbf{f} = (f_j)_{j=1}^m \in L^1(\mathbb{R}^{n_1}) \times ... \times L^1(\mathbb{R}^{n_m})$, we define the *Brascamp–Lieb functional* as

$$\mathrm{BL}(\mathbf{L}, \mathbf{p}; \mathbf{f}) := \frac{\int_{\mathbb{R}^n} \prod_{j=1}^m (f_j \circ L_j)^{p_j}}{\prod_{j=1}^m \left(\int_{\mathbb{R}^{n_j}} f_j \right)^{p_j}}. \tag{1.1.2}$$

We may then write $\mathrm{BL}(\mathbf{L}, \mathbf{p}) = \sup_{\mathbf{f}} \mathrm{BL}(\mathbf{L}, \mathbf{p}; \mathbf{f})$. The Brascamp–Lieb inequalities are a natural generalisation of many classical multilinear inequalities that commonly arise in analysis, examples of which include Hölder's inequality, Young's convolution inequality, and the somewhat lesser-known Loomis–Whitney inequality, which we shall now define. Let $\pi_j : \mathbb{R}^n \to \mathbb{R}^{n-1}$ denote the projection map onto the hyperplane $\langle e_j \rangle^\perp$, where $\langle e_j \rangle$ denotes the span of the j^{th} standard unit vector e_j. The Loomis–Whitney inequality states that the following holds for all non-negative $f_j \in L^1(\mathbb{R}^{n-1})$.

$$\int_{\mathbb{R}^n} \prod_{j=1}^n f_j \circ \pi_j^{\frac{1}{n-1}} \leqslant \prod_{j=1}^n \left(\int_{\mathbb{R}^{n-1}} f_j \right)^{\frac{1}{n-1}}$$

The Brascamp–Lieb inequalities have had a significant impact on a broad range of areas of mathematics. It was developments in the study of Brascamp–Lieb inequalities that led to the resolution of the century-old Vinagradov mean value conjecture [21], which is now a celebrated theorem in analytic number theory. Other deep number-theoretic connections were established by Christ et al. [31], who proved that the algorithmic construction of the set of Brascamp–Lieb data whose associated constant is finite is equivalent to the affirmative solution of Hilbert's tenth problem for rational polynomials. It should also be noted that Gowers norms, which have become an object of great interest in additive combinatorics [19,42,67], may be estimated from above via a suitable discrete version of a Brascamp–Lieb inequality. Furthermore, the Brascamp–Lieb inequalities have been found to arise in convex geometry as generalisations of Brunn-Minkowski type inequalities [2], in the study of entropy inequalities for many-body systems of particles [30], and have been used as a framework for finding effective solution algorithms for a broad class of optimisation problems arising in computer science [41].

The most immediate question in the theory of linear Brascamp–Lieb inequalities is of course that of finding the necessary and sufficient conditions for BL($\mathbf{L}$,$\mathbf{p}$) to be finite. We begin with the observation that, by an elementary scaling argument, the following is a necessary condition for finiteness:

$$\sum_{j=1}^{m} p_j n_j = n. \tag{1.1.3}$$

It was first proved by Barthe, later reproved by Carlen, Lieb, and Loss in [30], that this condition together with a spanning condition on the surjections L_j forms a necessary and sufficient condition for finiteness in the rank-one case, i.e. when $n_j = 1$ for all $j \in \{1, ..., m\}$ [2].

Another important and related question is that of finding necessary and sufficient conditions for the extremisability of Brascamp–Lieb inequalities, and to find a characterisation of the extremisers should they exist. As we shall discuss later on, if a Brascamp–Lieb inequality admits an extremiser, then it must admit a gaussian extremiser, a result that is related to the following theorem due to Lieb. Given the importance of gaussians in the context of Brascamp–Lieb inequalities, it shall at times be useful to tailor our notation specifically for them. Let $(\mathbf{L}, \mathbf{p})$ be a Brascamp–Lieb datum and let $\mathbf{G} = (G_j)_{j=1}^m$ be an m-tuple of gaussians of the form $G_j(x) := \exp(-\pi\langle A_j x, x\rangle)$, where each $A_j \in \mathbb{R}^{n_j \times n_j}$ is in the cone of real-valued $n_j \times n_j$ symmetric positive-definite matrices, which we denote by $\mathrm{Sym}_+(\mathbb{R}^{n_j})$. We shall refer to such an m-tuple of symmetric positive definite matrices as a *gaussian input*, and we let $\mathcal{G} := \mathrm{Sym}_+(\mathbb{R}^{n_1}) \times \ldots \times \mathrm{Sym}_+(\mathbb{R}^{n_m})$ denote the set of all gaussian inputs. Using the above definitions, we then define $\mathrm{BL_g}(\mathbf{L}, \mathbf{p}; \mathbf{A}) := \mathrm{BL}(\mathbf{L}, \mathbf{p}; \mathbf{G})$. Of course, since integrals of gaussians may be computed in terms of their underlying matrices, we have access to the following explicit formula:

$$\mathrm{BL_g}(\mathbf{L}, \mathbf{p}; \mathbf{A}) = \frac{\prod_{j=1}^m \det(A_j)^{p_j/2}}{\det\left(\sum_{j=1}^m p_j L_j^* A_j L_j\right)^{1/2}}$$

Theorem 1.1.4 (Lieb's Theorem [55]) *Given any Brascamp–Lieb datum* $(\mathbf{L}, \mathbf{p})$*, the associated Brascamp–Lieb inequality is exhausted by gaussians, that is to say*

$$\sup_{\mathbf{A} \in \mathcal{G}} \mathrm{BL_g}(\mathbf{L}, \mathbf{p}; \mathbf{A}) = \mathrm{BL}(\mathbf{L}, \mathbf{p}).$$

Only needing to test on gaussians makes the problem of establishing whether or not BL($\mathbf{L}$,$\mathbf{p}$) is finite significantly more tractable, and based upon this result, necessary and sufficient conditions for finiteness were proved by Bennett, Carbery, Christ, and Tao in [14]. They also establish necessary and sufficient conditions for both gaussian-extremisability and for when such a gaussian extremiser is unique up to rescaling. Before we give a statement of their theorem, we shall need to state some preliminary definitions.

Definition 1.1.5 *Let* $(\mathbf{L}, \mathbf{p})$ *be a Brascamp–Lieb datum. We say that the datum* $(\mathbf{L}, \mathbf{p})$ *is* feasible *if it satisfies the scaling condition (1.1.3), and that for all subspaces* $V \leqslant \mathbb{R}^n$*,*

$$\dim(V) \leqslant \sum_{j=1}^{m} p_j \dim(L_j V). \tag{1.1.6}$$

Definition 1.1.7 *Given* $(\mathbf{L}, \mathbf{p})$*, we say that a proper non-trivial subspace* $V \leqslant \mathbb{R}^n$ *is* critical *if it satisfies (1.1.6) with equality, and that the datum* $(\mathbf{L}, \mathbf{p})$ *is* simple *if it admits no critical subspaces.*

The significance of critical subspaces is that, if we were to restrict the domains of the surjections L_j to a critical subspace V, and their codomains to L_jV, then we would obtain a restricted datum that is itself feasible. Moreover, the orthogonal complement of a critical subspace is itself critical [14]. As a result, the Brascamp–Lieb datum, in the presence of critical subspaces, exhibits a certain splitting phenomenon, where it may be decomposed along orthogonal pairs of critical subspaces; and so, in a certain sense, similarly to the role that simple groups play in group theory, simple Brascamp–Lieb data may be treated as algebraically fundamental objects, from which one may build larger classes of Brascamp–Lieb data. An in-depth discussion of such structural considerations can be found in [14] and [72]. The most immediate such construction is via taking term-wise direct sums of simple data, and leads to the concept of 'semi-simple' data.

Definition 1.1.8 *We say that a Brascamp–Lieb datum* $(\mathbf{L}, \mathbf{p})$ *is* semi-simple *if and only if there exist invertible matrices* $C \in GL_n(\mathbb{R})$*, and* $C_j \in GL_{n_j}(\mathbb{R})$ *for each* $j \in \{1, ..., m\}$*, as well as simple Brascamp–Lieb data* $(\mathbf{L}^{(1)}, \mathbf{p}), ..., (\mathbf{L}^{(k)}, \mathbf{p})$ *where* $\mathbf{L}^{(r)} = (L_j^{(r)})_{j=1}^m$ *and*

$L_j^{(r)} : \mathbb{R}^{n^{(r)}} \to \mathbb{R}^{n_j^{(r)}}$, *such that for each* $j \in \{1, ..., m\}$, L_j *may be written as*

$$L_j = C_j^{-1} L_j^{(1)} \oplus ... \oplus L_j^{(k)} C.$$

Semi-simple data arise quite naturally, indeed Hölder's inequality, the Loomis–Whitney inequality, and certain cases of Young's convolution inequality are important examples of Brascamp–Lieb inequalities associated with semi-simple data.

Theorem 1.1.9 (Bennett, Carbery, Christ, Tao (2007) [14]) *The following three statements are true for all Brascamp–Lieb data* $(\mathbf{L}, \mathbf{p})$.

1. $\mathrm{BL}(\mathbf{L}, \mathbf{p}) < \infty$ *if and only if* $(\mathbf{L}, \mathbf{p})$ *is feasible.*
2. $\mathrm{BL}(\mathbf{L}, \mathbf{p}; \cdot)$ *is gaussian-extremisable if and only if* $(\mathbf{L}, \mathbf{p})$ *is semi-simple,*
3. $\mathrm{BL}(\mathbf{L}, \mathbf{p}; \cdot)$ *is uniquely gaussian-extremisable up to rescaling if and only if* $(\mathbf{L}, \mathbf{p})$ *is simple.*

The qualititative questions of finiteness and extremisability now largely settled, we shall now turn our attention to the regularity properties of the Brascamp–Lieb constant. This subject enjoys its own surprisingly rich theory in the literature; it was Bennett, Bez, Flock and Lee who first established that the Brascamp–Lieb constant was locally bounded [11] on the set of m-tuples $\mathbf{L} = (L_j)_{j=1}^m$ such that $(\mathbf{L}, \mathbf{p})$ is feasible, which we denote by $\mathcal{F}$, from the analysis of which it may be observed that $\mathcal{F}$ is open in $\mathbb{R}^{n_1 \times n} \times ... \times \mathbb{R}^{n_m \times n}$.

Theorem 1.1.10 (Bennett, Bez, Cowling, Flock (2017) [10]) *The mapping* $\mathrm{BL}(\cdot, \mathbf{p}) : \mathcal{F} \to \mathbb{R}$ *is continuous, but not differentiable.*

Later, Buschenhenke and the above authors further refine this statement in [9], where they prove that the Brascamp–Lieb constant is locally Hölder continuous on the set of feasible data. Let $\mathcal{S}$ denote the set of m-tuples $\mathbf{L} = (L_j)_{j=1}^m$ such that $(\mathbf{L}, \mathbf{p})$ is simple, then in light of the third part of Theorem 1.1.9, we know there exists a unique map $\mathbf{Y} : \mathcal{S} \to \mathcal{G}$ such that $\mathrm{BL}_{\mathrm{g}}(\mathbf{L}, \mathbf{p}; \mathbf{Y}(\mathbf{L})) = \mathrm{BL}(\mathbf{L}, \mathbf{p})$ and such that each component $Y_j(\mathbf{L})$ has unit determinant (we impose this constraint for the sake of uniqueness).

Theorem 1.1.11 (Valdimarsson (2010) [73]) *The set $\mathcal{S}$ is open in $\mathcal{F}$, and the map $\mathbf{Y}$ is smooth, whence the Brascamp–Lieb constant is also smooth on $\mathcal{S}$.*

The above result will be crucial to the analysis in Chapter 2, as this will allow us to construct a system of 'local extremisers' in the nonlinear regime that varies smoothly over the domain, although for the general case we treat in Chapter 3 we shall need to construct a substitute to Theorem 1.1.11, since in general Brascamp–Lieb inequalities are not extremisable. We do however know, due to Lieb's theorem, that for any $\delta > 0$ and any feasible datum $(\mathbf{L}, \mathbf{p})$ there exists a gaussian input $\mathbf{A}$ such that $\mathrm{BL}_{\mathrm{g}}(\mathbf{L}, \mathbf{p}; \mathbf{A}) \geqslant (1-\delta)\mathrm{BL}(\mathbf{L}, \mathbf{p})$ (we shall refer to such a $\mathbf{g}$ as a δ-*near extremiser* for $(\mathbf{L}, \mathbf{p})$), however we do not have any a priori information about the norms of its defining matrices or whether or not this choice may be made smoothly in $\mathbf{L}$. As we shall be dealing with data that is in general non-extremisible, we shall be interested in proving a well-quantified version of Lieb's theorem. More specifically, for each $\delta > 0$, we shall need to construct a map $\mathbf{Y}_\delta : \mathcal{F} \to \mathcal{G}$ that sends a given feasible Brascamp–Lieb datum to an associated δ-near gaussian extremiser, and is such that $\|\mathbf{Y}_\delta\|_{W^{1,\infty}}$ does not blow up too quickly as $\delta \to 0$ (the rationale for this choice of norm shall become clear later on). The construction of this map shall be the content of the forthcoming Theorem 3.1.15.

Our exposition of the linear theory now complete, in the next section we turn our attention to the main focus of this thesis, this being nonlinear Brascamp–Lieb inequalities.

1.2 Nonlinear Brascamp–Lieb Inequalities

Nonlinear Brascamp–Lieb inequalities are a relatively recent further generalisation of the linear Brascamp–Lieb inequalities, where the linear surjections L_j are allowed to be general submersions $B_j : M \to M_j$ between Riemannian manifolds. Given an m-tuple of exponents $\mathbf{p} = (p_j)_{j=1}^m$, we shall consider the corresponding inequality:

$$\int_M \prod_{j=1}^m (f_j \circ B_j)^{p_j} \leqslant C \prod_{j=1}^m \left(\int_{M_j} f_j \right)^{p_j} .$$

We shall refer to the pair $(\mathbf{B}, \mathbf{p})$ as a *nonlinear Brascamp–Lieb datum.* Inequalities of this type arise quite naturally in PDE and Fourier restriction contexts, as evidenced in [4,5,53] and [6,10,16] respectively. Early results of significance include a Sobolev variant of the nonlinear Brascamp–Lieb inequality [12] and a nonlinear $C^{1,\theta}$ perturbation of the Loomis–Whitney inequality [16], which was later extended to the C^1 case by Carbery, Hänninen, and Valdimarsson via multilinear factorisation [25]. Significant progress in this area was made recently by Bennett, Bez, Buschenhenke, Cowling, and Flock in [9], where they employ a tight induction-on-scales method that utilises techniques from convex optimisation to prove the following very general local nonlinear Brascamp–Lieb inequality.

Theorem 1.2.1 (Local Nonlinear Brascamp–Lieb Inequality (2018) [9]) *Let $\varepsilon > 0$, and suppose that $(\mathbf{B}, \mathbf{p})$ is a C^2 nonlinear Brascamp–Lieb datum defined over some neighbourhood $\widetilde{U}$ of a point $x_0 \in \mathbb{R}^n$. There exists a neighbourhood $U \subset \widetilde{U}$ of x_0 such that the following inequality holds for all $f_j \in L^1(\mathbb{R}^{n_j})$:*

$$\int_U \prod_{j=1}^m f_j \circ B_j(x)^{p_j} dx \leqslant (1+\varepsilon)\mathrm{BL}(\mathbf{dB}(x_0), \mathbf{p}) \prod_{j=1}^m \left(\int_{\mathbb{R}^{n_j}} f_j \right)^{p_j} . \tag{1.2.2}$$

It has been shown in a preprint of Bennett and Bez, to appear in the publications of the Research Institute of Mathematical Sciences (RIMS) [8], that Theorem 1.2.1 implies an equivalence between three statements that are, while at first glance unrelated, each manifestations of an underlying Brascamp–Lieb type notion of higher-order transversality.

Theorem 1.2.3 *Let $S_1, ..., S_m$ be a collection of compact submanifolds of $\mathbb{R}^n$, each equipped with their natural volume measures $\sigma_1, ..., \sigma_m$ respectively. Suppose that $q_1, ..., q_m \in [1, \infty)$ are Lebesgue exponents satisfying the following scaling condition,*

$$\sum_{j=1}^{m} \frac{\dim(S_j)}{q_j} = n \tag{1.2.4}$$

then, the following statements are equivalent.

- *(Transversality) For all $V \leqslant \mathbb{R}^n$ and all $(x_1, ..., x_m) \in S_1 \times ... \times S_m$,*

$$\dim(V) \leqslant \sum_{j=1}^{m} \frac{\dim(V \cap T_{x_j} S_j)}{q_j}.$$

- *(Convolution) For all $f_j \in L^{q_j}(S_j)$,*

$$\|f_1 d\sigma_1 * ... * f_m d\sigma_m\|_{L^\infty(\mathbb{R}^n)} \lesssim \prod_{j=1}^{m} \|f_j\|_{L^{q_j}}.$$

- *(Restriction) For all $\varepsilon > 0$ and $g_j \in L^2(S_j)$,*

$$\int_{B(0,R)} \prod_{j=1}^{m} |\widehat{g_j d\sigma_j}|^{2/q_j} \lesssim_\varepsilon R^\varepsilon \prod_{j=1}^{m} \|g_j\|_{L^2}^{2/q_j}.$$

We refer to operators of the form $\mathcal{E}g := \widehat{gd\sigma}$, where σ is a singular measure supported on a submanifold of $\mathbb{R}^n$ as a *Fourier extension operator*, see [46, 62, 66, 69] for further reading on this topic. It is natural to ask the question of whether or not there is a more general formulation of Theorem 1.2.1 that does not include an ε-loss, since we know that certain sharp results hold on the sphere, as established by Carlen, Lieb and Loss in [30], which were later generalised to the setting of compact homogeneous spaces by Bramati in [23]; we discuss conjectural sharp Brascamp–Lieb inequalities more generally in Section 6.2. Some other interesting results for compact domains depart from the usual transversality assumptions of the aforementioned authors, instead requiring some sort of bracket-spanning type curvature condition. This includes L^p-improving estimates for multilinear Radon-like transforms, explored by Tao and Wright in the bilinear setting in [70] then generalised by Stovall to the fully multilinear setting in [65]. We shall not investigate curvature considerations of this type in this thesis.

While some of the central questions of the local theory of nonlinear Brascamp–Lieb inequalities have been addressed, in the global setting many interesting questions remain open, and progress is largely still at the early stage of the analysis of special cases. Examples include inequalities for certain homogeneous data of degree one [13], a global weighted nonlinear Loomis–Whitney inequality in $\mathbb{R}^3$ [53], and some results in the context of integration spaces [29]. This thesis in part represents a small step towards a general theory of global nonlinear Brascamp–Lieb inequalities. In particular, the main results of Chapters 2 and 3 may be viewed as global nonlinear alternatives to the heat-flow monotonicity properties of the linear Brascamp–Lieb inequality, which we shall discuss in the next section.

1.3 Heat-flow Monotonicity

Establishing that an inequality enjoys some sort of monotonicity property under heat-flow has been shown to be the basis of an effective proof strategy in a variety of contexts. Schematically, the manner in which such a strategy works is that if one wishes to prove an inequality of the form $A(f) \leqslant B(f)$ for all f in some class of functions, where A and B are functionals defined on this class, it is enough to prove that there exists a semigroup

S^t acting on this class such that $A(f) \leqslant \liminf_{t\to 0} A(S^t f)$, $A(S^t f)$ is increasing in t, and that $\limsup_{t\to\infty} A(S^t f) \leqslant B(f)$. Carlen, Lieb and Loss exploit heat-flow monotonicity to great effect in their proof of the rank-one case of the Brascamp–Lieb inequality [30], generalisations of which can be found in [14, 23]. Heat-flow techniques were also used by Bennett, Carbery and Tao to great effect in their treatment of the multilinear Kakeya and restriction problems [15], later generalised by Tao in [68]. Methods that exploit heat-flow monotonicity are often referred to as 'semigroup interpolation' methods (see an article of Ledoux for further reading [54]), and a systematic study of the generation of monotone quantities for the heat equation can be found in [7]. An interesting manifestation of heat-flow monotonicity for the Brascamp–Lieb functional arises from the following inequality due to Keith Ball.

Lemma 1.3.1 (Ball's inequality [3, 6]) *Let* $(\mathbf{L}, \mathbf{p})$ *be a Brascamp–Lieb datum and let* $\mathbf{f} = (f_j)_{j=1}^m, \mathbf{g} = (g_j)_{j=1}^m \in L^1(\mathbb{R}^{n_1}) \times \ldots \times L^1(\mathbb{R}^{n_m})$*. Given* $x \in \mathbb{R}^n$*, we define* $\mathbf{h}^x := (f_j(\cdot)g_j(L_j(x) - \cdot))_{j=1}^m$*. For all choices of inputs* $\mathbf{f}$ *and* $\mathbf{g}$*, the following inequality holds.*

$$\mathrm{BL}(\mathbf{L}, \mathbf{p}; \mathbf{f})\mathrm{BL}(\mathbf{L}, \mathbf{p}; \mathbf{g}) \leqslant \sup_{x\in\mathbb{R}^n} \mathrm{BL}(\mathbf{L}, \mathbf{p}; \mathbf{h}^x)\mathrm{BL}(\mathbf{L}, \mathbf{p}; \mathbf{f} * \mathbf{g})$$

If we assume that $\mathrm{BL}(\mathbf{L}, \mathbf{p}) < \infty$ and that $\mathbf{g}$ is an extremising input, i.e. $\mathrm{BL}(\mathbf{L}, \mathbf{p}; \mathbf{g}) = \mathrm{BL}(\mathbf{L}, \mathbf{p})$, then this inequality implies the following two statements:

$$\mathrm{BL}(\mathbf{L}, \mathbf{p}; \mathbf{f}) \leqslant \mathrm{BL}(\mathbf{L}, \mathbf{p}; \mathbf{f} * \mathbf{g}) \tag{1.3.2}$$

$$\mathrm{BL}(\mathbf{L}, \mathbf{p}; \mathbf{f}) \leqslant \sup_{x\in\mathbb{R}^n} \mathrm{BL}(\mathbf{L}, \mathbf{p}; \mathbf{h}^x) \tag{1.3.3}$$

An important consequence is that, if we further suppose that $\mathbf{f}$ is an extremiser, then (1.3.2) implies that the set of extremisers is closed under convolution. This, along with the the topological closure of extremisers, guarantees the existence of a gaussian extremiser given the existence of at least one extremiser, as we may convolve a given extremiser with itself iteratively and apply the central limit theorem to the resulting sequence to find that the limiting extremiser must be gaussian [14].

Suppose that $\mathbf{g}$ is a gaussian extremiser, and define its associated family of rescalings as $\mathbf{g}_\tau := (\tau^{-n_j/2} g_j(\tau^{-1/2} x))_{j=1}^m$ where $\tau > 0$. By the scale-invariance of the Brascamp–Lieb inequality, each $\mathbf{g}_\tau$ is also an extremiser, hence if we now substitute $\mathbf{g}_\tau$ into (1.3.2) then we see that (1.3.2) then states that the Brascamp–Lieb functional is monotone increasing as the inputs flow under the following diffusion equation:

$$\partial_t f_j = \nabla \cdot (A_j^{-1} \nabla f_j)$$

where A_j is the positive definite matrix such that $g_j := \exp(-\pi \langle A_j x, x\rangle)$. We shall now run the scheme outlined at the beginning of this section to derive the sharp finiteness and extremisability of the Brascamp–Lieb inequality from (1.3.2), as was carried out in a special case in [14].

Lemma 1.3.4 *Let* $(\mathbf{L}, \mathbf{p})$ *be a Brascamp–Lieb datum and assume that* (1.1.3) *holds. Let* $\mathbf{g}(x) := (g_j(x))_{j=1}^m := (\exp(-\pi\langle A_j x, x\rangle))_{j=1}^m$ *for all* $x \in \mathbb{R}^n$*, where* $A_j \in \mathbb{R}^{n_j \times n_j}$ *is positive definite. If (1.3.2) holds for all inputs* $\mathbf{f}$*, then* $\mathrm{BL}(\mathbf{L}, \mathbf{p}) < \infty$*, furthermore* $\mathbf{g}$ *extremises the Brascamp–Lieb functional* $\mathrm{BL}(\mathbf{L}, \mathbf{p}; \cdot)$*.*

Proof. By homogeneity and scale-invariance of the Brascamp–Lieb functional, we may assume without loss of generality that $\int_{\mathbb{R}^{n_j}} f_j = 1$ and $\int_{\mathbb{R}^{n_j}} g_j = 1$ for each $j \in \{1, ..., m\}$. Given $\tau > 0$, we may define an m-tuple of anisotropic heat kernels $\mathbf{g}_\tau$:

$$\mathbf{g}_\tau(x) := (g_{j,\tau}(x))_{j=1}^m = (\tau^{-n_j} \exp(-\pi \tau^{-1/2} \langle A_j x, x\rangle))_{j=1}^m.$$

Observe that for all $\tau > 0$,

$$\begin{aligned}\tau^{n_j/2} f_j * g_{\tau,j}(L_j(\tau^{1/2}x)) &= \int_{\mathbb{R}^{n_j}} f_j(z)\exp(-\pi\tau^{-1}\langle A_j(\tau^{1/2}L_j(x)-z), \tau^{1/2}L_j(x)-z\rangle)dz \\ &= \int_{\mathbb{R}^{n_j}} f_j(z)\exp(-\pi|A_j^{1/2}L_j(x)|^2 + 2\pi\tau^{-1/2}\langle A_jL_j(x), z\rangle - \pi\tau^{-1}|z|^2)dz \\ &\underset{\tau\to\infty}{\longrightarrow} \exp(-\pi|A_j^{1/2}L_j(x)|^2)\int_{\mathbb{R}^{n_j}} f_j(z)dz = g_j \circ L_j(x).\end{aligned}$$

Combining this limit with (3.1.3) via the dominated convergence theorem then gives us that

$$\begin{aligned}\mathrm{BL}(\mathbf{L}, \mathbf{p}; \mathbf{f}) = \int_{\mathbb{R}^n}\prod_{j=1}^m f_j \circ L_j(x)^{p_j}dx &\leqslant \int_{\mathbb{R}^n}\prod_{j=1}^m (f_j * g_{j,\tau}) \circ L_j(x)^{p_j}dx \\ &= \int_{\mathbb{R}^n}\prod_{j=1}^m (f_j * g_{j,\tau}) \circ L_j(\tau x)^{p_j}\tau^{n/2}dx \\ &= \int_{\mathbb{R}^n}\prod_{j=1}^m \tau^{p_jn_j/2}(f_j * g_{j,\tau}) \circ L_j(\tau x)^{p_j}dx \\ &\underset{\tau\to\infty}{\longrightarrow} \int_{\mathbb{R}^n}\prod_{j=1}^m g_j \circ L_j(x)^{p_j}dx \\ &= \mathrm{BL}(\mathbf{L}, \mathbf{p}; \mathbf{g}).\end{aligned}$$

□

Taking the supremum in all $\mathbf{f}$ with unit mass, then implies that

$$\mathrm{BL}(\mathbf{L}, \mathbf{p}) = \sup_{\mathbf{f}} \mathrm{BL}(\mathbf{L}, \mathbf{p}; \mathbf{f}) \leqslant \mathrm{BL}(\mathbf{L}, \mathbf{p}; \mathbf{g}) \leqslant \mathrm{BL}(\mathbf{L}, \mathbf{p}),$$

hence $\mathbf{g}$ is an extremiser, whence we may read off the sharp constant.

$$\mathrm{BL}(\mathbf{L}, \mathbf{p}) = \det\left(\sum_{j=1}^m L_j^* A_j L_j\right)^{-1/2}$$

Observing this equivalence between heat-flow monotonicity and extremisability, it is then natural to consider whether or not, for some suitable choice of nonlinear Brascamp–Lieb datum, there exists a variable coefficient heat-flow for which the associated nonlinear Brascamp–Lieb functional is monotone, and if so whether or not this would imply that the inequality holds with finite constant. Indeed, this is the approach that was taken in both [23] and [30] to prove nonlinear Brascamp–Lieb inequalities in certain geometrically symmetric settings, so it is then plausible to suppose that a generalisation of such a monotonicity property could hold in a broader class of contexts. The inequalities (1.3.2) and (1.3.3) express an amenability of the linear Brascamp–Lieb functional to two distinct processes, the former being smoothing via heat-flow and the latter being localisation via gaussian extremisers, as we may think of h_j^x as an essentially truncated version of f_j, whose essential support is contained within a ball centred at $L_j(x)$. The proof strategy of [9] was to find a nonlinear version of (1.3.3) that would serve as a way to bound the left-hand side of (1.2.2) above by a supremum of similar integrals over smaller domains, so that if used recursively this would form the engine of an induction-on-scales argument. In Chapters 2 and 3 we establish a corresponding nonlinear version of (1.3.2), although admittedly we only establish heat-flow near-monotonicity for small times. At its core it is still an induction-on-scales argument, where we tightly bound the possible error between times that are close to one another so that when we string these inequalities together we are left with an error that is well-controlled.

1.4 Nonlinear Multilinear Kakeya Inequalities

In this thesis, the notation '$A \lesssim B$' shall denote that there exists a $C > 0$ depending only on the underlying dimensions, manifolds, and exponents such that $A \leqslant CB$, and '$A \simeq B'$ shall denote that $A \lesssim B \lesssim A$. Any additional dependence shall be indicated by a subscript.

The tools we will be using in Chapter 4 trace their lineage back to the multilinear Kakeya inequality, proved with an ε-loss by Bennett, Carbery, and Tao in [15], later established without losses by Guth in [45].

Theorem 1.4.1 (Guth [45]) *For each* $1 \leqslant j \leqslant n$*, let* $\mathbb{T}_j$ *be a collection of straight doubly infinite tubes* $T_j \subset \mathbb{R}^n$ *of unit width. Denote the direction of a tube* $T_j \in \mathbb{T}_j$ *by* $e(T_j)$*, and suppose that there exists* $\theta > 0$ *such that, for any configuration of tubes* $(T_1, ..., T_n) \in \mathbb{T}_1 \times ... \times \mathbb{T}_n$*, we have the uniform transversality bound* $|\bigwedge_{j=1}^n e(T_j)| > \theta$*, then the following inequality holds:*

$$\int_{\mathbb{R}^n} \left(\prod_{j=1}^n \sum_{T_j \in \mathbb{T}_j} \chi_{T_j} \right)^{\frac{1}{n-1}} dx \lesssim \theta^{-\frac{1}{n-1}} \prod_{j=1}^n (\#\mathbb{T})^{\frac{1}{n-1}} \tag{1.4.2}$$

Remarkably, the proof of this theorem relies heavily on sophisticated techniques from algebraic topology. If we suppose that each $T_j \in \mathbb{T}_j$ is parallel to the j-th axis, then we may interpret the tubes T_j as preimages of balls $V_j \subset \mathbb{R}^{n-1}$ under the projection π_j onto the orthogonal complement of the j-th coordinate axis, as such we may write $\sum_{T_j \in \mathbb{T}_j} \chi_{T_j} = \sum_{V_j \in \mathcal{V}_j} \chi_{V_j} \circ \pi_j$ for some collection $\mathcal{V}_j$ of unit balls V_j in $\mathbb{R}^{n-1}$, from which we recover the Loomis–Whitney inequality via rescaling and applying a standard density argument.

Similar statements hold for collections of nonlinear tubes, these being δ-neighbourhoods of smooth curves in $\mathbb{R}^n$, although admittedly with an $\delta^{-\varepsilon}$-loss at the endpoint. The transversality condition that such statements require generalises the linear case, in that we require that the tangent vectors to the central curves of these tubes to always be sufficiently transversal, a property we shall now define concretely.

Definition 1.4.3 *Fix* $\delta, \nu > 0$ *and let* $\mathbb{T}_1, ..., \mathbb{T}_n$ *each be collections of* δ*-neighbourhoods of smooth curves (a 1-dimensional submanifold) in* $\mathbb{R}^n$*. For each* $T_j \in \mathbb{T}_j$*, let* $c(T_j)$ *denote the central curve of* T_j*, and let* $e(T_j)(x)$ *be a unit tangent vector to* $c(T_j)$ *at* $x_j \in c(T_j)$*.*

We say that $\mathbb{T}_1, ..., \mathbb{T}_n$ *are* ν-transversal *if and only if, for any configuration of points* $(x_1, ..., x_n) \in c(T_1) \times ... \times c(T_m)$*, the uniform transversality bound* $|\bigwedge_{j=1}^{n} e(T_j)(x_j)| > \nu$ *holds.*

Theorem 1.4.4 (Bennett-Carbery-Tao [15]) *Let* $\varepsilon, \delta > 0$*, and if* $\mathbb{T}_1, ..., \mathbb{T}_n$ *are* ν-*tranversal collections of* δ*-neighbourhoods of smooth curves in* $\mathbb{R}^n$*, then, for all* $q > \frac{2n}{n-1}$ *the following estimate holds:*

$$\int_{\mathbb{R}^n} \prod_{j=1}^{n} \left(\sum_{T_j \in \mathbb{T}_j} \chi_{T_j} \right)^{q/n} \lesssim \delta^n \prod_{j=1}^{n} \#\mathbb{T}_j^{q/n} \tag{1.4.5}$$

Moreover, for $q = \frac{2n}{n-1}$*, we have that*

$$\int_{\mathbb{R}^n} \prod_{j=1}^{n} \left(\sum_{T_j \in \mathbb{T}_j} \chi_{T_j} \right)^{q/n} \lesssim_{\varepsilon} \delta^{n-\varepsilon} \prod_{j=1}^{n} \#\mathbb{T}_j^{q/n} \tag{1.4.6}$$

Motivated by seeking a more simple proof of Theorem 1.4.1, Carbery and Valdimarsson established the following affine-invariant generalisation via the Borsuk–Ulam theorem [27].

Theorem 1.4.7 (Carbery-Valdimarsson (2013) [27]) *Let* $1 \leqslant m \leqslant n$*. For each* $1 \leqslant j \leqslant m$*, let* $\mathbb{T}_j$ *be a collection of straight doubly infinite tubes* T_j *of unit width. Then, the following inequality holds:*

$$\int_{\mathbb{R}^n} \left(\sum_{(T_1,...,T_m) \in \mathbb{T}_1 \times ... \times \mathbb{T}_m} |\bigwedge_{j=1}^{m} e(T_j)| \chi_{T_1 \cap ... \cap T_m} \right)^{\frac{1}{m-1}} dx \lesssim \prod_{j=1}^{m} (\#\mathbb{T}_j)^{\frac{1}{m-1}} \tag{1.4.8}$$

If we can uniformly bound the weight $|\bigwedge_{j=1}^{m} e(T_j)|$ below by some $\theta > 0$, then this will allow us to factorise the integrand on the left-hand side of (1.4.8) in such a manner that we then recover Theorem 1.4.1.

Given the existence of affine-invariant versions of multilinear Kakeya inequalities, these being Theorem 1.7.1 and Theorem 1.4.7 respectively, it is then reasonable to suppose that there might hold a nonlinear affine-invariant multilinear Kakeya inequality that generalises Theorem 1.4.7. Most such variants substitute straight tubes, i.e. neighbourhoods of lines, with neighbourhoods of *algebraic varieties*, which are sets that are, while in general nonlinear, still defined using the algebraic structure of $\mathbb{R}^n$.

Definition 1.4.9 *Let $\mathbb{R}[x_1, ..., x_n]$ denote the ring of polynomials over the reals with variables $x_1, ..., x_m$. A subset $H \subset \mathbb{R}^n$ is an algebraic variety in $\mathbb{R}^n$ if and only if there exists a finite collection of polynomials $\mathcal{P} \subset \mathbb{R}[x_1, ..., x_n]$ such that*

$$H = \{x \in \mathbb{R}^n : p(x) = 0 \ \forall p \in \mathcal{P}\} \tag{1.4.10}$$

We then define the degree *of H to be the minimum of the quantity $\max_{p\in\mathcal{P}} \deg p$ as $\mathcal{P}$ ranges over all collections of polynomials such that (1.4.10) holds.*

For instance, any finite set of points is an algebraic variety, and its degree is equal to its cardinality. By the implicit function theorem, if M admits a defining vector-valued polynomial $\mathbf{p} = (p_1, ..., p_{n-d}) : \mathbb{R}^n \to \mathbb{R}^{n-d}$ whose derivative has full rank at a point $x \in M$, then M is locally a d-dimensional manifold near x, and we refer to such x as non-singular points of M. If the non-singular points of M form an open and dense subset of M, we shall refer to M as a d-dimensional algebraic variety. We remark that, while being perfectly suitable for our purposes, this is a restricted definition of an algebraic variety, and would be more widely referred to as the definition of a real affine variety. A more general definition of an algebraic variety can be found in [49] for example. An early example of nonlinear multilinear Kakeya inequalities involving varieties was offered by Bourgain and Guth in [22], where they proved a trilinear inequality for algebraic curves (1-dimensional algebraic varieties) in $\mathbb{R}^4$.

Theorem 1.4.11 (Bourgain-Guth 2011 [22]) *Suppose that $\Gamma_i \subset \mathbb{R}^4$ is an algebraic curve restricted to the unit 4-ball with degree $\lesssim 1$ and C^2 norm $\lesssim 1$. Let T_i denote the δ-neighbourhood of an algebraic curve Γ_i and let $\mathbb{T}$ be an arbitrary finite set of such T_i. For each $x \in T_i \in \mathbb{T}$, define an approximate tangent vector $v_i(x) \in \mathbb{R}^4$ by choosing a point $x' \in \Gamma_i \cap U_\delta(x)$ and setting $v_i(x)$ equal to a unit vector tangent to Γ_i at x'. The following estimate holds:*

$$\int_{U_1(0)} \left(\sum_{(T_i,T_j,T_k)\in\mathbb{T}^3} |v_i(x) \wedge v_j(x) \wedge v_k(x)| \chi_{T_i \cap T_j \cap T_k}(x) \right)^{\frac{1}{2}} dx \lesssim \delta^4 (\#\mathbb{T})^{\frac{3}{2}} \tag{1.4.12}$$

There are higher-dimensional generalisations of this inequality due to Zhang and Zorin-Kranich, but before we state them, we remark that any higher-dimensional analogue of (1.4.12) must involve some suitable generalisation of the wedge term in the integrand that tracks the transversality of the varieties in a similar manner. One such generalisation involves a weight that takes the form of a 'wedge product' of the tangent spaces of the varieties, which we shall now define.

Definition 1.4.13 *Let $W_1, ..., W_m$ be a collection of subspaces of $\mathbb{R}^n$, and for each W_j choose an orthonormal basis $w_1^j, ..., w_{k_j}^j$. Observing that the $\sum_{j=1}^m k_j$-dimensional volume of the parallelepiped generated by the union of these bases, given by $|\bigwedge_{j=1}^m \bigwedge_{i=1}^{k_i} w_i^j|$, does not depend on the choice of bases, we denote this quantity by $|\bigwedge_{j=1}^m W_j|$.*

Theorem 1.4.14 (k_j-variety theorem, Zhang 2015 [76]) *Assume that $\sum_{j=1}^m k_j = n$. For each $j \in \{1, ..., m\}$, let H_j be an open subset of a k_j-dimensional algebraic subvariety in $\mathbb{R}^n$, and let σ_j denote the k_j-dimensional Hausdorff measure on H_j, then,*

$$\int_{\mathbb{R}^n} \left(\int_{H_1 \cap U_1(x) \times ... \times H_m \cap U_1(x)} |\bigwedge_{i=1}^m T_{y_j} H_j| d\sigma_1(y_1)...d\sigma_m(y_m) \right)^{\frac{1}{m-1}} dx \lesssim \prod_{j=1}^m \deg(H_j)^{\frac{1}{m-1}} \tag{1.4.15}$$

While at first glance this inequality appears to have a very different form to (1.4.8) and (1.4.12), one may view the inner integral as a weighted bump function supported in the intersection of the unit neighbourhoods of the varieties $H_1, ..., H_m$, where this weight is a higher-dimensional generalisation of the wedge of tangent vectors arising in (1.4.12). We should remark that, in the same paper, Zhang does prove a stronger theorem than the above that accounts for more general configurations of dimensions and exponents, wherein the weight explicitly takes the form of a Brascamp–Lieb constant. Later, Zorin-Kranich devised a reformulation of this generalised theorem that makes use of Fremlin tensor product norms, and this is the version we shall be using to prove Theorem 4.1.3.

Definition 1.4.16 *Given measure spaces $X_1, ..., X_m$ and $p_j \in [1, \infty]$, define the* Fremlin tensor product *norm $\|F\|_{\overline{\bigotimes}_{j=1}^m L^{p_j}(X_j)}$ of a measurable function $F : X_1 \times ... \times X_m \to \mathbb{R}$ by*

$$\|F\|_{\overline{\bigotimes}_{j=1}^m L^{p_j}(X_j)} := \inf \left\{ \prod_{j=1}^m \|F_j\|_{L^{p_j}(X_j)} : F_j \in L^{p_j}(X_j), |F| \leqslant |F_1| \otimes ... \otimes |F_m| \right\}$$

We define the Fremlin tensor product space $\overline{\bigotimes}_{j=1}^m L^{p_j}(X_j)$ to be the completion of the normed space of all measurable functions F such that $\|F\|_{\overline{\bigotimes}_{j=1}^m L^{p_j}(X_j)} < \infty$.

The Fremlin tensor product norm is indeed a norm, as the subadditive property was proved in Theorem 2.2 of [60], and the reader may quickly verify that the point separation and absolute homogeneity axioms follow trivially from its definition. Zorin-Kranich also makes use of a non-standard regime for defining Brascamp–Lieb inequalities that takes, as data, collections of subspaces as opposed to linear maps, one that we shall now define. Given a collection of subspaces $W_1, ..., W_m \leqslant \mathbb{R}^n$ such that $\dim(W_j) = k_j$, with a corresponding collection of exponents $p_1, ..., p_m > 0$, the associated 'Brascamp–Lieb inequality' is defined as follows over all $f_j \in L^1(\mathbb{R}^n/W_j)$:

$$\int_{\mathbb{R}^n} \prod_{j=1}^m f_j(x+W_j)^{p_j} dx \leqslant C \prod_{j=1}^m \left(\int_{\mathbb{R}^n/W_j} f_j \right)^{p_j} \tag{1.4.17}$$

Following the notation of [77], we then write $\overrightarrow{W_j} = (W_1, ..., W_m)$, $\mathbf{p} := (p_1, ..., p_m)$, and denote the best constant $C > 0$ in the above inequality by $BL(\overrightarrow{W_j}, \mathbf{p})$; to be explicit, in this thesis we shall always use italics to refer to this subspace formulation of Brascamp–Lieb constants, and non-italics to refer to the standard one. In his paper, Zorin-Kranich makes use of local versions of the Brascamp–Lieb constants, which allows for exponents to lie outside of the polytope defined by the scaling condition $\sum_{j=1}^m p_j n_j = n$. We shall however state a version of Zorin-Kranich's theorem that assumes such a scaling condition, but nonetheless is more general than Theorem 1.4.14.

Theorem 1.4.18 (Zorin-Kranich 2017 [77]) *Let $\mathcal{Q} := \mathbb{Z}^n + [0,1)^n$ be a decomposition of $\mathbb{R}^n$ into unit cubes and for each $1 \leqslant j \leqslant m$, let $H_j \subset \mathbb{R}^n$ be an open subset of a k_j-dimensional algebraic variety and $p_j \in [0,1]$ be chosen such that $\sum_{j=1}^m p_j(n-k_j) = n$. Suppose that $P := \sum_{j=1}^m p_j \geqslant 1$, then the following inequality holds:*

$$\sum_{Q \in \mathcal{Q}} \| BL(\overrightarrow{T_{x_j} H_j}, \mathbf{p})^{-\frac{1}{P}} \|^P_{\bigotimes_{j=1}^m L^{P/p_j}_{x_j}(H_j \cap Q)} \lesssim \prod_{j=1}^m \deg(H_j)^{p_j} \tag{1.4.19}$$

Consequently, averaging over all translations of $\mathcal{Q}$ and rescaling by a factor of 2 via the forthcoming Lemma 4.3.1, we obtain the following inequality under the same conditions:

$$\int_{\mathbb{R}^n} \| BL(\overrightarrow{T_{x_j} H_j}, \mathbf{p})^{-\frac{1}{P}} \|^P_{\bigotimes_{j=1}^m L^{P/p_j}_{x_j}(H_j \cap U_1(x))} dx \lesssim \prod_{j=1}^m \deg(H_j)^{p_j}, \tag{1.4.20}$$

where $U_r(x) \subset \mathbb{R}^n$ denotes the open ball of radius r around $x \in \mathbb{R}^n$. This integral reformulation is the form we shall be using in this thesis. In analogy with the discussion following the statement of Theorems 1.4.1 and 1.4.7, it is natural to suppose heuristically that one might be able to derive a corresponding Brascamp–Lieb inequality from Theorems 1.4.14 or 1.4.18 by formally running the same argument as in the linear case; viewing the left-hand side as an integral of a weighted product of indicator functions associated to tubular neighbourhoods of varieties in $\mathbb{R}^n$, which we would then want to write as pullbacks of indicator functions associated to balls under some suitable nonlinear submersions, thereby obtaining a Brascamp–Lieb form that would extend to general functions via density. However, given a submersion $B : M \to N$ between Riemannian manifolds M and N, the preimage under B_j of a ball cannot in general be written directly as, for some $z \in M$, a tubular neighbourhood of a set of the form $B^{-1}(\{z\})$, which we refer to as a *fibre* of B_j, hence we cannot immediately run the same density argument as before. We therefore need to use a more detailed construction, where we cover these preimages by a union of many very thin tubular neighbourhoods of fibres, paying careful attention to how they overlap (see figure 4.3, Section 4.2.3); addressing these issues forms the main content of Chapter 4.

1.5 The Linear Theory of Oscillatory Integrals

In Chapter 5, we shall investigate multilinear Lebesgue estimates on generalisations of what Stein refers to as 'oscillatory integrals of the second kind' [62], which we shall put in context by first discussing some of their linear theory. Let $n \in \mathbb{N}$, and let $\phi, \psi : \mathbb{R}^{n-1} \times \mathbb{R}^n \to \mathbb{R}$ be C^2 functions. Consider the following one-parameter family of operators mapping functions on $\mathbb{R}^{n-1}$ to functions on $\mathbb{R}^n$:

$$S^\lambda f(\xi) := \int_{\mathbb{R}^{n-1}} e^{i\lambda\phi(x,\xi)}\psi(x,\xi)f(x)dx, \qquad \lambda > 1. \tag{1.5.1}$$

We refer to S^λ as an *oscillatory integral operator*, to ϕ as a *phase function*, and to ψ as an *amplitude function* or *cut-off function*, usually having compact support in one or more variables. In the special case where $\phi(x,\xi) = \rho(x)\cdot\xi$, S^λ coincides with the Fourier extension operator associated to the graph of ρ. In general, oscillatory integral operators enjoy good L^p mapping properties, provided that they satisfy a certain non-vanishing curvature condition due to Hörmander.

Definition 1.5.2 *We say that S^λ is a* Hörmander-type *operator, or is of* Hörmander type*, if the following holds.*

1. $\mathrm{supp}(\psi)$ *is contained in the unit ball in $\mathbb{R}^{n-1}\times\mathbb{R}^n$.*

2. *For all $(x,\xi)\in\mathbb{R}^{n-1}\times\mathbb{R}^n$, the matrix $\nabla_x\nabla_\xi\phi(x,\xi)$ is of full rank $n-1$.*

3. *Given that this is the case, we may define the associated (non-normalised) Gauss map $G:\mathbb{R}^{n-1}\times\mathbb{R}^n\to\Lambda^{n-1}(\mathbb{R}^n)\cong\mathbb{R}^n$ as follows:*

$$G(x,\xi) := \bigwedge_{j=1}^{n-1}\partial_{x_j}\nabla_\xi\phi(x,\xi) \tag{1.5.3}$$

We require that, for all $(x,\xi)\in\mathbb{R}^{n-1}\times\mathbb{R}^n$,

$$\det\nabla^2_{\xi\xi}(\langle\partial_\xi\phi(x,\xi),G(x,\xi)\rangle)\neq 0. \tag{1.5.4}$$

One may interpret the condition (1.5.4) in the above definition as a generalised non-vanishing curvature condition, for the reason that, in the extension case, this condition may be interpreted as requiring that the underlying manifold has non-vanishing sectional curvature.

Theorem 1.5.5 (Stein [63], Bourgain-Guth [22]) *Let S^λ be a Hörmander type op-*

erator, and suppose that the exponent $p \in [1, \infty]$ *falls within the following range:*

$$p \geqslant \frac{2(n+1)}{n-1} \qquad \text{if } n \text{ is odd} \tag{1.5.6}$$

$$p \geqslant \frac{2(n+2)}{n} \qquad \text{if } n \text{ is even} \tag{1.5.7}$$

Then, for all $\varepsilon > 0$*, the following estimate is satisfied uniformly in* $\lambda \geqslant 1$

$$\|S^\lambda f\|_{L^p(\mathbb{R}^n)} \lesssim_\varepsilon \lambda^{\varepsilon - \frac{n-1}{p}} \|f\|_{L^p} \tag{1.5.8}$$

In fact, Stein proved that the even dimensional case holds without an ε-loss. The matter of the L^p-mapping properties of Hörmander-type operators in the case when (1.5.4) is positive was later setted by Guth, Hickman, and Iliopoulou, who established L^p bounds for all p outside the range for which there are known counterexamples [47]. This does not however settle the restriction conjecture for positively curved hypersurfaces, as the restriction conjecture enjoys a larger range of exponents than general Hörmander operators do, due to a family of counterexamples discovered by Bourgain [20].

1.6 The Wavepacket Decomposition for Oscillatory Integrals

There is a deep connection between oscillatory integrals and nonlinear Kakeya inequalities that generalises a well-known connection between the Fourier restriction and Kakeya problems. This is that we may view $S^\lambda f$ as a superposition of modulated cut-off functions adapted to curvilinear tubes, which we refer to as 'wavepackets'. Hence, we may then view nonlinear Kakeya inequalities as non-oscillatory versions of estimates of the form (1.5.8) (see [74, 75] for further reading on this topic). In order to understand this connection, we must first define a wavepacket decomposition for the oscillatory integral operators we consider. Heuristically, what this involves is splitting an arbitrary L^2 function f into pieces that are essentially orthogonal and localised in both space and frequency, albeit to reciprocal scales due to uncertainty principle related phenomena.

Fix $\lambda > 1$ and let $\mathcal{Q}$ be a boundedly overlapping cover of $\mathbb{R}^{n-1}$ via open cubes of size $\lambda^{-1/2}$, say for instance $\mathcal{Q} := (9\lambda^{-1/2}/10)\mathbb{Z}^{n-1} + (0, \lambda^{-1/2})^{n-1}$. Let $\{\psi_Q\}_{Q\in\mathcal{Q}}$ be a partition of unity subordinate to $\mathcal{Q}$ such that $\|\nabla^k \psi_Q\|_{L^\infty(Q)} \lesssim \lambda^{k/2}$ for all $k \in \mathbb{N}$. We shall give an explicit construction of such a partition of unity for our specific choice of $\mathcal{Q}$. Let $s : \mathbb{R} \to [0, 1]$ be a smooth bump function that attains 1 on the interval $[1/10, 9/10]$ and attains 0 outside of the interval $(0, 1)$. Given $Q = 9\lambda^{-1/2}v/10 + (0, \lambda^{-1/2})$ where $v \in \mathbb{Z}^{n-1}$, Define the function $\widetilde{\psi}_Q : \mathbb{R}^{n-1} \to \mathbb{R}$ by $\widetilde{\psi}_Q(x_1, ..., x_{n-1}) := \prod_{i=1}^{n-1} s(v_i + \lambda^{1/2}x_i)$, and let $\psi_Q(x) := (\sum_Q \widetilde{\psi}_Q(x))^{-1}\widetilde{\psi}_Q(x)$. Differentiating $\psi_Q(x)$ we see that $|\nabla^k\psi_Q(x)| = \lambda^{k/2}|\nabla^k(\psi_Q(\lambda^{-1/2}\cdot))[\lambda^{1/2}x]| \lesssim \lambda^{k/2}$. Given $\omega \in \lambda^{1/2}\mathbb{Z}^{n-1}$, we then let $a_{Q,\omega} \in \mathbb{C}$ be the ω^{th} Fourier coefficient in the Fourier series of $f\psi_Q$, hence, defining $e_{Q,\omega}(x) := e^{-2\pi i x\cdot\omega}\psi_Q(x)$, we then have that

$$f = \sum_{Q\in\mathcal{Q}} f\psi_Q = \sum_{Q\in\mathcal{Q}} \sum_{\omega\in\lambda^{1/2}\mathbb{Z}^{n-1}} a_{Q,\omega}e_{Q,\omega},$$

Note that, by the bounded overlap of the supports of the functions $e_{Q,\omega}$ in Q and their L^2-orthogonality in ω, we have that

$$\|f\|_{L^2} \simeq \sum_{Q\in\mathcal{Q}} \sum_{\omega\in\lambda^{1/2}\mathbb{Z}^{n-1}} |a_{Q,\omega}|^2\lambda^{-\frac{n-1}{2}}.$$

We refer to a function of the form $S^\lambda e_{Q,\omega}$ as a *wavepacket*, and to the following as a *wavepacket decomposition* of $S^\lambda f$:

$$S^\lambda f = \sum_{Q \in \mathcal{Q}} \sum_{\omega \in \lambda^{1/2}\mathbb{Z}^{n-1}} a_{Q,\omega} S^\lambda e_{Q,\omega}$$

We now consider the support of a given wavepacket, the geometry of which is determined by the phase ϕ. First of all, fix a small $\delta > 0$, then, given a pair $(Q, \omega) \in \mathcal{Q} \times \lambda^{1/2}\mathbb{Z}^{n-1}$, we define an associated tube:

$$T^\lambda_{Q,\omega} := \{\xi \in \mathbb{R}^n : |\nabla_x \phi(x, \xi) - \omega| \leqslant \lambda^{\delta - 1/2} \;\; (x, \xi) \in \mathrm{supp}(\psi)\}$$

We raise the exponent in the definition of $T^\lambda_{Q,\omega}$ by δ in order to ensure that $\|S^\lambda e_{Q,\omega}\|_{L^\infty(\mathbb{R}^n \setminus T^\lambda_{Q,\omega})}$ has good decay as $\lambda \to \infty$. Via a standard stationary phase argument, one may show that $S^\lambda e_{Q,\omega}$ is essentially supported in $T^\lambda_{Q,\omega}$, and that $|S^\lambda e_{Q,\omega}| \gtrsim 1$ on $T^\lambda_{Q,\omega}$. One is therefore justified in viewing an L^p bound on S^λ as, in some sense, a 'modulated' nonlinear Kakeya inequality, where rather than bounding an L^p norm of a sum of indicator functions associated to tubes, we instead are interested in bounding an L^p norm of a sum of modulated indicator functions associated to tubes. In fact, via a now standard Rademacher function argument, one may derive nonlinear Kakeya inequalities from oscillatory integral inequalities. Naturally, this relationship between oscillatory integrals and nonlinear Kakeya transfers into the multilinear setting, and so we shall find that an improved understanding of multilinear Kakeya-type inequalities of the type discussed in Section 1.4 leads to an improvement in the corresponding oscillatory problem, in fact this is the content of the main theorem of Chapter 5, Theorem 5.1.1.

1.7 The Multilinear Theory of Oscillatory Integrals

In recent years, there has been much interest in multilinear versions of inequalities of the type discussed in the previous section, in no small part due to the fact that approaches to these problems via a suitable multilinear version have proved to be highly effective [22,69]. They are in some ways easier to work with than the linear version because they, similarly to the multilinear Kakeya inequality, often benefit from transversality hypotheses. In fact, under such transversality hypotheses, curvature hypotheses such as (1.5.4) may even be disregarded altogether, as in such a case we do not need each contribution to the product to even be integrable in order for their product to be well-behaved.

Let $\phi_1, ..., \phi_m : \mathbb{R}^{n-1} \times \mathbb{R}^n \to \mathbb{R}$ be a collection of phases, and let $\psi_1, ..., \psi_m : \mathbb{R}^{n-1} \times \mathbb{R}^n \to \mathbb{R}$ be a collection of amplitudes with compact support in both variables. Define the following oscillatory integral operators

$$S_j^\lambda f_j(\xi) := \int_{\mathbb{R}^n} e^{i\lambda\phi_j(x,\xi)}\psi_j(x,\xi)f_j(x)dx$$

For each $j \in \{1, ..., m\}$, let $G_j(x, \xi)$ denote the non-normalised Gauss map (as defined in (1.5.3)) associated to the phase ϕ_j. We say that the collection of oscillatory integral operators $S_1^\lambda, ..., S_n^\lambda$ is ν-transversal if, for all $x_1, ..., x_n \in \mathbb{R}^{n-1}$ and $\xi \in \mathbb{R}^n$ such that $(x_j, \xi) \in \text{supp}(\psi_j)$,

$$\det(G_1(x_1, \xi), ..., G_n(x_n, \xi)) > \nu$$

Theorem 1.7.1 (Bennett-Carbery-Tao) *Suppose that $S_1^\lambda, ..., S_n^\lambda$ are ν-transversal, then* *the following inequality holds uniformly in $\lambda > 1$ for $p \geqslant \frac{2}{n-1}$ and $q' \leqslant p(n-1)$.*

$$\left\| \prod_{j=1}^{n} S_j^\lambda f_j \right\|_{L^p(\mathbb{R}^n)} \lesssim_\varepsilon \lambda^{\varepsilon - \frac{n}{p}} \prod_{j=1}^{n} \|f_j\|_{L^q(\mathbb{R}^{n-1})} \tag{1.7.2}$$

In fact they prove that this statement holds without an ε-loss away from the endpoint. The proof of this theorem centres around the multilinear Kakeya theorem they prove in the same paper, stated earlier in this chapter as Theorem 1.4.4. The argument is fundamentally an induction-on-scales, where they first apply a wavepacket decomposition to $S_j^\lambda f_j$ for each $j \in \{1, ..., n\}$, so that they then have a corresponding collection of tubes $\mathbb{T}_j^\lambda$ on which the wavepackets of $S_j^\lambda f_j$ are supported. One may show that the ν-transversality hypothesis on the collection $S_1^\lambda, ..., S_n^\lambda$ implies that the families of tubes $\mathbb{T}_1^\lambda, ..., \mathbb{T}_n^\lambda$ are themselves mutually ν-transversal. They then partition the domain $\mathbb{R}^n$ into cubes of scale $\lambda^{1/2}$, such that the contribution from each can be bounded by an inductive hypothesis, and then observe that, since the contributions to any given cube can only arise from the wavepackets that pass through it, one may therefore interpret the resulting upper bound on each of the cubes as a product of sums of indicator functions associated to these tubes. In which case, one may then view the sum of these upper bounds over all of the cubes as the left-hand side of the multilinear Kakeya inequality associated to the families $\mathbb{T}_j^\lambda$ of underlying tubes on which the wavepackets are essentially supported.

They then apply a suitable curvilinear multilinear Kakeya inequality to these tubes, and obtain a restriction estimate that improves on our inductive hypothesis, but with an ε-loss in the exponent of λ that is roughly half of that of the hypothesis. Hence, iterating this argument we then may make this loss arbitrarily small, thus proving the theorem. In Chapter 5, we globalise this argument and extend it to higher-dimensional regimes using a framework that generalises both nonlinear multilinear Kakeya inequalities and nonlinear Brascamp–Lieb inequalities.

1.8 Guide to the Thesis

Chapters 2 and 3 are about a certain heat-flow near-monotonicity property enjoyed by nonlinear Brascamp–Lieb inequalities. In the former, we study what we refer to as the 'simple' case, which serves to introduce some of the techniques we use in the latter chapter to prove the theorem in general. In Chapter 4, we prove a global nonlinear Brascamp–Lieb inequality for what we refer to as 'quasialgebraic' nonlinear data, incorporating a natural weight that dampens local degeneracies, which in doing so imparts a diffeomorphism-invariance property to the inequality. We also use similar techniques to prove two alternate multilinear Kakeya versions of this statement. We depart from the main theme of nonlinear Brascamp–Lieb inequalities to the related topic of multilinear oscillatory integrals in Chapter 5, where we prove a global L^2 multilinear oscillatory integral estimate in general dimensions. Lastly, in Chapter 6 we discuss some of the further research topics and conjectures that lead on from the results in this thesis.

Chapter 2

A Nonlinear Variant of Ball's Inequality: The Simple Case

In this chapter, we shall prove a near-monotone global nonlinear version of the heat-flow monotonicity property enjoyed by simple linear Brascamp–Lieb data. The most natural nonlinear candidate to consider would be nonlinear Brascamp–Lieb data $(\mathbf{B}, \mathbf{p})$ that is both suitably smooth and 'locally simple', in the sense that $(\mathbf{dB}(x), \mathbf{p})$ is a simple Brascamp–Lieb datum for each x in the domain of $\mathbf{B}$. The reason why this is the most natural case to first consider is that, by Theorem 1.1.9, each simple Brascamp–Lieb datum has a unique extremiser up to rescaling, thus our nonlinear datum comes ready equipped with 'local extremisers' from which we may construct our heat-flow. Although a similar near-monotonicity statement holds for general nonlinear Brascamp–Lieb data on Riemannian manifolds, the core argument for both is essentially the same, so it is instructive to first consider just the simple case in the euclidean setting.

2.1 Preliminaries

We say that a nonlinear Brascamp–Lieb datum $(\mathbf{B}, \mathbf{p})$ over U is a *simple nonlinear Brascamp–Lieb datum over* U if $\mathbf{dB}(u) \in \mathcal{S}$ for all $u \in U$, where $\mathcal{S}$ denotes the set of simple Brascamp–Lieb data as in Section 1.1. By Theorem 1.1.9, for each simple nonlinear datum there then exists a unique (up to rescaling) family of extremising gaussian inputs $\{\mathbf{g}_u := (g_{u,j})_{j=1}^m\}_{u\in U}$ such that, for each $u \in U$, $\mathbf{g}_u$ is an extremiser for the inequality associated to the datum $(\mathbf{dB}(u), p)$. Moreover, by scale-invariance of the linear inequality, each of its L^1-rescalings $\mathbf{g}_{u,\delta}(x) := (g_{u,\delta,j}(x))_{j=1}^m := (\delta^{-n_j} g_{u,j}(\delta^{-1}x))_{j=1}^m$ are also extremisers for $(\mathbf{dB}(x), \mathbf{p})$. We shall think of these rescaled gaussians as heat kernels, even though strictly speaking a genuine heat kernel would have $\delta^{1/2}$ in the place of where we have written δ. Each $g_{u,j}$ may be written explicitly as

$$g_{u,j} := e^{-\pi\langle A_j(u)x,x\rangle} = e^{-\pi|x|^2_{u,j}}$$

where $A_j : U \to \mathbb{R}^{n_j\times n_j}$ assigns to each u a symmetric positive-definite matrix and $|\cdot|_{u,j} := \langle A_j(u)\cdot,\cdot\rangle$. There is another symmetric positive-definite matrix-valued function $M : U \to \mathbb{R}^{n\times n}$ that shall be of importance to us, defined as

$$M(u) = \sum_{j=1}^m p_j dB_j(u)^* A_j(u) dB_j(u)$$

From this definition we have the identity

$$\frac{1}{\mathrm{BL}(\mathbf{dB}(u),\mathbf{p})}\prod_{j=1}^m g_{u,j}\circ dB_j(u)(x)^{p_j} = (\det M(u))^{1/2} e^{-\pi|x|^2_u} \tag{2.1.1}$$

where $|\cdot|_u := \langle M(u)\cdot,\cdot\rangle^{\frac{1}{2}}$ [14].

We shall need to impose some additional uniformity conditions on the nonlinear datum

$\mathbf{B}$, in particular we need the associated family of gaussian extremisers to have bounded eccentricity and obey a uniform Hölder continuity property.

Definition 2.1.2 *Given an open set $U \subset \mathbb{R}^n$, we say that a function $f : U \to \mathbb{R}^k$ is uniformly $C^{1,\theta}$ over U if and only if for all $\varepsilon > 0$ there exists $\delta > 0$ such that, if $|x-y| \leqslant \delta$ and $x \neq y$, then $|f(x)-f(y)|+|\nabla f(x)-\nabla f(y)||x-y|^{-\theta} \leqslant \varepsilon$. Let $\mathcal{S}^\theta(U)$ denote the set of uniformly $C^{1,\theta}$ simple nonlinear Brascamp–Lieb data $(\mathbf{B}, \mathbf{p})$ over U such that the closure of the set $\{(\mathbf{dB}(x), \mathbf{p}) : x \in U\}$ is contained in $\mathcal{S}$.*

For example, $\mathcal{S} \subset \mathcal{S}^\theta(U)$, and moreover, by openness of $\mathcal{S}$, all sufficiently small $C^{1,\theta}$ perturbations of a member of $\mathcal{S}$ is also in $\mathcal{S}^\theta(U)$, the reader is encouraged to bear this example in mind over the course of this chapter.

Part of the reason for imposing uniform Hölder regularity on the nonlinear datum $(\mathbf{B}, \mathbf{p})$ is that our argument requires that the associated matrix-valued functions A_j and M are also uniformly Hölder continuous. Fortunately, this follows in a fairly straightforward manner from the regularity of $\mathbf{B}$.

Proposition 2.1.3 *For any open $U \subset \mathbb{R}^n$ and $\boldsymbol{p} \in [0,1]^m$, if $\boldsymbol{B} \in \mathcal{S}^\theta(U)$, then we may choose the corresponding matrix-valued functions A_j, M such that they are uniformly $C^{0,\theta}$ bounded on U, and that moreover, $A_j^{-1} := A_j(\cdot)^{-1}$ and $M^{-1} := M(\cdot)^{-1}$ are L^∞ bounded.*

Proof. Let $K := \overline{\mathbf{dB}(U)}$, and recall the map $\mathbf{G}$ in Theorem 1.1.11 that sends a simple datum to its unique (up to rescaling) associated gaussian extremiser; we may therefore write $(A_j)_{j=1}^m =: \mathbf{A} = \mathbf{G} \circ \mathbf{dB}$. Since $K \subset \mathcal{S}$ and K is compact, by the extreme value theorem $0 < \inf_{\mathbf{L} \in K} |\mathbf{G}(\mathbf{L})| \leqslant |A_j(x)| \leqslant \sup_{\mathbf{L} \in K} |\mathbf{G}(\mathbf{L})| < \infty$, hence A_j and A_j^{-1} are L^∞ bounded, and therefore since M is then a sum of products of L^∞ functions, M is L^∞ bounded. By smoothness of $\mathbf{G}$ and the fundamental theorem of calculus, we know that there for any $x, y \in U$, and let $d_K(\mathbf{L}_1, \mathbf{L}_2)$ denote the infimum of the lengths of piecewise

C^1 contained in K with endpoints $\mathbf{L}_1 \in K$ and $\mathbf{L}_2 \in K$. It is clear that $d_K : K \times K \to \mathbb{R}$ defines a metric on K and that it is continuous with respect to the ambient euclidean metric, so by compactness of K, $d_K(\mathbf{L}_1, \mathbf{L}_2) \lesssim |\mathbf{L}_1 - \mathbf{L}_2|$.

$$|A_j(x) - A_j(y)| = |\mathbf{G} \circ \mathbf{dB}(x) - \mathbf{G} \circ \mathbf{dB}(y)| \tag{2.1.4}$$

$$\leqslant \|\mathbf{dG}\|_{L^\infty(K)} d_K(\mathbf{dB}(x), \mathbf{dB}(y)) \tag{2.1.5}$$

$$\lesssim |\mathbf{dB}(x) - \mathbf{dB}(y)| \lesssim |x - y|^\theta \tag{2.1.6}$$

hence A_j is uniformly $C^{0,\theta}$, and therefore so is M, as it is a sum of products of $C^{0,\theta}$ functions. □

It is this result that shall give us the necessary uniform control to obtain a near-monotonicity property for the functional $\int_{\mathbb{R}^n} \prod_{j=1}^m f_j \circ B_j(x)^{p_j} dx$ under a certain regularisation process that, although akin to heat-flow, includes some truncation in the kernel and dependence upon the variable y, as we shall need to pointwise adapt this process to the local behaviour of the submersions B_j.

2.2 Notation

Before we state the main result of this chapter we shall need to introduce some notation. In this chapter, we shall use the notation $x \lesssim y$ to denote $x \leqslant Cy$ where $C > 0$ depends only on the underlying Brascamp–Lieb datum $\mathbf{B} \in \mathcal{S}^\theta(U)$, as well as the underlying dimensions and exponents. We will find that we need to truncate the gaussians $g_{u,\tau,j}$ outside of a certain ball depending on τ in order to address local-constancy issues arising from the rapid decay in their tails. Let $\tau > 0$, $y \in \mathbb{R}^n$, $w \in \mathbb{R}^{n_j}$, and define the following families of balls.

$$V_\tau(x) := \left\{ y \in \mathbb{R}^n : |x - y| \leqslant \tau \log\left(\frac{1}{\tau}\right) \right\},$$

$$V_{\tau,j}(z) := \left\{ y \in \mathbb{R}^{n_j} : |y - z| \leqslant ||dB_j||_{L^\infty} \tau \log\left(\frac{1}{\tau}\right) \right\},$$

We find that this logarithmic radius is perfectly suitable for the purposes of this chapter, however in Chapter 3 the setup is a lot more sensitive to truncation and so we shall need to use a polynomial factor instead, being careful to select an appropriate exponent. Letting $\kappa > 0$, we shall denote the centred dilate of $V_\tau(x)$ and $V_{\tau,j}(z)$ by a factor of κ as $\kappa V_\tau(x)$ and $\kappa V_{\tau,j}(z)$ respectively. We include the factor of $\|dB_j\|_{L^\infty}$ in the definition of $V_{\tau,j}(z)$ so that we impose the convenient property that, for all $x \in \mathbb{R}^n$, $dB_j(x)(V_\tau(0)) \subset V_{\tau,j}(0)$.

2.3 Statement of the Theorem

We are now in a position to state our main theorem.

Theorem 2.3.1 *For each $\boldsymbol{B} \in \mathcal{S}^\theta(\mathbb{R}^n)$ and any $\alpha \in (0, \theta)$, there exists a $\nu \simeq_\alpha 1$ such that, for all $\tau \in (0, \nu)$ the following inequality holds over all $f_j \in L^1(\mathbb{R}^{n_j})$.*

$$\int_{\mathbb{R}^n} \prod_{j=1}^m f_j \circ B_j(x)^{p_j} dx \leqslant (1 + \tau^\alpha) \int_{\mathbb{R}^n} \prod_{j=1}^m f_j * (g_{x,\tau,j} \mathbb{1}_{V_{\tau,j}(0)}) \circ B_j(x)^{p_j} dx. \tag{2.3.2}$$

We may think of this as a heat-flow near-monotonicity statement, since the one-parameter family of linear operators $H_{x,t,j}$ defined by $H_{x,t,j} f_j := f_j * g_{x,t^{1/2},j}$ is the solution semi-group of the Cauchy problem for the anisotropic heat equation $\partial_t u(y,t) = \nabla \cdot (A_j^{-1}(x) \nabla u(y,t))$ with initial data $u(y, 0) = f_j(y)$. A benefit of the truncation built into Theorem 2.3.1 is that it allows one to impose some *local-constancy* on arbitrary $f_j \in L^1(\mathbb{R}^n)$, a notion that we now define explicitly.

Definition 2.3.3 *Let X be a metric space and $f : X \to (0, \infty)$. Given $\kappa, \mu > 0$, we say that f is κ-constant at scale μ if and only if $f(x) \leqslant \kappa f(y)$ for all $x, y \in X$ such that $d(x, y) \leqslant \mu$.*

The lack of local-constancy of arbitrary L^1 functions is a central difficulty in the study of nonlinear Brascamp–Lieb inequalities. Many proofs for known nonlinear Brascamp–Lieb inequalities are based around addressing this issue in some manner, for instance, the induction-on-scales arguments used in [17] and [9] are inductions on the scale of constancy of the input functions f_j. Some authors even dispense with arbitrary L^1 functions entirely, instead imposing some a priori local constancy as in [56,77], or Sobolev regularity as in [12]. The regularised inputs $f_j * \tilde{g}_{y,\tau,j}(z)$ enjoy a local-constancy property uniform both in y and z, this being the content of Lemmas 2.5.8 and 2.5.9 respectively. It is unfortunate therefore that gaussians are not locally constant at any scale, due to their rapid decay. This means that in general $f_j * g_{y,\tau,j}$ will also not be locally constant at any scale, however we remedy this by truncating these gaussians outside of a sufficiently large ball centred at the origin. From now on, the truncated gaussians we shall be using shall be denoted by, given $\tau > 0$ and $u \in \mathbb{R}^n$,

$$\tilde{g}_{u,\tau,j} := g_{u,\tau,j} \mathbb{1}_{V_{\tau,j}(0)}.$$

2.4 Outline of the proof of Theorem 2.3.1

The proof of Theorem 2.3.1 has similar features to an induction-on-scales type argument, in the sense that we bound the best constant associated to a weaker inequality above in a self-similar manner, then iterate to obtain a bound on the best constant associated to (2.3.2), taking care to show that the resulting bounds are well controlled under this iteration. Let $(\mathbf{B}, \mathbf{p}) \in \mathcal{S}^\theta(\mathbb{R}^n)$ and consider the following inequality for some $0 < s < t$.

$$\int_{\mathbb{R}^n} \prod_{j=1}^m f_j * \tilde{g}_{x,s,j} \circ B_j(x)^{p_j} dx \leqslant C \int_{\mathbb{R}^n} \prod_{j=1}^m f_j * \tilde{g}_{x,t,j} \circ B_j(x)^{p_j} dx \tag{2.4.1}$$

Define $C(s,t)$ to be the infimum over all constants $C \in (0,\infty]$ such that (2.4.1) is satisfied by all non-negative $f_j \in L^1(\mathbb{R}^{n_j})$. Notice that for all $0 < r < s < t$,

$$\int_{\mathbb{R}^n} \prod_{j=1}^m f_j * \tilde{g}_{x,r,j} \circ B_j(x)^{p_j} dx \leqslant C(r,s) \int_{\mathbb{R}^n} \prod_{j=1}^m f_j * \tilde{g}_{x,s,j} \circ B_j(x)^{p_j} dx$$
$$\leqslant C(r,s)C(s,t) \int_{\mathbb{R}^n} \prod_{j=1}^m f_j * \tilde{g}_{x,t,j} \circ B_j(x)^{p_j} dx. \qquad (2.4.2)$$

The inequality (2.4.2) yields the following simple yet important statement which will serve to bound the constant in the self-similar manner alluded to earlier, which, although immediate, we refer to as a proposition for structural reasons.

Proposition 2.4.3 *For all* $0 < r < s < t$,

$$C(r,t) \leqslant C(r,s)C(s,t). \qquad (2.4.4)$$

The idea will be to use Proposition 2.4.3 to break down the constant in (2.4.1) into a product of constants for times that are much closer together. If we take these times to be sufficiently close, then we have an explicit tight bound on the associated constant.

Proposition 2.4.5 *Given* $\beta \in (0,\theta)$*, there exists* $\tilde{\nu} \simeq_\beta 1$ *such that for all* $0 < \tau < \tilde{\nu}$*.*

$$C(\tau, \sqrt{2}\tau) \leqslant (1 + \tau^\beta). \qquad (2.4.6)$$

Along with some minor technical considerations, these two propositions are the only ingredients we need to prove Theorem 2.3.1, as we shall now demonstrate.

Proof of Theorem 2.3.1 given Proposition 2.4.5. Let $\beta \in (\alpha,\theta)$, $\nu > 0$, and $0 < \tau < \nu$, where for now we only require that $\nu \leqslant \sqrt{2}\tilde{\nu}$. Define the geometric sequence $\tau_k := 2^{-\frac{k}{2}}\tau$ and let $K \in \mathbb{N}$. By repeatedly applying Proposition 2.4.3, we split the constant $C(\tau_K, \tau)$

into pieces that can be dealt with by Proposition 2.4.5.

$$\begin{aligned}
C(\tau_K, \tau) &\leqslant C(\tau_K, \tau_{K-1}) C(\tau_{K-1}, \tau) \\
&\leqslant C(\tau_K, \tau_{K-1}) C(\tau_{K-1}, \tau_{K-2}) C(\tau_{K-2}, \tau) \\
&\leqslant \ldots \leqslant \prod_{k=1}^{K} C(\tau_k, \tau_{k-1}) \leqslant \prod_{k=1}^{K} (1 + \tau_k^{\beta})
\end{aligned}$$

Taking logarithms of the above inequality, we obtain that

$$\begin{aligned}
\log(C(\tau_K, \tau)) &\leqslant \sum_{k=1}^{K} \log(1 + \tau_k^{\beta}) \\
&\leqslant \sum_{k=1}^{\infty} \tau_k^{\beta} = \frac{\tau^{\beta}}{2^{\beta/2} - 1}.
\end{aligned}$$

It then follows, by a routine application of Taylor's theorem to the exponential map, absorbing constants and making ν accordingly smaller if necessary, that $C(\tau_K, \tau) \leqslant \exp(\frac{\tau^{\beta}}{2^{\beta/2}-1}) \leqslant (1 + \tau^{\alpha})$. Having obtained a bound on $C(\tau_K, \tau)$ uniform in K, we then complete the proof by considering (2.4.1) with $s = \tau_K$ and $t = \tau$ and taking the limit as $K \to \infty$, since Fatou's lemma then implies that

$$\begin{aligned}
\int_{\mathbb{R}^n} \prod_{j=1}^{m} f_j \circ B_j(x)^{p_j} dx &\leqslant \lim_{K \to \infty} \int_{\mathbb{R}^n} \prod_{j=1}^{m} f_j * \tilde{g}_{x, \tau_K, j} \circ B_j(x)^{p_j} dx \\
&\leqslant \lim_{K \to \infty} C(\tau_K, \tau) \int_{\mathbb{R}^n} \prod_{j=1}^{m} f_j * \tilde{g}_{x, \tau, j} \circ B_j(x)^{p_j} dx \\
&\leqslant (1 + \tau^{\alpha}) \int_{\mathbb{R}^n} \prod_{j=1}^{m} f_j * \tilde{g}_{x, \tau, j} \circ B_j(x)^{p_j} dx. \qquad \square
\end{aligned}$$

Now, all that remains to prove Theorem 2.3.1 is to prove Proposition 2.4.5, but before we do that we need to understand more about the properties of the various gaussian kernels involved.

2.5 Gaussian Lemmas

Here we shall state and prove the technical results concerning the gaussians $g_{y,\tau,j}$ that we require to prove Proposition 2.4.5. Throughout this section, we shall assume that $\mathbf{B} \in \mathcal{S}^{\theta}(\mathbb{R}^n)$. We will make use of the parameter $\eta \in (\beta, \theta)$, which we shall regard as fixed, and, in light of Proposition 2.1.3, we shall denote the Hölder seminorms of M and A_j by μ and μ_j respectively. We shall denote the induced 2-norm of a matrix S by $|S| := \sup_{|v|=1} |Sv|$.

Lemma 2.5.1 (General Truncation of Gaussians) *Let $n \in \mathbb{N}$ and $A \in \mathbb{R}^{n\times n}$ be a positive definite matrix. For each $\tau > 0$, define $g_\tau : \mathbb{R}^n \to \mathbb{R}$ to be the gaussian $g_\tau(x) := \tau^{-n}\exp(-\pi\tau^{-2}\langle Ax, x\rangle)$. Let $c, \kappa > 0$, and suppose that $|A^{-1}| \leqslant c$. Given $\gamma > 0$, there exists a $\nu > 0$ depending only on n, κ, γ, and c such that for all $\tau \in (0, \nu)$*

$$\int_{\mathbb{R}^n} g_\tau \leqslant (1 + \tau^\gamma) \int_{\kappa V_\tau(0)} g_\tau \tag{2.5.2}$$

Proof. In this proof we shall break from our convention and the relation $\lesssim$ shall denote that the implicit constant depends only upon c, n, and m. We shall first prove the claim assuming $\kappa = 1$, then use a rescaling argument to obtain the general result. If we assume that $g_{u,\tau}$ is L^1-normalised, then it suffices to show that there exists $\nu > 0$ such that, for all $\tau \in (0, \nu)$,

$$\int_{\mathbb{R}^n \setminus V_\tau(0)} g_\tau \lesssim \tau^{2\gamma} \tag{2.5.3}$$

This is because then there would then exist a $c' \simeq 1$ such that

$$\int_{V_\tau(0)} g_\tau = 1 - \int_{\mathbb{R}^n \setminus V_\tau(0)} g_\tau \geqslant 1 - c'\tau^{2\gamma} \geqslant (1 + \tau^\gamma)^{-1}$$

provided that τ is sufficiently small. To estimate the left hand side of (2.5.3), we shall partition the domain of integration $\mathbb{R}^n \backslash V_\tau(0)$ into annuli, and bound the resulting infinite sum above by a lacunary series. We take $\tau \in (0, \nu)$, where $\nu \in (0,1)$ is chosen such that $\frac{\pi}{2} c^{-2} \log(\frac{1}{\nu})^2 \geqslant 2\gamma$.

$$\begin{aligned}
\int_{\mathbb{R}^n \backslash V_\tau(0)} g_\tau &= \int_{|x| \geqslant \log(1/\tau)} \exp(-\pi |Ax|^2) dx \\
&= \sum_{k=0}^{\infty} \int_{2^k \log(1/\tau) \leqslant |x| \leqslant 2^{k+1} \log(1/\tau)} \exp(-\pi |Ax|^2) dx \\
&\leqslant \sum_{k=0}^{\infty} \sup_{2^k \log(1/\tau) = |x|} (\exp(-\pi |Ax|^2)) Vol(\{2^k \log(1/\tau) \leqslant |x| \leqslant 2^{k+1} \log(1/\tau)\}) \\
&\leqslant \log(1/\tau)^n \sum_{k=0}^{\infty} 2^{nk} \exp(-\pi |A^{-1}|^{-2} 2^{2k} \log(1/\tau)^2) \\
&\lesssim \log(1/\tau)^n \sum_{k=0}^{\infty} 2^{nk} \tau^{\pi c^{-2} 2^{2k} \log(1/\tau)} \\
&\lesssim \sum_{k=0}^{\infty} \tau^{\pi c^{-2} 2^{2k-1} \log(1/\tau)} \lesssim \sum_{k=0}^{\infty} \tau^{2\gamma 2^{2k}} \lesssim \tau^{2\gamma}
\end{aligned}$$

Above we removed a factor of 2^{-1} from the exponent to absorb the logarithmic and geometric factors. Now we turn our attention to the case when $\kappa \neq 1$. By what has been established, for all $\kappa > 0$ there exists a $\nu \simeq_{\gamma,\kappa} 1$ such that for all $\tau \in (0, \nu)$,

$$1 \leqslant (1 + \tau^\gamma) \int_{V_{\tau^\kappa}(0)} g_{\tau^\kappa}.$$

Rescaling the right-hand side we obtain the desired result.

$$1 \leqslant (1 + \tau^\gamma) \tau^{n(1-\kappa)} \int_{|x| \leqslant \tau^{1-\kappa} \tau^\kappa \log(1/\tau^\kappa)} g_{\tau^\kappa}(\tau^{\kappa-1} x) dx = (1 + \tau^\gamma) \int_{\kappa V_\tau(0)} g_\tau \tag{2.5.4}$$

□

We now apply this result to the families of gaussians $g_{x,\tau,j}$ and $\prod_{j=1}^m g_{x,\tau,j} \circ dB_j(x)^{p_j}$

Lemma 2.5.5 (Truncation of extremising Gaussians) *There exists* $\nu_1 \simeq_\eta 1$ *such* *that for all* $\tau \in (0, \nu_1)$ *and* $x \in \mathbb{R}^n$*,*

$$\int_{\mathbb{R}^n} \prod_{j=1}^m g_{x,\tau,j} \circ dB_j(x)(y-x)^{p_j} dy \leqslant (1+\tau^\eta) \int_{0.1V_\tau(x)} \prod_{j=1}^m g_{x,\tau,j} \circ dB_j(x)(y-x)^{p_j} dy$$

and that for each $j \in \{1, ..., m\}$*,*

$$\int_{\mathbb{R}^{n_j}} g_{y,\tau,j}(z) dz \leqslant (1+\tau^\eta) \int_{V_{\tau,j}(0)} g_{y,\tau,j}(z) dz$$

Proof. This follows from (2.1.1), observing that Lemma 2.5.1 may be applied in a uniform manner to each gaussian, taking $c = \|A_j^{-1}\|_{L^\infty}$ and $c = \|M^{-1}\|_{L^\infty}$ for the respective cases.□

The reader should observe that if we instead work with balls of radius $\simeq \tau$ rather than $\simeq \tau \log(1/\tau)$ we are only able to obtain a uniform bound on the left hand side of (2.5.3), hence tightness requires that we work with these non-standard radii.

Lemma 2.5.6 (Local switching of Gaussians) *There exists a* $\nu_2 \simeq_\eta 1$ *such that given* $\tau \in (0, \nu_2)$*,* $v \in \mathbb{R}^n$*, and* $x, y \in \mathbb{R}^n$ *such that* $y \in V_\tau(x)$

$$\frac{1}{\mathrm{BL}(\mathbf{dB}(y), \mathbf{p})} \prod_{j=1}^m g_{y,\tau,j} \circ dB_j(y)(x-y)^{p_j} \leqslant \frac{1+\tau^\eta}{\mathrm{BL}(\mathbf{dB}(x), \mathbf{p})} \prod_{j=1}^m g_{x,\tau,j} \circ dB_j(x)(x-y)^{p_j}$$

Proof. Let $0 < \tau < \nu_2$, where ν_2 is to be later determined, and $y \in \kappa V_\tau(x)$. It follows from the definition of M that we may write

$$\frac{1}{\mathrm{BL}(\mathbf{dB}(y), \mathbf{p})} \prod_{j=1}^m g_{y,\tau,j} \circ dB_j(y)(v)^{p_j} = \det(M(y))^{1/2} \tau^{-n} \exp\left(-\frac{\pi}{\tau^2}|v|_y^2\right). \qquad (2.5.7)$$

For reasons that will become clear, we shall now prove the claim that $\log \circ \det \circ M$ is uniformly Hölder continuous. By Proposition 2.1.3, M is uniformly Hölder continuous and M^{-1} is bounded so, for all $x, y \in \mathbb{R}^n$,

$$|M(x)M(y)^{-1} - I| \leqslant \|M^{-1}\|_{L^\infty}|M(x) - M(y)| \lesssim |x-y|^\theta$$

Because $\log \circ \det$ is smooth, we may take its first order Taylor expansion around I to find that there exist $\delta \simeq 1$ such that for $|x - y| < \delta$,

$$|\log \circ \det(M(x)) - \log \circ \det(M(y))| = |\log \circ \det(I + (M(x)M(y)^{-1} - I))| \lesssim |x-y|^\theta,$$

The condition that $|x-y| \leqslant \delta$ can then be dropped by observing that by the fact that $|M|$ and $|M^{-1}|$ are bounded above implies that $|\det \circ M|$ is both bounded above and bounded away from zero, hence $\log \circ \det \circ M$ is bounded, so the claim holds for all $x, y \in \mathbb{R}^n$ provided we enlarge the implicit constant in the above inequality. Taking the logarithm of the ratio of (2.5.7) and itself with y replaced with x, then applying the Hölder continuity of M and $\log \circ \det \circ M$, we obtain that, provided $x \neq y$.

$$\begin{aligned} &\log\left(\frac{\mathrm{BL}(\mathbf{dB}(x), \mathbf{p}) \prod_{j=1}^m g_{y,\tau,j} \circ dB_j(y)(x-y)^{p_j}}{\mathrm{BL}(\mathbf{dB}(y), \mathbf{p}) \prod_{j=1}^m g_{x,\tau,j} \circ dB_j(x)(x-y)^{p_j}}\right) = \log\left(\frac{\det(M(y))^{1/2}\exp\left(-\frac{\pi}{\tau^2}|x-y|_y^2\right)}{\det(M(x))^{1/2}\exp\left(-\frac{\pi}{\tau^2}|x-y|_x^2\right)}\right) \\ &\quad\leqslant \log(\det(M(y)M(x)^{-1}) + \frac{\pi}{\tau^2}(x-y)^T(M(y) - M(x))(x-y) \\ &\quad\lesssim_\kappa \tau^\theta \log(1/\tau)^{2+\theta}. \end{aligned}$$

The above implies an upper bound $\exp(c\tau^\theta \log(1/\tau)^{2+\theta})$ on the error factor for some $c \simeq_\kappa 1$, hence, for a sufficiently small choice of ν_2, this error is at most $(1 + \tau^\eta)$. □

Lemma 2.5.8 (Stability of heat-flow under local switching) *There exists $\nu_3 \simeq_\eta 1$, such that for all $\tau \in (0, \nu_3)$ and $x, y \in \mathbb{R}^n$ such that $y \in V_\tau(x)$,*

$$\tilde{g}_{y,\tau,j} \leqslant (1 + \tau^\eta)\tilde{g}_{x,\tau,j}.$$

hence, for each non-negative $f_j \in L^1(\mathbb{R}^{n_j})$ *and* $0 < \tau < \nu_3$

$$f_j * \tilde{g}_{y,\tau,j} \leqslant (1 + \tau^\eta) f_j * \tilde{g}_{x,\tau,j}.$$

Proof. Let $\tau > 0$ and $y \in V_\tau(x)$. First of all, for each $j \in \{1, ..., m\}$ and $w \in \mathbb{R}^{n_j}$,

$$g_{y,\tau,j}(w) = \exp\left(-\frac{\pi}{\tau^2}\langle (A_j(y) - A_j(x))w, w\rangle\right) g_{x,\tau,j}(w)$$

Using the Hölder continuity of $A_{u,j}$, it then follows that for all $w \in V_{\tau,j}(0)$ and $\tau \leqslant \nu_3 \simeq 1$,

$$\begin{aligned} g_{y,\tau,j}(w) &\leqslant \exp\left(\frac{\pi}{\tau^2}|A_j(y) - A_j(x)|\|dB_j\|_{L^\infty}^2 \tau^2 \log(\tau^{-1})^2\right) g_{x,\tau,j}(w) \\ &\leqslant \exp\left(\pi\mu_j \|dB_j\|_{L^\infty}^2 \tau^\theta \log(\tau^{-1})^{2+\theta}\right) g_{x,\tau,j}(w) \\ &\leqslant (1 + \tau^\eta) g_{x,\tau,j}(w) \end{aligned}$$

□

The need for truncated gaussians within our setup is made apparent in the proof of the previous lemma, as we may observe that we cannot obtain a similar result for non-truncated gaussians, since the tails of the gaussians $g_{x,\tau,j}$ and $g_{y,\tau,j}$ decay so rapidly that their ratios will diverge exponentially as we move away from the origin. Thankfully, truncating the gaussians outside of an appropriately sized ball, small enough to obtain sufficient local constancy, but large enough for the truncation error to decay polynomially as $\tau \to 0$, solves these issues. It should also be said that truncating the gaussians is required in order for them to be suitable cutoff functions for the partition of unity argument that we will ultimately use to prove Proposition 2.4.5.

For the final lemma, we shall need to dilate the radius of truncation of $\tilde{g}_{x,\tau,j}$ by a small factor slightly larger than one. In general, given $\kappa > 0$, we shall denote the gaussian truncated to the ball $\kappa V_{\tau,j}(0)$ by

$$\tilde{g}^{\kappa}_{y,\tau,j} := g_{y,\tau,j}\mathbb{1}_{\kappa V_{\tau,j}(0)}$$

Lemma 2.5.9 (Improvement of local constancy under heat flow) *Let $\gamma \in (0, \theta)$, then there exists a $\nu_4 \simeq_{\eta,\gamma} 1$ such that for all $\tau \in (0, \nu_4)$, $x \in \mathbb{R}^n$ and $z \in V_{\tau,j}(0)$,*

$$\tilde{g}_{x,\tau,j}(z) \leqslant (1+\tau^{\eta})\tilde{g}^{1.1}_{x,\tau,j}(\tilde{z}) \quad \textit{for all} \quad |z - \tilde{z}| \leqslant \tau^{1+\gamma}$$

Hence, it follows that for each non-negative $f_j \in L^1(\mathbb{R}^{n_j})$, $0 < \tau < \nu_4$, $x \in \mathbb{R}^n$ and $z, \tilde{z} \in \mathbb{R}^{n_j}$,

$$f_j * \tilde{g}_{x,\tau,j}(z) \leqslant (1+\tau^{\eta}) f_j * \tilde{g}^{1.1}_{x,\tau,j}(\tilde{z}) \quad \textit{for all} \quad |z - \tilde{z}| \leqslant \tau^{1+\gamma}$$

Proof. Let $0 < \tau < \nu_3$, where ν_3 is yet to be determined. Firstly, for each $j \in \{1, ..., m\}$, $x \in \mathbb{R}^n$, and $z, \tilde{z} \in \mathbb{R}^{n_j}$

$$\begin{aligned} g_{x,\tau,j}(\tilde{z}) &= \exp(-\pi\tau^{-2}(|z|^2_{x,j} + 2\langle z, z-\tilde{z}\rangle_{x,j} + |z-\tilde{z}|^2_{x,j})) \\ &\geqslant \exp(-\pi\tau^{-2}(|z|^2_{x,j} + 2|z|_{x,j}|z-\tilde{z}|_{x,j} + |z-\tilde{z}|^2_{x,j})) \\ &\geqslant g_{x,\tau,j}(z)\exp(-3\pi\tau^{-2}|z|_{x,j}|z-\tilde{z}|_{x,j}). \end{aligned}$$

Now suppose that $z \in V_{\tau,j}(0)$ and $|z-\tilde{z}| \leqslant \tau^{1+\gamma}$, then, choosing ν_4 to be sufficiently small,

$$|\tilde{z}| \leqslant |z| + |z-\tilde{z}| \leqslant 1.1\|dB_j\|_{L^\infty}\tau\log\left(\frac{1}{\tau}\right)$$

so $\tilde{z} \in 1.1V_{\tau,j}(0) = \text{supp}\left(\tilde{g}^{1.1}_{x,\tau,j}\right)$, hence

$$\begin{aligned}\tilde{g}_{x,\tau,j}(z) = g_{x,\tau,j}(z) &\leqslant \exp(3\pi\tau^{-2}|z|_{x,j}|z-\tilde{z}|_{x,j})g_{x,\tau,j}(\tilde{z}) \\ &\leqslant \exp\left(3\pi||A_j||_{L^\infty}\tau^{-1}\log(1/\tau)|z-\tilde{z}|\right)g_{x,\tau,j}(\tilde{z}) \\ &\leqslant \exp\left(3\pi||A_j||_{L^\infty}\tau^{\gamma}\log(1/\tau)\right)g_{x,\tau,j}(\tilde{z}) \\ &\leqslant (1+\tau^\eta)g_{x,\tau,j}(\tilde{z}) = (1+\tau^\eta)\tilde{g}^{1.1}_{x,\tau,j}(\tilde{z}) \end{aligned}$$ □

2.6 Proof of Proposition 2.4.5

This proof shall draw heavily from that of Ball's linear inequality. We introduce the truncated gaussians that we want to convolve our inputs with as a partition of unity. This partition will also conveniently serve to split up the integral into a continuum of localised pieces, which will allow us to exploit local constancy and Holder regularity to perturb the B_j to an affine approximation on each of those pieces. We may then apply the linear Brascamp–Lieb inequality piecewise, and in doing so, obtain the desired form on the right-hand side.

Proof. Let $\tilde{\nu} \leqslant \min\{\nu_1, \nu_2, \nu_3, \nu_4\}$ and take some $0 < \tau < \tilde{\nu}$. Later on we will retrospectively impose some trivially stricter assumptions on the size of $\tilde{\nu}$, a statement of which we omit here for the sakes of readability. For all $y \in U$, by definition of $\mathbf{g}_{y,\tau}$, and Lemma 2.5.5,

$$\text{BL}(\mathbf{dB}(y), \mathbf{p}) \leqslant (1+\tau^\eta)\int_{0.1V_\tau(y)}\prod_{j=1}^m \tilde{g}_{y,\tau,j} \circ dB_j(y)(x-y)^{p_j}dx.$$

We may then multiply the left-hand-side of (2.4.1) by the right-hand-side of the above inequality, then exchange orders of integration, to obtain that for every non-negative

$f_j \in L^1(\mathbb{R}^{n_j})$,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \prod_{j=1}^m f_j * \tilde{g}_{y,\tau,j} \circ B_j(y)^{p_j} dy \\
&\leqslant (1+\tau^\eta) \int_{\mathbb{R}^n} \frac{1}{\mathrm{BL}(\mathbf{dB}(y), \mathbf{p})} \int_{0.1V_\tau(y)} \prod_{j=1}^m f_j * \tilde{g}_{y,\tau,j} \circ B_j(y)^{p_j} g_{y,\tau,j} \circ dB_j(y)(x-y)^{p_j} dxdy \\
&\leqslant (1+\tau^\eta) \int_{\mathbb{R}^n} \int_{0.1V_\tau(x)} \frac{1}{\mathrm{BL}(\mathbf{dB}(y), \mathbf{p})} \prod_{j=1}^m f_j * \tilde{g}_{y,\tau,j} \circ B_j(y)^{p_j} g_{y,\tau,j} \circ dB_j(y)(x-y)^{p_j} dydx \\
&\leqslant (1+\tau^\eta) \int_{\mathbb{R}^n} \int_{0.1V_\tau(x)} \frac{1}{\mathrm{BL}(\mathbf{dB}(y), \mathbf{p})} \prod_{j=1}^m f_j * \tilde{g}_{y,\tau,j} \circ B_j(y)^{p_j} g_{y,\tau,j} \circ dB_j(y)(x-y)^{p_j} dydx.
\end{aligned}$$

Having truncated appropriately, we may now apply Lemma 2.5.6 and Lemma 2.5.8 to interchange some of the instances of the variables x and y in the integrand, incurring an error factor of at most $(1+\tau^\eta)^{1+\sigma}$, where $\sigma = \sum_{j=1}^m p_j$.

$$\begin{aligned}
&\int_{\mathbb{R}^n} \prod_{j=1}^m f_j * \tilde{g}_{y,\tau,j} \circ B_j(y)^{p_j} dy \\
&\leqslant (1+\tau^\eta)^{2+\sigma} \int_{\mathbb{R}^n} \frac{1}{\mathrm{BL}(\mathbf{dB}(x), \mathbf{p})} \int_{0.1V_\tau(x)} \prod_{j=1}^m f_j * \tilde{g}_{x,\tau,j} \circ B_j(y)^{p_j} g_{x,\tau,j} \circ dB_j(x)(x-y)^{p_j} dydx
\end{aligned}$$

By the $C^{1,\theta}$ regularity of the data, $|B_j(x) - B_j(y) - dB_j(y)(x-y)| \leqslant \mu|x-y|^{1+\theta} \leqslant \mu(0.1\tau\log(1/\tau))^{1+\theta}$, so for a perhaps smaller choice of $\tilde{\nu}$, $|B_j(x) - B_j(y) - dB_j(y)(x-y)| \leqslant \tau^{1+\gamma}$ for all $y \in 0.1V_\tau(x)$. We may then apply Lemma 2.5.9, and replace each $B_j(y)$ with its first-order affine approximation centred at x, which we shall denote by $L_j^x := B_j(x) + dB_j(x)(y-x)$, at the cost of an error factor of $(1+\tau^\eta)^\sigma$ and a fattening of the support of $\tilde{g}_{x,\tau,j}$.

$$
\begin{aligned}
&\int_{U+V_\tau(0)} \prod_{j=1}^m f_j * \tilde{g}_{y,\tau,j} \circ B_j(y)^{p_j} dy \\
&\leqslant (1+\tau^\eta)^{2+2\sigma} \int_{U+V_{\sqrt{2}\tau}(0)} \frac{\int_{0.1V_\tau(x)} \prod_{j=1}^m f_j * \tilde{g}^{1.1}_{x,\tau,j} \circ L_j^x(y)^{p_j} g_{x,\tau,j} \circ dB_j(x)(x-y)^{p_j} dy}{\mathrm{BL}(\mathbf{dB}(x), \mathbf{p})} \\
&= (1+\tau^\eta)^{2+2\sigma} \int_{U+V_{\sqrt{2}\tau}(0)} \frac{\int_{0.1V_\tau(0)} \prod_{j=1}^m f_j * \tilde{g}^{1.1}_{x,\tau,j}(B_j(x) - dB_j(x)y)^{p_j} g_{x,\tau,j} \circ dB_j(x)(y)^{p_j} dy}{\mathrm{BL}(\mathbf{dB}(x), \mathbf{p})} dx \\
&\leqslant (1+\tau^\eta)^{2+2\sigma} \int_{U+V_{\sqrt{2}\tau}(0)} \frac{\int_{\mathbb{R}^n} \prod_{j=1}^m f_j * \tilde{g}^{1.1}_{x,\tau,j}(B_j(x) - dB_j(x)y)^{p_j} \tilde{g}^{0.1}_{x,\tau,j} \circ dB_j(x)(y)^{p_j} dy}{\mathrm{BL}(\mathbf{dB}(x), \mathbf{p})} dx
\end{aligned}
$$

The last line follows from the observation that $0.1V_\tau(0) \subset \bigcap_{j=1}^m dB_j(x)^{-1}(0.1V_{\tau,j}(0))$ for all $x \in \mathbb{R}^n$. We are now in a position to apply the linear inequality.

$$
\begin{aligned}
&\leqslant (1+\tau^\eta)^{2+2\sigma} \int_{\mathbb{R}^n} \prod_{j=1}^m \left(\int_{\mathbb{R}^{n_j}} f_j * \tilde{g}^{1.1}_{x,\tau,j}(B_j(x) - z) \tilde{g}^{0.1}_{x,\tau,j}(z) dz \right)^{p_j} dx \\
&\leqslant (1+\tau^\eta)^{2+2\sigma} \int_{\mathbb{R}^n} \prod_{j=1}^m f_j * \tilde{g}^{1.1}_{x,\tau,j} * \tilde{g}^{0.1}_{x,\tau,j} \circ B_j(x)^{p_j} dx
\end{aligned}
$$

We need to prove the claim that, if $\tilde{\nu}$ is chosen to be sufficiently small, then $\tilde{g}^{1.1}_{x,\tau,j} * \tilde{g}^{0.1}_{x,\tau,j} \leqslant \tilde{g}_{x,\sqrt{2}\tau,j}$ for each $j \in \{1, ..., m\}$. Since $1.2 \leqslant \sqrt{2}$, if τ is sufficiently small, then $1.1\tau \log(1/\tau) + 0.1\tau \log(1/\tau) = \tau \log(1/\tau^{1.2}) \leqslant \sqrt{2}\tau \log(1/\tau^{1.2/\sqrt{2}}) \leqslant \sqrt{2}\tau \log(1/\sqrt{2}\tau)$, which implies that

$$
\begin{aligned}
\operatorname{supp}\left(\tilde{g}^{1.1}_{x,\tau,j} * \tilde{g}^{0.1}_{x,\tau,j}\right) &\subset \operatorname{supp}\left(\tilde{g}^{1.1}_{x,\tau,j}\right) + \operatorname{supp}\left(\tilde{g}^{0.1}_{x,\tau,j}\right) \\
&= 1.1V_{\tau,j}(0) + 0.1V_{\tau,j}(0) \subset V_{\sqrt{2}\tau,j}(0).
\end{aligned}
$$

By the semigroup property for heat equations, $g_{y,\tau,j} * g_{y,\tau,j} = g_{x,\sqrt{2}\tau,j}$, hence the claim follows from combining (8) and the pointwise bound $\tilde{g}^{1.1}_{x,\tau,j} * \tilde{g}^{0.1}_{x,\tau,j} \leqslant g_{x,\tau,j} * g_{x,\tau,j} = g_{x,\sqrt{2}\tau,j}$. At this point, we are essentially done, provided that we have chosen $\tilde{\nu}$ to be small enough for $(1+\tau^\eta)^{2+2\sigma} \leqslant (1+\tau^\beta)$ to hold over all $\tau \in (0,\tilde{\nu})$.

$$\begin{aligned}\int_{\mathbb{R}^n} \prod_{j=1}^m f_j * \tilde{g}_{y,\tau,j} \circ B_j(y)^{p_j} dy &\leqslant (1+\tau^\eta)^{2+2\sigma} \int_{\mathbb{R}^n} \prod_{j=1}^m \left(f_j * \tilde{g}_{x,\sqrt{2}\tau,j} \circ B_j(x)\right)^{p_j} dx \\ &\leqslant (1+\tau^\beta) \int_{\mathbb{R}^n} \prod_{j=1}^m \left(f_j * \tilde{g}_{x\sqrt{2}\tau,j} \circ B_j(x)\right)^{p_j} dx \qquad \square\end{aligned}$$

Chapter 3

A Nonlinear Variant of Ball's Inequality: The General Case

This chapter is a re-edited version of the preprint 'A Nonlinear Version of Ball's Inequality' [40].

3.1 Setup and Notation

In this chapter, we shall consider fixed complete Riemannian manifolds (without boundary) M, M_1,...,M_m of dimensions n, n_1,...,n_m. We require at least a Riemannian manifold structure for a number of reasons, for instance so that the notion of a gaussian defined on a tangent space make sense. We shall refer to the exponential map based at a point x on a manifold $e_x : T_xN \to N$. The injectivity radius of a point $x \in N$ is the largest number $\rho_x > 0$ such that e_x restricts to a diffeomorphism on the ball of radius ρ_x around $0 \in T_xN$. We shall assume that the manifolds we consider have bounded geometry, by which we mean that they have injectivity radii uniformly bounded below, by a number $\rho > 0$ which we now fix, and also that both the Riemannian curvature and its covariant derivative are uniformly bounded above. These are standard conditions for the type of global setting we shall be considering that exist to ensure that we may apply exponential maps in a uniform manner. For further reading about analysis on manifolds with bounded geometry, see [50, 61].

We shall refer to a ball centred at a point $x \in M$ of radius $r > 0$ on a manifold M by $U_r(x)$, and refer to a ball centred at a point $v \in T_xM$ of radius $r > 0$ by $V_r(v)$ (the tangent space that this ball belongs to should always be clear from context, if it is not stated explicitly). We shall consider submersions $B_j : M \to M_j$ ($j \in \{1, ..., m\}$) that may be viewed as fixed for the entirety, and are assumed to have at least L^∞ bounded derivative maps. Noting this, we shall denote a ball centred at $z \in M_j$ of radius $r\|dB_j\|_{L^\infty}$ by $U_{r,j}(z)$, and similarly a ball centred at $w \in T_zM_j$ of radius $r\|dB_j\|_{L^\infty}$ by $V_{r,j}(w)$, simply for the technical reason that then $\bigcap_{j=1}^m dB_j(x)^{-1}(V_{r,j}(0)) \subset V_r(0)$, a property that shall prove to be useful later on. Similarly to the linear case, we refer to the pair $(\mathbf{B}, \mathbf{p})$ as a nonlinear Brascamp–Lieb datum. We shall also make use of a fixed parameter $\gamma \in (0, 1)$ close to 1, The exact choice of value here is not particularly important, the reader may take γ to be 0.9, say, however we refrain from doing this for the sakes of clarity and good book-keeping.

We shall always use a single bar to denote a finite dimensional norm, usually a 2-norm, and double bars to denote an infinite dimensional norm, which we shall always specify with a subscript. In the case where we are taking a norm of a matrix, we shall assume that this is the induced 2-norm unless stated otherwise. Furthermore, if y is some variable, Q is a normed space valued function of y, and f is a real valued function of y, then we shall use the the notation $Q(y) = \mathcal{O}(f(y))$ to denote that $\|Q(y)\| \lesssim f(y)$.

3.1.1 Statements of Results

Before we state our nonlinear version of (1.3.2), we must first preliminarily define our 'heat-flow'. The construction thereof is rather involved, however the resulting flow operator $H_{x,\tau,j}$ may nonetheless be written essentially as a convolution with a gaussian kernel $G_{x,\tau,j} : T_{B_j(x)}M_j \to \mathbb{R}$, the key properties of which we now state as a proposition.

Proposition 3.1.1 *Suppose that* $(\mathbf{B}, \mathbf{p})$ *is a nonlinear Brascamp–Lieb datum such that each* $B_j : M \to M_j$ *is* C^2 *and that there exists* $C > 0$ *such that* $\|\mathbf{dB}\|_{W^{1,\infty}}, \|\mathrm{BL}(\mathbf{dB}, \mathbf{p})\|_{L^\infty} \leqslant C$. *Then, there exists an* $\varepsilon > 0$ *such that, for* $\tau > 0$ *sufficiently small, there exists a smooth family of gaussian inputs* $\mathbf{G}_{x,\tau} := (G_{x,\tau,j})_{j=1}^m$ *parametrised by* $x \in M$ *satisfying the following properties:*

1. *Each gaussian* $G_{x,\tau,j}$ *is of unit mass and is defined by a corresponding* τ*-dependent positive definite matrix* $A_{\tau,j}(x)$, *in the sense that*

$$G_{x,\tau,j}(z) := \tau^{-n_j} \det(A_{\tau,j}(x))^{1/2} \exp(-\pi\tau^{-2}\langle A_{\tau,j}(x)z, z\rangle).$$

2. $\mathbf{G}_{x,\tau}$ *is a* τ^ε*-near extremiser for the datum* $(\mathbf{dB}(x), \mathbf{p})$.

3. $\|A_{\tau,j}\|_{W^{1,\infty}(M)}, \|\det A_{\tau,j}\|_{W^{1,\infty}(M)} \leqslant \tau^{-\varepsilon}$ *for all* $j \in \{1, ..., m\}$.

The construction of this $G_{x,\tau,j}$ is carried out in detail in Section 3.1.3, whence (1) automatically follows, see the end of Section 3.1.4 and the remark after Lemma 3.2.1 for the proof of properties (2) and (3) respectively. We may now define the corresponding flow operator, wherein we include some truncation to allow us to map locally to the tangent space on which $G_{x,\tau,j}$ is defined.

$$\begin{aligned} H_{x,\tau,j} &: L^1(M_j) \to L^1(U_{\rho-\tau^\gamma}(B_j(x))) \\ H_{x,\tau,j} f_j(z) &:= \int_{U_{\tau^\gamma,j}(z)} f_j(w) G_{x,\tau,j}(e^{-1}_{B_j(x)}(z) - e^{-1}_{B_j(x)}(w)) dw \end{aligned}$$

We now state our near-monotonicity result, which is the main theorem of this chapter.

Theorem 3.1.2 (Nonlinear Ball's Inequality) *Suppose that* $(\mathbf{B}, \mathbf{p})$ *is a nonlinear Brascamp–Lieb datum such that each* $B_j : M \to M_j$ *is a twice continuously differentiable submersion, and that there exists* $C > 0$ *such that* $\|\mathbf{dB}\|_{W^{1,\infty}}, \|\mathrm{BL}(\mathbf{dB}, \mathbf{p})\|_{L^\infty} \leqslant C$.
Then, there exists a $\beta > 0$ *such that for* $\tau > 0$ *sufficiently small, for all non-negative* $f_j \in L^1(M_j)$,

$$\int_M \prod_{j=1}^m f_j \circ B_j(x)^{p_j} dx \leqslant (1 + \tau^\beta) \int_M \prod_{j=1}^m H_{x,\tau,j} f_j \circ B_j(x)^{p_j} dx. \tag{3.1.3}$$

Of course, in the euclidean case we may identify our domain with every tangent space, so (3.1.3) then takes the following more familiar form:

$$\int_{\mathbb{R}^n} \prod_{j=1}^m f_j \circ B_j(x)^{p_j} dx \leqslant (1 + \tau^\beta) \int_{\mathbb{R}^n} \prod_{j=1}^m f_j * (G_{x,\tau,j} \chi_{U_{\tau^\gamma,j}(0)}) \circ B_j(x)^{p_j} dx.$$

which of course implies a non-truncated, genuine heat-flow near-monotonicity statement.

$$\int_{\mathbb{R}^n} \prod_{j=1}^m f_j \circ B_j(x)^{p_j} dx \leqslant (1 + \tau^\beta) \int_{\mathbb{R}^n} \prod_{j=1}^m f_j * G_{x,\tau,j} \circ B_j(x)^{p_j} dx.$$

Often nonlinear Brascamp–Lieb inequalities exhibit sufficient diffeomorphism invariance that to prove them in manifold settings it suffices to only consider the euclidean one. We should therefore make clear that in order to obtain an inequality at the level of generality of (3.1.3), it is vital for our analysis that we work in the manifold setting at every stage in the proof, as this inequality is not sufficiently diffeomorphism-invariant to be reducible to the euclidean case, even in the case where $M \cong \mathbb{R}^n$.

Example 3.1.4 (Multilinear Radon-like transforms) *Given functions* $f_1, ..., f_m \in L^1(\mathbb{R}^n)$, *we may write their n-fold convolution as an integral over an affine subspace.*

$$f_1 * ... * f_n(y) := \int_{\{x_1+x_2+...+x_m=y\}} \prod_{j=1}^{m} f_j(x_j) d\sigma(x_1, ..., x_m) \tag{3.1.5}$$

We may be generalise this notion by nonlinearly perturbing this subspace. Let $\phi : \mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_m} \to \mathbb{R}^n$ *be a smooth function and suppose that* $y \in \mathbb{R}^n$ *is a regular value of* ϕ, *and consider the following multilinear operator* T.

$$T(f_1, ..., f_m)(y) := \int_{\phi^{-1}(\{y\})} \prod_{j=1}^{m} f_j(x_j) d\sigma(x_1, ..., x_m) \tag{3.1.6}$$

Let $N := \sum_{j=1}^{m} n_j$ *and* $p_j := (N-n)/N$, *for all* $j \in \{1, ..., m\}$ *and denote the natural projection map from* $T_x\phi^{-1}(\{y\}) \subset \mathbb{R}^N = \mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_m}$ *to* $\mathbb{R}^{n_j}$ *by* L_j^x. *Then, provided that* ϕ *satisfies the condition that* $\mathrm{BL}(\mathbf{L}^x, \mathbf{p}) \lesssim 1$ *for all* $y \in \mathbb{R}^n$, *and all* $(x_1, ..., x_m) \in \phi^{-1}(\{y\})$, *then there exists a* $\beta > 0$ *and a family of gaussians* $(G_{x,\tau,j})_{j=1}^m$ *as in Proposition 3.1.1 such that for all* $f_j \in L^1(\mathbb{R}^{n_j})$ *and* $\tau > 0$ *sufficiently small,*

$$T(f_1^{\frac{N-n}{N}}, ..., f_m^{\frac{N-n}{N}})(y) \leqslant (1+\tau^\beta) \int_{\phi^{-1}(\{y\})} \prod_{j=1}^{m} (f_j * G_{x,\tau,j})^{\frac{N-n}{N}}(x_j) d\sigma(x_1, ..., x_m). \tag{3.1.7}$$

The main upshot of Theorem 3.1.2 is that one may use the local-constancy of $H_{x,\tau,j} f_j$ to perturb the argument in the right-hand side of (3.1.3), either at small scales as in Corollary 3.1.8, which yields a slightly improved version of the local nonlinear Brascamp–Lieb inequality first proved in [9] that better quantifies the relationship between the ε-loss in the constant and the size of the domain on the left-hand side, or at large scales as in Corollary 3.1.9, which states that finiteness is stable under L^∞ perturbations.

Corollary 3.1.8 *Let* $(\mathbf{B}, \mathbf{p})$ *be a nonlinear Brascamp–Lieb datum satisfying the same conditions as in Theorem 3.1.2, then there exists a* $\beta > 0$ *such that for each* $x_0 \in M$ *and* *all $\tau > 0$ sufficiently small,*

$$\int_{U_\tau(x_0)} \prod_{j=1}^{m} f_j \circ B_j(x)^{p_j} dx \leqslant (1+\tau^\beta)\mathrm{BL}(\mathbf{dB}(x_0), \mathbf{p}) \prod_{j=1}^{m} \left(\int_{M_j} f_j \right)^{p_j}$$

Corollary 3.1.9 *Suppose that $B_j, \widetilde{B}_j : \mathbb{R}^n \to \mathbb{R}^{n_j}$ for all $j \in \{1, ...m\}$ and $(\mathbf{B}, \mathbf{p})$ is a nonlinear datum satisfying the conditions of Theorem 3.1.2, that the inequality associated with $(\widetilde{\mathbf{B}}, \mathbf{p})$ holds with finite constant, and that $\|\mathbf{B} - \widetilde{\mathbf{B}}\|_{L^\infty} < \infty$, then the inequality associated with $(\mathbf{B}, \mathbf{p})$ holds with finite constant.*

In particular, this corollary implies that any inequality associated with a nonlinear L^∞ perturbation of a feasible linear datum, provided that it satisfies the conditions of Theorem 3.1.2, must hold with finite constant. It would be reasonable to suggest that a similar result would hold in the non-euclidean setting, however, due to certain technical geometric complications, this appears to fall beyond the scope of this thesis.

3.1.2 Reduction of Theorem 3.1.2

Let $C(s,t)$ denote the best constant $C \in (0, \infty]$ for the following inequality.

$$\int_M \prod_{j=1}^{m} H_{x,s,j} f_j \circ B_j(x)^{p_j} dx \leqslant C(s,t) \int_M \prod_{j=1}^{m} H_{x,t,j} f_j \circ B_j(x)^{p_j} dx \qquad (3.1.10)$$

It is easy to see that $C(s,t)$ enjoys the submultiplicative property $C(r,t) \leqslant C(r,s)C(s,t)$. We claim that this together with the following proposition is sufficient to prove Theorem 3.1.2.

Proposition 3.1.11 *There exist $\beta, \nu > 0$ such that, for all $\tau \in (0, \nu)$,*

$$C(\tau, \sqrt{2}\tau) \leqslant (1 + \tau^\beta).$$

Proof of Theorem 3.1.2 given Proposition 3.1.11. Setting $\tau_0 = \tau$, define the geometric sequence $\tau_k := 2^{-k/2}\tau_0$ and let $K \in \mathbb{N}$. We can split the constant $C(\tau_K, \tau)$ into pieces that can be dealt with by Proposition 3.1.11.

$$\begin{aligned} C(\tau_K, \tau) &\leqslant C(\tau_K, \tau_{K-1})C(\tau_{K-1}, \tau) \\ &\leqslant C(\tau_K, \tau_{K-1})C(\tau_{K-1}, \tau_{K-2})C(\tau_{K-2}, \tau) \\ &\leqslant ... \leqslant \prod_{k=1}^{K} C(\tau_k, \tau_{k-1}) \leqslant \prod_{k=1}^{K}(1 + \tau_k^\beta) \end{aligned}$$

Taking logarithms of the above inequality, we obtain that

$$\begin{aligned} \log(C(\tau_K, \tau)) &\leqslant \sum_{k=1}^{K} \log(1 + \tau_k^\beta) \\ &\leqslant \sum_{k=1}^{\infty} \tau_k^\beta = \frac{\tau^\beta}{2^{\beta/2} - 1} \end{aligned}$$

It then follows that, making τ accordingly smaller if necessary, that $C(\tau_K, \tau) \leqslant \exp(\frac{\tau^\beta}{2^{\beta/2}-1}) \leqslant (1 + \tau^{\beta/2})$. For each $j \in \{1, ..., m\}$, let $f_j \in C_0^\infty(M_j)$ be a non-negative function. By the forthcoming Lemma 3.2.11, we know that $H_{x,\tau,j} f_j \circ B_j(x) \to f_j \circ B_j(x)$ as $\tau \to 0$ for all $x \in M$, hence we may apply Fatou's lemma and consider (3.1.10) with $s = \tau_K$ and $t = \tau$, taking the limit as $K \to \infty$.

$$\begin{aligned} \int_M \prod_{j=1}^{m} f_j \circ B_j(x)^{p_j} dx &\leqslant \liminf_{K\to\infty} \int_M \prod_{j=1}^{m} H_{x,\tau_K,j} f_j \circ B_j(x)^{p_j} dx \\ &\leqslant \liminf_{K\to\infty} C(\tau_K, \tau) \int_M \prod_{j=1}^{m} H_{x,\tau,j} f_j \circ B_j(x)^{p_j} dx \\ &\leqslant (1 + \tau^{\beta/2}) \int_M \prod_{j=1}^{m} H_{x,\tau,j} f_j \circ B_j(x)^{p_j} dx \qquad (3.1.12) \end{aligned}$$

This implies the theorem since we may extend this inequality by density to general non-negative $f_j \in L^1(M_j)$. □

This initial reduction complete, we now turn our attention to the task of constructing the family of near-extremising gaussians $G_{x,\tau,j}$, but, as we briefly discussed at the end of Section 1.1, in order to do this we shall first need to establish a slight improvement of the effective version of Lieb's theorem (Theorem 1.1.4) first proved in [9].

3.1.3 A Regularised Effective Lieb's Theorem

An issue with constructing a suitable heat-flow outside of the case where $(\mathbf{dB}(x), \mathbf{p})$ is simple is that we do not then have a natural choice of gaussian extremiser to use as our heat kernel, in fact, generally speaking $(\mathbf{dB}(x), \mathbf{p})$ may not admit a gaussian extremiser at all. While Lieb's theorem does guarantee the existence of a δ-near gaussian extremiser for any $\delta > 0$, i.e. there exists a gaussian input $\mathbf{A}$ such that $\mathrm{BL}_{\mathrm{g}}(\mathbf{dB}(x), \mathbf{p}; \mathbf{A}) \geqslant (1-\delta)\mathrm{BL}(\mathbf{dB}(x), \mathbf{p})$, it does not offer any quantitative information about this gaussian. The authors of [9] overcame these problems by establishing an effective version of Lieb's theorem that tracks how the family of δ-near extremisers for a given Brascamp–Lieb datum degenerates as $\delta \to 0$. We now state a simplified version of their result.

Theorem 3.1.13 (Effective Lieb's theorem [9]) *There exists $N \in \mathbb{N}$ depending only on the dimensions and exponents such that the following holds: For any given $D > 0$ there exists $\delta_0 > 0$ such that for every $\delta \in (0, \delta_0)$ and any feasible datum $(\mathbf{L}, \mathbf{p})$ such that $\mathrm{BL}(\mathbf{L}, \mathbf{p}), |\mathbf{L}| \leqslant D$,*

$$\sup_{|A|,|A^{-1}| \leqslant \delta^{-N}} \mathrm{BL}_{\mathrm{g}}(\mathbf{L}, \mathbf{p}; \mathbf{A}) \geqslant (1-\delta)\mathrm{BL}(\mathbf{L}, \mathbf{p}). \tag{3.1.14}$$

This theorem, in other words, establishes the existence of a function $\mathbf{Y}_\delta$ from the set of feasible Brascamp–Lieb data to the set of gaussian inputs such that $\mathbf{Y}_\delta(\mathbf{L})$ is a δ-near extremiser for $(\mathbf{L}, \mathbf{p})$ and both $\|\mathbf{Y}_\delta\|_{L^\infty}$ and $\|\mathbf{Y}_\delta^{-1}\|_{L^\infty}$ are bounded above by δ^{-N} (to clarify, $\mathbf{Y}_\delta^{-1}(\mathbf{L})$ refers to the gaussian input whose jth entry is the inverse of the jth entry of $\mathbf{Y}_\delta(\mathbf{L})$). It says nothing however about the existence of a smooth, let alone continuous, function with such properties. Unfortunately, we require $\mathbf{Y}_\delta$ to be $W^{1,\infty}$ bounded for our analysis, moreover, we require that its $W^{1,\infty}$ norm is bounded polynomially in δ.

Theorem 3.1.15 *Recall the definition of $\mathcal{F}$ and $\mathcal{G}$ from Section 1.1. There exists an $N \in \mathbb{N}$ depending only on the dimensions and exponents such that the following holds: For all open $\Omega \Subset \mathcal{F}$ there exists a $\nu > 0$ such that for all $\delta \in (0, \nu)$, there exists a smooth function $\mathbf{Y}_\delta : \Omega \to \mathcal{G}$ such that $\det(\mathbf{Y}_\delta(\mathbf{L})_j) = 1$ for all $j \in \{1, ..., m\}$, $\|\mathbf{Y}_\delta\|_{W^{1,\infty}(\Omega)}, \|\mathbf{Y}_\delta^{-1}\|_{L^\infty(\Omega)} \leqslant \delta^{-N}$ and, for each $\mathbf{L} \in \Omega$, $\mathbf{Y}_\delta(\mathbf{L})$ is a δ-near extremiser for $(\mathbf{L}, \mathbf{p})$, i.e., that $\mathrm{BL_g}(\mathbf{L}, \mathbf{p}; \mathbf{Y}_\delta(\mathbf{L})) \geqslant (1-\delta)\mathrm{BL}(\mathbf{L}, \mathbf{p})$.*

Fortunately, the authors of [9] in the same paper establish the Hölder continuity of the Brascamp–Lieb constant as a consequence of their effective Lieb's theorem, which, as it shall turn out, is an essential ingredient for proving Theorem 3.1.15.

Proposition 3.1.16 ([9]) *There exists a number $\theta \in (0, 1)$ and a constant C_0 depending on the dimensions $(n_j)_{j=1}^m$ and exponents $(p_j)_{j=1}^m$ such that the following holds: Given data $\mathbf{L}, \mathbf{L}'$ such that $|\mathbf{L}|, |\mathbf{L}'| \leqslant C_1$ and $\mathrm{BL}(\mathbf{L}, \mathbf{p}), \mathrm{BL}(\mathbf{L}', \mathbf{p}) \leqslant C_2$, we then have*

$$|\mathrm{BL}(\mathbf{L}, \mathbf{p}) - \mathrm{BL}(\mathbf{L}', \mathbf{p})| \leqslant C_0 C_1^{n+\theta(n-1)} C_2^3 |\mathbf{L} - \mathbf{L}'|^\theta. \tag{3.1.17}$$

Proof of Theorem 3.1.15. The proof strategy is to locally average the potentially discontinuous function given by Theorem 3.1.13 in such a way that we both preserve its good properties and impose on it some additional regularity. We will be averaging via a discrete cover of Ω, which we shall now define. Let $\theta \in (0, 1)$ be an exponent to be determined later, and let $E \subset \Omega$ be the following discrete grid of points:

$$E := \Omega \cap \left(\left(\frac{\delta}{100} \right)^{\frac{1}{\theta}} \mathbb{Z}^{n_1 \times n} \times ... \times \mathbb{Z}^{n_m \times n} \right). \tag{3.1.18}$$

Now, let $\mathcal{I}$ be an indexing set for E so that we may write $E = \{\mathbf{L}_i\}_{i\in\mathcal{I}}$, then let $\mathcal{Q} := \{Q_i\}_{i\in\mathcal{I}}$ be a cover of Ω via axis-parallel cubes of width equal to $(\delta/10)^{\frac{1}{\theta}}$, with each Q_i centred at $\mathbf{L}_i$. One should note as a matter of technicality that we may need to take δ to be very small for $\mathcal{Q}$ to genuinely be a cover of Ω.

By Theorem 3.1.13, there exists an $N \in \mathbb{N}$ such that for sufficiently small $\delta > 0$ there exists a function $\mathbf{Y}_\delta^0 : \Omega \to \mathcal{G}$ such that $\|\mathbf{Y}_\delta^0\|_{L^\infty(\Omega)}, \|(\mathbf{Y}_\delta^0)^{-1}\|_{L^\infty(\Omega)} \leqslant \delta^{-N}$ and $\mathbf{Y}_\delta^0(\mathbf{L})$ is a $\delta/2$-near extremiser for $(\mathbf{L}, \mathbf{p}) \in \Omega$. We begin by showing that, for a suitable choice of θ and provided that δ is chosen to be sufficiently small, for all $i \in \mathcal{I}$, $\mathbf{Y}_\delta^0(\mathbf{L}_i)$ is also a δ-near extremiser for any $(\mathbf{L}, \mathbf{p})$ such that $\mathbf{L} \in Q_i \cap \Omega$. By compactness of Ω and smoothness of the Brascamp–Lieb functional in $\mathbf{L}$ on $\mathcal{F}$, there exists a $\nu_1 \in (0, 1)$ such that for $\eta \in (0, \nu_1)$ and all $\mathbf{L}, \mathbf{L}' \in \Omega$ satisfying $|\mathbf{L} - \mathbf{L}'| \leqslant \eta^2$, we have

$$\mathrm{BL_g}(\mathbf{L}', \mathbf{p}; \mathbf{Y}_\delta^0(\mathbf{L})) \geqslant (1 - \eta)\mathrm{BL_g}(\mathbf{L}, \mathbf{p}; \mathbf{Y}_\delta^0(\mathbf{L})). \tag{3.1.19}$$

The presence of exponent in the bound η^2 here is merely for absorbing constants. By Proposition 3.1.16, we may choose $\theta \in (0, 1/2)$ such that the following holds: There exists $\nu_2 \in (0, 1)$ such that for $\eta \in (0, \nu_2)$ and $|\mathbf{L} - \mathbf{L}'| \leqslant \eta^{\frac{1}{\theta}}$, we have that

$$\mathrm{BL}(\mathbf{L}, \mathbf{p}) \geqslant (1 - \eta)\mathrm{BL}(\mathbf{L}', \mathbf{p}). \tag{3.1.20}$$

Again we have used some freedom in our choice in θ to absorb the constants that arise in (3.1.17). Choose δ such that $0 < \delta \leqslant \min\{\nu_1, \nu_2, 1\}$, then for all $i \in \mathcal{I}$ and all $\mathbf{L} \in Q_i \cap \Omega$, since $|\mathbf{L} - \mathbf{L}_i| < \delta^{\frac{1}{\theta}}/10^{\frac{1}{\theta}} \leqslant \delta^2/100$, we may apply (3.1.19) and (3.1.20) together with the fact that $\mathbf{Y}^0_\delta(\mathbf{L}_i)$ is a $\delta/2$-near extremiser for $(\mathbf{L}_i, \mathbf{p})$ to prove the claim.

$$\begin{aligned}\mathrm{BL_g}(\mathbf{L}, \mathbf{p}; \mathbf{Y}^0_\delta(\mathbf{L}_i)) &\geqslant (1-\delta/10)\mathrm{BL_g}(\mathbf{L}_i, \mathbf{p}; \mathbf{Y}^0_\delta(\mathbf{L}_i)) \\ &\geqslant (1-\delta/2)(1-\delta/10)\mathrm{BL_g}(\mathbf{L}_i, \mathbf{p}) \\ &\geqslant (1-\delta/2)(1-\delta/10)^2\mathrm{BL_g}(\mathbf{L}, \mathbf{p}) \\ &\geqslant (1-\delta)\mathrm{BL}(\mathbf{L}, \mathbf{p}) \end{aligned} \tag{3.1.21}$$

Now, let $\{\rho_i\}_{i\in\mathcal{I}}$ be a smooth partition of unity subordinate to $\mathcal{Q}$ with indexing set $\mathcal{I}$ such that $\|d\rho_i\|_{L^\infty} \lesssim \delta^{-\frac{1}{\theta}}$ (we may use a similar construction to that in section 1.6), and define the function $\mathbf{Y}^1_\delta : \Omega \to \mathcal{G}$.

$$\mathbf{Y}^1_\delta(\mathbf{L}) := \left(\sum_{i\in\mathcal{I}} \rho_i(\mathbf{L})\mathbf{Y}^0_\delta(\mathbf{L}_i)^{-1}\right)^{-1} \tag{3.1.22}$$

Again, we clarify that inversions are defined component-wise. We claim that, for any $\mathbf{L} \in \Omega$, $\mathbf{Y}^1_\delta(\mathbf{L})$ is an $\mathcal{O}(\delta)$-near extremiser for $(\mathbf{L}, \mathbf{p})$. Firstly, by the homogeneity of the Brascamp–Lieb functional and (3.1.21), each $\rho_i(\mathbf{L})^{-1}\mathbf{Y}^0_\delta(\mathbf{L}_i)$ is a δ-near extremiser for all $(\mathbf{L}, \mathbf{p})$ such that $\mathbf{L} \in Q_i \cap \Omega$. Consider now a generic δ_1-near and δ_2-near extremiser, call them $\mathbf{A}_1$ and $\mathbf{A}_2$ respectively, for some generic linear datum $(\mathbf{L}, \mathbf{p})$, then by Ball's inequality,

$$\begin{aligned} & \mathrm{BL_g}(\mathbf{L}, \mathbf{p}; \mathbf{A}_1)\mathrm{BL_g}(\mathbf{L}, \mathbf{p}; \mathbf{A}_2) \leqslant \mathrm{BL}(\mathbf{L}, \mathbf{p})\mathrm{BL_g}(\mathbf{L}, \mathbf{p}; (\mathbf{A}_1^{-1} + \mathbf{A}_2^{-1})^{-1}) \\ \implies \quad & (1-\delta_1)(1-\delta_2)\mathrm{BL}(\mathbf{L}, \mathbf{p})^2 \leqslant \mathrm{BL}(\mathbf{L}, \mathbf{p})\mathrm{BL_g}(\mathbf{L}, \mathbf{p}; (\mathbf{A}_1^{-1} + \mathbf{A}_2^{-1})^{-1}) \\ \implies \quad & (1-\delta_1)(1-\delta_2)\mathrm{BL}(\mathbf{L}, \mathbf{p}) \leqslant \mathrm{BL_g}(\mathbf{L}, \mathbf{p}; (\mathbf{A}_1^{-1} + \mathbf{A}_2^{-1})^{-1}) \\ \implies \quad & (1-\delta_1-\delta_2)\mathrm{BL}(\mathbf{L}, \mathbf{p}) \leqslant \mathrm{BL_g}(\mathbf{L}, \mathbf{p}; (\mathbf{A}_1^{-1} + \mathbf{A}_2^{-1})^{-1}) \end{aligned} \tag{3.1.23}$$

hence $(\mathbf{A}_1^{-1} + \mathbf{A}_2^{-1})^{-1}$ is a $(\delta_1 + \delta_2)$-near extremiser for $(\mathbf{L}, \mathbf{p})$. Since we are pointwise only ever summing boundedly many (by which we mean $\lesssim 1$) contributions in (3.1.22), by iterating (3.1.23), we find that $\mathbf{Y}_\delta^1(\mathbf{L})$ is an $\mathcal{O}(\delta)$-near extremiser for $(\mathbf{L}, \mathbf{p})$ (similar observations about the closure of extremisers under harmonic addition were made in [14]). We may of course remove the implicit constant here by a simple substitution, so we shall proceed assuming that $\mathbf{Y}_\delta^1(\mathbf{L})$ is a δ-near extremiser for $(\mathbf{L}, \mathbf{p})$, for all $\mathbf{L} \in \Omega$.

It remains to prove that $\mathbf{Y}_\delta^1$ satisfies the necessary L^∞ and $W^{1,\infty}$ bounds. We shall start with the L^∞ bounds. One bound is trivial, namely that

$$|\mathbf{Y}_\delta^1(\mathbf{L})^{-1}| \leqslant \max_{\alpha:\mathbf{L}\in Q_\alpha} |\mathbf{Y}_\delta^0(\mathbf{L}_i)^{-1}| \leqslant \delta^{-N}.$$

The other requires the elementary fact that, for all symmetric positive definite matrices $A, B \in \mathbb{R}^{n\times n}$, $|(A^{-1}+B^{-1})^{-1}| \lesssim |A|+|B|$, which follows from the fact that, for all $|v| = 1$,

$$|(A^{-1} + B^{-1})v| \geqslant (|A^{-1}v|^2 + |B^{-1}v|^2)^{1/2} \geqslant |A|^{-2} + |B|^{-2} \gtrsim (|A| + |B|)^{-1} \qquad (3.1.24)$$

which then gives us that

$$|\mathbf{Y}_\delta^1(\mathbf{L})| \lesssim \max_{i:\mathbf{L}\in Q_i} |\mathbf{Y}_\delta^0(\mathbf{L}_i)| \leqslant \delta^{-N}.$$

It remains to prove the L^∞ bound on the derivative $\mathbf{dY}_\delta^1$. We use the chain rule to deal with the matrix inversions, apply the above established bounds on $|\mathbf{Y}_\delta^1(\mathbf{L})|$, then apply the triangle inequality to show that the derivative is at most polynomially bounded. Let

$W \in \mathbb{R}^{n_1 \times n} \times \ldots \times \mathbb{R}^{n_m \times n}$ be some unit vector, then

$$\begin{aligned}|\mathbf{dY}^1_\delta[\mathbf{L}](W)| &= |\mathbf{Y}^1_\delta(\mathbf{L})\mathbf{d}((\mathbf{Y}^1_\delta)^{-1})[\mathbf{L}](W)\mathbf{Y}^1_\delta(\mathbf{L})| \\ &\leqslant \delta^{-2N}|\mathbf{d}((\mathbf{Y}^1_\delta)^{-1})[\mathbf{L}](W)| \\ &\leqslant \delta^{-2N}\sum_{i\in\mathcal{I}}|\nabla\rho_i(\mathbf{L})||\mathbf{Y}^0_\delta(\mathbf{L}_i)| \\ &\lesssim \delta^{-3N-\frac{1}{\theta}}.\end{aligned}$$

Changing our choice of N and absorbing constants as appropriate, we may assume for the rest of the proof that $\|\mathbf{dY}^1_\delta\|_{L^\infty} \leqslant \delta^{-N}$. Finally, we obtain the desired function $\mathbf{Y}_\delta$ by renormalising the determinant, which, by the homogeneity of the Brascamp–Lieb functional, does not affect the property of being a δ-near extremiser. We shall use the polynomial bounds for $\mathbf{Y}^1_\delta =: (Y^1_{\delta,j})^m_{j=1}$ to help us establish polynomial bounds for $\mathbf{Y}_\delta =: (Y_{\delta,j})^m_{j=1}$, which we define below.

$$\mathbf{Y}_\delta(\mathbf{L}) := (\det(Y^1_{\delta,j}(\mathbf{L}))^{-1/n_j}Y^1_{\delta,j}(\mathbf{L}))^m_{j=1}$$

Since the 2-norm of a real symmetric matrix is its maximal eigenvalue, and the determinant of a matrix is the product of its eigenvalues, we know that $\det \mathbf{Y}^1_\delta(\mathbf{L}) \leqslant |\mathbf{Y}^1_\delta(\mathbf{L})|^{n_j} \leqslant \delta^{-n_jN}$, similarly, $\det \mathbf{Y}^1_\delta(\mathbf{L})^{-1} \leqslant |\mathbf{Y}^1_\delta(\mathbf{L})^{-1}|^{n_j} \leqslant \delta^{-n_jN}$. Finally, we must bound the derivative $\mathbf{dY}_\delta[\mathbf{L}]$. Now, in the case when $n_j = 1$, $\det(Y^1_{\delta,j}) = Y^1_{\delta,j}$, so $Y_{\delta,j}$ is identically one, hence the claim of the theorem is trivial, so shall proceed assuming that $n_j > 1$. Letting $\mathbf{L} \in \Omega$, by the chain rule and Jacobi's formula, we know that $d(\det Y^1_{\delta,j})[\mathbf{L}](W) = \mathrm{adj}(Y^1_{\delta,j}(\mathbf{L}))^* : dY^1_{\delta,j}[\mathbf{L}](W)$, where adj denotes an adjugate and the colon denotes the frobenius inner product $A : B := \sum_{ij} A_{ij}B_{ij}$, so, taking some $|W| = 1$,

$$
\begin{aligned}
|\mathbf{dY}_\delta[\mathbf{L}](W)| &\leqslant |d(\det(Y^1_{\delta,j})^{-1/n_j}))[\mathbf{L}](W)\mathbf{Y}_\delta(\mathbf{L})| + |\det(Y^1_{\delta,j}[\mathbf{L}])^{-1/n_j} dY^1_{\delta,j}[\mathbf{L}](W)| \\
&\leqslant \delta^{-N}\left(|d(\det(Y^1_{\delta,j})^{-1/n_j})[\mathbf{L}]| + \delta^{-2N}\right) \\
&\leqslant \delta^{-N}\left(\frac{1}{n_j}|d(\det Y^1_{\delta,j})[\mathbf{L}]|^{1-1/n_j} + \delta^{-2N}\right) \\
&\leqslant \delta^{-N}\left(\frac{1}{n_j}|\mathrm{adj}(Y^1_{\delta,j})|^{1-1/n_j}_{Frob}|d(\mathbf{Y}^1_\delta)_j[\mathbf{L}]|^{1-1/n_j}_{Frob} + \delta^{-2N}\right) \\
&\lesssim \delta^{-N}\left(\frac{1}{n_j}|Y^1_{\delta,j}|^{(n_j-1)(1-1/n_j)}|d(\mathbf{Y}^1_\delta)_j[\mathbf{L}]|^{1-1/n_j} + \delta^{-2N}\right) \lesssim \delta^{-cN}.
\end{aligned}
$$

For some $c \simeq 1$. Above we used the fact that the adjugate is a homogeneous polynomial of degree $n_j - 1$ to obtain the bound $|\mathrm{adj}(Y^1_{\delta,j})|_{Frob} \lesssim |\mathrm{adj}(Y^1_{\delta,j})|_\infty \lesssim |Y^1_{\delta,j}|^{n_j-1}$. □

3.1.4 Definition of $G_{x,\tau,j}$

We shall now define the gaussian arising in the statement of Theorem 3.1.2 using Theorem 3.1.15. In order to do this, we need to find a way of globally applying Theorem 3.1.15 to our manifold context, and to this end, we define $\mathcal{BL}_x$ to be the set of feasible Brascamp–Lieb data with domain T_xM and codomains $T_{B_1(x)}M_1, ..., T_{B_m(x)}M_m$, and we consider the following set

$$\Omega_x := \{\mathbf{L} \in \mathcal{BL}_x : |\mathbf{L}|, \mathrm{BL}(\mathbf{L}, \mathbf{p}) < C\}$$

We remark that $\mathcal{BL}_M := \bigsqcup_{x\in M} \mathcal{BL}_x$ then defines a fibre bundle over M, with natural projection map $\pi_{\mathcal{BL}} : \mathcal{BL}_M \to M$ and $\Omega_M := \bigsqcup_{x\in M} \Omega_x$ defines a fibre subbundle of $\mathcal{BL}_M$ containing $\bigsqcup_{x\in M}\{\mathbf{dB}(x)\}$, although we do not rigorously justify these claims as it is not necessary for the proof. Let $\mathcal{U}$ be a boundedly overlapping cover of M via small balls of the same radius, let $\{\phi_U : U \to \mathbb{R}^n\}_{U\in\mathcal{U}}$ be a normal atlas and $\{\phi_{B_j(U)} : B_j(U) \to \mathbb{R}^{n_j}\}_{B_j(U)\in\mathcal{U}}$ be an atlas for $B_j(M)$ consisting of restrictions of normal charts. We may use them to define define a system of local trivialisations for $\mathcal{BL}_M$.

$$\psi_U : \pi_{\mathcal{BL}}^{-1}(U) \to U \times \mathcal{BL}$$
$$\psi_U(x, \mathbf{L}) := (x, (d\phi_{B_j(U)}[B_j(x)] \circ L_j \circ d\phi_U[x]^{-1})_{j=1}^m)$$

By our bounded geometry assumptions, the forthcoming Lemma 3.2.4 implies that the exponential map has has bounded first and second derivatives, hence our normal atlases may be chosen such that $\bigcup_{x \in M} \psi_U(\pi_{\mathcal{BL}}^{-1}(U) \cap \Omega_M) \subset U \times \Omega$, where $\Omega := \{\mathbf{L} \in \mathcal{BL} : |\mathbf{L}|, \mathrm{BL}(\mathbf{L}, \mathbf{p}) < 2C\}$. The set Ω is open and relatively compactly contained in $\mathcal{BL}$, therefore there exists a $\mathbf{Y}_\delta : \Omega \to \mathcal{G}$ as in Theorem 3.1.15 for this choice of Ω. Let $\{\rho_U\}_{U \in \mathcal{U}}$ be a partition of unity subordinate to M with uniformly bounded derivatives, and define the following gaussian input-valued function:

$$\mathbf{a}_\tau(x) := \left(\sum_{U \in \mathcal{U}} \rho_U(x) \left(\mathbf{C}_U(x)^* \mathbf{Y}_{\tau^\alpha} \circ \pi_2 \circ \psi_U(x, \mathbf{dB}(x)) \mathbf{C}_U(x) \right)^{-1} \right)^{-1}, \qquad (3.1.25)$$

where π_2 denotes projection onto the second component, $\mathbf{C}_U(x) := (d\phi_{B_j(U)}(B_j(x)))_{j=1}^m$, and $\alpha \in (0,1)$ is a small exponent to be later determined, which we shall use to control the blow-up of $\mathbf{a}_\tau$ under various norms. By scale-invariance of the Brascamp–Lieb inequality and the closure of δ-near extremisers under invertible linear changes of co-ordinates, each term of the form $\mathbf{C}_U(x)^* \mathbf{Y}_{\tau^\alpha} \circ \pi_2 \circ \psi_U(x, \mathbf{dB}(x)) \mathbf{C}_U(x)$ in (3.1.25) is a τ^α-near extremiser for $(\mathbf{dB}(x), \mathbf{p})$, therefore iterating (3.1.23) implies that $\mathbf{a}_\tau(x)$ is a $\mathcal{O}(\tau^\alpha)$-near extremiser for $(\mathbf{dB}(x), \mathbf{p})$. Moreover, following the same reasoning as in the proof of Theorem 3.1.15, we may derive that $\|\mathbf{a}_\tau\|_{L^\infty(M)}, \|\mathbf{a}_\tau^{-1}\|_{L^\infty(M)} \lesssim \tau^{-\alpha N}$. In both instances, we may ignore the implicit constants that arise by simply raising the exponent α a qualitatively small amount. We may then define a gaussian $g_{x,\tau,j} : T_{B_j(x)}M_j \to \mathbb{R}$ as

$$g_{x,\tau,j}(z) = \tau^{-n_j} \exp\left(-\frac{\pi}{\tau^2}\langle a_{\tau,j}(x)z, z\rangle\right).$$

Implicitly, we may view this gaussian as the fundamental solution of the following anisotropic heat equation at time $t = \tau^2$.

$$\partial_t u(z,t) = \nabla_z \cdot (a_{\tau,j}(x)^{-1}\nabla_z u(z,t))$$

At last, we define our gaussian kernel $G_{x,\tau,j}$ as the following infinite convolution.

$$G_{x,\tau,j} := \ast_{k=1}^{\infty} g_{x,2^{-k/2}\tau,j}$$

We shall now show that $G_{x,\tau,j}$ is well-defined if $\alpha < 2N^{-1}$, where this $N \in \mathbb{N}$ is the one that arises in Theorem 3.1.15. To see this, we consider the partial convolution

$$G^{(K)}_{x,\tau,j} := g_{x,2^{-1/2}\tau,j} * ... * g_{x,2^{-K/2}\tau,j} = \tau^{-n_j}\det(C_K)^{1/2}\exp(-\pi\tau^2\langle C_K v, v\rangle),$$

where $C_K := (\sum_{k=1}^{k=K} 2^{-k} a_{2^{-k/2}\tau,j}(x)^{-1})^{-1}$ (this formula may be checked by an application of the Fourier transform). We now just need to show that C_K converges as $K \to \infty$, since then $G^{(K)}_{x,\tau,j}$ converges pointwise. Let $l \in \mathbb{N}$, then by the fact that $\|a^{-1}_{2^{-k/2}\tau,j}\|_{L^\infty(M)} \leqslant 2^{k\alpha N/2}\tau^{-\alpha N}$ for all $k > 0$,

$$\begin{aligned}|C_{K+l}^{-1} - C_K^{-1}| &\leqslant \sum_{k=K+1}^{K+l} 2^{-k}|a_{2^{-k/2}\tau,j}(x)^{-1}| \\ &\leqslant \sum_{k=K+1}^{K+l} 2^{-k}2^{k\alpha N/2}\tau^{-\alpha N} \\ &\leqslant 2^{(\alpha N/2-1)K}\tau^{-\alpha N}\sum_{k=1}^{l} 2^{(\alpha N/2-1)k}\end{aligned}$$

By our choice of α, $|C_{K+l}^{-1} - C_K^{-1}| \to 0$ as $K \to \infty$ uniformly in l, so C_K^{-1} is a Cauchy sequence, and therefore converges. By continuity of matrix inversion, the limit of C_K then exists provided that $\lim_{K\to\infty}(C_K^{-1}) \in GL_{n_j}(\mathbb{R})$, otherwise C_K must be unbounded, since if it were not it must admit a convergent subsequence, which would have to converge to the inverse of the limit of C_K^{-1}, resulting in a contradiction. We therefore only need to check that C_K is bounded, whence $G_{x,\tau,j}^{(K)} \to G_{x,\tau,j}$ pointwise, which follows from applying (3.1.24) and the L^∞ bound on $\mathbf{a}_\tau$.

$$|C_K| \lesssim \sum_{k=1}^{K} 2^{-k} |a_{2^{-k/2}\tau,j}(x)| \leqslant \sum_{k=1}^{\infty} 2^{(\alpha N/2-1)k} \tau^{-\alpha N} \lesssim_{\alpha,N} \tau^{-\alpha N} < \infty,$$

If we denote the limit of C_K by $A_{\tau,j}(x)$, then we may write $G_{x,\tau,j}(z)$ explicitly as

$$G_{x,\tau,j}(z) = \tau^{-n_j} \det(A_{\tau,j}(x))^{1/2} \exp(-\pi\tau^{-2}\langle A_{\tau,j}(x)z, z\rangle).$$

It is worth noting that by using infinitely many applications of (3.1.23), we see that $\mathbf{A}_\tau(x) := (A_{\tau,j}(x))_{j=1}^m$ is an $\mathcal{O}(\tau^\alpha)$-near extremiser for $(\mathbf{dB}(x), \mathbf{p})$, establishing property (2) of Proposition 3.1.1.

$$\frac{\mathrm{BL_g}(\mathbf{dB}(x), \mathbf{p}; \mathbf{A}_\tau(x))}{\mathrm{BL}(\mathbf{L}, \mathbf{p})} \geqslant 1 - \tau^\alpha \sum_{k=1}^{\infty} 2^{-k\alpha/2} = 1 - \tau^\alpha (2^{\alpha/2} - 1)^{-1}$$

Observe that in the case where $(\mathbf{dB}(x), \mathbf{p})$ is simple, we may forego Theorem 3.1.15 and use an exact extremiser for our definition of $g_{x,\tau,j}$, in which case $\mathbf{a}_\tau$ is constant in $\tau > 0$, and we would then have the identifications $\mathbf{A}_\tau = \mathbf{a}_\tau$ and $G_{x,\tau,j} = g_{x,\tau,j}$. Of course, if the reader wanted to run our argument in the simple case with exact extremisers, then they would need to take care to ensure that these exact extremisers satisfy appropriate $W^{1,\infty}$ boundedness of the type we shall prove for our near-extremisers in the next section.

3.2 Gaussian Lemmas

This section is, for the most part, dedicated to establishing the properties we require of our gaussians $G_{x,\tau,j}$ and $g_{x,\tau,j}$ in order to prove Proposition 3.1.11, which, as we have shown in Section 3.1.2, implies Theorem 3.1.2. We need to quantify how these gaussians behave under small perturbations in a number of variables, and for this purpose we shall first need to prove various bounds on norms the underlying gaussian input-valued functions $\mathbf{a}_\tau$ and $\mathbf{A}_\tau$.

Lemma 3.2.1 *For any $\varepsilon > 0$, provided α is chosen such that $\alpha < \min\{\frac{2}{3N}, \frac{\varepsilon}{N}\}$, there exists a $\nu > 0$ such that for every $\tau \in (0, \nu)$, $\|\mathbf{A}_\tau\|_{W^{1,\infty}}, \|\det \mathbf{A}_\tau\|_{W^{1,\infty}}, \|\mathbf{A}_\tau^{-1}\|_{L^\infty} \leqslant \tau^{-\varepsilon}$.*

Proof. The proof is similar to that of Lemma 3.1.15, as it amounts to a straightforward application of the triangle inequality and an application of the bounds on $a_{2^{-k/2}\tau,j}(x)$ that immediately follow from Theorem 3.1.15, taking $\nu > 0$ small enough so that we may bound any constants that arise from above by $\tau^{\alpha N - \varepsilon}$, for all $\tau \in (0, \nu)$.

$$|A_{\tau,j}(x)^{-1}| \leqslant \sum_{k=1}^{\infty} 2^{-k} |a_{2^{-k/2}\tau,j}(x)^{-1}| \leqslant \sum_{k=1}^{\infty} 2^{(\alpha N/2-1)k} \tau^{-\alpha N} \leqslant \tau^{-\varepsilon}$$
$$|A_{\tau,j}(x)| \lesssim \sum_{k=1}^{\infty} 2^{-k} |a_{2^{-k/2}\tau,j}(x)| \leqslant \sum_{k=1}^{\infty} 2^{(\alpha N/2-1)k} \tau^{-\alpha N} \leqslant \tau^{-\varepsilon}$$

Now, take $W \in T_{B_j(x)} M_j$ such that $|W| = 1$, then the bound on $dA_{\tau,j}(x)$ follows from the

L^∞ boundedness of $\mathbf{Y}_\delta^{-1}$ and the $W^{1,\infty}$ boundedness of $\mathbf{Y}_\delta$

$$
\begin{aligned}
|dA_{\tau,j}(x)(W)| &= |A_{\tau,j}(x)d(A_{\tau,j}^{-1})(x)(W)A_{\tau,j}(x)| \\
&\leqslant \tau^{-2\varepsilon}\left|\sum_{k=1}^{\infty} 2^{-k} d(a_{2^{-k/2}\tau,j})^{-1}(x)(W)\right| \\
&\lesssim \tau^{-2\varepsilon}\sum_{k=1}^{\infty} 2^{-k}\|\mathbf{d}(\mathbf{Y}^{-1}_{2^{-\alpha k/2}\tau^{\alpha}})\|_{L^\infty}\|\mathbf{d}^2\mathbf{B}\|_{L^\infty} \\
&\lesssim \tau^{-2\varepsilon}\sum_{k=1}^{\infty} 2^{-k}\|\mathbf{Y}^{-1}_{2^{-\alpha k/2}\tau^{\alpha}}\|_{L^\infty}^2\|\mathbf{d}\mathbf{Y}_{2^{-\alpha k/2}\tau^{\alpha}}\|_{L^\infty} \\
&\leqslant \tau^{-2\varepsilon}\sum_{k=1}^{\infty} 2^{k(3\alpha N/2-1)}\tau^{-2\alpha N} \lesssim \tau^{-4\varepsilon}
\end{aligned}
$$

We now turn our attention to the $W^{1,\infty}$ bound for $\det(A_{\tau,j}(x))$. First of all, $|\det A_{\tau,j}(x)| \leqslant |A_{\tau,j}(x)|^{n_j} \leqslant \tau^{-\varepsilon n_j}$ for all $\tau \in (0,\nu)$, so we have the bound $|A_{\tau,j}(x)| \leqslant \tau^{-\varepsilon}$, similarly $|A_{\tau,j}(x)^{-1}| \leqslant \tau^{-2\varepsilon}$ for all such τ. all that remains is to establish the L^∞ bound on $d(\det A_{\tau,j})$. The case when $n_j = 1$ has already been established, since then $\det A_{\tau,j} = A_{\tau,j}$, so suppose then that $n_j > 1$. Taking any $x \in M$ and $w \in T_xM$ such that $|w| = 1$, then by Jacobi's formula, the chain rule, the Cauchy-Schwarz inequality, and the equivalence of finite dimensional norms, we have that

$$
\begin{aligned}
|d(\det A_{\tau,j})[x](W)| &= |\mathrm{adj}(A_{\tau,j}(x))^* : dA_{\tau,j}(x)(w)| \\
&\lesssim |A_{\tau,j}(x)|^{n_j-1}|dA_{\tau,j}(x)| \lesssim \tau^{-2(n_j-1)\varepsilon}
\end{aligned}
$$

This proves the claim, since we may adjust ε accordingly. □

We shall henceforth consider $\varepsilon \in (0,(1-\gamma)/2)$ and $\alpha \in (0,\varepsilon/2N)$ as fixed parameters, and we also note at this point that we have now proved property (3) of Proposition 3.1.1, completing its proof.

Lemma 3.2.2 *For all* $\eta \in (0,\min\{\gamma-2\varepsilon, 0.9\gamma-\varepsilon, 3\gamma-2-\varepsilon\})$*, there exists a* $\nu > 0$ *such* *that the following holds: for all $\tau \in (0, \nu)$ and $x, y \in M$ such that $d(x,y) \leqslant \tau^\gamma$, and $z \in M_j$ such that $d(z, B_j(x)) \leqslant \tau^\gamma$, for all $f_j \in L^1(M_j)$,*

$$H_{y,\tau,j} f_j(z) \leqslant (1 + \tau^\eta) H_{x,\tau,j} f_j(z) \tag{3.2.3}$$

In order to prove this statement, we shall need a geometric lemma, the proof of which may be found in the appendix.

Lemma 3.2.4 *Suppose that M is a Riemannian manifold with bounded geometry, then given $x \in M$, then the norms of the covariant derivatives (up to second order) of the exponential map based at $p \in M$ may be bounded above uniformly in p in the open unit ball.*

Proof of Lemma 3.2.2. Let $\tau > 0$ be small, let $x, y \in M$ satisfy $d(x,y) \leqslant \tau^\gamma$, and take some $z \in M_j$ such that $d(z, B_j(x)) \leqslant \tau^\gamma$. First of all, by the chain rule, for any $v \in T_x M$, $d(e^{-1}_{B_j(y)} \circ e_{B_j(x)})[v] = d(e^{-1}_{B_j(y)})[e_{B_j(x)}(v)] de_{B_j(x)}[v]$. Given $w \in U_{\tau,j}(z)$, by Taylor's theorem, we may approximate $v_y := e^{-1}_{B_j(y)}(z) - e^{-1}_{B_j(y)}(w)$ in terms of $v_x := e^{-1}_{B_j(x)}(z) - e^{-1}_{B_j(x)}(w)$ in the following manner:

$$\begin{aligned} v_y &= e^{-1}_{B_j(y)} \circ e_{B_j(x)} \circ e^{-1}_{B_j(x)}(z) - e^{-1}_{B_j(y)} \circ e_{B_j(x)} \circ e^{-1}_{B_j(x)}(w) \\ &= d(e^{-1}_{B_j(y)} \circ e_{B_j(x)})[e^{-1}_{B_j(x)}(z)](v_x) + \mathcal{O}(|v_x|^2) \\ &= d(e^{-1}_{B_j(y)})[z] de_{B_j(x)}[e^{-1}_{B_j(x)}(z)](v_x) + \mathcal{O}(|v_x|^2). \end{aligned} \tag{3.2.5}$$

Above, we use Lemma 3.2.4 to uniformly bound the higher derivatives. Define the linear map $T_{x,y} := d(e^{-1}_{B_j(y)})[z] de_{B_j(x)}[e^{-1}_{B_j(x)}(z)]$, then it follows that

$$\begin{aligned} |A_{\tau,j}(y)^{1/2} v_y|^2 &= |A_{\tau,j}(y)^{1/2} (T_{x,y} v_x + \mathcal{O}(|v_x|^2))|^2 \\ &\leqslant |A_{\tau,j}(y)^{1/2} (T_{x,y} v_x)|^2 + \tau^{2.9\gamma - \varepsilon}, \end{aligned} \tag{3.2.6}$$

for sufficiently small $\tau > 0$. Now, by the uniform bounds on $\det A_{\tau,j}$ established in Lemma 3.2.1, we have that

$$\begin{aligned} |\log(\det A_{\tau,j}(x)) - \log(\det A_{\tau,j}(y))| &\leqslant \frac{|\det A_{\tau,j}(x) - \det A_{\tau,j}(y)|}{\min\{|\det A_{\tau,j}(x)|, |\det A_{\tau,j}(y)|\}} \\ &\leqslant \tau^{-\varepsilon} \|d(\det A_{\tau,j})\|_{L^\infty} d(x,y) \\ &\leqslant \tau^{\gamma-2\varepsilon}. \end{aligned}$$

Together with (3.2.6), this implies the bound $G_{y,\tau,j}(v_y) \leqslant (1+\tau^\eta) G_{x,\tau,j}(v_x)$ for sufficiently small $\tau > 0$.

$$\begin{aligned} \frac{G_{y,\tau,j}(v_y)}{G_{x,\tau,j}(v_x)} &= \frac{\det(A_{\tau,j}(y))}{\det(A_{\tau,j}(x))} \exp(\pi\tau^{-2}(|A_{\tau,j}(x)^{1/2}v_x|^2 - |A_{\tau,j}(y)^{1/2}v_y|)) \\ &\leqslant \exp(\tau^{\gamma-2\varepsilon} + \pi\tau^{0.9\gamma-\varepsilon} + \pi\tau^{-2}(|A_{\tau,j}(x)^{1/2}v_x|^2 - |A_{\tau,j}(y)^{1/2}T_{x,y}v_x|^2)) \\ &\leqslant \exp(\tau^{\gamma-2\varepsilon} + \pi\tau^{0.9\gamma-\varepsilon} + \pi\tau^{-2}\langle (A_{\tau,j}(x) - T_{x,y}^* A_{\tau,j}(y)T_{x,y})v_x, v_x\rangle) \\ &\leqslant \exp(\tau^{\gamma-2\varepsilon} + \pi\tau^{0.9\gamma-\varepsilon} + \pi\tau^{-2}|A_{\tau,j}(x) - T_{x,y}^* A_{\tau,j}(y)T_{x,y}||v_x|^2) \\ &\leqslant \exp(\tau^{\gamma-2\varepsilon} + \pi\tau^{0.9\gamma-\varepsilon} + 2\pi\tau^{-2}\|dA_{\tau,j}\|_{L^\infty}\tau^{3\gamma}) \\ &\leqslant \exp(\tau^{\gamma-2\varepsilon} + \pi\tau^{0.9\gamma-\varepsilon} + 2\pi\tau^{3\gamma-2-\varepsilon}) \leqslant 1 + \tau^\eta \end{aligned}$$

In the penultimate line we applied the mean value theorem in to obtain $|A_{\tau,j}(x) - T_{x,y}^* A_{\tau,j}(y)T_{x,y}| \leqslant 2\|dA_{\tau,j}\|_{L^\infty} d(x,y)$. The claim then easily follows from the definition of $H_{x,\tau,j}$.

$$\begin{aligned} H_{y,\tau,j} f_j(z) &:= \int_{U_{\tau,j}(z)} f_j(w) G_{x,\tau,j}(e^{-1}_{B_j(x)}(z) - e^{-1}_{B_j(x)}(w)) dw \\ &\leqslant (1+\tau^\eta) \int_{U_{\tau,j}(z)} f_j(w) G_{x,\tau,j}(e^{-1}_{B_j(x)}(z) - e_{B_j(x)^{-1}}(w) dw = (1+\tau^\eta) H_{y,\tau,j} f_j(z) \end{aligned} \tag{3.2.7}$$

□

Lemma 3.2.8 (General Truncation of Gaussians) *Let $m, n \in \mathbb{N}$, $\kappa \simeq 1$, and for each $\tau > 0$ let $A_\tau \in \mathbb{R}^{n\times n}$ be a positive definite matrix. Define $g_\tau : \mathbb{R}^n \to \mathbb{R}$ to be the gaussian $g_\tau(x) := \tau^{-n} \exp(-\pi\tau^{-2}\langle A_\tau x, x\rangle)$. Let $\gamma > 0$, and suppose that $|\det(A_\tau)|^{-1}, |A_\tau^{-1}| \leqslant \tau^{-\varepsilon}$ for some $\varepsilon \in (0,(1-\gamma)/2)$. There exists a $\nu > 0$ depending only on n, m, ε, and γ such that for all $\tau \in (0,\nu)$*

$$\det(A_\tau)^{-1/2} = \int_{\mathbb{R}^n} g_\tau \leqslant (1+\tau^\varepsilon) \int_{U_{\kappa\tau^\gamma}(0)} g_\tau. \tag{3.2.9}$$

Proof. By the freedom of choice we have in γ, if the claim holds for $\kappa = 1$ and we let $\gamma' = \gamma - \eta$ for some small $\eta > 0$, we obtain the general result by enlarging the domain of integration on the right-hand side of (3.2.9) from U_{τ^γ} to $U_{\kappa\tau^{\gamma'}}(0)$, taking $\tau \leqslant \kappa^{-1/\eta}$. It is sufficient to show that there exists $\nu > 0$ such that, for all $\tau \in (0,\nu)$,

$$\int_{\mathbb{R}^n\backslash U_{\tau^\gamma}(0)} g_\tau \leqslant c\tau^{2\varepsilon}. \tag{3.2.10}$$

for some $c \simeq 1$. To see this we simply split the integral of g_τ into $U_{\tau^\gamma}(0)$ and $\mathbb{R}^n\backslash U_{\tau^\gamma}(0)$.

$$\begin{aligned}
\det(A_\tau)^{-1/2} &= \int_{\mathbb{R}^n\backslash U_{\tau^\gamma}(0)} g_\tau + \int_{U_{\tau^\gamma}(0)} g_\tau \\
&\leqslant c\tau^{2\varepsilon} + \int_{U_{\tau^\gamma}(0)} g_\tau \\
\det(A_\tau)^{-1/2} - c\tau^{2\varepsilon} &\leqslant \int_{U_{\tau^\gamma}(0)} g_\tau \\
\det(A_\tau)^{-1/2} &\leqslant (1 - c\det(A_\tau)^{-1/2}\tau^{2\varepsilon})^{-1} \int_{U_{\tau^\gamma}(0)} g_\tau \leqslant (1 - c\tau^{3\varepsilon/2})^{-1} \int_{U_{\tau^\gamma}(0)} g_\tau
\end{aligned}$$

Which of course implies (3.2.9) if τ is taken to be sufficiently small. To estimate the left hand side of (3.2.10), we shall partition the domain of integration $\mathbb{R}^n\backslash U_\tau(0)$ into annuli, and bound the resulting infinite sum above by a lacunary series. We take $\tau \in (0,\nu)$, where

$\nu \in (0,1)$ is chosen such that $\frac{\pi}{2}\nu^{2\varepsilon+\gamma-1} \geqslant 2\varepsilon$.

$$\begin{aligned}
\int_{\mathbb{R}^n \setminus U_\tau(0)} g_\tau &= \int_{|x| \geqslant \tau^{\gamma-1}} \exp(-\pi |A_\tau^{1/2} x|^2) dx \\
&= \sum_{k=0}^{\infty} \int_{2^k \tau^{\gamma-1} \leqslant |x| \leqslant 2^{k+1}\tau^{\gamma-1}} \exp(-\pi |A_\tau^{1/2} x|^2) dx \\
&\leqslant \sum_{k=0}^{\infty} \sup_{2^k \log(1/\tau) = |x|} (\exp(-\pi |A_\tau x|^2)) Vol(\{2^k \tau^{\gamma-1} \leqslant |x| \leqslant 2^{k+1}\tau^{\gamma-1}\}) \\
&\leqslant \sigma_{n-1} \tau^{n(1-\gamma)} \sum_{k=0}^{\infty} 2^{nk} \exp(-\pi |A_\tau^{-1}|^{-2} 2^{2k} \tau^{2(\gamma-1)}) \\
&\lesssim \sum_{k=0}^{\infty} \tau^{\frac{\pi}{2}\tau^{2\varepsilon} 2^{2k} \tau^{\gamma-1}} \lesssim \sum_{k=0}^{\infty} \tau^{\frac{\pi}{2}\nu^{2\varepsilon+\gamma-1} 2^{2k}} \lesssim \sum_{k=0}^{\infty} \tau^{2\varepsilon 2^{2k}} \lesssim \tau^{2\varepsilon},
\end{aligned}$$

□

We may now prove the pointwise convergence to initial data for $H_{x,\tau,j} f_j \circ B_j$, a fact the reader will recall that we needed to prove that Proposition 3.1.11 implied Theorem 3.1.2.

Lemma 3.2.11 (Pointwise convergence to initial data) *For each $j \in \{1, ..., m\}$, let $f_j \in C_0(M_j)$ and $x \in M$, then,*

$$\lim_{\tau \to 0} H_{x,\tau,j} f_j \circ B_j(x) = f_j \circ B_j(x). \tag{3.2.12}$$

We give a proof of this lemma in the appendix.

Lemma 3.2.13 (Switching) *For all $\eta \in (0, \alpha)$, there exists $\nu > 0$ such that for $\tau \in (0, \nu)$ and $x, y \in M$ such that $d(x,y) \leqslant \tau^\gamma$,*

$$\frac{1}{\mathrm{BL}(\mathbf{dB}(y), \mathbf{p})} \prod_{j=1}^{m} g_{y,\tau,j} \circ dB_j(y)(e_y^{-1}(x))^{p_j} \leqslant \frac{1+\tau^\eta}{\mathrm{BL}(\mathbf{dB}(x), \mathbf{p})} \prod_{j=1}^{m} g_{x,\tau,j} \circ dB_j(x)(e_x^{-1}(y))^{p_j}. \tag{3.2.14}$$

Proof. Let $0 < \tau < \rho/10$, and define the positive-definite symmetric matrix field $M_\tau \in$

$\Gamma(TM \otimes T^*M)$.

$$M_\tau(x) := \sum_{j=1}^{m} p_j dB_j(x)^* a_{\tau,j}(x) dB_j(x)$$

It follows from the definition of M_τ that

$$\begin{aligned}\prod_{j=1}^{m} g_{x,\tau,j} \circ dB_j(x)(v)^{p_j} &= \prod_{j=1}^{m} \exp(p_j \langle a_{\tau,j}(x) dB_j(x) v, dB_j(x) v\rangle) \\ &= \exp(-\pi\tau^{-2}|M_\tau(x)^{1/2}v|^2)\end{aligned}$$

Hence, by the fact that $\mathbf{a}_{\tau,j}(x)$ is a τ^α-near extremiser for $(\mathbf{dB}(x), \mathbf{p})$,

$$(1-\tau^\alpha)\mathrm{BL}(\mathbf{dB}(x), \mathbf{p}) \leqslant \mathrm{BL}_{\mathrm{g}}(\mathbf{dB}(x), \mathbf{p}; \mathbf{a}_{\tau,j}(x)) = \det(M_\tau(x))^{-1/2} \leqslant \mathrm{BL}(\mathbf{dB}(x), \mathbf{p}),$$

hence

$$\begin{aligned}(1-\tau^\alpha)\det(M_\tau(x))^{1/2}\tau^{-n}\exp\left(-\frac{\pi}{\tau^2}|M_\tau(x)^{1/2}v|^2\right) &\leqslant \frac{\prod_{j=1}^{m} g_{x,\tau,j} \circ dB_j(x)(v)^{p_j}}{\mathrm{BL}(\mathbf{dB}(x), \mathbf{p})} \\ &\leqslant \det(M_\tau(x))^{1/2}\tau^{-n}\exp\left(-\frac{\pi}{\tau^2}|M_\tau(x)^{1/2}v|^2\right)\end{aligned}$$

Taking logarithms of the ratio of the two quantities arising on either side of (3.2.14) reveals that the logarithm of the error factor in (3.2.14) is polynomial in τ.

$$\begin{aligned}&\log\left(\frac{\mathrm{BL}(\mathbf{dB}(x), \mathbf{p})\prod_{j=1}^{m} g_{y,\tau,j} \circ dB_j(y)(e_y^{-1}(x))^{p_j}}{\mathrm{BL}(\mathbf{dB}(y), \mathbf{p})\prod_{j=1}^{m} g_{x,\tau,j} \circ dB_j(x)(e_x^{-1}(y))^{p_j}}\right) \\ &\leqslant \log\left(\frac{\exp\left(-\pi\tau^{-2}|M_\tau(y)^{1/2}e_y^{-1}(x)|^2\right)\det(M_\tau(y))^{1/2}}{(1-\tau^\alpha)\exp\left(-\pi\tau^{-2}|M_\tau(x)^{1/2}e_x^{-1}(y)|^2\right)\det(M_\tau(x))^{1/2}}\right) \\ &\leqslant \pi\tau^{-2}\left(|M_\tau(y)^{1/2}e_y^{-1}(x)|^2 - |M_{\tau,j}(x)^{1/2}e_x^{-1}(y)|^2\right) + \log(\det(M_\tau(y)M_\tau(x)^{-1})) - \log(1-\tau^\alpha)\end{aligned}$$

Let $\sigma : I \to M$ be a geodesic such that $\sigma(0) = x$ and $\sigma(1) = y$. Let $P_\sigma : T_xM \to T_yM$

denote parallel transport along σ. It is straightforward to check that $e_y^{-1}(x) := -P_\sigma e_x^{-1}(y)$, hence we may collate the two squares in the first term, allowing us to bound the resulting quantity using the mean value theorem.

$$\log\left(\frac{\mathrm{BL}(\mathbf{dB}(x),\mathbf{p})\prod_{j=1}^m g_{y,\tau,j}\circ dB_j(y)(e_y^{-1}(x))^{p_j}}{\mathrm{BL}(\mathbf{dB}(y),\mathbf{p})\prod_{j=1}^m g_{x,\tau,j}\circ dB_j(x)(e_x^{-1}(y))^{p_j}}\right)$$
$$\leqslant \pi\tau^{-2}\langle (P_\sigma^{-1}M_\tau(y)P_\sigma - M_\tau(x))e_x^{-1}(y), e_x^{-1}(y)\rangle + \log(\det(M_\tau(y)M_\tau(x)^{-1})) - \log(1-\tau^\alpha)$$
$$\lesssim \tau^{\gamma-2}\|dM_\tau\|_{L^\infty}|e_x^{-1}(y)|^2 + \tau^{\gamma-2\varepsilon} + \tau^\alpha \lesssim \tau^{3\gamma-2-\varepsilon} + \tau^{\gamma-2\varepsilon} + \tau^\alpha \lesssim \tau^\alpha.$$

Hence, provided τ is taken to be sufficiently small, we obtain the desired upper bound.

$$\frac{\mathrm{BL}(\mathbf{dB}(x),\mathbf{p})\prod_{j=1}^m g_{y,\tau,j}\circ dB_j(y)(e_y^{-1}(x))^{p_j}}{\mathrm{BL}(\mathbf{dB}(y),\mathbf{p})\prod_{j=1}^m g_{x,j,\tau}\circ dB_j(x)(e_x^{-1}(y))^{p_j}} \leqslant \exp(c\tau^\alpha) \leqslant 1+\tau^\eta,$$

where $c \simeq 1$. □

The next lemma ensures that the we may perturb the operators $H_{x,\tau,j}$ in x at the expense of a quantitatively small multiplicative error, and it will be a key tool not only for proving our theorem but also for proving Corollaries 3.1.8 and 3.1.9. It shall become clear why, like in the previous chapter, it is essential that we truncate the gaussians $G_{x,\tau,j}$, as gaussians are not locally constant at any scale unless restricted to a ball of suitable size with respect to the scale of mollification, and the choice of τ^γ is well suited to our purposes, unlike the radius of $\tau\log(1/\tau)$ of the previous chapter, which is too large to cope with the blow-up in eccentricity of the gaussians as $\tau \to 0$ that we have introduced here.

Again, like the previous chapter, in order to perturb to a nearby gaussian the radius of truncation needs to be slightly increased, and so we therefore shall need to define a minor modification of $H_{x,\tau,j}$, where the radius of the domain of integration is multiplied by a factor of 1.1. This factor is of course chosen arbitrarily, but since this consideration is a minor technicality we simply choose a value for the sakes of concreteness.

$$H^{1.1}_{x,\tau,j} : L^1(M_j) \to L^1(U_{\rho-1.1\tau^\gamma}(B_j(x)))$$
$$H^{1.1}_{x,\tau,j} f_j(z) := \int_{U_{1.1\tau^\gamma,j}(z)} f_j(w) G_{x,\tau,j}(e^{-1}_{B_j(x)}(z) - e^{-1}_{B_j(x)}(w)) dw$$

Lemma 3.2.15 (Local-constancy) *For any $\eta \in (0, \gamma - \varepsilon)$, there exists a $\nu > 0$ such that the following holds for all $\tau \in (0, \nu)$: Let $x \in M$, then given $z, \tilde{z} \in U_{\tau,j}(B_j(x))$ such that $d(z, \tilde{z}) \lesssim \tau^2$ we have that for all $f_j \in L^1(M_j)$,*

$$H_{x,\tau,j} f_j(z) \leqslant (1 + \tau^\eta) H^{1.1}_{x,\tau,j} f_j(\tilde{z}) \tag{3.2.16}$$

Proof. First of all we need to prove a similar claim for the kernel $G_{x,\tau,j}$. Suppose that $v, w \in T_{B_j(x)} M_j$ are such that $|v - w| \leqslant \kappa\tau^2$ for some $\kappa \simeq 1$ and $v, w \in V_{\tau^\gamma,j}(0)$.

$$\begin{aligned}
\frac{G_{x,\tau,j}(v)}{G_{x,\tau,j}(w)} &= \exp(\pi\tau^{-2}(|A_{\tau,j}(x)^{1/2} v|^2 - |A_{\tau,j}(x)^{1/2} w|^2)) \\
&= \exp(\pi\tau^{-2}\langle A_{\tau,j}(x)(v - w), v + w\rangle) \\
&\leqslant \exp(\pi\tau^{-2}\|A_{\tau,j}\||v - w||v + w|) \\
&\leqslant \exp(2C^2\kappa\pi\tau^{-2}\tau^{-\varepsilon}\tau^2\tau^\gamma) \\
&= \exp(2C^2\kappa\pi\tau^{\gamma-\varepsilon})
\end{aligned}$$

Hence it follows that, for all $\tau > 0$ sufficiently small, $d(z, \tilde{z}) \lesssim \tau^2$, and $w \in U_{\tau,j}(z)$,

$$G_{x,\tau,j}(e^{-1}_{B_j(x)}(z) - e^{-1}_{B_j(x)}(w)) \leqslant (1 + \tau^\eta) G_{x,\tau,j}(e^{-1}_{B_j(x)}(\tilde{z}) - e^{-1}_{B_j(x)}(w)).$$

The lemma then follows from applying this bound directly to the definition of $H^{1.1}_{x,\tau,j} f_j$.

$$\begin{aligned} H_{x,\tau,j} f_j(z) &= \int_{U_{\tau^\gamma,j}(z)} f_j(w) G_{x,\tau,j}(e^{-1}_{B_j(x)}(z) - e^{-1}_{B_j(x)}(w)) dw \\ &\leqslant (1+\tau^\eta) \int_{U_{1.1\tau^\gamma,j}(z)} f_j(w) G_{x,\tau,j}(e^{-1}_{B_j(x)}(\tilde{z}) - e^{-1}_{B_j(x)}(w)) dw \\ &= (1+\tau^\eta) H^{1.1}_{x,\tau,j} f_j(\tilde{z}) \end{aligned}$$

□

3.3 Proof of Proposition 3.1.11

Our proof strategy is to use the near-extremising gaussians $g_{x,\tau,j}$ to construct a partition of unity for the integral on the left-hand side of (3.1.3), subordinate to balls of scale τ^γ. At this scale, we may apply our lemmas from the previous section to perturb the integral, so that we may then apply the linear Brascamp–Lieb inequality locally, thereby obtaining the desired form on the right-hand side. Gaussian partitions of unity were also used in [9], and, notably, more recently in the context of decoupling for the parabola by Guth, Maldague, and Wang [48].

Proof. For each $j \in \{1, ..., m\}$, take some arbitrary $f_j \in L^1(M_j)$. Let $\eta \in (0, \min\{\alpha, 0.9\gamma - \varepsilon, \gamma - 2\varepsilon, 3\gamma - 2 - \varepsilon\})$ and choose $\nu > 0$ such that (3.2.3), (3.2.9), (A.9), (3.2.14), and (3.2.16) hold for $\tau \in (0, \nu)$. Consider the following collection of truncated gaussians.

$$\left\{ \frac{\chi_{V_{0.1\tau^\gamma}(0)} \prod_{j=1}^m g_{y,\tau,j} \circ dB_j(y)^{p_j}}{\mathrm{BL}(\mathbf{dB}(y), \mathbf{p})} \right\}_{y \in M} \tag{3.3.1}$$

By Lemma 3.2.8 and the fact that $\mathbf{a}_\tau(y)$ is a τ^α-near extremiser for $(\mathbf{dB}(y), \mathbf{p})$, we know that for $\tau > 0$ sufficiently small,

$$\mathrm{BL}_{\mathrm{g}}(\mathbf{dB}(y), \mathbf{p}; \mathbf{a}_\tau(y)) \leqslant (1+\tau^\eta) \int_{V_{0.1\tau^\gamma}(0)} \prod_{j=1}^m g_{y,\tau,j} \circ dB_j(y)(v)^{p_j}$$

$$\mathrm{BL}(\mathbf{dB}(y), \mathbf{p}) \leqslant (1+\tau^\eta)^2 \int_{V_{0.1\tau^\gamma}(0)} \prod_{j=1}^m g_{y,\tau,j} \circ dB_j(y)(v)^{p_j},$$

hence we may continuously split up the integral on the left-hand side of (3.1.3) by introducing (3.3.1) as one might a partition of unity.

$$\begin{aligned}
&\int_M \prod_{j=1}^m H_{y,\tau,j} f_j \circ B_j(y)^{p_j} dy \\
&\quad \leqslant (1+\tau^\eta)^2 \int_M \int_{V_{0.1\tau^\gamma}(y)} \prod_{j=1}^m H_{y,\tau,j} f_j \circ B_j(y)^{p_j} g_{y,\tau,j} \circ dB_j(y)(v)^{p_j} dv \frac{dy}{\mathrm{BL}(\mathbf{dB}(y), \mathbf{p})} \\
&\quad \leqslant (1+\tau^\eta)^3 \int_M \int_{U_{0.1\tau^\gamma}(y)} \prod_{j=1}^m H_{y,\tau,j} f_j \circ B_j(y)^{p_j} g_{y,\tau,j} \circ dB_j(y)(e_y^{-1}(x))^{p_j} dx \frac{dy}{\mathrm{BL}(\mathbf{dB}(y), \mathbf{p})} \\
&\quad = (1+\tau^\eta)^3 \int_M \int_{U_{0.1\tau^\gamma}(x)} \prod_{j=1}^m H_{y,\tau,j} f_j \circ B_j(y)^{p_j} g_{y,j,\tau} \circ dB_j(y)(e_y^{-1}(x))^{p_j} \frac{dy}{\mathrm{BL}(\mathbf{dB}(y), \mathbf{p})} dx
\end{aligned}$$

We want to perturb the inner integral to a linear Brascamp–Lieb inequality in y. To do this, we first apply Lemma 3.2.2 and Lemma 3.2.13 to remove some of the unwanted y-dependence. Let $P := \sum_{j=1}^m p_j$, then

$$\begin{aligned}
&\int_M \prod_{j=1}^m H_{y,\tau,j} f_j \circ B_j(y)^{p_j} dy \\
&\quad \leqslant (1+\tau^\eta)^{3+P} \int_M \int_{U_{0.1\tau^\gamma}(x)} \prod_{j=1}^m H_{x,\tau,j} f_j \circ B_j(y)^{p_j} g_{y,\tau,j} \circ dB[y](e_y^{-1}(x))^{p_j} \frac{dy}{\mathrm{BL}(\mathbf{dB}(y), \mathbf{p})} dx \\
&\quad \leqslant (1+\tau^\eta)^{3+2P} \int_M \int_{U_{0.1\tau^\gamma}(x)} \prod_{j=1}^m H_{x,\tau,j} f_j \circ B_j(y)^{p_j} g_{x,\tau,j} \circ dB[x](e_x^{-1}(y))^{p_j} dy \frac{dx}{\mathrm{BL}(\mathbf{dB}(x), \mathbf{p})} \\
&\quad \leqslant (1+\tau^\eta)^{3+3P} \int_M \int_{V_{0.1\tau^\gamma}(x)} \prod_{j=1}^m H_{x,\tau,j} f_j \circ B_j(e_x(v))^{p_j} g_{x,\tau,j} \circ dB[x](v)^{p_j} dv \frac{dx}{\mathrm{BL}(\mathbf{dB}(x), \mathbf{p})}.
\end{aligned}$$

We may then use Lemma 3.2.15 to replace the instance of $B_j(e_x(v))$ with its affine approximation around x, given by $L_j^x(v) := e_{B_j(x)}(dB_j(x)v)$.

$$\leqslant (1+\tau^\eta)^{3+4P}\int_M\int_{V_{0.1\tau^\gamma}(x)}\prod_{j=1}^m H^{1.1}_{x,\tau,j} f_j\circ L_j^x(v)^{p_j} g_{x,\tau,j}\circ dB_j(x)(v)^{p_j}dv\frac{dx}{\mathrm{BL}(\mathbf{dB}(x),\mathbf{p})}$$
$$\leqslant (1+\tau^\eta)^{3+4P}\int_M\int_{T_xM}\prod_{j=1}^m H^{1.1}_{x,\tau,j} f_j\circ L_j^x(v)^{p_j} g_{x,\tau,j}\chi_{V_{0.1\tau^\gamma,j}(0)}\circ dB_j(x)(v)^{p_j}dv\frac{dx}{\mathrm{BL}(\mathbf{dB}(x),\mathbf{p})}.$$

Above we used the fact that, for all $x\in M$ $V_{0.1\tau^\gamma}(0)\subset\bigcap_{j=1}^m dB_j(x)^{-1}V_{0.1\tau^\gamma,j}(0)$. At this point we may apply the linear Brascamp–Lieb inequality $(\mathbf{dB}(x),\mathbf{p})$ to the inner integral.

$$\leqslant (1+\tau^\eta)^{3+4P}\int_M\prod_{j=1}^m\left(\int_{V_{0.1\tau^\gamma,j}(0)} H^{1.1}_{x,\tau,j} f_j(e_{B_j(x)}(v_j))g_{x,\tau,j}(v_j)dv_j\right)^{p_j}dx \tag{3.3.2}$$

The resulting integrals in (3.3.2) may be then be bounded by a convolution.

$$\begin{aligned}\int_{V_{0.1\tau^\gamma,j}(0)} & H^{1.1}_{x,\tau,j} f_j(e_{B_j(x)}(v_j))g_{x,\tau,j}(v_j)dv_j\\ &= \int_{V_{0.1\tau^\gamma,j}(0)}\int_{U_{1.1\tau^\gamma,j}(B_j(x))} f_j(z)G_{x,\tau,j}(v_j-e^{-1}_{B_j(x)}(z))g_{x,\tau,j}(v_j)dzdv_j\\ &\leqslant (1+\tau^\eta)\int_{V_{0.1\tau^\gamma,j}(0)}\int_{V_{1.1\tau^\gamma,j}(B_j(x))} f_j\circ e_{B_j(x)}(w)G_{x,\tau,j}(v_j-w)g_{x,\tau,j}(v_j)dwdv_j\\ &= G_{x,\tau,j}\chi_{V_{1.1\tau^\gamma}(0)} * g_{x,\tau,j}\chi_{V_{0.1\tau^\gamma}(0)} * f_j\circ e_{B_j(x)}(0)\end{aligned} \tag{3.3.3}$$

Now, $G_{x,\tau,j} * g_{x,\tau,j} = G_{x,2^{1/2}\tau,j}$ by definition of $G_{x,\tau,j}$, and the support of $\chi_{V_{1.1\tau^\gamma}(0)} * \chi_{V_{0.1\tau^\gamma}(0)}$ is the ball around the origin of radius $1.2\tau^\gamma$, which is less than $2^{\gamma/2}\tau^\gamma$ provided that $\gamma\geqslant 2\log_2(1.2)\approx 0.526....$ This implies that $\mathrm{supp}(G_{x,\tau,j}\chi_{V_{1.1\tau^\gamma}(0)} * g_{x,\tau,j}\chi_{V_{0.1\tau^\gamma}(0)})\subset V_{2^{\gamma/2}\tau^\gamma}(0)$, hence

$$G_{x,\tau,j}\chi_{V_{1.1\tau^\gamma}(0)} * g_{x,\tau,j}\chi_{V_{0.1\tau^\gamma}(0)}\leqslant (G_{x,\tau,j} * g_{x,\tau,j})\chi_{V_{2^\gamma\tau^\gamma}(0)} = G_{x,2^{1/2}\tau,j}\chi_{V_{2^{\gamma/2}\tau^\gamma}(0)}$$

We may then bound (3.3.3) as follows:

$$\int_{V_{0.1\tau^\gamma,j}(0)} H^{1.1}_{x,\tau,j} f_j(e_{B_j(x)}(v_j)) g_{x,\tau,j}(v_j) dv_j \leqslant H_{x,2^{1/2}\tau,j} f_j \circ B_j(x). \tag{3.3.4}$$

Finally, we complete the proof by combining (3.3.2) with (3.3.4) and taking $\beta \in (0, \eta)$. ▫

3.4 Proof of Corollaries 3.1.8 and 3.1.9

Proof of Corollary 3.1.8. Take some arbitrary $f_j \in L^1(M_j)$ for all $j \in \{1, ..., m\}$. By Theorem 3.1.2, there exists a $\beta > 0$ such that for $\tau > 0$ sufficiently small

$$\begin{aligned}
\int_{U_{\tau^\gamma}(x_0)} \prod_{j=1}^m f_j \circ B_j(x)^{p_j} dx &\leqslant \int_M \prod_{j=1}^m f_j \chi_{U_{\tau^\gamma,j}(x_0)} \circ B_j(x)^{p_j} dx \\
&\leqslant (1+\tau^\beta) \int_M \prod_{j=1}^m H_{x,\tau,j}(f_j \chi_{U_{\tau^\gamma,j}(x_0)}) \circ B_j(x)^{p_j} dx \\
&\leqslant (1+\tau^\beta) \int_{U_{2\tau^\gamma}(x_0)} \prod_{j=1}^m H_{x,\tau,j} f_j \circ B_j(x)^{p_j} dx
\end{aligned}$$

Take η and ν as in the proof of Proposition 3.1.11, if we take $\tau \in (0, \nu)$, then we may apply Lemma 3.2.13 to perturb $H_{x,\tau,j}$ to $H_{x_0,\tau,j}$ and Lemma 3.2.15 to perturb $B_j(x)$ to $L_j^{x_0}(x)$, at which point we may apply the linear inequality to complete the proof.

$$\begin{aligned}
&\leqslant (1+\tau^\beta)(1+\tau^\eta)^P \int_{U_{2\tau^\gamma}(x_0)} \prod_{j=1}^m H_{x_0,\tau,j} f_j \circ B_j(x)^{p_j} dx \\
&\leqslant (1+\tau^\beta)(1+\tau^\eta)^{2P} \int_{U_{2\tau^\gamma}(x_0)} \prod_{j=1}^m H^{1.1}_{x_0,\tau,j} f_j \circ L_j^{x_0}(x)^{p_j} dx \\
&\leqslant (1+\tau^\beta)(1+\tau^\eta)^{2P} \mathrm{BL}(\mathbf{dB}(x_0), \mathbf{p}) \prod_{j=1}^m \left(\int_{U_{2\tau^\gamma,j}(0)} H^{1.1}_{x_0,\tau,j} f_j \circ e_{B_j(x)} \right)^{p_j} \\
&\leqslant (1+\tau^\beta)(1+\tau^\eta)^{3P} \mathrm{BL}(\mathbf{dB}(x_0), \mathbf{p}) \prod_{j=1}^m \left(\int_{M_j} f_j \right)^{p_j}
\end{aligned}$$

▫

Proof of Corollary 3.1.9. Fix some $\tau > 0$ small enough so that (3.1.3) holds for the non-linear datum $(\mathbf{B}, \mathbf{p})$. Let $R := \|\mathbf{B} - \widetilde{\mathbf{B}}\|_{L^\infty(\mathbb{R}^n)}$ and take some $v \in V_{\tau^\gamma}(B_j(x))$, then

$$
\begin{aligned}
\frac{G_{x,\tau,j}(B_j(x) - v)}{G_{x,\tau,j}(\widetilde{B}_j(x) - v)} &\leqslant \exp(\pi\tau^{-2}\|A_{\tau,j}\||\widetilde{B}_j(x) - B_j(x)||\widetilde{B}_j(x) + B_j(x) - 2v|) \\
&\leqslant \exp(\pi R\tau^{-2-\varepsilon}|\widetilde{B}_j(x) + B_j(x) - 2v|) \\
&\leqslant \exp(\pi R\tau^{-2-\varepsilon}(2|B_j(x) - v| + |\widetilde{B}_j(x) - B_j(x)|)) \\
&\lesssim_{R,\tau} 1. \qquad (3.4.1)
\end{aligned}
$$

Define the following convolution operator:

$$
H_{\tau,j} f_j(y) := \int_{\mathbb{R}^n} f_j(z) \exp(-\pi\tau^{\varepsilon-2}|y - z|^2) dz. \qquad (3.4.2)
$$

Since $G_{x,\tau,j}(z) \leqslant \exp(-\pi\tau^{\varepsilon-2}|z|^2)$ by Lemma 3.2.1, $H_{x,\tau,j} f_j \leqslant H_{\tau,j} f_j$, so combining this with (3.4.1), we may bound $H_{x,\tau,j} f_j \circ B_j(x)$ as follows,

$$
\begin{aligned}
H_{x,\tau,j} f_j \circ B_j(x) &= \int_{V_{\tau^\gamma}(B_j(x))} f_j(z) G_{x,\tau,j}(B_j(x) - z) dz \\
&\lesssim_{R,\tau} \int_{V_{\tau^\gamma}(B_j(x))} f_j(z) G_{x,\tau,j}(\widetilde{B}_j(x) - z) dz \\
&\leqslant \tau^{-n_j} \int_{\mathbb{R}^n} f_j(z) \exp(-\pi\tau^{\varepsilon-2}|\widetilde{B}_j(x) - z|^2) dz \\
&= H_{\tau,j} f_j \circ \widetilde{B}_j(x).
\end{aligned}
$$

The finiteness of $(\mathbf{B},\mathbf{p})$ then follows easily from (3.1.3) and the finiteness of $(\widetilde{\mathbf{B}},\mathbf{p})$.

$$\begin{aligned}\int_{\mathbb{R}^n}\prod_{j=1}^m f_j\circ B_j(x)^{p_j}\,dx &\leqslant (1+\tau^\beta)\int_{\mathbb{R}^n}\prod_{j=1}^m H_{x,\tau,j}f_j\circ B_j(x)^{p_j}\,dx\\ &\lesssim_\tau \int_{\mathbb{R}^n}\prod_{j=1}^m H_{\tau,j}f_j\circ \widetilde{B}_j(x)^{p_j}\,dx\\ &\lesssim_{\widetilde{\mathbf{B}}} \prod_{j=1}^m\left(\int_{\mathbb{R}^{n_j}} H_{\tau,j}f_j\right)^{p_j}\\ &\lesssim \prod_{j=1}^m\left(\int_{\mathbb{R}^{n_j}} f_j\right)^{p_j}\end{aligned}$$

□

Chapter 4

An Algebraic Brascamp–Lieb Inequality

This chapter is a re-edited version of the article 'An Algebraic Brascamp–Lieb inequality' [39].

4.1 Introduction

As stated at the beginning of the thesis, a common feature of many problems studied in modern harmonic analysis is the presence of some underlying geometric object, examples including Kakeya inequalities, Fourier restriction theory, and generalised Radon transforms. Usually, this object is equipped with a measure that does not detect geometric features such as curvature or transversality, properties that are often highly relevant in the contexts we are considering. It has many times been found that incorporating a weight that tracks these geometric features in a suitable manner yields inequalities that require few geometric hypotheses and exhibit additional uniformity properties (in the context of generalised Radon-transforms and convolution with measures supported on submanifolds, see for example [32, 36, 37, 43, 44, 57, 64], or in the context of Fourier restriction [1, 18, 26, 28, 35, 38, 51, 58]). In particular, one often finds that if the geometric object in question may be parametrised by polynomials or rational functions, then the associated bounds will usually only depend on their degree, as observed in [32, 35–37, 64].

The main theorem of this chapter is another instance of this phenomenon, and is set in the context of a global nonlinear Brascamp–Lieb inequality. The underlying object in question is a collection of maps that have a certain algebraic structure that generalises that enjoyed by polynomial, rational, and algebraic maps. Like polynomials, these maps have a well-defined notion of degree, and the bounds for the corresponding nonlinear Brascamp–Lieb inequalities we obtain depend only on these degrees, the underlying dimensions, and exponents.

It was first suggested in [16] that a global Brascamp–Lieb inequality should include an appropriate weight factor in order to compensate for local degeneracies, and it is upon this suggestion that we include a weight factor of the form $\mathrm{BL}(\mathbf{dB}(x), \mathbf{p})^{-1}$ in our inequality. It was also discussed in the same paper that even with an appropriate weight factor one cannot expect a global nonlinear Brascamp–Lieb inequality to hold with only local hypotheses, due to reasons relating to infinite failure of injectivity. We address this issue by imposing that our nonlinear maps are *quasialgebraic*, a property we define in the following section, which entails that the fibres of our maps can only intersect one another boundedly often, thereby precluding such injectivity-related counterexamples.

4.1.1 Preliminary Definitions and Notation

Definition 4.1.1 *Let $M \subset \mathbb{R}^n$ be an open subset of a d-dimensional algebraic variety and let N be a Riemannian manifold. We say that a map $F : M \to N$ that is C^∞ on an open dense subset of M is* quasialgebraic *if its fibres are open subsets of algebraic varieties. We define the degree of F to be the maximum degree of its fibres (this may be infinite).*

The author is not aware of this notion of a quasialgebraic map being discussed anywhere in the literature, however this is not to pretend that it is an innovative concept, merely one that is very much tailored to our purposes. As remarked earlier, the class of quasialgebraic maps encompasses many important classes of maps, as ordered below.

$$\{polynomial\ maps\} \subset \{rational\ maps\} \subset \{algebraic\ maps\} \subset \{quasialgebraic\ maps\}$$

As one would hope, the notion of degree in Definition 4.1.1 coincides with the conventional notion of degree for each of the above classes. It is easy to check that, unlike the classes of polynomial, rational, and algebraic maps, the class of quasialgebraic maps is 'closed' under diffeomorphism, in the sense that given a quasialgebraic map $F : M \to N$, and a diffeomorphism $\phi : N \to N'$, the map $F' := \phi \circ F : M \to N'$ is a quasialgebraic map of the same degree as F.

Given a manifold X, We let $U_r(x)$ denote an open ball of radius $r > 0$ centred at a point $x \in X$, and we denote the centred dilate of a ball V by a factor $c > 0$ by cV. Notice that at some points either dB_j will not be defined or will fail to be surjective; in such cases we set $\mathrm{BL}(\mathbf{dB}(x), \mathbf{p}) = \infty$. Given a Brascamp–Lieb datum $(\mathbf{L}, \mathbf{p})$ such that $L_j : V \to V_j$ and a subspace $W \leqslant V$, we let $\mathrm{BL}_W(\mathbf{L}, \mathbf{p})$ denote the best constant $C > 0$ in the following 'restricted' Brascamp–Lieb inequality.

$$\int_W \prod_{j=1}^m f_j \circ L_j(x)^{p_j} d\lambda_W(x) \leqslant C \prod_{j=1}^m \left(\int_{L_jW} f_j(x_j) d\lambda_{L_jW}(x_j) \right)^{p_j}. \tag{4.1.2}$$

Lastly, we shall denote the zero-set of a polynomial map $p : \mathbb{R}^n \to \mathbb{R}^k$ by $Z(p) := \{x \in \mathbb{R}^n : p(x) = 0\}$.

4.1.2 Main Results

We shall now state our main theorem.

Theorem 4.1.3 (Quasialgebraic Brascamp–Lieb Inequality) *Let $d, m, n \in \mathbb{N}$ and, for each $1 \leqslant j \leqslant m$, let $n_j \in \mathbb{N}$ and $p_j \in [0,1]$. Assume that the scaling condition $\sum_{j=1}^{m} p_j n_j = d$ is satisfied. Let $M \subset \mathbb{R}^n$ be an open subset of a d-dimensional algebraic variety, and for each $j \in \{1, ..., m\}$, let M_j be an n_j-dimensional Riemannian manifold.*

We consider quasialgebraic maps $B_j : M \to M_j$ that extend to quasialgebraic maps on some open set $A \subset \mathbb{R}^n$. Setting $\mathbf{p} := (p_1, ..., p_m)$ and equipping each M_j with the measure μ_j induced by its Riemannian metric, the following inequality holds for all non-negative $f_j \in L^1(M_j)$:

$$\int_M \prod_{j=1}^{m} f_j \circ B_j(x)^{p_j} \frac{d\sigma(x)}{\mathrm{BL}_{T_x M}(\mathbf{dB}(x), \mathbf{p})} \lesssim \deg(M) \prod_{j=1}^{m} \left(\deg(B_j) \int_{M_j} f_j(x_j) d\mu_j(x_j) \right)^{p_j}, \tag{4.1.4}$$

where σ is the induced d-dimensional Hausdorff measure on M.

The reader should note the similar thrust shared by this theorem and Theorem 1 of [43]. We also remark that if M is not an algebraic variety then the notion of degree no longer makes sense in this context, and therefore this structural condition is necessary. On the other hand, while the Riemannian structure on M_j is a convenient setting for our analysis, as it immediately gives us suitable notions of differentiability, measure, and distance, it is however plausible that one might be able to generalise this inequality to some broader class of topological spaces for which these notions may be defined, although we shall not pursue this level of generality in this thesis. Unsurprisingly, Theorem 4.1.3 immediately gives us a less powerful, but more concisely stated weighted nonlinear Brascamp–Lieb inequality for polynomial maps.

Corollary 4.1.5 (Polynomial Brascamp–Lieb Inequality) *Let the dimensions and exponents be as in Theorem 4.1.3, and let* $B_j : \mathbb{R}^d \to \mathbb{R}^{n_j}$ *be polynomial maps. The following inequality holds over all non-negative* $f_j \in L^1(\mathbb{R}^{n_j})$*:*

$$\int_{\mathbb{R}^d} \prod_{j=1}^m f_j \circ B_j(x)^{p_j} \frac{dx}{\mathrm{BL}(\mathbf{dB}(x), \mathbf{p})} \lesssim \prod_{j=1}^m \left(\deg(B_j) \int_{\mathbb{R}^{n_j}} f_j(x_j) dx_j \right)^{p_j}. \tag{4.1.6}$$

Notice that we impose no local condition on the maps B_j, not even that they are submersions. This is allowed because the weight we have incorporated on the left-hand side vanishes when the maps B_j degenerate, hence we do not have to worry about counterexamples such as where the functions f_j concentrate at critical values of B_j. In this sense, despite the rigidity of the algebraic structure required by Theorem 4.1.3, in applications the lack of uniform boundedness requirements, of the kind specified by Theorem 3.1.2, in some aspects make it very robust by comparison. We shall now demonstrate this flexibility with a concrete example.

Example 4.1.7 *Let* $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ *denote the unit d-sphere, let* $V_1, ..., V_m \leqslant \mathbb{R}^{d+1}$ *be a collection of subspaces of* $\mathbb{R}^{d+1}$ *such that* $\sum_{j=1}^m \mathrm{codim}(V_m) = d$*, let* $\omega := \bigwedge_{j=1}^m \bigwedge_{k=1}^{\mathrm{codim}(V_j)} e_{j,k} \in \Lambda^d(\mathbb{R}^{d+1})$*, where* $\{e_{j,k}\}_{k=1}^{\mathrm{codim}(V_j)}$ *is an orthogonal basis of* $V_j^\perp$ *for each* $j \in \{1, ..., m\}$*, and let* $L_j : \mathbb{R}^{d+1} \to V_j$ *denote the projection onto* V_j*. Letting* $\mathbf{p} := \left(\frac{1}{m-1}, ..., \frac{1}{m-1}\right)$*, by Proposition 1.2 of [6] and the forthcoming Lemma 4.2.2, we know that* $\mathrm{BL}_{T_x\mathbb{S}^d}(\mathbf{L}, \mathbf{p}) = \mathrm{BL}((\mathbf{L}, \pi_{(T_x\mathbb{S}^d)^\perp}), (\mathbf{p}, 1)) = |x \wedge \omega|^{\frac{-1}{m-1}}$*, where* $\pi_{(T_x\mathbb{S}^d)^\perp} : \mathbb{R}^{d+1} \to (T_x\mathbb{S}^d)^\perp$ *denotes the natural projection map, so then applying Theorem 4.1.3 yields the following inequality for all* $f_j \in L^1(V_j)$*:*

$$\left\| \left(\prod_{j=1}^m f_j \circ L_j(x) \right) |x \wedge \omega| \right\|_{L^{\frac{1}{m-1}}(\mathbb{S}^d)} \lesssim \prod_{j=1}^m \|f_j\|_{L^1(V_j)}$$

Notice that way in which Theorem 4.1.3 is stated means we do not need to remove the *points $x \in \mathbb{S}^n$ that lie in one of the subspaces V_j, these being the points at which the corresponding projection L_j fails to be a submersion when restricted to $\mathbb{S}^d$.*

Brascamp–Lieb inequalities were first studied as a generalisation of Young's convolution inequality on $\mathbb{R}^n$ in [24], it is therefore fitting that one may view Theorem 4.1.3 as a generalisation of Young's convolution inequality on algebraic groups, those being algebraic varieties equipped with a group structure such that the associated multiplication and inversion maps are 'morphisms' of varieties, i.e. restrictions of polynomial maps.

Corollary 4.1.8 *Let G be an algebraic group, with left-invariant Haar measure $d\mu$. We let $\Delta : G \to (0, \infty)$ be the modular character associated to (G, μ), which is the unique homomorphism such that for all measurable $f : G \to \mathbb{R}$,*

$$\int_G f(x)d\mu(x) = \Delta(g) \int_G f(xg)d\mu(x).$$

We define left-convolution as follows:

$$f * g(x) := \int_G f(xy^{-1})g(y)d\mu(y)$$

The inequality (4.1.9) holds for all $p_1, ..., p_m, r \in [1, \infty]$ such that $\frac{1}{r'} = \sum_{j=1}^m \frac{1}{p_j'}$, and all $f_j \in L^{p_j}(G)$,

$$\left\| *_{j=1}^m f_j \Delta^{\sum_{l=1}^{j-1} \frac{1}{p_l'}} \right\|_{L^r(G)} \lesssim \deg(G) \deg(m_G)^\sigma \prod_{j=1}^m \|f_j\|_{L^{p_j}(G)} \tag{4.1.9}$$

where $m_G : G \times G \to G$ is the multiplication operation, and $\sigma := \sum_{j=1}^m \frac{1}{p_j}$.

We give a proof of this corollary in the appendix. It is important to note that since the best constant for Young's inequality on locally compact topological groups is always less than or equal to one [71], Corollary 4.1.8 does not offer any improvement to the theory, however it is nonetheless included in this chapter for the sake of context; we refer the reader to [34, 52, 59, 71] for further details on Young's inequality in abstract settings. We remarked earlier on that Theorem 4.1.3 is an example of an affine-invariant inequality, in the sense that the left-hand side is invariant under the natural action $A : B_j \mapsto B_j \circ A$ of $GL_n(\mathbb{R})$ on the class of quasialgebraic data, however this inequality in fact exhibits a more general diffeomorphism-invariance property, as described by the following proposition.

Proposition 4.1.10 *Let the dimensions and exponents be as in Theorem 4.1.3. Let M and $\widetilde{M}$ be d-dimensional Riemannian manifolds equipped with induced measures μ and $\widetilde{\mu}$, and, for each $1 \leqslant j \leqslant m$, let M_j be an n_j-dimensional Riemannian manifold. Let $B_j : M \to M_j$ be a.e. C^1, and $\phi : \widetilde{M} \to M$ be a diffeomorphism. Defining $\widetilde{\mathbf{B}} = (\tilde{B}_j)_{j=1}^m = (B_j \circ \phi)_{j=1}^m$, the following then holds for all $f_j \in L^1(M_j)$:*

$$\int_{\widetilde{M}} \prod_{j=1}^m f_j \circ \widetilde{B}_j(x)^{p_j} \frac{d\widetilde{\mu}(x)}{\mathrm{BL}_{T_x\widetilde{M}}(\mathbf{d}\widetilde{\mathbf{B}}(x), \mathbf{p})} = \int_M \prod_{j=1}^m f_j \circ B_j(x)^{p_j} \frac{d\mu(x)}{\mathrm{BL}_{T_x M}(\mathbf{dB}(x), \mathbf{p})}$$

Proof. By the chain rule and Lemma 3.3 of [14], for almost every $x \in M$,

$$\mathrm{BL}_{T_x\widetilde{M}}(\mathbf{d}\widetilde{\mathbf{B}}(x), \mathbf{p}) = \mathrm{BL}_{T_x\widetilde{M}}(\mathbf{dB}(\phi(x))d\phi(x), \mathbf{p}) = \mathrm{BL}_{T_{\phi(x)}M}(\mathbf{dB}(\phi(x)), \mathbf{p}) \det(d\phi(x))^{-1}.$$

Hence, by changing variables we obtain that

$$\begin{aligned} \int_{\widetilde{M}} \prod_{j=1}^m f_j \circ \widetilde{B}_j(x)^{p_j} \frac{dx}{\mathrm{BL}_{T_x\widetilde{M}}(\mathbf{d}\widetilde{\mathbf{B}}(x), \mathbf{p})} &= \int_{\widetilde{M}} \prod_{j=1}^m f_j \circ \widetilde{B}_j(x)^{p_j} \frac{\det(d\phi(x))dx}{\mathrm{BL}_{T_{\phi(x)}M}(\mathbf{dB}(\phi(x)), \mathbf{p})} \\ &= \int_M \prod_{j=1}^m f_j \circ B_j(x)^{p_j} \frac{dx}{\mathrm{BL}_{T_x M}(\mathbf{dB}(x), \mathbf{p})}. \end{aligned}$$ □

In light of Proposition 4.1.10, one may extend Theorem 4.1.3 to any m-tuple of maps $(B_j)_{j=1}^m$ that may each be written as a composition of a quasialgebraic map with a common diffeomorphism ϕ, however we shall leave this as a remark. The proof strategy for Theorem 4.1.3 will be to appeal to a generalised endpoint multilinear curvilinear Kakeya inequality, which we will view as a discrete version of (4.1.4), and run a limiting argument in order to recover the full inequality.

4.2 Setup for the Proof of Theorem 4.1.3

4.2.1 Reductions

We shall assume for the remainder of the chapter without loss of generality that the maps B_j have finite degree, since the case of infinite degree holds vacuously, and that $\mathrm{BL}_{T_xM}(\mathbf{dB}(x), \mathbf{p}) < \infty$ for all $x \in M$, in particular that B_j is a submersion on M. We may do this firstly because we may remove the set of non-smooth points harmlessly since it is closed and null, so M is still an open subset of an algebraic variety, and secondly we may remove the set of smooth points at which the weight arising in (4.1.3) vanishes, i.e. those $x \in M$ such that $\mathrm{BL}(\mathbf{dB}(x), \mathbf{p}) = \infty$, since this set is closed by continuity of the reciprocal of the Brascamp–Lieb constant (Theorem 5.2 of [9]).

We shall begin by reducing to the case where $d = n$, i.e. where M is an open subset of $\mathbb{R}^n$. We begin with a standard geometric lemma.

Lemma 4.2.1 *Let N be an $(n-d)$-dimensional Riemannian manifold and let $\chi_\delta : N \to \mathbb{R}$ be the normalised characteristic function associated to the δ-ball centred at some fixed $z_0 \in N$, defined by $\chi_\delta(z) := \delta^{n-d}\chi_{U_\delta(z_0)}$. Given an open set $A \subset \mathbb{R}^n$ and a submersion $B : A \to N$, then for any continuous and integrable $f : A \to \mathbb{R}$ the following holds:*

$$\int_A f(x)\chi_\delta \circ B(x)dx \xrightarrow{\delta \to 0} \int_{A \cap B^{-1}(\{z_0\})} f(x) \det(dB(x)dB(x)^*)^{-\frac{1}{2}} d\sigma(x),$$

where $d\sigma$ denotes the induced d-dimensional Hausdorff measure.

We also require the following identity of Brascamp–Lieb constants, which may be regarded as a crude example of a Brascamp–Lieb constant splitting through a critical subspace, a phenomenon that was studied in its full generality in [14].

Lemma 4.2.2 *Let $d, n, m \in \mathbb{N}$, $n_1, ..., n_m \in \mathbb{N}$ and write $n_{m+1} = n-d$. For $1 \leqslant j \leqslant m+1$, we consider linear surjections $L_j : \mathbb{R}^n \to \mathbb{R}^{n_j}$ such that, for $1 \leqslant j \leqslant m$, L_j restricts to a surjection on the subspace $V := \ker(L_{m+1})$. Let $p_j \in [0,1]$ for $1 \leqslant j \leqslant m$ and $p_{m+1} = 1$, and assume that the scaling condition $\sum_{j=1}^{m+1} p_j n_j = n$ is satisfied. Let $\widetilde{\mathbf{L}} := (L_j)_{j=1}^{m+1}$ and $\widetilde{\mathbf{p}} := (p_j)_{j=1}^{m+1}$. Then, the scaling condition $d = \sum_{j=1}^{m} p_j n_j$ holds. Furthermore, if we let $\mathbf{L} := (L|_V)_{j=1}^{m}$ and $\mathbf{p} := (p_j)_{j=1}^{m}$, we then have the following identity:*

$$\mathrm{BL}(\widetilde{\mathbf{L}}, \widetilde{\mathbf{p}}) = \det(L_{m+1}L_{m+1}^*)^{-\frac{1}{2}}\mathrm{BL}(\mathbf{L}, \mathbf{p}).$$

The proofs of these lemmas are given in the appendix. Combining them with Theorem 4.1.3 in the euclidean case then yields the general case.

Proposition 4.2.3 *If Theorem 4.1.3 holds for $d = n$, then Theorem 4.1.3 holds for general d.*

Proof. Let $B_{m+1} : \mathbb{R}^n \to \mathbb{R}^{n-d}$ be a polynomial map such that M is an open subset of $Z(B_{m+1})$, and that $\deg(B_{m+1}) = \deg(M)$. Let $A \subset \mathbb{R}^n$ be any bounded open set such that B_{m+1} restricts to a submersion on $A \cap M$. Recall the definition of χ_δ from Lemma 4.2.1. By Lemmas 4.2.1 and 4.2.2, we know that given any $f_j \in C_0^\infty(M_j)$,

$$\begin{aligned}\int_{A\cap M} \prod_{j=1}^{m} f_j \circ B_j(x)^{p_j} \frac{d\sigma(x)}{\mathrm{BL}_{T_xM}(\mathbf{dB}(x), \mathbf{p})} &= \int_{A\cap M} \prod_{j=1}^{m} f_j \circ B_j(x)^{p_j} \frac{\det(dB_{m+1}(x)dB_{m+1}(x)^*)^{-\frac{1}{2}}d\sigma(x)}{\mathrm{BL}(\widetilde{\mathbf{dB}}(x), \widetilde{\mathbf{p}})} \\ &= \lim_{\delta\to 0}\int_{A} \prod_{j=1}^{m} f_j \circ B_j(x)^{p_j} \frac{\chi_\delta \circ B_{m+1}(x)dx}{\mathrm{BL}(\widetilde{\mathbf{dB}}(x), \widetilde{\mathbf{p}})}.\end{aligned}$$

Applying Theorem 4.1.3 inside the limit on the right-hand side we then obtain

$$\int_{A\cap M}\prod_{j=1}^{m} f_j\circ B_j(x)^{p_j}\frac{dx}{\mathrm{BL}_{T_xM}(\mathbf{dB}(x),\mathbf{p})}$$
$$\lesssim \deg(B_{m+1})\lim_{\delta\to 0}\left(\int_{\mathbb{R}^{n-d}}\chi_\delta(z)dz\right)\prod_{j=1}^{m}\left(\deg(B_j)\int_{M_j} f_j(x_j)d\mu_j(x_j)\right)^{p_j}$$
$$\simeq \deg(M)\prod_{j=1}^{m}\left(\deg(B_j)\int_{M_j} f_j(x_j)d\mu_j(x_j)\right)^{p_j},$$

which yields the desired inequality, since the right-hand side is uniform in the choice of A, and extends to arbitrary $f_j \in L^1(M_j)$ via density. □

We shall henceforth assume that our domain is of full dimension, and to emphasise this, for the remainder of the proof we shall denote the domain of B_j by $U \subset \mathbb{R}^n$ instead of M.

Having reduced Theorem 4.1.3 to the euclidean case, we shall further reduce Theorem 4.1.3 to a more discrete inequality, where the domain U is replaced with a compact subset $\Omega \subset U$, and the arbitrary L^1 functions f_j are specifically sums of characteristic functions associated to small balls on M_j.

Proposition 4.2.4 *For every compact set $\Omega \subset U$, there exists a $\nu > 0$ such that, for all $\delta \in (0,\nu)$ and all collections $\mathcal{V}_j$ (allowing duplicates) of δ-balls in M_j, the following holds:*

$$\int_\Omega \prod_{j=1}^{m}\left(\sum_{V_j\in\mathcal{V}_j}\chi_{V_j}\circ B_j(x)\right)^{p_j}\frac{dx}{\mathrm{BL}(\mathbf{dB}(x),\mathbf{p})} \lesssim \prod_{j=1}^{m}\left(\deg(B_j)\delta^{n_j}\#\mathcal{V}_j\right)^{p_j}. \tag{4.2.5}$$

We shall now derive Theorem 4.1.3 from Proposition 4.2.4 via a standard limiting argument.

Proof of Theorem 4.1.3 given Proposition 4.2.4. The idea of this proof is to take an increasing sequence of compact domains Ω_k whose union is U, and for each term in the sequence apply (4.2.5), choosing the collections of balls $\mathcal{V}_j$ such that the sums of their indicator functions approximate f_j from below, up to a constant, from which (4.1.4) follows by the monotone convergence theorem.

For each $k \in \mathbb{N}$, let $\Omega_k \Subset U$ be compact subset such that $\Omega_k \subset \Omega_{k+1}$ and $\bigcup_{k=1}^{\infty} \Omega_k = U$. Let $\mathcal{Q}_j^{(k)}$ be an essentially disjoint cover of M_j such that for each Q_j there exists a δ_k-ball $V(Q_j)$ containing Q_j and $|Q| \simeq \delta_k^{n_j}$. For each $j \in \{1, ..., m\}$, let $f_j \in C_0^{\infty}(M_j) \cap L^1(M_j)$ be a non-negative function and, for each $Q_j \in \mathcal{Q}_j^{(k)}$, let $c_j^{(k)}(Q_j) \in \mathbb{N}$ be chosen such that for all $z \in M_j$

$$\inf_{z \in Q_j} f_j(z)k - 1 \leqslant c_j^{(k)}(Q_j) \leqslant \inf_{z \in Q_j} f_j(z)k.$$

At least one such choice exists since the upper and lower bounds are separated by 1. By construction, we have the pointwise limit $\frac{1}{k}\sum_{Q_j \in \mathcal{Q}_j^{(k)}} c_j^{(k)}(Q_j)\chi_{Q_j} \underset{k\to\infty}{\nearrow} f_j$, so in particular, for all $x \in \mathbb{R}^n$, by the monotone convergence theorem,

$$\int_{\Omega_k} \prod_{j=1}^{m} \left(\frac{1}{k} \sum_{Q_j \in \mathcal{Q}_j^{(k)}} c_j^{(k)}(Q_j)\chi_{Q_j} \circ B_j(x) \right)^{p_j} \frac{dx}{\mathrm{BL}(\mathbf{dB}(x), \mathbf{p})}$$
$$\underset{k\to\infty}{\longrightarrow} \int_U \prod_{j=1}^{m} f_j \circ B_j(x)^{p_j} \frac{dx}{\mathrm{BL}(\mathbf{dB}(x), \mathbf{p})} \quad (4.2.6)$$

On the other hand, provided that each δ_k is chosen to be sufficiently small with respect to Ω_k, we may apply (4.2.5) to the multiset consisting of $c_j^{(k)}(Q_j)$ copies of each ball $V(Q_j)$

for each $Q_j \in \mathcal{Q}_j^{(k)}$.

$$\begin{aligned}
&\int_{\Omega_k} \prod_{j=1}^m \left(\frac{1}{k} \sum_{Q_j \in \mathcal{Q}_j^{(k)}} c_j^{(k)}(Q_j) \chi_{Q_j} \circ B_j(x) \right)^{p_j} \frac{dx}{\mathrm{BL}(\mathbf{dB}(x), \mathbf{p})} \\
&\qquad \leqslant \frac{1}{k^P} \int_{\Omega_k} \prod_{j=1}^m \left(\sum_{Q_j \in \mathcal{Q}_j^{(k)}} c_j^{(k)}(Q_j) \chi_{V(Q_j)} \circ B_j(x) \right)^{p_j} \frac{dx}{\mathrm{BL}(\mathbf{dB}(x), \mathbf{p})} \\
&\qquad \lesssim \frac{1}{k^P} \prod_{j=1}^m \left(\deg(B_j) \delta_k^{n_j} \sum_{Q_j \in \mathcal{Q}_j^{(k)}} c_j^{(k)}(Q_j) \right)^{p_j} \\
&\qquad \underset{k \to \infty}{\longrightarrow} \prod_{j=1}^m \left(\deg(B_j) \int_{M_j} f_j(x_j) d\mu_j(x_j) \right)^{p_j}. \qquad (4.2.7)
\end{aligned}$$

The last line is also a consequence of the monotone convergence theorem. Theorem 4.1.3 then follows by combining (4.2.6) with (4.2.7), since by density this argument improves to arbitrary $f_j \in L^1(M_j)$. □

4.2.2 Central Constructions

The strategy for proving Proposition 4.2.4 is based on appealing to Theorem 1.4.18, in particular finding a collection of open subsets $H_1, ..., H_m$ of algebraic varieties such that, if substituted into (1.4.20), then the resulting inequality would yield (4.2.5). These manifolds may be thought of as the unions of 'discrete foliations' of the preimages $B_j^{-1}(V_j)$ via the fibres of B_j.

We shall now carry out this construction. Fix Ω and let $\delta > 0$ and $\mathcal{V}_j$ be a finite collection of δ-balls in M_j. Let $\alpha > 1$, for each $V_j \in \mathcal{V}_j$ let x_{V_j} denote the centre of V_j, and choose an orthonormal basis $\partial_1, ..., \partial_{n_j} \in T_{x_{V_j}} M_j$. Given $\varepsilon > 0$, we define the discrete ε-grid $\Lambda_{V_j}^{\varepsilon} := \bigoplus_{i=1}^{n_j} \varepsilon \mathbb{Z} \partial_i$, and we consider the intersection of a dilation of V_j with the image of this grid under the exponential map:

$$\Gamma(V_j) := \exp_{x_{V_j}}\left(\Lambda_{V_j}^{\delta^\alpha}\right) \cap 2V_j.$$

We have dilated the balls V_j by a factor of 2 for technical reasons that will become

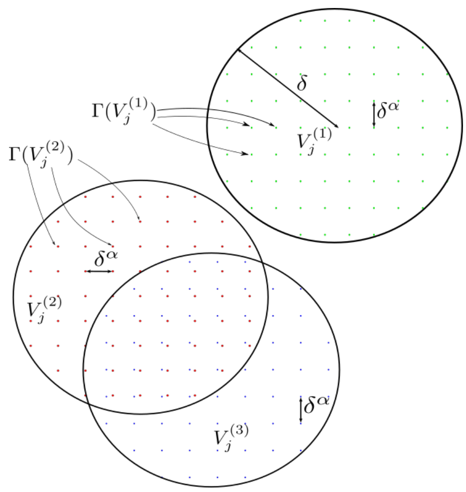

Figure 4.1: The specific case when $\mathcal{V}_j = \{V_j^{(1)}, V_j^{(2)}, V_j^{(3)}\}$

apparent in the proof of Lemma 4.3.9, the reader is encouraged to ignore it upon first reading. In order to track multiplicities, it shall be important that for each $V_j, V_j' \in \mathcal{V}_j$, we have $\Gamma(V_j) \cap \Gamma(V_j') = \varnothing$, however this is not guaranteed by our construction as it stands, hence if there exists $z \in \Gamma(V_j) \cap \Gamma(V_j')$, then we shall remedy this by simply translating one of these discrete sets by a negligible non-zero distance of, say, $\delta^{\alpha^{100}}$.

We shall now use the assumption that B_j is quasialgebraic. For each $z \in M_j$ there exists a polynomial map $p_j^z : \mathbb{R}^n \to \mathbb{R}^{n_j}$ such that $B_j^{-1}(\{z\})$ is an open subset of $Z(p_j^z)$ and $\deg(p_j^z) \leqslant \deg(B_j)$. Define the following polynomial map:

$$S_j := \prod_{V_j \in \mathcal{V}_j} \prod_{z \in \Gamma(V_j)} p_j^z,$$

and let $Z(S_j)$ be its zero-set. By our assumption that B_j is a submersion, we may assume that $Z(S_j)$ is an $(n-n_j)$-dimensional variety, and contains the following open subset that will serve as our aforementioned 'discrete' foliation:

$$H_j := \bigcup_{V_j \in \mathcal{V}_j} B_j^{-1}(\mathit{\Gamma}(V_j)) \subset Z(S_j).$$

Observe that if $\delta > 0$ is chosen to be sufficiently small, then $\#\mathit{\Gamma}(V_j) \simeq \delta^{-\alpha n_j}|V_j| \simeq \delta^{(1-\alpha)n_j}$, hence we may bound the degree of $Z(S_j)$ as follows:

$$\deg(Z(S_j)) \leqslant \sum_{V_j \in \mathcal{V}_j} \sum_{z \in \mathit{\Gamma}(V_j)} \deg(p_j^z) \leqslant \deg(B_j) \sum_{V_j \in \mathcal{V}_j} \#\mathit{\Gamma}(V_j) \simeq \deg(B_j)\delta^{(1-\alpha)n_j} \#\mathcal{V}_j \quad (4.2.8)$$

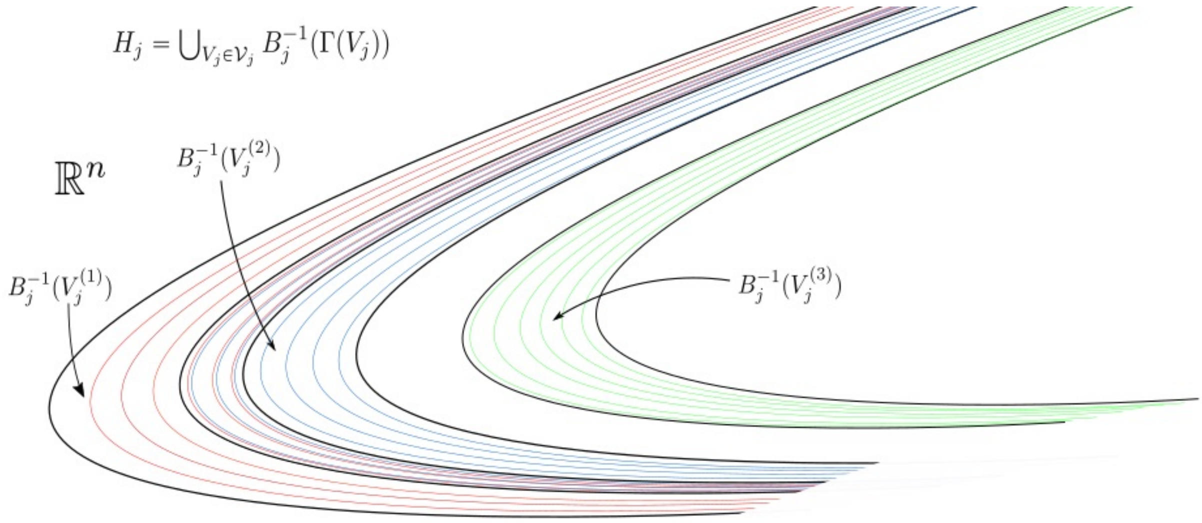

Figure 4.2: Picture of H_j

4.2.3 Heuristic Explanation of Proof Strategy

Let $f_j := \sum_{V_j \in \mathcal{V}_j} \chi_{V_j}$, and observe that the right-hand side of (4.2.8) is equal to $\deg(B_j)\delta^{-\alpha n_j} \int_{\mathbb{R}^{n_j}} f_j$, so provided we cancel the factor of $\delta^{\alpha n_j}$ at some stage, it then seems promising to substitute $H_1, ..., H_m$ into (1.4.20), and try to obtain (4.2.5) from that.

Morally, we may view the left-hand side of (1.4.20) as measuring the size of the intersections of tubular neighbourhoods of the varieties H_j of unit thickness, weighted by their mutual transversality. By rescaling we may reduce the size of these neighbourhoods to an arbitrarily small scale, for technical reasons we will reduce the thickness of the tubes to near δ^{β}-scale, where $\alpha < \beta < 1$.

If we now substitute the varieties $H_1, ..., H_m$ into (1.4.20), assuming our meshes $\Gamma(V_j)$ are sufficiently fine with respect to the size of V_j, then the left-hand side would essentially be measuring the size of the set

$$\bigcap_{j=1}^{m} \bigcup_{V_j \in \mathcal{V}_j} \bigcup_{z \in \Gamma(V_j)} (B_j^{-1}(\{z\}) + U_{\delta^{\beta}}(0)). \tag{4.2.9}$$

which we claim contains $\bigcap_{j=1}^{m} \bigcup_{V_j \in \mathcal{V}_j} B_j^{-1}(V_j) \cap \Omega$, and it is this set that the left-hand side of (4.2.5) is measuring, so all we need to make sure of is that the two measures in question essentially coincide.

The measure being applied to (4.2.9) is the Lebesgue measure weighted not only by the transversality of the leaves $B_j^{-1}(\{z\})$ comprising H_j, as imparted by the integrand $BL(\overrightarrow{T_{x_j} H_j}, \mathbf{p})$, but also, for each j, by a combinatorial factor that counts, given $x \in \bigcap_{j=1}^{m} \bigcup_{V_j \in \mathcal{V}_j} B_j^{-1}(V_j) \cap \Omega$, the number of δ^{β}–neighbourhoods that x lies in, and this factor is given by $\sum_{z \in \Gamma(V_j)} \chi_{B_j^{-1}(\{z\}) + U_{\delta^{\beta}}(0)}(x)$. As the forthcoming Lemma 4.3.9 demonstrates, this factor itself splits into two factors: one counts the number of preimages $B_j^{-1}(V_j)$ that x lies in, which is exactly given by $\sum_{V_j \in \mathcal{V}_j} \chi_{V_j} \circ B_j(x)$, and the other is a factor that counts the amount of overlap between tubes associated with the same ball V_j at a point $x \in U$. This factor will be large when the tubes are tightly packed, and low when the tubes are more spaced out. These situations correspond to the derivative map $dB_j(x)$ having respectively large and small 'volume', which is quantified by the function $|R_j(x)|$, which we define in the next section. It is due to the content of Lemma 4.3.4 that these additional

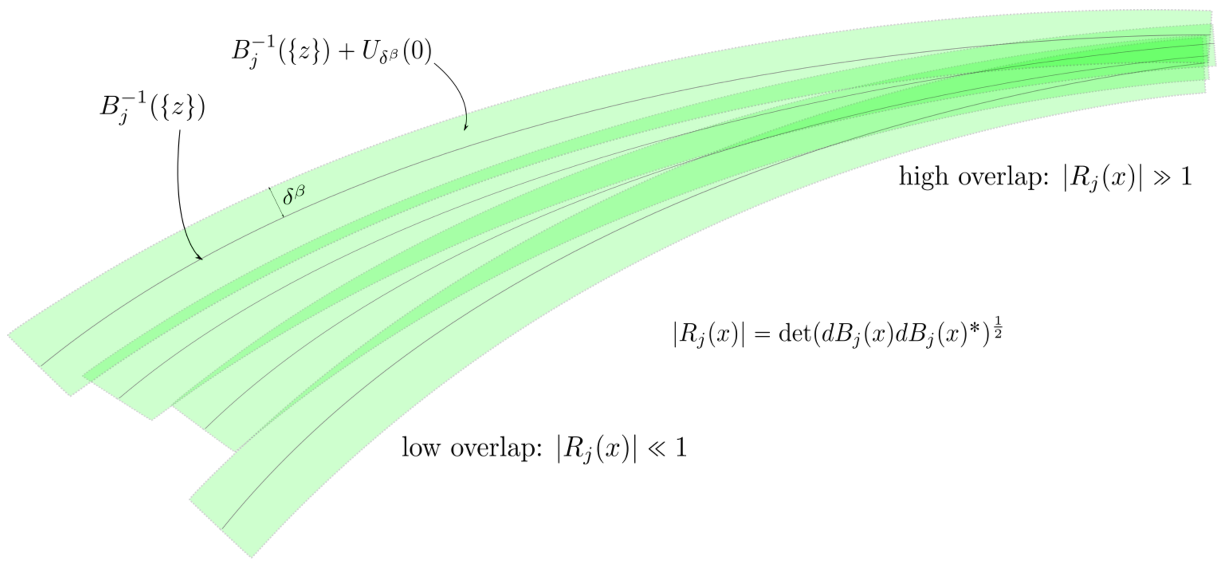

Figure 4.3: overlapping δ^β-tubes

$|R_j(x)|$-factors will allow us to move from BL-factors to BL-factors, which finally gives us the left-hand side of (4.2.5).

4.3 Lemmas

Here we shall prove the results that form the ingredients we need to prove Proposition 4.2.4. First of all, we shall investigate how Fremlin tensor product norms behave under rescaling.

Lemma 4.3.1 *Let $X_1, .., X_m \subset \mathbb{R}^n$ be smooth submanifolds such that* $\dim(X_j) = k_j$*, let* $q_1, ..., q_m \geqslant 1$*, and let* $F \in \bigotimes_{j=1}^m L^{q_j}(X_j)$*. Then, for all* $\varepsilon > 0$*,*

$$\|BL(\overrightarrow{T_{x_j}X_j}, \mathbf{p})\|_{\overline{\bigotimes}_{j=1}^m L^{q_j}_{x_j}(X_j)} = \varepsilon^{\sum k_j/q_j} \|BL(\overrightarrow{T_{x_j}(\varepsilon^{-1}X_j)}, \mathbf{p})\|_{\overline{\bigotimes}_{j=1}^m L^{q_j}_{x_j}(\varepsilon^{-1}X_j)}. \tag{4.3.2}$$

Proof. First of all, since dilation is a conformal mapping, it must preserve tangent spaces of submanifolds, so in particular $T_{\varepsilon x_j}X_j = T_{x_j}(\varepsilon^{-1}X_j)$. For each $j \in \{1, ..., m\}$, let $F_j \in L^{q_j}(X_j)$ be an arbitrary function satisfying $F_j \geqslant 0$ and $BL(\overrightarrow{T_{x_j}X_j}, \mathbf{p}) \leqslant F_1(x_1)...F_m(x_m)$

a.e. pointwise. By the definition of a Fremlin tensor product norm, it then suffices that

$$\prod_{j=1}^{m} \|F_j\|_{L^{q_j}(X_j)} = \varepsilon^{\sum k_j/q_j} \prod_{j=1}^{m} \|F_j(\varepsilon\cdot)\|_{L^{q_j}(\varepsilon^{-1}X_j)}, \tag{4.3.3}$$

which follows immediately from rescaling the L^{q_j} norms. □

A necessary ingredient for proving Proposition 4.2.4 is a formula relating the standard BL-constants with the nonstandard BL-constants arising in (1.4.20). We find that we may derive an explicit factorisation that makes explicit the dual role that the BL-constant plays, in both measuring the mutual transversality of the kernels of the L_j and measuring how close the maps L_j come to being non-surjective.

Lemma 4.3.4 *Let* $(\mathbf{L}, \mathbf{p})$ *be a Brascamp–Lieb datum such that each map* $L_j : V \to V_j$ *is surjective, and let* $R_j \in \Lambda^{n_j}(V)$ *denote the* n_j*-fold wedge product of the rows of* L_j*, then*

$$\mathrm{BL}(\mathbf{L}, \mathbf{p}) = BL(\overrightarrow{\ker(L_j)}, \mathbf{p}) \prod_{j=1}^{m} |R_j|^{-p_j}. \tag{4.3.5}$$

Proof. For the sakes of concreteness, we shall assume that the domains of the surjections L_j is $\mathbb{R}^n$ equipped with the standard inner product. By the first isomorphism theorem, for each $j \in \{1, ..., m\}$ there exists an isomorphism $\phi_j : \mathbb{R}^n/\ker(L_j) \to V_j$ such that $L_j = \phi_j \circ \pi_j$, where $\pi_j : \mathbb{R}^n \to \mathbb{R}^n/\ker(L_j)$ is the canonical projection map.
First of all, we claim that $|\det(\phi_j)| = |R_j|$. To see this, observe that $|L_j[0,1]^n| = |\phi_j \circ \pi_j[0,1]^n| = |\det(\phi)|$, so the claim then follows provided we can show that $|L_j[0,1]^n| = |R_j|$.

$$|L_j[0,1]^n| = |(L_j[0,1]^n) \times [0,1]^{n-n_j}| = |M^\top [0,1]^n| = |\det(M)|,$$

where $M \in \mathbb{R}^{n\times n}$ is the matrix whose first n_j rows are the rows of L_j and the last $n - n_j$

rows are $e_{n_j+1}, ..., e_n$, where $e_1, ..., e_n$ is an orthonormal basis of $\mathbb{R}^n$ such that $e_1, ..., e_{n_j}$ span $\ker(L_j)^\perp$ and $e_{n_j+1}, ..., e_n$ spans $\ker(L_j)$. Since $R_j = \pm|R_j| \bigwedge_{j=1}^{n_j} e_j$, the claim then quickly follows:

$$|L_j[0,1]^n| = |\det M| = |R_j \wedge (\bigwedge_{j=n_j+1}^{n} e_j)| = |R_j||\bigwedge_{j=1}^{n} e_j| = |R_j|.$$

Now, let $f_j \in L^1(V_j)$ be arbitrary and $\widetilde{f_j} := f_j \circ \phi_j$. We may then change variables and rewrite the left-hand side of the Brascamp–Lieb inequality associated to $(\mathbf{L}, \mathbf{p})$ as follows.

$$\int_{\mathbb{R}^n} \prod_{j=1}^{m} f_j \circ L_j(x)^{p_j} dx = \int_{\mathbb{R}^n} \prod_{j=1}^{m} \widetilde{f_j} \circ \pi_j(x)^{p_j} dx = \int_{\mathbb{R}^n} \prod_{j=1}^{m} \widetilde{f_j}(x + \ker(L_j))^{p_j} dx \qquad (4.3.6)$$

Moreover, $\int_{\mathbb{R}^n/\ker(L_j)} \widetilde{f_j} = |\det(\phi_j)|^{-1} \int_{H_j} f_j = |R_j|^{-1} \int_{H_j} f_j$, hence combining this with (4.3.6) we obtain that

$$\int_{\mathbb{R}^n} \prod_{j=1}^{m} f_j \circ L_j(x)^{p_j} dx \leqslant BL(\overrightarrow{\ker(L_j)}, \mathbf{p}) \prod_{j=1}^{m} \left(|R_j|^{-1} \int_{H_j} f_j \right)^{p_j}. \qquad (4.3.7)$$

Therefore $\mathrm{BL}(\mathbf{L}, \mathbf{p}) \leqslant BL(\overrightarrow{\ker(L_j)}, \mathbf{p}) \prod_{j=1}^{m} |R_j|^{-p_j}$. Furthermore, observing that (4.3.7) is sharp, by the definitions of BL and BL, this automatically improves to the desired formula $\mathrm{BL}(\mathbf{L}, \mathbf{p}) = BL(\overrightarrow{\ker(L_j)}, \mathbf{p}) \prod_{j=1}^{m} |R_j|^{-p_j}$. □

We remark that $|R_j|$ may also be written as $\det(L_j L_j^*)^{1/2}$, since $|R_j|^2 = \langle R_j, R_j \rangle_{\Lambda^{n_j}(\mathbb{R}^n)} = \det((r_{j,k} \cdot r_{j,l})_{k,l=1}^{n}) = \det(L_j L_j^*)$, where $r_{j,k}$ is the k^{th} row of L_j. As one would expect, the formula (4.3.6) also allows us to carry stability properties from the standard BL-constants to the BL-constants arising in (1.4.20), which we state more precisely in the following corollary.

Corollary 4.3.8 *Let $\Omega \subset U$ be compact. Writing $\mathbf{x} := (x_1, ..., x_m)$, the weight function*

$g : \Omega^m \to \mathbb{R}$ defined by

$$g(\mathbf{x}) := BL(\overrightarrow{\ker dB_j(x_j)}, \mathbf{p})^{-1}$$

is uniformly continuous and locally constant at a sufficiently small scale, that is to say for $\varepsilon > 0$ sufficiently small depending on Ω, for all $\mathbf{x}, \mathbf{y} \in \Omega^m$,

$$|\mathbf{x} - \mathbf{y}| < \varepsilon \Longrightarrow g(\mathbf{x}) \lesssim g(\mathbf{y}).$$

Proof. For each $x_j \in \Omega$, let $\Pi_j^{x_j} : \mathbb{R}^n \to \ker(dB_j(x_j))^\perp$ denote the projection map onto $\ker(dB_j(x_j))^\perp$, and let $\phi_j^{x_j} : \mathbb{R}^{n_j} \to \ker(dB_j(x_j))^\perp$ be a family of isometric isomorphisms that varies continuously in x_j. Define the family of surjections $L_j^{x_j} : \mathbb{R}^n \to \mathbb{R}^{n_j}$ by $L_j^{x_j} := (\phi_j^{x_j})^{-1} \circ \Pi_j^{x_j}$, and let $\mathbf{L}^\mathbf{x} := (L_j^{x_j})_{j=1}^m$. By Lemma 4.3.4, $\mathrm{BL}(\mathbf{L}^\mathbf{x}, \mathbf{p})^{-1} = g(\mathbf{x})^{-1} \prod_{j=1}^m |\det(\phi_j^{x_j})|^{p_j} = g(\mathbf{x})$, hence continuity of g follows from the continuity of the reciprocal of the Brascamp–Lieb constant over Ω, which was established in [10]. By compactness of Ω and the positivity of g, $g \otimes g^{-1}$ is then uniformly continuous on $(\Omega^m)^2$, so because $g \otimes g^{-1}(\mathbf{x}; \mathbf{x}) = 1$ for all $\mathbf{x} \in \Omega^m$, there exists $\varepsilon > 0$ such that for all $\mathbf{x}, \mathbf{y} \in \Omega^m$, $g \otimes g^{-1}(\mathbf{x}; \mathbf{y}) < 2$ provided that $|\mathbf{x} - \mathbf{y}| < \varepsilon$, completing the proof. □

The next proposition will allow us to simultaneously cover the preimages $B_j^{-1}(V_j)$ of the Balls V_j by tubular neighbourhoods of the varieties comprising H_j, and account for the missing factor in the weight $\mathrm{BL}(\mathbf{dB}(x), \mathbf{p})^{-1}$, as alluded to in Section 4.2.3.

Lemma 4.3.9 *Let $\Omega \subset U$ be compact and fix $j \in \{1, ..., m\}$. Let $R_j(x) \in \Lambda^{n_j}(\mathbb{R}^n)$ denote the n_j-fold wedge product of the rows of $dB_j(x)$, then for a sufficiently small choice of $\delta > 0$ depending on Ω, over all $x \in \Omega$,*

$$|R_j(x)|\chi_{V_j} \circ B_j(x) \lesssim \delta^{(\alpha-\beta)n_j} \sum_{z \in \Gamma(V_j)} \chi_{B_j^{-1}(\{z\}) + U_{\delta^\beta}(0)}(x). \tag{4.3.10}$$

To prove this lemma, we shall need to establish the following intuitive geometric fact that shall allow us to deal with the nonlinearity present in the quasialgebraic maps B_j.

Lemma 4.3.11 *Given the same hypotheses as Lemma 4.3.9, for a sufficiently small choice of $\delta > 0$ depending on Ω, $L_j^x(U_{\delta/2}(x)) \subset B_j(U_\delta(x))$ for all $x \in \Omega$, where $L_j^x(y) := \exp_{B_j(x)}(dB_j(x)(y-x))$ is the first-order approximation of B_j about x (not to be confused with the notation used in Corollary 4.3.8).*

We give the proof of this lemma in the appendix.

Proof of Lemma 4.3.9. We immediately have that for each $x \in \Omega$,

$$\begin{aligned}\sum_{z \in \mathit{\Gamma}(V_j)} \chi_{B_j^{-1}(\{z\})+U_{\delta^\beta}(0)}(x) &= \#\{z \in \mathit{\Gamma}(V_j) : d(x, B_j^{-1}(\{z\})) \leqslant \delta^\beta\} \\ &= \#\left(\mathit{\Gamma}(V_j) \cap B_j(U_{\delta^\beta}(x))\right) \\ &= \#\left(\exp_{x_{V_j}}\left(\Lambda_{V_j}^{\delta^\alpha}\right) \cap 2V_j \cap B_j(U_{\delta^\beta}(x))\right).\end{aligned}$$

By Lemma 4.3.11 we then, for $\delta > 0$ sufficiently small, have the bound

$$\sum_{z \in \mathit{\Gamma}(V_j)} \chi_{B_j^{-1}(\{z\})+U_{\delta^\beta}(0)}(x) \geqslant \#\left(\exp_{x_{V_j}}\left(\Lambda_{V_j}^{\delta^\alpha}\right) \cap 2V_j \cap L_j^x(U_{\delta^\beta/2}(x))\right). \tag{4.3.12}$$

Recall that we denote the centre of V_j by $x_{V_j} \in M_j$. $|dB_j(x)|$ is uniformly bounded over $x \in \Omega$, so provided that $x \in B_j^{-1}(V_j)$, then for all $y \in U_{\delta^\beta/2}(x)$, $d(L_j^x(y), x_{V_j}) \leqslant d(B_j(x), x_{V_j}) + \|dB_j(x)\|_{L^\infty(\Omega)}|y-x| \leqslant \delta + \|dB\|\delta^\beta/2 < 2\delta$, if we take $\delta > 0$ to be sufficiently small. This implies that if $x \in B_j^{-1}(V_j) \cap \Omega$, then for $\delta > 0$ sufficiently small,

$L_j^x(U_{\delta^\beta/2}(x)) \subset 2V_j$, which together with (4.3.12) yields that

$$\sum_{z\in\Gamma(V_j)} \chi_{B_j^{-1}(\{z\})+U_{\delta^\beta}(0)}(x) \geqslant \#\left(\exp_{x_{V_j}}\left(\Lambda_{V_j}^{\delta^\alpha}\right)\cap L_j^x(U_{\delta^\beta/2}(x))\right)\chi_{B_j^{-1}(V_j)}(x)$$

$$\simeq \#\left(\exp_{x_{V_j}}\left(\Lambda_{V_j}^{\delta^{\alpha-\beta}}\right)\cap L_j^x(U_1(x))\right)\chi_{B_j^{-1}(V_j)}(x). \qquad (4.3.13)$$

Given $\varepsilon > 0$, define $\mathcal{Q}_j^\varepsilon$ to be the cubic decomposition of $T_{x_{V_j}}M_j$ into ε-cubes whose sides are axis parallel and whose corresponding set of centres is $\Lambda_{V_j}^\varepsilon$, and recall the definition of $c > 0$ from the proof of Lemma 4.3.11. If we take $\delta^{\alpha-\beta} < c/10$, then for all $x \in \Omega$ and $Q \in \mathcal{Q}_j^{\delta^{\alpha-\beta}}$ such that $Q \cap L_j^x(U_{1/2}(x)) \neq \varnothing$, we must have that $Q \subset L_j^x(U_1(x))$, since otherwise there would exist a point outside of $L_j^x(U_1(x))$ within a distance $c/2$ of $B_j(x)$, which implies that $dB_j(x)|_{\ker dB_j(x)^\perp}$ has an eigenvalue with absolute value less than c, which is of course a contradiction. Since the map that takes a cube in $\mathcal{Q}_j^{\delta^{\alpha-\beta}}$ to its centre then defines an injection from $D := \{Q \in \mathcal{Q}_j^{\delta^{\alpha-\beta}} : Q \cap L_j^x(U_{1/2}(x)) \neq \varnothing\}$ to $\exp_{x_{V_j}}\left(\Lambda_{V_j}^{\delta^{\alpha-\beta}}\right) \cap L_j^x(U_1(x))$, we obtain the following bound:

$$\begin{aligned}\sum_{z\in\Gamma(V_j)} \chi_{B_j^{-1}(\{z\})+U_{\delta^\beta}(0)}(x) &\geqslant (\#D)\chi_{B_j^{-1}(V_j)}(x)\\
&= \Big|\bigcup_{Q\in D} Q\Big|\big|[0,\delta^{\alpha-\beta}]^{n_j}\big|^{-1}\chi_{B_j^{-1}(V_j)}(x)\\
&\geqslant |L_j^x(U_{1/2}(x))|\delta^{(\beta-\alpha)n_j}\chi_{B_j^{-1}(V_j)}(x)\\
&\simeq |dB_j(x)[0,1]^n|\delta^{(\beta-\alpha)n_j}\chi_{B_j^{-1}(V_j)}(x).\end{aligned}$$

Since $\chi_{B_j^{-1}(V_j)} = \chi_{V_j} \circ B_j$, the claim then follows from the fact that $|dB_j(x)[0,1]^n| = |R_j(x)|$, which follows from an inspection of the proof that $|L_j[0,1]^n| = |R_j|$ in Lemma 4.3.4. □

Finally, we need a technical lemma that will allow us to bound the volumes of intersections of balls with varieties below by the characteristic functions arising on the right-hand side of (4.3.10).

Lemma 4.3.14 *Let $\Omega \subset U$ be compact, and fix $j \in \{1, ..., m\}$. Then, for a sufficiently small choice of $\delta > 0$ depending on Ω, the following holds for all $x \in \Omega$ and $z \in M_j$:*

$$\delta^{\beta(n-n_j)}\chi_{B_j^{-1}(\{z\})+U_{\delta^\beta}(0)}(x) \lesssim |B_j^{-1}(\{z\}) \cap U_{2\delta^\beta}(x)|. \tag{4.3.15}$$

Proof. We shall begin with some reductions. First of all, we fix $z \in M_j$, making sure in what comes after that our choice $\delta > 0$ does not depend on this particular choice of $z \in M_j$. Suppose that for each choice of $x_0 \in \Omega$, there exists a corresponding choice of $\delta_{x_0} > 0$ such that (4.3.15) holds for each $x \in U_{\delta_{x_0}^2}(x_0)$ and $0 < \delta \leqslant \delta_{x_0}$. The set $\{U_{\delta_{x_0}^2}(x_0) : x_0 \in \Omega\}$ is then an open cover of Ω, so by compactness of Ω we may take a finite subcover $\mathcal{U}$. The minimal radius among the balls in $\mathcal{U}$, which we shall denote by $\tilde{\delta}$, is such that (4.3.15) holds for all $\delta \in (0, \tilde{\delta})$ and $x \in \Omega$, so the lemma would then hold. It therefore suffices to fix $x_0 \in \Omega$ and prove the claim that there exists a δ_{x_0} such that (4.3.15) holds for each $x \in U_{\delta_{x_0}^2}(x_0)$ and $0 < \delta \leqslant \delta_{x_0}$.

Furthermore, we may assume that M_j is an open subset of $\mathbb{R}^{n_j}$. To justify this, by compactness of Ω and continuity of B_j, we may choose a $\delta > 0$ sufficiently small such that $\exp_x$ is a diffeomorphism on $U_\delta(0) \subset T_y M_j$ for each $y \in B_j(\Omega)$. We then restrict B_j to $B_j^{-1}(U_\delta(z))$ and prove that the claim holds with B_j replaced with $\widetilde{B}_j := \exp_z^{-1} \circ B_j$, and z replaced with $0 \in \mathbb{R}^{n_j}$, since in this case $\widetilde{B}_j^{-1}(\{0\}) = B_j^{-1}(\{z\})$, hence we would obtain the claim for our original choice of B_j.

Fix $x_0 \in \Omega$, recall the definition of $L_j^{x_0}$ from Lemma 4.3.9 and let $A \in SO(n)$ be a rotation such that $A \ker dB_j(x_0) = \mathbb{R}^{n-n_j} \times \{0\}^{n_j}$. Since B_j is a submersion on Ω, $dB_j(x_0)$ is surjective, hence it admits a right inverse, call it S. Let $\psi := B_j - dB_j(x_0)$. We define the function $\phi : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\phi(y) := A(y + S\psi(y))).$$

For all $y \in B_j^{-1}(\{z\})$, $z = B_j(y) = dB_j(x_0)y + \psi(y) = dB_j(x_0)(y + S\psi(y)) = dB_j(x_0)(A^{-1}\phi(y))$, so $A^{-1}\phi(y) \in dB_j(x_0)^{-1}(\{z\})$, hence $A^{-1}\phi(y) - Sz \in \ker dB_j(x_0)$, so $\phi(y) \in \mathbb{R}^{n-n_j} \times \{0\}^{n_j} + ASz$. We have now shown that $\phi(B_j^{-1}(\{z\})) \subset \mathbb{R}^{n-n_j} \times \{0\}^{n_j} + ASz$. Moreover, one quickly verifies that $d\phi(x_0) = A(I + Sd\psi(x_0)) = A$, hence ϕ is a diffeomorphism in a sufficiently small ball around x_0, therefore by taking δ to be sufficiently small, we may assume that, for all $x \in U_{\delta^{2\beta}}(x_0)$, $U_{\frac{3\delta^\beta}{2}}(\phi(x)) \subset \phi(U_{2\delta^\beta}(x))$ and $\det(d\phi|_{B_j^{-1}(\{z\})}(y)) \simeq 1$ for all $y \in U_\delta(x)$, from which it follows that, for all $x \in U_{\delta^{2\beta}}(x_0)$,

$$\begin{aligned} |B_j^{-1}(\{z\}) \cap U_{2\delta^\beta}(x)| &= \int_{(\mathbb{R}^{n-n_j} \times \{0\} + ASz) \cap \phi(U_{2\delta^\beta}(x))} \det(d\phi|_{B_j^{-1}(\{z\})}(y))^{-1} dy \\ &\simeq |(\mathbb{R}^{n-n_j} \times \{0\}^{n_j} + ASz) \cap \phi(U_{2\delta^\beta}(x))| \qquad (4.3.16) \\ &\geqslant |(\mathbb{R}^{n-n_j} \times \{0\}^{n_j} + ASz) \cap U_{\frac{3\delta^\beta}{2}}(\phi(x))| \\ &\geqslant |(\mathbb{R}^{n-n_j} \times \{0\}^{n_j} + ASz) \cap U_{\frac{3\delta^\beta}{2}}(\phi(x))| \chi_{B_j^{-1}(\{z\}) + U_{\delta^\beta}(0)}(x). \qquad (4.3.17) \end{aligned}$$

Since ϕ is smooth and $d\phi(x_0) = A$ is an isometry, if $x_0 \in B_j^{-1}(\{z\}) + U_{\delta^\beta}(0)$ and δ is sufficiently small then by Taylor's theorem we know that for all $x \in U_{\delta^{2\beta}}(x_0)$, $\phi(x) \in \phi(B_j^{-1}(\{z\})) + U_{\frac{5\delta^\beta}{4}}(0) = (\mathbb{R}^{n-n_j} \times \{0\}^{n_j} + ASz) + U_{\frac{5\delta^\beta}{4}}(0)$. In other words, $\operatorname{dist}(\phi(x), (\mathbb{R}^{n-n_j} \times \{0\}^{n_j} + ASz)) \leqslant \frac{5\delta^\beta}{4}$, hence $(\mathbb{R}^{n-n_j} \times \{0\}^{n_j} + ASz) \cap U_{\frac{3\delta^\beta}{2}}(\phi(x)))$ is an $(n - n_j)$-disc of radius at least $\sqrt{\frac{9\delta^{2\beta}}{4} - \frac{25\delta^{2\beta}}{16}} \simeq \delta^\beta$, therefore

$$|(\mathbb{R}^{n-n_j} \times \{0\}^{n_j} + ASz) \cap U_{\frac{3\delta^\beta}{2}}(\phi(x)))| \chi_{B_j^{-1}(\{z\}) + U_{\delta^\beta}(0)}(x) \gtrsim \delta^{\beta(n-n_j)} \chi_{B_j^{-1}(\{z\}) + U_{\delta^\beta}(0)}. \qquad (4.3.18)$$

This bound together with (4.3.17) then yields the claim. □

4.4 Proof of Proposition 4.2.4

Proof. Let $\Omega \subset U$ and choose $\delta > 0$ so that we may apply Corollary 4.3.8, Lemma 4.3.9, and Lemma 4.3.14 to Ω. After first applying Lemma 4.3.4, they yield the following pointwise estimate for all $x \in \Omega$,

$$\begin{aligned}
&\mathrm{BL}(\mathbf{dB}(x), \mathbf{p})^{-1} \prod_{j=1}^{m} \left(\sum_{V_j \in \mathcal{V}_j} \chi_{V_j} \circ B_j(x) \right)^{p_j} \\
&= BL(\overrightarrow{\ker dB_j(x)}, \mathbf{p})^{-1} \prod_{j=1}^{m} \left(\sum_{V_j \in \mathcal{V}_j} |R_j(x)| \chi_{V_j} \circ B_j(x) \right)^{p_j} \\
&\lesssim BL(\overrightarrow{\ker dB_j(x)}, \mathbf{p})^{-1} \prod_{j=1}^{m} \delta^{(\alpha-\beta)p_j n_j} \left(\sum_{V_j \in \mathcal{V}_j} \sum_{z \in \Gamma(V_j)} \chi_{B_j^{-1}(\{z\}) + U_{\delta^\beta}(0)}(x) \right)^{p_j} \\
&= \delta^{(\alpha-\beta)n} BL(\overrightarrow{\ker dB_j(x)}, \mathbf{p})^{-1} \prod_{j=1}^{m} \left(\sum_{V_j \in \mathcal{V}_j} \sum_{z \in \Gamma(V_j)} \chi_{B_j^{-1}(\{z\}) + U_{\delta^\beta}(0)}(x) \right)^{p_j} \\
&\lesssim \delta^{(\alpha-\beta)n} BL(\overrightarrow{\ker dB_j(x)}, \mathbf{p})^{-1} \prod_{j=1}^{m} \delta^{-\beta p_j (n-n_j)} \left(\sum_{V_j \in \mathcal{V}_j} \sum_{z \in \Gamma(V_j)} |B_j^{-1}(\{z\}) \cap U_{2\delta^\beta}(x)| \right)^{p_j} \\
&= \delta^{(\alpha-\beta P)n} BL(\overrightarrow{\ker dB_j(x)}, \mathbf{p})^{-1} \prod_{j=1}^{m} |H_j \cap U_{2\delta^\beta}(x)|^{p_j}. \qquad (4.4.1)
\end{aligned}$$

Above we used the scaling condition $\sum_{j=1} p_j n_j = n$ to pull out the power of δ from the product. By Corollary 4.3.8, for all $x \in \Omega$ and $x_1, ..., x_m \in H_j \cap U_{2\delta^\beta}(x)$,

$$BL(\overrightarrow{\ker dB_j(x)}, \mathbf{p})^{-\frac{1}{P}} \simeq BL(\overrightarrow{T_{x_j} H_j}, \mathbf{p})^{-\frac{1}{P}}. \qquad (4.4.2)$$

To speak in general terms momentarily, for each $j \in \{1, ..., m\}$ let $q_j \in [1, \infty]$ and let X_j be a finite measure space. Suppose that a function $F \in \bigotimes_{j=1}^{m} L^{q_j}(X_j)$ satisfies $|F| \leqslant C$ for some $C > 0$ pointwise almost everywhere, then since constant functions are elementary tensors, it is immediate from the definition of the Fremlin tensor product, discussed in Section 1.4, that $\|F\|_{\overline{\bigotimes}_{j=1}^m L^{q_j}(X_j)} \leqslant C \prod_{j=1}^m |X_j|^{1/q_j}$. Now returning to our specific case, we may therefore average (4.4.2) via the Fremlin tensor product norm to find that find that

$$BL(\overrightarrow{\ker dB_j(x)}, \mathbf{p})^{-1} \prod_{j=1}^m |H_j \cap U_{2\delta^\beta}(x)|^{p_j} \simeq \|BL(\overrightarrow{T_{x_j} H_j}, \mathbf{p})^{\frac{-1}{P}}\|^P_{\overline{\bigotimes}_{j=1}^m L_{x_j}^{P/p_j}(H_j \cap U_{2\delta^\beta}(x))}. \tag{4.4.3}$$

We then integrate the inequality (4.4.1) combined with (4.4.3) with respect to x over Ω.

$$\int_\Omega \prod_{j=1}^m \left(\sum_{V_j \in \mathcal{V}_j} \chi_{V_j} \circ B_j(x) \right)^{p_j} \frac{dx}{\mathrm{BL}(\mathbf{dB}(x), \mathbf{p})} \lesssim \delta^{(\alpha - \beta P)n} \int_\Omega \|BL(\overrightarrow{T_{x_j} H_j}, \mathbf{p})^{\frac{-1}{P}}\|^P_{\overline{\bigotimes}_{j=1}^m L_{x_j}^{P/p_j}(H_j \cap U_{2\delta^\beta}(x))} \tag{4.4.4}$$

At this point we then apply Lemma 4.3.1 to rescale the inner integral so that we may then apply Theorem 1.4.18. Finally, using the bound on the degree of $Z(S_j) \supset H_j$ 4.2.8, we obtain (4.2.5), completing the proof.

$$\begin{aligned}
&\int_{\Omega}\prod_{j=1}^{m}\left(\sum_{V_j\in\mathcal{V}_j}\chi_{V_j}\circ B_j(x)\right)^{p_j}\frac{dx}{\mathrm{BL}(\mathbf{dB}(x),\mathbf{p})}\\
&\lesssim \delta^{\beta\sum_{j=1}^{m}p_j(n-n_j)}\delta^{(\alpha-\beta(P-1))n}\int_{\frac{\delta^{-\beta}}{2}\Omega}\|BL(\overrightarrow{T_{x_j}\left(\tfrac{\delta^{-\beta}}{2}H_j\right)},\mathbf{p})^{\frac{-1}{P}}\|^{P}_{\bigotimes_{j=1}^{m}L^{P/p_j}_{x_j}\left(\frac{\delta^{-\beta}}{2}H_j\cap U_1(x)\right)}dx\\
&\leqslant \delta^{\alpha n}\int_{\mathbb{R}^n}\|BL(\overrightarrow{T_{x_j}\left(\tfrac{\delta^{-\beta}}{2}H_j\right)},\mathbf{p})^{\frac{-1}{P}}\|^{P}_{\bigotimes_{j=1}^{m}L^{P/p_j}_{x_j}\left(\frac{\delta^{-\beta}}{2}H_j\cap U_1(x)\right)}dx\\
&\leqslant \delta^{\alpha n}\int_{\mathbb{R}^n}\|BL(\overrightarrow{T_{x_j}\left(\tfrac{\delta^{-\beta}}{2}Z(S_j)\right)},\mathbf{p})^{\frac{-1}{P}}\|^{P}_{\bigotimes_{j=1}^{m}L^{P/p_j}_{x_j}\left(\frac{\delta^{-\beta}}{2}Z(S_j)\cap U_1(x)\right)}dx\\
&\lesssim \delta^{\alpha n}\prod_{j=1}^{m}(\deg Z(S_j))^{p_j}\\
&\lesssim \delta^{\alpha n}\prod_{j=1}^{m}\left(\deg(B_j)\delta^{(1-\alpha)n_j}\#\mathcal{V}_j\right)^{p_j}=\prod_{j=1}^{m}(\deg(B_j)\delta^{n_j}\#\mathcal{V}_j)^{p_j}
\end{aligned}$$

□

4.5 Two Kakeya–Brascamp–Lieb Versions

In this section, we shall apply the same techniques we used to prove Theorem 4.1.3 to prove two distinct Kakeya–Brascamp–Lieb inequalities. Let $d, m, n \in \mathbb{N}$ and, for each $1 \leqslant j \leqslant m$, let $n_j \in \mathbb{N}$ and $p_j \in [0,1]$. Assume that the scaling condition $\sum_{j=1}^{m} p_j n_j = d$ is satisfied. Let $M \subset \mathbb{R}^n$ be an open subset of a d-dimensional algebraic variety, and for each $j \in \{1, ..., m\}$, let M_j be an n_j-dimensional Riemannian manifold.

We consider collections $\mathbb{B}_j = \{B_j\}$ of quasialgebraic maps $B_j : M \to M_j$ that extend to quasialgebraic maps on some open set $A \subset \mathbb{R}^n$. Setting $\mathbf{p} := (p_1, ..., p_m)$ where $\sum_{j=1}^{m} p_j \dim(M_j) = \dim(M)$ and equipping each M_j with the measure μ_j induced by its Riemannian metric, then we have two Kakeya–Brascamp–Lieb versions of Theorem 4.1.3. The first of which, like Theorem 4.1.3, is also a geometrically invariant inequality, however we can only formulate it when each exponent p_j is equal to $\frac{1}{m-1}$.

Theorem 4.5.1 (Invariant Quasialgebraic Kakeya–Brascamp–Lieb) *If each $p_j = \frac{1}{m-1}$, then the following inequality holds for all $f_{B_j} \in L^1(M_j)$:*

$$\int_M \left(\sum_{(B_1,...,B_m) \in \mathbb{B}_1 \times ... \times \mathbb{B}_m} \frac{\prod_{j=1}^m f_{B_j} \circ B_j(x)}{\mathrm{BL}(\mathbf{dB}(x), \mathbf{p})^{m-1}} \right)^{\frac{1}{m-1}} d\sigma(x)$$

$$\lesssim \deg(M) \prod_{j=1}^m \left(\sum_{B_j \in \mathbb{B}_j} \deg(B_j) \int_{M_j} f_{B_j}(x_j) d\mu_j(x_j) \right)^{\frac{1}{m-1}} \tag{4.5.2}$$

where σ is the induced d-dimensional Hausdorff measure on M.

One may run a similar argument to Proposition 4.1.10 in order to show that this integral satisfies appropriate diffeomorphism-invariance properties. The second applies to any configuration of dimensions and exponents that satisfy the appropriate scaling condition, however this is at the expense of removing the invariant weight factor, and therefore require some additional uniformity asssumptions.

Theorem 4.5.3 (Non-invariant Quasialgebraic Kakeya–Brascamp–Lieb) *If there exists $C > 0$ such that $|dB_j(x)|, \mathrm{BL}(\mathbf{dB}(x), \mathbf{p}) \simeq C$ for all $x \in M$ and $j \in \{1, ..., m\}$, then the following inequality holds for all $f_{B_j} \in L^1(M_j)$:*

$$\int_M \prod_{j=1}^m \left(\sum_{B_j \in \mathbb{B}_j} f_{B_j} \circ B_j(x) \right)^{p_j} d\sigma(x) \lesssim_C \deg(M) \prod_{j=1}^m \left(\sum_{B_j \in \mathbb{B}_j} \deg(B_j) \int_{M_j} f_{B_j}(x_j) d\mu_j(x_j) \right)^{p_j}, \tag{4.5.4}$$

We shall use this theorem later in Chapter 5 to prove a corollary of the forthcoming Theorem 5.1.1. Given the dichotomy between Theorems 4.5.1 and 4.5.3, it is natural to suppose that there exists an invariant Kakeya–Brascamp–Lieb inequality for general exponents that generalises both (4.5.2) and (4.5.4), however it is not clear yet what such an inequality would look like, although we discuss this line of enquiry in Section 6.4.

4.5.1 Setup

We need to modify our definition of the varieties $H_1, ..., H_m$ to suit the more geometrically complex inequalities (4.5.2) and (4.5.4). We now consider the collections of quasialgebraic maps $\mathbb{B}_j$ to be fixed. For each $B_j \in \mathbb{B}_j$, let $\mathcal{V}_{B_j}$ be a collection of δ-balls in M_j, and given $V_j \in \mathcal{V}_j$, recall the definition of $\Gamma(V_j)$, from Section 3.1. Similarly to the earlier construction, if two $\Gamma(V_j)$ happen to have non-empty intersection, then we may translate them by some qualitatively small amount without affecting the rest of the proof. Now define the following algebraic varieties for each $j \in \{1, ..., m\}$:

$$H_j := \bigcup_{B_j \in \mathbb{B}_j} \bigcup_{V_j \in \mathcal{V}_{B_j}} B_j^{-1}(\Gamma(V_j))$$

Similarly to before, since $\#(\Gamma(V_j)) \simeq \delta^{(1-\alpha)n_j}$ for each $V_j \in \mathcal{V}_{B_j}$, then we see that

$$\begin{aligned} \deg(H_j) = \deg\left(\bigcup_{B_j \in \mathbb{B}_j} \bigcup_{V_{B_j} \in \mathcal{V}_{B_j}} B_j^{-1}(\Gamma(V_{B_j}))\right) &= \sum_{B_j \in \mathbb{B}_j} \sum_{V_{B_j} \in \mathcal{V}_{B_j}} \deg(B_j^{-1}(\Gamma(V_{B_j}))) \\ &\leqslant \deg(B_j) \sum_{B_j \in \mathbb{B}_j} \sum_{V_{B_j} \in \mathcal{V}_{B_j}} \#(\Gamma(V_{B_j})) \\ &\lesssim \deg(B_j)\delta^{(1-\alpha)n_j} \sum_{B_j \in \mathbb{B}_j} \#\mathcal{V}_{B_j} \qquad (4.5.5) \end{aligned}$$

4.5.2 Reduction to a discrete inequality

First of all, we need to show that the following proposition is sufficient to prove Theorem 4.5.1.

Proposition 4.5.6 *Let $\mathbb{B}_j$ be as in Theorem 4.5.1. For all compact $\Omega \Subset U$, there exists a $\delta_\Omega > 0$ such that, for $\delta \in (0, \delta_\Omega)$, for all collections (allowing duplicates) $\mathcal{V}_{B_j}$ of δ-balls*

$V_{B_j} \subset \mathcal{V}_{B_j}$*, the following inequality holds:*

$$\int_\Omega \left(\sum_{(B_1,\ldots,B_m)\in\mathbb{B}_1\times\ldots\times\mathbb{B}_m} \frac{\prod_{j=1}^m \sum_{V_{B_j}\in\mathcal{V}_{B_j}} \chi_{V_{B_j}} \circ B_j(x)}{\mathrm{BL}(\mathbf{dB}(x),\mathbf{p})} \right)^{\frac{1}{m-1}} dx$$
$$\lesssim \deg(M) \prod_{j=1}^m \left(\sum_{B_j\in\mathbb{B}_j} \deg(B_j)\delta^{n_j} \#\mathcal{V}_{B_j} \right)^{\frac{1}{m-1}}, \tag{4.5.7}$$

Proof of Theorem 4.5.1 given Proposition 4.5.6. Recall the definition of $\Omega^{(k)}$, $\mathcal{Q}_j^{(k)}$, and $V(Q_j)$ from Section 4.2.1. For each $B_j \in \mathbb{B}_j$, let $f_{B_j} \in C_0^\infty(M_j)$ be a non-negative function and, for each $Q_j \in \mathcal{Q}_j^{(k)}$, let $c_{B_j}^{(k)}(Q_j) \in \mathbb{N}$ be chosen such that

$$\inf_{z\in Q_j} f_{B_j}(z)k - 1 \leqslant c_{B_j}^{(k)}(Q_j) \leqslant \inf_{z\in Q_j} f_{B_j}(z)k.$$

By construction, we have the pointwise limit $\frac{1}{k}\sum_{Q_j\in\mathcal{Q}_j^{(k)}} c(Q_j)\chi_{Q_j} \underset{k\to\infty}{\nearrow} f_{B_j}$, so in particular, for all $x \in \mathbb{R}^n$, by the monotone convergence theorem,

$$\int_{\Omega_k} \left(k^{-m} \sum_{\mathbf{B}\in\mathbb{B}_1\times\ldots\times\mathbb{B}_m} \frac{\prod_{j=1}^m \sum_{V_j\in\mathcal{V}_j} c_{B_j}^{(k)}(Q_j)\chi_{V_{B_j}} \circ B_j(x)}{\mathrm{BL}(\mathbf{dB}(x),\mathbf{p})^{m-1}} \right)^{\frac{1}{m-1}} dx$$
$$\underset{k\to\infty}{\longrightarrow} \int_U \prod_{j=1}^m \left(\sum_{\mathbf{B}\in\mathbb{B}} \frac{\prod_{j=1}^m f_{B_j}\circ B_j(x)}{\mathrm{BL}(\mathbf{dB}(x),\mathbf{p})^{m-1}} \right)^{\frac{1}{m-1}} dx \tag{4.5.8}$$

On the other hand, provided that $\delta_k < \delta_{\Omega_k}$ for each $k \in \mathbb{N}$, we may apply (4.5.7) to the multiset $\mathcal{V}_{B_j}^{(k)}$ consisting of $c_{B_j}(V_{B_j})$ copies of each ball $V(Q_j)$ for each $Q_j \in \mathcal{Q}_j^{(k)}$.

$$\begin{aligned}
&\int_{\Omega_k} \left(k^{-m} \sum_{\mathbf{B} \in \mathbb{B}_1 \times \ldots \times \mathbb{B}_m} \frac{\prod_{j=1}^m \sum_{Q_j \in \mathcal{Q}_j^{(k)}} c_{B_j}^{(k)}(Q_j) \chi_{Q_j} \circ B_j(x)}{\mathrm{BL}(\mathbf{dB}(x), \mathbf{p})^{m-1}} \right)^{\frac{1}{m-1}} dx \\
&\leqslant \int_{\Omega_k} \left(k^{-m} \sum_{\mathbf{B} \in \mathbb{B}_1 \times \ldots \times \mathbb{B}_m} \frac{\prod_{j=1}^m \sum_{Q_j \in \mathcal{Q}_j^{(k)}} c_{B_j}^{(k)}(Q_j) \chi_{V(Q_j)} \circ B_j(x)}{\mathrm{BL}(\mathbf{dB}(x), \mathbf{p})^{m-1}} \right)^{\frac{1}{m-1}} dx \\
&\lesssim k^{-\frac{m}{m-1}} \prod_{j=1}^m \left(\delta^{n_j} \sum_{B_j \in \mathbb{B}_j} \deg(B_j) \sum_{Q_j \in \mathcal{Q}_j} c_{B_j}^{(k)}(Q_j) \right)^{\frac{1}{m-1}} \\
&\simeq \prod_{j=1}^m \left(\frac{1}{k} \sum_{B_j \in \mathbb{B}_j} \deg(B_j) \int_{M_j} \sum_{Q_j \in \mathcal{Q}_j^{(k)}} c_{B_j}^{(k)}(Q_j) \chi_{Q_j}(x_j) d\mu_j(x_j) \right)^{\frac{1}{m-1}} \\
&\xrightarrow[k \to \infty]{} \prod_{j=1}^m \left(\sum_{B_j \in \mathbb{B}_j} \deg(B_j) \int_{M_j} f_{B_j}(x_j) d\mu_j(x_j) \right)^{\frac{1}{m-1}}. \qquad (4.5.9)
\end{aligned}$$

The last line follows from the monotone convergence theorem, and combining it with (4.5.8) then yields Theorem 4.5.1. □

We shall now reduce Theorem 4.5.3 to its corresponding discrete version in a similar manner.

Proposition 4.5.10 *Let $\mathbb{B}_j$ be as in Theorem 4.5.3. For all compact $\Omega \Subset U$, there exists a $\delta_\Omega > 0$ such that, for $\delta \in (0, \delta_\Omega)$, for all collections (allowing duplicates) $\mathcal{V}_{B_j}$ of δ-balls $V_{B_j} \subset \mathcal{V}_{B_j}$, the following inequality holds:*

$$\int_\Omega \prod_{j=1}^m \left(\sum_{B_j \in \mathbb{B}_j} \sum_{V_{B_j} \in \mathcal{V}_{B_j}} \chi_{V_{B_j}} \circ B_j(x) \right)^{p_j} d\sigma(x) \lesssim_C \deg(M) \prod_{j=1}^m \left(\sum_{B_j \in \mathbb{B}_j} \deg(B_j) \delta^{n_j} \# \mathcal{V}_{B_j} \right)^{p_j}, \qquad (4.5.11)$$

Proof of Theorem 4.5.3 given Proposition 4.5.10. Take $V(Q_j)$, Ω_k, $\mathcal{Q}^{(k)}$, and c_{B_j} as before

$$\inf_{z \in Q_j} f_{B_j}(z)k - 1 \leqslant c^{(k)}_{B_j}(Q_j) \leqslant \inf_{z \in Q_j} f_{B_j}(z)k.$$

By construction, we have the pointwise limit $\frac{1}{k}\sum_{Q_j \in \mathcal{Q}^{(k)}_j} c_{B_j}(Q_j)\chi_{Q_j} \underset{k\to\infty}{\nearrow} f_{B_j}$, so in particular, for all $x \in \mathbb{R}^n$, by the monotone convergence theorem,

$$\int_{\Omega_k} \prod_{j=1}^m \left(\frac{1}{k} \sum_{B_j \in \mathbb{B}_j} \sum_{Q_j \in \mathcal{Q}^{(k)}_j} c^{(k)}_{B_j}(Q_j)\chi_{Q_j} \circ B_j(x) \right)^{p_j} dx \underset{k\to\infty}{\longrightarrow} \int_U \prod_{j=1}^m \left(\sum_{B_j \in \mathbb{B}_j} f_{B_j} \circ B_j(x) \right)^{p_j} dx \tag{4.5.12}$$

On the other hand, provided that $\delta_k < \delta_{\Omega_k}$ for each $k \in \mathbb{N}$, we may apply (4.5.11) to the multiset consisting of $c^{(k)}_{B_j}(Q_j)$ copies of each ball $V(Q_j)$ for each $Q_j \in \mathcal{Q}^{(k)}_j$ to obtain the desired bound.

$$\begin{aligned}
&\int_{\Omega_k} \prod_{j=1}^m \left(\frac{1}{k} \sum_{B_j \in \mathbb{B}_j} \sum_{Q_j \in \mathcal{Q}^{(k)}_j} c^{(k)}_{B_j}(Q_j)\chi_{Q_j} \circ B_j(x) \right)^{p_j} dx \\
&\quad \leqslant \int_{\Omega_k} \prod_{j=1}^m \left(\frac{1}{k} \sum_{B_j \in \mathbb{B}_j} \sum_{Q_j \in \mathcal{Q}^{(k)}_j} c^{(k)}_{B_j}(Q_j)\chi_{V(Q_j)} \circ B_j(x) \right)^{p_j} dx \\
&\quad \lesssim_C \frac{1}{k^P} \prod_{j=1}^m \left(\delta^{n_j} \sum_{B_j \in \mathbb{B}_j} \deg(B_j) \sum_{Q_j \in \mathcal{Q}^{(k)}_j} c^{(k)}_{B_j}(Q_j) \right)^{p_j} \\
&\quad \simeq \prod_{j=1}^m \left(\frac{1}{k} \sum_{B_j \in \mathbb{B}_j} \deg(B_j) \int_{M_j} \sum_{Q_j \in \mathcal{Q}^{(k)}_j} c^{(k)}_{B_j}(Q_j)\chi_{Q_j}(x_j) d\mu_j(x_j) \right)^{p_j} \\
&\quad \underset{k\to\infty}{\longrightarrow} \prod_{j=1}^m \left(\sum_{B_j \in \mathbb{B}_j} \deg(B_j) \int_{M_j} f_{B_j}(x_j) d\mu_j(x_j) \right)^{p_j}.
\end{aligned} \tag{4.5.13}$$

Again, the last line is also a consequence of the monotone convergence theorem. Combining this with (4.5.12) yields Theorem 4.5.3. □

4.5.3 Proof of Propositions 4.5.6 and 4.5.10

Proof of Proposition 4.5.6. This proof follows a very similar argument to the proof of Proposition 4.2.4, where we apply Lemma 4.3.9 followed by Lemma 4.3.14 to dominate the left-hand-side of 4.5.7 by an integral that takes the form of a δ^{β}-scale version the left-hand side of (1.4.15).

$$\begin{aligned}
&\sum_{(B_1,\ldots,B_m)\in\mathbb{B}_1\times\ldots\times\mathbb{B}_m} \frac{\prod_{j=1}^m \sum_{V_j\in\mathcal{V}_j} \chi_{V_j}\circ B_j(x)}{\mathrm{BL}(\mathbf{dB}(x),\mathbf{p})^{m-1}} \\
&\lesssim \delta^{(\alpha-\beta)N} \sum_{(B_1,\ldots,B_m)\in\mathbb{B}_1\times\ldots\times\mathbb{B}_m} \frac{\prod_{j=1}^m |dB_j(x)dB_j(x)^*|^{1/2} \sum_{V_j\in\mathcal{V}_j}\sum_{z_j\in\Gamma(V_j)} \chi_{B_j^{-1}(\{z_j\})+U_{\delta^\beta}(0)}(x)}{\mathrm{BL}(\mathbf{dB}(x),\mathbf{p})^{m-1}} \\
&\lesssim \delta^{(\alpha-\beta)N} \sum_{(B_1,\ldots,B_m)\in\mathbb{B}_1\times\ldots\times\mathbb{B}_m} \frac{\prod_{j=1}^m \sum_{V_j\in\mathcal{V}_j}\sum_{z_j\in\Gamma(V_j)} \chi_{B_j^{-1}(\{z_j\})+U_{\delta^\beta}(0)}(x)}{\mathrm{BL}(\overrightarrow{\ker(dB_j(x))},\mathbf{p})^{m-1}} \\
&\lesssim \delta^{\alpha N-\beta mn} \sum_{(B_1,\ldots,B_m)\in\mathbb{B}_1\times\ldots\times\mathbb{B}_m} \frac{\prod_{j=1}^m \sum_{V_j\in\mathcal{V}_j}\sum_{z_j\in\Gamma(V_j)} |B_j(\{z_j\})+U_{2\delta^\beta}(0)|}{\mathrm{BL}(\overrightarrow{\ker(dB_j(x))},\mathbf{p})^{m-1}} \\
&\lesssim \delta^{\alpha N-\beta mn} \sum_{(B_1,\ldots,B_m)\in\mathbb{B}_1\times\ldots\times\mathbb{B}_m} \frac{\prod_{j=1}^m |H_{B_j}\cap U_{2\delta^\beta}(x)|}{\mathrm{BL}(\overrightarrow{\ker(dB_j(x))},\mathbf{p})^{m-1}} \qquad (4.5.14)
\end{aligned}$$

By the continuity of the Brascamp–Lieb constant, for sufficiently small $\delta > 0$ depending on Ω, for all $x \in \Omega$, we may estimate the summand above by a locally averaged version

$$\frac{\prod_{j=1}^m |H_{B_j}\cap U_{2\delta^\beta}(x)|}{\mathrm{BL}(\overrightarrow{\ker(dB_j(x))},\mathbf{p})^{m-1}} \simeq \int_{H_{B_1}\cap U_{2\delta^\beta}(x)\times\ldots\times H_{B_m}\cap U_{2\delta^\beta}(x)} \frac{d\sigma_1(x_1)\ldots d\sigma_m(x_m)}{\mathrm{BL}(\overrightarrow{T_{x_j}H_{B_j}},\mathbf{p})^{m-1}} \qquad (4.5.15)$$

We then integrate this inequality over Ω to obtain a δ^β-scale version of the left-hand side of (1.4.15)

$$\begin{aligned}
&\lesssim \delta^{\alpha N-\beta mn} \sum_{(B_1,\dots,B_m)\in\mathbb{B}_1\times\dots\times\mathbb{B}_m} \int_{H_{B_1}\cap U_{2\delta^\beta}(x)\times\dots\times H_{B_m}\cap U_{2\delta^\beta}(x)} \frac{d\sigma_1(x_1)\dots d\sigma_m(x_m)}{\mathrm{BL}(\overrightarrow{T_{x_j}H_{B_j}},\mathbf{p})^{m-1}}\\
&= \delta^{\alpha N-\beta mn} \int_{H_1\cap U_{2\delta^\beta}(x)\times\dots\times H_m\cap U_{2\delta^\beta}(x)} \frac{d\sigma_1(x_1)\dots d\sigma_m(x_m)}{\mathrm{BL}(\overrightarrow{T_{x_j}H_j},\mathbf{p})^{m-1}}
\end{aligned}$$

To bring everything up to unit scale, we then rescale the inner and outer integrals, using the fact that dilation is a conformal mapping.

$$\begin{aligned}
&\int_\Omega \left(\sum_{(B_1,\dots,B_m)\in\mathbb{B}_1\times\dots\times\mathbb{B}_m} \frac{\prod_{j=1}^m \sum_{V_j\in\mathcal{V}_j} \chi_{V_j}\circ B_j(x)}{\mathrm{BL}(\mathbf{dB}(x),\mathbf{p})^{m-1}} \right)^{\frac{1}{m-1}} dx\\
&\quad\lesssim \delta^{\alpha n-\beta\frac{mn}{m-1}} \int_\Omega \left(\int_{H_1\cap U_{\delta^\beta}(x)\times\dots\times H_m\cap U_{\delta^\beta}(x)} \frac{d\sigma_1(x_1)\dots d\sigma_m(x_m)}{BL(\overrightarrow{T_{x_j}H_j},\mathbf{p})^{m-1}} \right)^{\frac{1}{m-1}} dx\\
&\quad= \delta^{\alpha n-\beta\frac{mn}{m-1}} \delta^{\sum_{j=1}^m \frac{n-n_j}{m-1}} \int_\Omega \left(\int_{\delta^{-\beta}(H_1\cap U_{\delta^\beta}(x)\times\dots\times H_m\cap U_{\delta^\beta}(x))} \frac{d\sigma_1(x_1)\dots d\sigma_m(x_m)}{BL(\overrightarrow{T_{x_j}(\delta^{-\beta}H_j)},\mathbf{p})^{m-1}} \right)^{\frac{1}{m-1}} dx\\
&\quad= \delta^{\alpha n-\beta n} \int_\Omega \left(\int_{(\delta^{-\beta}H_1)\cap U_1(\delta^{-\beta}x)\times\dots\times(\delta^{-\beta}H_m)\cap U_1(\delta^{-\beta}x))} \frac{d\sigma_1(x_1)\dots d\sigma_m(x_m)}{BL(\overrightarrow{T_{x_j}(\delta^{-\beta}H_j)},\mathbf{p})^{m-1}} \right)^{\frac{1}{m-1}} dx\\
&\quad= \delta^{\alpha n} \int_{\frac{\delta^{-\beta}}{2}\Omega} \left(\int_{(\delta^{-\beta}H_1)\cap U_1(x)\times\dots\times(\delta^{-\beta}H_1)\cap U_1(x)} \frac{d\sigma_1(x_1)\dots d\sigma_m(x_m)}{BL(\overrightarrow{T_{x_j}(\delta^{-\beta}H_j)},\mathbf{p})^{m-1}} \right)^{\frac{1}{m-1}} dx
\end{aligned}$$

Proposition 1.2 of [6] gives us that

$$BL(\overrightarrow{T_{x_j}\left(\delta^{-\beta}/2\right)H_j},\mathbf{p}) = |\bigwedge_{j=1}^m T_{x_j}(\delta^{-\beta}H_j)|^{-\frac{1}{m-1}}.$$

Hence, we may then apply (1.4.15) followed by (4.5.5) to obtain the desired bound.

$$\delta^{\alpha n}\int_{\mathbb{R}^n}\left(\int_{(\delta^{-\beta}H_1)\cap U_1(x)\times\ldots\times(\delta^{-\beta}H_1)\cap U_1(x)}|\bigwedge_{j=1}^{m}T_{x_j}(\delta^{-\beta}H_j)|d\sigma_1(x_1)\ldots d\sigma_m(x_m)\right)^{\frac{1}{m-1}}dx$$
$$\lesssim \delta^{\alpha n}\prod_{j=1}^{m}\deg(H_j)^{\frac{1}{m-1}}\lesssim\prod_{j=1}^{m}\left(\sum_{B_j\in\mathbb{B}_j}\deg(B_j)\delta^{n_j}\#\mathcal{V}_{B_j}\right)^{\frac{1}{m-1}}\qquad\square$$

We now turn our attention to Proposition 4.5.10, which again follows a similar proof strategy to Proposition 4.2.4.

Proof of Proposition 4.5.10. Again, we apply Lemma 4.3.9 and Lemma 4.3.14, each time absorbing the derivative-dependent contributions into the implicit constant.

$$\prod_{j=1}^{m}\left(\sum_{B_j\in\mathbb{B}_j}\sum_{V_{B_j}\in\mathcal{V}_{B_j}}\chi_{V_{B_j}}\circ B_j(x)\right)^{p_j}\lesssim_C\delta^{(\alpha-\beta)n}\prod_{j=1}^{m}\left(\sum_{B_j\in\mathbb{B}_j}\sum_{V_{B_j}\in\mathcal{V}_{B_j}}\sum_{z_j\in\Gamma(V_{B_j})}\chi_{B_j^{-1}(\{z_j\})+U_{\delta^\beta}(0)}(x)\right)^{p_j}$$
$$\lesssim\delta^{(\alpha-\beta)n}\delta^{-\beta\sum_{j=1}^{m}p_j(n-n_j)}\prod_{j=1}^{m}\left(\sum_{B_j\in\mathbb{B}_j}\sum_{V_{B_j}\in\mathcal{V}_{B_j}}\sum_{z_j\in\Gamma(V_{B_j})}|B_j(\{z_j\})\cap U_{2\delta^\beta}(x)|\right)^{p_j}$$
$$=\delta^{(\alpha-\beta P)n}\prod_{j=1}^{m}|H_j\cap U_{2\delta^\beta}(x)|^{p_j}$$

We now may rescale these volumes by a factor of $\delta^{-\beta}/2$ to bring them up to unit scale.

$$=\delta^{(\alpha-\beta P)n}\delta^{\beta\sum_{j=1}^{m}p_j(n-n_j)}\prod_{j=1}^{m}\left|\left(\frac{\delta^{-\beta}}{2}H_j\right)\cap U_1\left(\frac{\delta^{-\beta}}{2}x\right)\right|^{p_j}$$
$$=\delta^{(\alpha-\beta)n}\prod_{j=1}^{m}\left|\left(\frac{\delta^{-\beta}}{2}H_j\right)\cap U_1\left(\frac{\delta^{-\beta}}{2}x\right)\right|^{p_j}$$

Integrating in x, we may then apply (1.4.20) and (4.2.8) to obtain the desired bound.

$$\begin{aligned}
\int_{\mathbb{R}^n} \prod_{j=1}^{m} \left(\sum_{B_j \in \mathbb{B}_j} \sum_{V_{B_j} \in \mathcal{V}_{B_j}} \chi_{V_j} \circ B_j(x) \right)^{p_j} &\lesssim_C \delta^{(\alpha-\beta)n} \int_{\mathbb{R}^n} \prod_{j=1}^{m} \left| \left(\frac{\delta^{-\beta}}{2} H_j \right) \cap U_1 \left(\frac{\delta^{-\beta}}{2} x \right) \right|^{p_j} dx \\
&= \delta^{\alpha n} \int_{\mathbb{R}^n} \prod_{j=1}^{m} \left| \left(\frac{\delta^{-\beta}}{2} H_j \right) \cap U_1(x) \right|^{p_j} dx \\
&\lesssim_C \delta^{\alpha n} \prod_{j=1}^{m} \deg(H_j) \\
&\lesssim \prod_{j=1}^{m} \left(\sum_{B_j \in \mathbb{B}_j} \deg(B_j) \delta^{n_j} \# \mathcal{V}_{B_j} \right)^{p_j}
\end{aligned}$$

□

Chapter 5

Global L^2-bounds for Multilinear Oscillatory Integrals

In this chapter, we will prove a global generalisation of an L^2 multilinear integral estimate due to Bennett, Carbery, Tao [15], stated earlier as Theorem 1.7.1. The proof methodology draws heavily from their induction-on-scales approach, dealing with the fine scale oscillation by induction and organising the resulting cube-wise bounds using a more general version of the multilinear Kakeya inequality, which we now define.

Definition 5.0.1 (Nonlinear Kakeya–Brascamp–Lieb Inequality) *Let M be an n-dimensional manifold and, for each $j \in \{1, ..., m\}$, let M_j be an n_j-dimensional Riemannian manifold. For each $j \in \{1, ..., m\}$, let $\mathbb{B}_j$ be a finite collection of submersions $B_j : M \to M_j$, and for each B_j, let $f_{B_j} \in L^1(M_j)$, then the associated nonlinear Kakeya–Brascamp–Lieb inequality is as follows:*

$$\int_M \prod_{j=1}^m \left(\sum_{B_j \in \mathbb{B}_j} f_{B_j} \circ B_j(\xi) \right)^{p_j} d\xi \leqslant \mathrm{NKBL}(\mathbb{B}, \mathbf{p}) \prod_{j=1}^m \left(\sum_{B_j \in \mathbb{B}_j} \int_{M_j} f_{B_j} \right)^{p_j}, \tag{5.0.2}$$

where $\mathbb{B} := (\mathbb{B}_j)_{j=1}^m$, and $\mathrm{NKBL}(\mathbb{B}, \mathbf{p}) \in (0, \infty]$ is the optimal constant.

The reader should note that throughout this chapter, like the previous one, the notation $A \lesssim B$ will denote that $A \leqslant CB$, where C depends only on the underlying dimensions and exponents, and any additional dependence will denoted by a subscript.

5.1 Statement of Results

Our theorem states that, provided the operators $S_1^\lambda, ..., S_m^\lambda$ are suitably 'transverse' in the sense that the geometry of their wavepackets may be organised by a nonlinear Kakeya–Brascamp–Lieb inequality, then we can obtain multilinear estimates, although admittedly with an ε-loss in the exponent of λ and requiring some polynomial decay in the amplitude. Before we state our theorem, we shall clarify that, given some $\Omega \subset \mathbb{R}^n$ the differential operator $\nabla^k : C^\infty(\Omega) \to C^\infty(\Omega; (\mathbb{R}^n \otimes ... \otimes \mathbb{R}^n)^*)$ sends a function f to the $(k,0)$-tensor that maps a k-tuple $(v_1, ..., v_k)$ to the k^{th} order directional derivative $\partial_{v_1}...\partial_{v_k} f$. Given a $(k,0)$-tensor T, we denote its operator norm by $\sup_{|v_i|=1, 1 \leqslant i \leqslant k} |T(v_1, ..., v_k)|$.

Theorem 5.1.1 (Multilinear L^2 Oscillatory Integral Estimate) *Let $\varepsilon > 0$. For each $j \in \{1, ..., m\}$, let $\phi_j : \mathbb{R}^{n_j} \times \mathbb{R}^n \to \mathbb{R}$ be a smooth phase and $\psi_j : \mathbb{R}^{n_j} \times \mathbb{R}^n \to \mathbb{R}$ such that $\operatorname{supp}(\psi_j) \subset K_j \times \mathbb{R}^n$ for some compact $K_j \subset \mathbb{R}^{n_j}$ with $|K_j| \lesssim 1$. Assume that the exponents $p_1, ..., p_m \in (0,1]$ satisfy the scaling condition $\sum_{j=1}^m p_j n_j = n$. We consider the following one-parameter family of oscillatory integral operators:*

$$S_j^\lambda f(\xi) := \int_{\mathbb{R}^{n_j}} e^{i\lambda\phi_j(x,\xi)} \psi_j(x,\xi) f(x) dx, \quad \lambda > 1 \tag{5.1.2}$$

Let $N_\varepsilon := \lceil n^2 \varepsilon^{-1} \min_{j \in \{1,...,m\}} (p_j)^{-1} \rceil + 1 \simeq_\varepsilon 1$, and suppose that the following conditions hold for some $a > 0$:

1. *(Regularity) For all $(x, \xi) \in K_j \times \mathbb{R}^n$ and $1 \leqslant k \leqslant N_\varepsilon + \lceil n_j/2 \rceil + 1$:*

   - $|\nabla_x^k \phi_j(x,\xi)| \lesssim \langle \xi \rangle^a$.

- $|\nabla_x \nabla_\xi \phi_j(x,\xi)| \lesssim 1$ *and* $|\nabla_x^k \nabla_\xi \phi_j(x,\xi)| \lesssim \langle \xi \rangle^a$ *for* $k \geqslant 2$.
- $|\nabla_x^k \psi_j(x,\xi)| \lesssim \langle \xi \rangle^{-(N_\varepsilon + 2ak)}$.

Here $\langle \xi \rangle := (1+|\xi|^2)^{1/2}$ *denotes the Japanese bracket.*

2. *(Transversality) Given a finite subset* $A_j \subset \mathbb{R}^{n_j}$ *for each* $j \in \{1, ..., m\}$, *set*

$$\mathbb{B}_j = \{B_j : \mathbb{R}^n \to \mathbb{R}^{n_j} \mid B_j(\xi) := \nabla_x \phi_j(x,\xi), x \in A_j\}$$

and $\mathbb{B} := (\mathbb{B}_j)_{j=1}^m$, *then* $\mathrm{NKBL}(\mathbb{B}, \mathbf{p}) \lesssim 1$.

Then, the following inequality holds for all $f_j \in L^2(\mathbb{R}^{n_j})$*:*

$$\int_{\mathbb{R}^n} \prod_{j=1}^m |S_j^\lambda f_j|^{2p_j} \lesssim_\varepsilon \lambda^{\varepsilon - n} \prod_{j=1}^m \|f_j\|_{L^2}^{2p_j}, \tag{5.1.3}$$

Observe that the exponents here coincide with those of the endpoint case of Theorem 1.7.1, and therefore this Theorem is indeed a generalisation of Theorem 1.7.1, where the condition that ψ must be compactly supported in ξ has been relaxed to only requiring polynomial decay. Essentially, Theorem 5.1.1 states that every nonlinear Kakeya–Brascamp–Lieb inequality implies a related class of multilinear oscillatory integral inequalities; for example, Theorem 4.5.3 implies that bounds of the form (5.1.3) hold if the phase admits a quasialgebraic structure.

Corollary 5.1.4 *Fix some* $\varepsilon > 0$. *For each* $j \in \{1, ..., m\}$, *suppose that* ϕ_j *is a phase and that* ψ_j *is an amplitude function satisfying the regularity conditions of Theorem 5.1.1, with the additional properties that* $\mathrm{BL}((\nabla_\xi \nabla_x \phi(x_j, \xi))_{j=1}^m, \mathbf{p}) \simeq 1$ *for all* $(x_j, \xi) \in \mathrm{supp}(\psi_j)$, *and that the mapping* $\xi \mapsto \nabla_x \psi_j(x,\xi)$ *is quasialgebraic of bounded degree for all* $x \in \mathbb{R}^{n_j}$, *then,* (5.1.3) *holds for all* $f_j \in L^2(\mathbb{R}^{n_j})$.

Proof. For each $j \in \{1, ..., m\}$, let $A_j \subset \mathbb{R}^{n_j}$ be a finite subset, and define $\mathbb{B}_{\mathbf{A}}$ as in Theorem 5.1.1. By the uniform boundedness assumptions of the corollary, Theorem 4.5.3 implies that $\mathrm{NKBL}(\mathbb{B}_{\mathbf{A}}, \mathbf{p}) \lesssim 1$, therefore the transversality condition holds. Since the regularity condition holds by assumption, (5.1.3) follows by the Theorem 5.1.1. □

The proof of Theorem 5.1.1 is a refined version of the proof of Proposition 6.9 in [15], and where it deviates from [15] is in the much more careful treatment of the tail contributions. In [15], they do not pose any issues since they have negligible mass, so Hölder's inequality will suffice, however in the global setting that we consider, due to the noncompact support of ψ_j in the second variable, the tails are possibly not even integrable, so we need to use something more sophisticated to bound them. First of all, rather than bounding the pointwise contributions from each tail of each wavepacket separately, we bound the tail contributions simultaneously, exploiting the cancellation between them in order to obtain some additional decay. Secondly, we have to use the transversality of the supports of the tails in order to obtain even a finite bound, and the only way of doing that is by appealing to a Kakeya–Brascamp–Lieb inequality, as we do to organise the main contributions. The multilinearity of the problem however means that the tails cannot in general be 'disentangled' from the main contributions, so we have to change the inductive hypothesis itself to a hybridised form that generalises both (5.0.2) and (5.1.3) in order to deal with the terms that are mixtures of both tails and main contributions.

5.2 Stationary Phase

First of all, we may assume that $\lambda > C$, for some large $C \simeq 1$, since we may then derive the case when $\lambda \in (1, C)$ by considering the phase $C^{-1}\phi_j$. Using a similar construction to the one used in Section 1.6, for each $j \in \{1, ..., m\}$, let $\mathcal{Q}_j$ be a boundedly overlapping open cover of the closure of $\bigcup_{\xi \in \mathbb{R}^n} \mathrm{supp}(\psi_j(\cdot, \xi))$ by cubes with centre x_Q of diameter less than $\lambda^{-1/2}$, and let $\{\chi_{Q_j}\}$ be a corresponding smooth partition of unity such that $|\nabla^k \chi_{Q_j}| \lesssim \lambda^{k/2}$ for all $k \in \mathbb{N}$. Taking the Fourier series of each $f_j \chi_{Q_j}$ and summing them together, we then obtain a wavepacket decomposition for f_j:

$$f_j := \sum_{Q_j \in \mathcal{Q}_j} \sum_{\omega_j \in \lambda^{-1/2}\mathbb{Z}^{n_j}} a_{Q_j,\omega_j} e_{Q_j,\omega_j}$$

where $a_{Q_j,\omega_j} \in \mathbb{C}$ and $e_{Q_j,\omega_j} := e^{-2\pi i \lambda x \cdot \omega_j} \chi_{Q_j}$. We now want to show that the essential support of each $S_j^\lambda e_{Q_j,\omega_j}$ lies in a certain tube, namely, fixing some small $\delta > 0$, and letting x_{Q_j} denote the centre of Q_j,

$$T^\lambda_{Q_j,\omega_j} := \{\xi \in \mathbb{R}^n : |\nabla_x \phi_j(x_{Q_j}, \xi) - 2\pi\omega_j| \leqslant \lambda^{\delta - 1/2} \textit{ and } (x_{Q_j}, \xi) \in \operatorname{supp}(\psi_j)\}.$$

Let $\mathcal{U}$ be a boundedly overlapping cover of $\mathbb{R}^n$ via $\lambda^{-1/2}$ balls. Given $U \in \mathcal{U}$, let $W_j^U := \{(Q_j, \omega_j) : T^\lambda_{Q_j,\omega_j} \cap U \neq \varnothing\}$, let $f_j^U := \sum_{(Q_j,\omega_j) \in W_j^U} a_{Q_j,\omega_j} e_{Q_j,\omega_j}$ and $f_j^{U^c} := f_j - f_j^U$. We view the former term as the dominant term, and it is the content of the following proposition that we may treat the latter as a tail term for $\xi \in U$.

Proposition 5.2.1 *For each $Q_j \in \mathcal{Q}_j$, let $B_{Q_j}(\xi) := \nabla_x \phi_j(x_{Q_j}, \xi)$, and for given $\omega_j \in \lambda^{-1/2}\mathbb{Z}^{n_j}$ and $\lambda > 1$, define the following continuous function:*

$$\rho^\lambda_{\omega_j} : \mathbb{R}^{n_j} \to \mathbb{R}$$

$$\rho^\lambda_{\omega_j}(x) := \begin{cases} \lambda^{\delta - 1/2} |x - 2\pi\omega_j|^{-1} & \textit{if } |x - 2\pi\omega_j| \geqslant \lambda^{\delta - 1/2} \\ 1 & \textit{otherwise} \end{cases}$$

For all $\xi \in U \in \mathcal{U}$, we have the following pointwise bound for all $N \leqslant N_\varepsilon$:

$$|S_j^\lambda f_j^{U^c}(\xi)| \lesssim_N \lambda^{-\delta N - n_j/4} \left(\sum_{Q_j,\omega_j} |a_{Q_j,\omega_j}|^2 \rho^\lambda_{\omega_j} \circ B_{Q_j}(\xi)^{2N} \right)^{1/2} \tag{5.2.2}$$

We shall need to use a vector calculus lemma in order to prove this proposition, a proof of which is given in the appendix. Given a multi-index $\alpha \in \mathbb{N}^d$ we write $|\alpha| := \sum_{i=1}^d \alpha_i$.

Lemma 5.2.3 *Let $d \in \mathbb{N}$ and $W : \mathbb{R}^d\backslash\{0\} \to \mathbb{R}^d$ be the vector field with mapping $W(z) := |z|^{-2}z$, then for all multi-indices $\alpha \in \mathbb{N}^d$, each component of $|z|^{2(|\alpha|+1)}\partial^\alpha W(z)$ is a homogeneous polynomial of degree $|\alpha|+1$. It follows that for all $k \in \mathbb{N}$ and $z \in \mathbb{R}^m\backslash\{0\}$,*

$$|\nabla^k W(z)| \lesssim |z|^{-(k+1)}. \tag{5.2.4}$$

Proof of Proposition 5.2.1. The proof of this claim is a non-standard stationary phase argument, where bound the whole tail contribution simultaneously via integration by parts, exploiting the cancellation between them. First of all, given $\xi \in \mathbb{R}^n$ and ω_j, define $K_{\xi,\omega_j} := K_j\backslash\{x \in K_j : \nabla_x\phi_j(x,\xi) = 2\pi\omega_j\}$. We claim that for any Q_j such that $(Q_j,\omega_j) \notin W_j^U$, $Q_j \subset K_{\xi,\omega_j}$. To see this, we show that the following quantity is bounded from below in $x \in Q_j$:

$$\log\left(\frac{|\nabla_x\phi_j(x,\xi) - 2\pi\omega_j|^2}{|\nabla_x\phi_j(x_{Q_j},\xi) - 2\pi\omega_j|^2}\right)$$

To do this we simply apply Taylor's theorem and use the regularity condition of Theorem 5.1.1 to bound the higher-order derivatives that arise.

$$\begin{aligned}\left|\log\left(\frac{|\nabla_x\phi_j(x,\xi) - 2\pi\omega_j|^2}{|\nabla_x\phi_j(x_{Q_j},\xi) - 2\pi\omega_j|^2}\right)\right| &= |\log(|\nabla_x\phi_j(x,\xi) - 2\pi\omega_j|^2) - \log(|\nabla_x\phi_j(x_{Q_j},\xi) - 2\pi\omega_j|^2)| \\ &\lesssim \frac{|\nabla_x^2\phi_j(x,\xi)||x - x_{Q_j}|}{|\nabla_x\phi_j(x,\xi) - 2\pi\omega_j|} \lesssim \langle\xi\rangle^a\lambda^{-\delta}\end{aligned}$$

Now, consider the following linear first-order differential operator:

$$\begin{aligned}D_{\xi,\omega_j} &: C^\infty(K_{\xi,\omega_j}) \to C^\infty(K_{\xi,\omega_j}) \\ D_{\xi,\omega_j}g(x) &:= \frac{-i\nabla\phi_j(x,\xi) - 2\pi\omega_j}{\lambda|\nabla\phi_j(x,\xi) - 2\pi\omega_j|^2}\cdot\nabla g(x)\end{aligned}$$

Differentiating $e^{i\lambda(\phi_j(x,\xi)-2\pi x\cdot\omega_j)}$ we see that this phase function is a fixed point for D_{ξ,ω_j}, hence it is also a fixed point for D^N_{ξ,ω_j}, for all $N \leqslant N_\varepsilon$. We may then substitute this fixed point equation into the definition of $S^\lambda_j f^{U^c}_j$ and take the L^2-adjoint of D^N_{ξ,ω_j} to move the derivatives onto $\psi_{Q_j}(x,\xi) := \psi_j(x,\xi)\chi_{Q_j}(x)$.

$$\begin{aligned}
|S^\lambda_j f^{U^c}_j(\xi)| &:= \left|\int_{\mathbb{R}^{n_j}} e^{i\lambda\phi_j(x,\xi)}\psi(x,\xi) f^{U^c}_j(x)dx\right| \\
&= \left|\sum_{(Q_j,\omega_j)\notin W^U_j} a_{Q_j,\omega_j}\int_{\mathbb{R}^{n_j}} e^{i\lambda(\phi_j(x,\xi)-2\pi x\cdot\omega_j)}\psi_{Q_j}(x,\xi)dx\right| \\
&\leqslant \sum_{Q_j}\left|\sum_{\omega_j:(Q_j,\omega_j)\notin W^U_j} a_{Q_j,\omega_j}\int_{Q_j} e^{i\lambda(\phi_j(x,\xi)-2\pi x\cdot\omega_j)}\psi_{Q_j}(x,\xi)dx\right| \\
&= \sum_{Q_j}\left|\sum_{\omega_j:(Q_j,\omega_j)\notin W^U_j} a_{Q_j,\omega_j}\int_{Q_j\cap K_{\xi,\omega_j}} D^N_{\xi,\omega_j} e^{i\lambda(\phi_j(x,\xi)-2\pi x\cdot\omega_j)}\psi_{Q_j}(x,\xi)dx\right| \\
&= \sum_{Q_j}\left|\sum_{\omega_j:(Q_j,\omega_j)\notin W^U_j} a_{Q_j,\omega_j}\int_{Q_j} e^{i\lambda(\phi_j(x,\xi)-2\pi x\cdot\omega_j)}(D^*_{\xi,\omega_j})^N\psi_{Q_j}(x,\xi)dx\right| \qquad (5.2.5)
\end{aligned}$$

Here, D^*_{ξ,ω_j} refers to the formal L^2 adjoint of D_{ξ,ω_j}, and integration by parts reveals that $D^*_{\xi,\omega_j}g(x) = -i\lambda^{-1}\nabla_x\cdot(v_{\omega_j}(x,\xi)g(x))$, where v_{ω_j} is defined as the following vector field,

$$v_{\omega_j}(x,\xi) := \frac{\nabla_x\phi_j(x,\xi)-\omega_j}{|\nabla_x\phi_j(x,\xi)-\omega_j|^2}.$$

Let $\psi^{(N)}_{Q_j,\omega_j}(x,\xi) := (D^*_{\xi,\omega_j})^N\psi_{Q_j}(x,\xi)$. We then apply the Cauchy-Schwarz inequality and rescale to bring the integral up to unit scale.

$$\begin{aligned}
|S_j^\lambda f_j^{U^c}(\xi)| &:= \sum_{Q_j} \left| \int_{Q_j} e^{i\lambda\phi_j(x,\xi)} \left(\sum_{\omega_j:(Q_j,\omega_j)\notin W_j^U} e^{-2\pi i\lambda x\cdot\omega_j} a_{Q_j,\omega_j} \psi^{(N)}_{Q_j,\omega_j}(x,\xi) \right) dx \right| \\
&\leqslant \lambda^{-n_j/4} \sum_{Q_j} \left\| \sum_{\omega_j:(Q_j,\omega_j)\notin W_j^U} e^{-2\pi i\lambda x\cdot\omega_j} a_{Q_j,\omega_j} \psi^{(N)}_{Q_j,\omega_j}(x,\xi) \right\|_{L^2_x(Q_j)} \\
&= \lambda^{-n_j/2} \sum_{Q_j} \left\| \sum_{\omega_j:(Q_j,\omega_j)\notin W_j^U} e^{-2\pi i\lambda^{1/2} x\cdot\omega_j} a_{Q_j,\omega_j} \psi^{(N)}_{Q_j,\omega_j}(\lambda^{-1/2}x,\xi) \right\|_{L^2_x(\lambda^{1/2}Q_j)} \\
&\leqslant \lambda^{-n_j/2} \sum_{Q_j} \left\| \sum_{\omega_j:(Q_j,\omega_j)\notin W_j^U} e^{-2\pi i\lambda^{1/2} x\cdot\omega_j} a_{Q_j,\omega_j} \psi^{(N)}_{Q_j,\omega_j}(\lambda^{-1/2}y,\xi) \right\|_{L^2_x L^\infty_y(\lambda^{-1/2}Q_j)}
\end{aligned}$$

By the Sobolev embedding of $W^{k,2}(\lambda^{1/2}Q_j)$ in $L^\infty(\lambda^{1/2}Q_j)$, where $k = \lceil n/2 \rceil + 1$, we then have that

$$\begin{aligned}
|S_j^\lambda f_j^{U^c}(\xi)| &\lesssim \lambda^{-n_j/2} \sum_{Q_j} \left\| \sum_{\omega_j:(Q_j,\omega_j)\notin W_j^U} e^{-2\pi i\lambda x\cdot\omega_j} a_{Q_j,\omega_j} \psi^{(N)}_{Q_j,\omega_j}(\lambda^{-1/2}y,\xi) \right\|_{L^2_x W^{k,2}_y(\lambda^{1/2}Q_j)} \\
&\simeq \lambda^{-k/2-n_j/2} \sum_{Q_j} \left(\int_{\lambda^{1/2}Q_j} \int_{\lambda^{1/2}Q_j} \left| \sum_{\omega_j:(Q_j,\omega_j)\notin W_j^U} e^{-2\pi i\lambda^{1/2} x\cdot\omega_j} a_{Q_j,\omega_j} \nabla^k \psi^{(N)}_{Q_j,\omega_j}(\lambda^{-1/2}y,\xi) \right|^2 dydx \right)^{1/2} \\
&= \lambda^{-k/2-n_j/2} \sum_{Q_j} \left(\int_{\lambda^{1/2}Q_j} \int_{\lambda^{1/2}Q_j} \left| \sum_{\tilde\omega_j\in\mathbb{Z}^{n_j}} e^{-2\pi i x\cdot\tilde\omega_j} \tilde a_{Q_j,\tilde\omega_j} \nabla^k \psi^{(N)}_{Q_j,\tilde\omega_j}(\lambda^{-1/2}y,\xi) \right|^2 dxdy \right)^{1/2}
\end{aligned}$$

where $\tilde a_{Q_j,\tilde\omega_j} := a_{Q_j,\lambda^{-1/2}\tilde\omega_j}$ for $(Q_j,\lambda^{-1/2}\tilde\omega_j) \notin W_j^U$ and vanishes otherwise. We may then apply Plancharel's theorem to the inner integral to obtain

$$\begin{aligned}|S_j^\lambda f_j^{U^c}(\xi)| &\lesssim \lambda^{-k/2-n_j/2}\sum_{Q_j}\left(\int_{\lambda^{1/2}Q_j}\sum_{\tilde{\omega}_j\in\mathbb{Z}^{n_j}}|\tilde{a}_{Q_j,\tilde{\omega}_j}|^2|\nabla_y^k\psi^{(N)}_{Q_j,\tilde{\omega}_j}(\lambda^{-1/2}y,\xi)|^2dy\right)^{1/2}\\ &= \lambda^{-k/2-n_j/4}\left(\sum_{(Q_j,\omega)\notin W_j^U}|a_{Q_j,\omega_j}|^2\|\nabla_y^k\psi^{(N)}_{Q_j,\omega_j}(\lambda^{-1/2}y,\xi)\|^2_{L^2_y(\lambda^{1/2}Q_j)}\right)^{1/2}\\ &= \lambda^{-k/2}\left(\sum_{(Q_j,\omega)\notin W_j^U}|a_{Q_j,\omega_j}|^2\|\nabla_y^k\psi^{(N)}_{Q_j,\omega_j}(y,\xi)\|^2_{L^2_y(Q_j)}\right)^{1/2}\end{aligned}$$

To prove the proposition it is therefore sufficient to show that

$$|\nabla_x^k\psi^{(N)}_{Q_j,\omega_j}(x,\xi)| \lesssim \lambda^{k/2-\delta N}\rho^\lambda_{\omega_j}\circ B_{Q_j}(\xi)^N\langle\xi\rangle^{-(N_\varepsilon-N+2ak)}$$

for all $(x,\xi)\in Q_j\times U$. We shall proceed by induction on N, and prove the claim for general $0\leqslant k\leqslant N_\varepsilon-N+\lceil n_j/2\rceil+1$. For the base case $N=0$, the claim holds for all $0\leqslant k\leqslant N_\varepsilon+\lceil n_j/2\rceil+1$ by the regularity hypothesis of Theorem 5.1.1 and the definition of $\rho^\lambda_{\omega_j}\circ B_{Q_j}$, so now assume for inductive hypothesis that the claim holds for all $0\leqslant k\leqslant N_\varepsilon-N+\lceil n_j/2\rceil+1$ for some $N\leqslant N_\varepsilon$. Before proceeding with the proof of the inductive step, we shall need to briefly define some notation related to multi-indices. Given multi-indices $\alpha,\beta\in\mathbb{N}^{n_j}$, we say that $\beta\leqslant\alpha$ if for each $i\in\{1,...,n_j\}$, $\beta_i\leqslant\alpha_i$, and we let $\binom{\alpha}{\beta}:=\prod_{i=1}^{n_j}\binom{\alpha_i}{\beta_i}$. Let $\alpha\in\mathbb{N}^{n_j}$ be a multi-index such that $|\alpha|=k$. By definition of $\psi^{(N)}_{Q_j,\omega_j}$ we have the recurrence relation

$$\begin{aligned}\partial_x^\alpha\psi^{(N+1)}_{Q_j,\omega_j}(x,\xi) &= \partial_x^\alpha D^*_{\xi,\omega_j}\psi^{(N)}_{Q_j,\omega_j}(x,\xi)\\ &= -i\lambda^{-1}\partial_x^\alpha(\nabla_x\cdot(v_{\omega_j}(x,\xi)\psi^{(N)}_{Q_j,\omega_j}(x,\xi))).\end{aligned}$$

By the product rule, for any C^{k+1} functions $v : \mathbb{R}^n \to \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$,

$$\partial^\alpha \nabla \cdot (vf) = \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \left((\partial^\beta \nabla \cdot v) \partial^{\alpha-\beta} f + \partial^\beta v \cdot \partial^{\alpha-\beta} \nabla f \right),$$

hence, by the triangle inequality we have that

$$\begin{aligned}
&|\partial_x^\alpha \psi_{Q_j,\omega_j}^{(N+1)}(x,\xi)| \\
&\quad \leqslant \lambda^{-1} \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \left(|\partial_x^\beta \nabla_x \cdot v_{\omega_j}(x,\xi) \partial_x^{\alpha-\beta} \psi_{Q_j,\omega_j}^{(N)}(x,\xi)| + |\partial_x^\beta v_{\omega_j}(x,\xi) \cdot \partial_x^{\alpha-\beta} \nabla_x \psi_{Q_j,\omega_j}^{(N)}(x,\xi)| \right) \\
&\quad \lesssim \lambda^{-1} \sum_{l=0}^{k} |\nabla_x^l v_{\omega_j}(x,\xi)| |\nabla_x^{k-l} \psi_{Q_j,\omega_j}^{(N)}(x,\xi)| + |\nabla_x^l v_{\omega_j}(x,\xi)| |\nabla_x^{k-l+1} \psi_{Q_j,\omega_j}^{(N)}(x,\xi)| \\
&\quad \leqslant \lambda^{-1-\delta N} \rho_{\omega_j}^\lambda \circ B_{Q_j}(\xi)^N \sum_{l=0}^{k} |\nabla_x^{l+1} v_{\omega_j}(x,\xi)| \lambda^{(k-l)/2} \langle \xi \rangle^{-(N_\varepsilon - N + 2a(k-l))} \\
&\qquad\qquad + |\nabla_x^l v_{\omega_j}(x,\xi)| \lambda^{(k-l+1)/2} \langle \xi \rangle^{-(N_\varepsilon - N + 2a(k-l+1))}
\end{aligned}$$

where above we used the inductive hypothesis. The claim therefore holds provided that $|\nabla^l v_{\omega_j}(x,\xi)| \lesssim \lambda^{(l+1)/2-\delta} \rho_{\omega_j}^\lambda \circ B_j^x(\xi) \langle \xi \rangle^{2l}$ for all $l \leqslant N_\varepsilon$. The case $l = 0$ holds vacuously, so without loss of generality assume that $l \geqslant 1$. Let $W : \mathbb{R}^{n_j} \setminus \{0\} \to \mathbb{R}^{n_j}$ denote the vector field with the mapping $W(z) := \frac{z}{|z|^2}$ and, suppressing the implicit dependence on ξ and ω_j, let $\Phi : Q_j \to \mathbb{R}^{n_j}$ denote the map $\Phi(z) := \nabla_x \phi_j(z,\xi) - 2\pi\omega_j$. Then, $v_{\omega_j} = W \circ \Phi$, so by the multivariate version of Faa di Bruno's theorem [33], we know that for a multi-index α, the derivatives of v_{ω_j} take the form

$$\partial_x^\alpha v_{\omega_j}(x,\xi) = \sum_{\beta \leqslant \alpha, \beta \neq 0} \partial^\beta W \circ \Phi(x) y_{\alpha,\beta}(x)$$

where $y_{\alpha,\beta}$ is a sum of $\mathcal{O}(1)$ terms of the form $\prod_{i=1}^{|\beta|} \partial^{\gamma_i} \Phi_{k_i}$, where $1 \leqslant \gamma_i \leqslant \beta$ for each

$i \in \{1, ..., |\beta|\}$. We may then use this formula to bound $|\nabla^l v_{\omega_j}(x, \xi)|$.

$$|\nabla^l v_{\omega_j}(x, \xi)| \lesssim \max_{1 \leqslant r \leqslant l} |\nabla^r W \circ \Phi_j(x)| \|\nabla_x \Phi(x, \xi)\|^r_{C^{r-1}_x}$$

By Lemma 5.2.3, we know that for all $r \in \mathbb{N}$ and $z \in \mathbb{R}^{n_j} \backslash \{0\}$, $|\nabla^r W(z)| \lesssim |z|^{-(r+1)}$, hence

$$\begin{aligned} |\nabla^l v_{\omega_j}(x, \xi)| &\lesssim \max_{1 \leqslant r \leqslant l} |\Phi(x)|^{-(r+1)} \|\nabla_x \Phi(x, \xi)\|^r_{C^{r-1}_x} \\ &\leqslant \max_{1 \leqslant r \leqslant l} |\Phi(x)|^{-(r+1)} \langle \xi \rangle^{ar} \\ &\leqslant |\nabla_x \phi_j(x, \xi) - \omega_j|^{-1} \langle \xi \rangle^{al} \end{aligned} \tag{5.2.6}$$

We now want to bound $|\nabla_x \phi_j(x, \xi) - \omega_j|$ from below by $|\nabla_x \phi_j(x_{Q_j}, \xi) - \omega_j|$. By the Taylor's theorem, we have that

$$|\nabla_x \phi_j(x_{Q_j}, \xi) - \omega_j|^2 - |\nabla_x \phi_j(x, \xi) - \omega_j|^2 \lesssim |\nabla_x \phi_j(x_{Q_j}, \xi) - \omega_j||\nabla^2_x \phi_j(x, \xi)||x - x_{Q_j}|$$
$$\frac{|\nabla_x \phi_j(x_{Q_j}, \xi) - \omega_j|^2}{|\nabla_x \phi_j(x, \xi) - \omega_j|^2} \leqslant 1 + \frac{\lambda^{-1/2} \langle \xi \rangle^a}{|\nabla_x \phi_j(x, \xi) - \omega_j|} \leqslant 1 + \lambda^{-\delta} \langle \xi \rangle^a,$$

hence $|\nabla_x \phi_j(x, \xi) - \omega_j|^{-1} \lesssim \lambda^{1/2-\delta} \rho_{\omega_j} \circ B_{Q_j}(\xi) \langle \xi \rangle^{a/2}$. Combining this with (5.2.6) then yields the desired bound.

$$|\nabla^l v_{\omega_j}(x, \xi)| \leqslant \lambda^{1/2-\delta} \rho_{\omega_j} \circ B_{Q_j}(\xi) \langle \xi \rangle^{a(l+1/2)} \leqslant \lambda^{(l+1)/2-\delta} \rho_{\omega_j} \circ B_{Q_j}(\xi) \langle \xi \rangle^{2al}$$

This closes the induction, completing the proof. □

5.3 Induction-on-Scales

We shall now set up our central induction-on-scales argument. We need to use a slightly stronger, hybridised version of (5.1.3) as our inductive hypothesis, which is that there exists an $\alpha > 0$ such that, for any ϕ_j and ψ_j satisfying the hypotheses of our theorem, for all $J_0 \subset \{1, ..., m\}$, the following inequality holds uniformly in all choices of finite sets $A_j \subset \mathbb{R}^{n_j}$ for $j \notin J_0$, letting $\mathbb{B}_j := \{B_j := \nabla_x \phi_j(x, \xi), x \in A_j\}$.

$$\int_{\mathbb{R}^n} \prod_{j \in J_0} |S_j^\lambda f_j(\xi)|^{2p_j} \prod_{j \notin J_0} \left(\sum_{B_j \in \mathbb{B}_j} g_{B_j} \circ B_j(x) \right)^{p_j} d\xi \lesssim_\alpha \lambda^{\alpha - P_{J_0}} \prod_{j \in J_0} \|f_j\|_{L^2}^{2p_j} \prod_{j \notin J_0} \left(\sum_{B_j \in \mathbb{B}_j} \int_{\mathbb{R}^{n_j}} g_{B_j} \right)^{p_j}. \tag{5.3.1}$$

Our theorem of course coincides with the case when $J_0 = \{1, ..., m\}$, and with (5.0.2) when $J_0 = \varnothing$. The reason for incorporating the non-oscillatory terms is that we may use Proposition 5.2.1 to absorb the tail contribution into them, as we now demonstrate. We shall abbreviate the non-oscillatory part of the left-hand-side of (5.3.1) by $G_{J_0}(\xi) := \prod_{j \notin J_0} \left(\sum_{B_j \in \mathbb{B}_j} g_{B_j} \circ B_j(\xi) \right)^{p_j}$

$$\begin{aligned} \int_{\mathbb{R}^n} \prod_{j \in J_0} |S_j^\lambda f_j|^{2p_j} G_{J_0}(\xi) d\xi &\leqslant \sum_{U \in \mathcal{U}} \int_U \prod_{j \in J_0} |S_j^\lambda f_j|^{2p_j} G_{J_0} \\ &\leqslant \sum_{U \in \mathcal{U}} \int_U \prod_{j \in J_0} (|S_j^\lambda f_j^U| + |S_j^\lambda f_j^{U^c}|)^{2p_j} G_{J_0} \end{aligned}$$

Now, for a given $U \in \mathcal{U}$ and $J \subset J_0$, let $U_J := \{\xi \in U : |S_j^\lambda f_j^U(\xi)| \geqslant |S_j^\lambda f_j^{U^c}(\xi)| \ \forall j \in J\}$. The collection $\{U_J\}_{J \subset J_0}$ defines a cover of U, hence we may write

$$\begin{aligned} \int_{\mathbb{R}^n} \prod_{j \in J_0} |S_j^\lambda f_j|^{2p_j} G_{J_0} &\leqslant \sum_{J \subset J_0} \sum_{U \in \mathcal{U}} \int_{U_J} \prod_{j \in J_0} (|S_j^\lambda f_j^U| + |S_j^\lambda f_j^{U^c}|)^{2p_j} G_{J_0} \\ &\lesssim \sum_{J \subset J_0} \sum_{U \in \mathcal{U}} \int_{U_J} \prod_{j \in J} |S_j^\lambda f_j^U|^{2p_j} \prod_{j \in J_0 \setminus J} |S_j^\lambda f_j^{U^c}|^{2p_j} G_{J_0} \\ &\leqslant \sum_{J \subset J_0} \sum_{U \in \mathcal{U}} \int_U \prod_{j \in J} |S_j^\lambda f_j^U|^{2p_j} G_{J_0}. \end{aligned}$$

Proposition 5.2.1 then yields the following upper bound, for all $N \leqslant N_\varepsilon$.

$$\int_{\mathbb{R}^n} \prod_{j \in J_0} |S_j^\lambda f_j|^{2p_j} G_{J_0} \lesssim_N \sum_{J \subset J_0} \lambda^{-P_{J_0 \setminus J}/2 - 2\delta N \sum_{j \in J_0 \setminus J} p_j} \sum_{U \in \mathcal{U}} \int_U \prod_{j \in J} |S_j^\lambda f_j^U|^{2p_j} \widetilde{G}_J^U, \tag{5.3.2}$$

where, for a given subset $S \subset \{1, ..., m\}$, $P_S := \sum_{j \in S} p_j n_j$, and we have summarised the non-oscillatory terms as

$$\widetilde{G}_J^U(\xi) := \prod_{j \notin J, j \in J_0} \left(\sum_{Q_j, \omega_j} |a_{Q_j,\omega_j}|^2 \rho_{\omega_j}^\lambda \circ B_{Q_j}(\xi)^{2N} dx \right)^{p_j} G_{J_0}(\xi).$$

We shall from now on assume that $N > n_j/2$ for each $j \in J_0$, so that $(\rho_{\omega_j}^\lambda)^{2N}$ is integrable and in particular $\int_{\mathbb{R}^{n_j}} (\rho_{\omega_j}^\lambda)^{2N} \lesssim \lambda^{(\delta - 1/2)n_j}$. Now, suppose that there exists $\beta > 0$ such that for each $U \in \mathcal{U}$ and $J \subset J_0$,

$$\int_U \prod_{j \in J} |S_j^\lambda f_j^U|^{2p_j} \widetilde{G}_J^U$$
$$\lesssim_\beta \lambda^{\beta - P_J} \prod_{j \in J} \|f_j^U\|_{L^2}^{2p_j} \prod_{j \in J_0 \setminus J} \left(\sum_{Q_j, \omega_j} |a_{Q_j,\omega_j}|^2 \int_{B_{Q_j}(U)} (\rho_{\omega_j}^\lambda)^{2N} \right)^{p_j} \prod_{j \notin J_0} \left(\sum_{B_j \in \mathbb{B}_j} \int_{B_{Q_j}(U)} g_{B_j} \right)^{p_j} \tag{5.3.3}$$

By the almost orthogonality of the e_{Q_j,ω_j}, we know that

$$\|f_j^U\|_{L^2}^2 \simeq \sum_{(Q_j,\omega_j) \in W_j^U} |a_{Q_j,\omega_j}|^2 \|e_{Q_j,\omega_j}\|_{L^2}^2 \simeq \lambda^{-n_j/2} \sum_{(Q_j,\omega_j) \in W_j^U} |a_{Q_j,\omega_j}|^2 \tag{5.3.4}$$

Take an $(Q_j, \omega_j) \in W_j^U$, then there exists a $\xi' \in T_{Q_j,\omega_j} \cap U$. By the triangle inequality,

$$\begin{aligned}|\nabla_x \phi(x_{Q_j}, \xi_U) - \omega_j| &\leqslant |\nabla_x \phi(x_{Q_j}, \xi_U) - \nabla_x \phi(x_{Q_j}, \xi')| + |\nabla_x \phi(x_{Q_j}, \xi') - \omega_j| \\ &\leqslant \lambda^{-1/2} |\nabla_\xi \nabla_x \phi(x_{Q_j}, \xi_U)| + \lambda^{\delta - 1/2} \\ &\leqslant 2\lambda^{\delta-1/2}\end{aligned}$$

where applied the fact that $\nabla_\xi \nabla_x \phi_j$ is L^∞ bounded and the fact that we may assume that λ is large. It therefore follows that for all $(Q_j, \omega_j) \in W_j^U$, $\rho_{\omega_j}^{\lambda/10} \circ B_j(\xi_U) = 1$, provided that δ is sufficiently small. Applying this to 5.3.4 we obtain the inequality

$$\|f_j^U\|_{L^2}^2 \leqslant \lambda^{-n_j/2} \sum_{(Q_j,\omega_j)} |a_{Q_j,\omega_j}|^2 \rho_{\omega_j}^{\lambda/10} \circ B_{Q_j}(\xi_U).$$

The mean-value theorem and the L^∞ boundedness of $\nabla_\xi \nabla_x \phi$ implies that $B_{Q_j}(U) \subset U_{c\lambda^{-1/2}}(B_{Q_j}(\xi_U))$ for some $c \simeq 1$. Let $\tilde{\rho}_{\omega_j}^\lambda(z) := \lambda^{n_j/2} \int_{U_{2\lambda^{-1/2}}(z)} (\rho_{\omega_j}^\lambda)^{2N}$, and $\tilde{g}_{B_j}(z) := \lambda^{n_j/2} \int_{U_{2\lambda^{-1/2}}(z)} g_{B_j}$. Summing (5.3.3) over U and averaging, we obtain a Kakeya–Brascamp–

Lieb form, which is bounded by the transversality hypothesis.

$$\sum_U \int_U \prod_{j\in J} |S_j^\lambda f_j^U|^{2p_j} G_J^U$$

$$\lesssim_\beta \lambda^{\beta-P_J} \sum_U \prod_{j\in J} \|f_j^U\|_{L^2}^{2p_j} \prod_{j\in J_0\setminus J} \left(\sum_{(Q_j,\omega_j)\in W_j^U} |a_{Q_j,\omega_j}|^2 \int_{B_{Q_j}(U)} (\rho_{\omega_j}^\lambda)^{2N}(\xi) dx \right)^{p_j} \prod_{j\notin J_0} \left(\sum_{B_j\in\mathbb{B}_j} \int_{B_{Q_j}(U)} g_{B_j} \right)^{p_j}$$

$$\lesssim \lambda^{\beta-P_J-n/2} \sum_U \prod_{j\in J} \left(\sum_{(Q_j,\omega_j)\in W_j^U} |a_{Q_j,\omega_j}|^2 \rho_{\omega_j}^\lambda \circ B_j^{x_{Q_j}}(\xi_U)^{2N} \right)$$

$$\times \prod_{j\in J_0\setminus J} \left(\sum_{Q_j,\omega_j} |a_{Q_j,\omega_j}|^2 \tilde{\rho}_{\omega_j}^\lambda \circ B_{Q_j}(\xi_U) \right)^{p_j} \prod_{j\notin J_0} \left(\sum_{B\in\mathbb{B}_j} \tilde{g}_{B_j} \circ B_j(\xi_U) \right)^{p_j}$$

$$\lesssim \lambda^{\beta-P_J} \int_{\mathbb{R}^n} \prod_{j\in J} \left(\sum_{Q_j,\omega_j} |a_{Q_j,\omega_j}|^2 \rho_{\omega_j}^\lambda \circ B_j^{x_{Q_j}}(\xi)^{2N} \right)^{p_j}$$

$$\times \prod_{j\in J_0\setminus J} \left(\sum_{Q_j,\omega_j} |a_{Q_j,\omega_j}|^2 \tilde{\rho}_{\omega_j}^\lambda \circ B_{Q_j}(\xi) \right)^{p_j} \prod_{j\notin J_0} \left(\sum_{B\in\mathbb{B}_j} \tilde{g}_{B_j} \circ B_j(\xi) \right)^{p_j} d\xi$$

$$\lesssim \lambda^{\beta-P_J} \prod_{j\in J} \left(\sum_{Q_j,\omega_j} |a_{Q_j,\omega_j}|^2 \lambda^{(\delta-1/2)n_j} \right)^{p_j} \prod_{j\in J_0\setminus J} \left(\sum_{Q_j,\omega_j} |a_{Q_j,\omega_j}|^2 \lambda^{(\delta-1/2)n_j} \right)^{p_j} \prod_{j\notin J_0} \left(\sum_{B_j\in\mathbb{B}_j} \int_{\mathbb{R}^{n_j}} \tilde{g}_{B_j} \right)^{p_j}$$

$$\lesssim \lambda^{\beta+\delta P_{J_0}-P_J} \prod_{j\in J_0} \|f_j\|_{L^2}^{2p_j} \prod_{j\notin J_0} \left(\sum_{B_j\in\mathbb{B}_j} \int_{\mathbb{R}^{n_j}} g_{B_j} \right)^{p_j}$$

Combining this with (5.3.2) then yields (5.3.1), provided that N may be chosen such that $N \geqslant n\delta^{-1}\min(p_j)^{-1}/2$.

$$\int_{\mathbb{R}^n} \prod_{j\in J_0} |S_j^\lambda f_j|^{2p_j} \prod_{j\notin J_0} \left(\sum_{B_j\in\mathbb{B}_j} \tilde{g}_j^x \circ B_j(\xi) \right)^{p_j} \leqslant \sum_{J\subset J_0} \lambda^{-n/2-2\delta N \sum_{j\in J_0\setminus J} p_j} \int_U \prod_{j\in J} |S_j^\lambda f_j^U|^{2p_j} \widetilde{G}_J^U$$

$$\lesssim_\beta \sum_{J\subset J_0} \lambda^{\beta+\delta(P_{J_0}-2N\sum_{j\in J_0\setminus J} p_j)-P_{J_0\setminus J}/2-P_J} \prod_{j\in J_0} \|f_j\|_{L^2}^{2p_j} \prod_{j\notin J_0} \left(\sum_{B_j\in\mathbb{B}_j} \int_{\mathbb{R}^{n_j}} g_{B_j} \right)^{p_j}$$

$$\lesssim_N \lambda^{\beta+\delta n-P_{J_0}} \prod_{j\in J_0} \|f_j\|_{L^2}^{2p_j} \prod_{j\notin J_0} \left(\sum_{B_j\in\mathbb{B}_j} \int_{\mathbb{R}^{n_j}} g_{B_j} \right)^{p_j} \qquad (5.3.5)$$

In summary, we have now shown that if (5.3.3) holds for some β, then (5.3.1) holds for $\alpha = \beta + \delta n$. We now want to show that there exists at least one $\beta = \beta_0$ such that that (5.3.3) holds, which we refer to as the 'base case', and that if (5.3.1) holds for some α then (5.3.3) holds for $\beta = \alpha/2$, which we refer to as the 'inductive step'. Iterating this chain of implications we find that (5.3.1) holds for $\alpha = 2^{-k}\beta_0 + \sum_{r=0}^{k} 2^{-r}\delta n$ for all $k \in \mathbb{N}$, hence (5.3.1) holds for all $\alpha > 2\delta n$. Choosing $\delta < \varepsilon/2n$ sufficiently close to $\varepsilon/2n$ so that $n\delta^{-1}\min(p_j)^{-1}/2 < N_\varepsilon$ then proves the theorem.

5.4 The Remaining Proof

All that remains now to prove Theorem 5.1.1 is to establish the two inequalities that we referred to at the end of the last section as the base case and the inductive step respectively.

Proof of Theorem 5.1.1. The base case follows from a crude size estimate on the oscillatory part, followed by an application of the transversality assumption. For each $j \in J$, choose any $B_j = \nabla_x \phi_j(x, \xi)$

$$\begin{aligned}
\int_U \prod_{j\in J} |S_j^\lambda f_j^U|^{2p_j} \widetilde{G}_J^U &\leqslant \int_U \prod_{j\in J} \left(\int_{\mathbb{R}^{n_j}} |\psi_j(x,\xi)||f_j^U(x)|dx \right)^{p_j} \widetilde{G}_J^U(\xi) d\xi \\
&\lesssim \prod_{j\in J} \|f_j^U\|_{L^2}^{2p_j} \int_U \widetilde{G}_J^U(\xi) d\xi \\
&= \prod_{j\in J} \|f_j^U\|_{L^2}^{2p_j} \int_U \prod_{j\in J} \chi_{B_j(U)} \circ B_j(\xi)^{p_j} \widetilde{G}_J^U(\xi) d\xi \\
&\lesssim \lambda^{-P_J/2} \prod_{j\in J} \|f_j^U\|_{L^2}^{2p_j} \prod_{j\in J_0\setminus J} \left(\sum_{Q_j,\omega_j} |a_{Q_j,\omega_j}|^2 \int_{B_{Q_j}(U)} (\rho_{\omega_j}^\lambda)^{2N} \right)^{p_j} \prod_{j\notin J_0} \left(\sum_{B_j \in \mathbb{B}_j} \int_{B_j(U)} g_{B_j} \right)^{p_j}
\end{aligned}$$

This shows that (5.3.3) holds for $\beta = n/2$, establishing the base case.

The inductive step involves rescaling the integral over each U by a factor of $\lambda^{1/2}$, so that they are up to unit scale, at which point we then apply the inductive hypothesis but now with a scaling parameter of value $\lambda^{1/2}$, thereby halving the resulting exponent as desired. First of all, let $\chi_{\lambda^{1/2}U}$ denote a smooth bump function that takes the value 1 in $\lambda^{1/2}U$ and 0 outside of $2\lambda^{1/2}U$. Since the ball $\lambda^{1/2}U$ is at unit scale, we may further assume that the derivatives of $\chi_{\lambda^{1/2}U}$ are uniformly bounded in λ. Denoting the centre of U by ξ_U, we then define the rescaled versions of the phase and cut-off functions to which we shall be applying the inductive hypothesis, $\phi_j^\lambda(x,\xi) := \lambda^{1/2}\phi_j(x,\lambda^{-1/2}\xi) - \lambda^{1/2}\phi_j(x,\xi_U)$ and $\psi_j^\lambda(x,\xi) := \psi_j(x,\lambda^{-1/2}\xi)\chi_{\lambda^{1/2}U}(\xi)$. We need to check that the hypotheses of our theorem hold for phases ϕ_j^λ and amplitudes ψ_j^λ uniformly in $\lambda > 0$. First of all, the transversality condition holds by invariance of (5.0.2) under translation and rescaling, so we just need to check the regularity condition. Let $\xi \in 2\lambda^{1/2}U$, $x \in \mathbb{R}^{n_j}$, and $k \leqslant 2N_\varepsilon$.

- $|\nabla_x^k\phi_j^\lambda(x,\xi)| = \lambda^{1/2}|\nabla_x^k\phi_j(x,\lambda^{-1/2}\xi) - \nabla_x^k\phi_j(x,\xi_U)| \lesssim \lambda^{1/2}|\nabla_x^k\nabla_\xi\phi(x,\xi_U)||\lambda^{-1/2}\xi - \xi_U| \lesssim \langle\xi\rangle^a$
- $|\nabla_x^k\nabla_\xi\phi_j^\lambda(x,\xi)| = \lambda^{-1/2}|\nabla_x^k\nabla_\xi(\phi_j(x,\lambda^{-1/2}\xi))| = |\nabla_x^k\nabla_\xi\phi_j(x,\lambda^{-1/2}\xi)| \lesssim \langle\xi\rangle^a$ for $k \geqslant 2$, and this also means that $|\nabla_x\nabla_\xi\phi_j^\lambda(x,\xi)| \lesssim 1$.
- $|\nabla_x^k\psi_j^\lambda(x,\xi)| = |\nabla_x^k\psi_j(x,\lambda^{-1/2}\xi)| \lesssim 1$

Now, suppose that the inductive hypothesis (5.3.1) holds for $\alpha = \beta$. We have shown that $\phi_1^\lambda, ...\phi_m^\lambda$ and $\psi_1^\lambda, ..., \psi_m^\lambda$ satisfy the hypotheses of Theorem 5.1.1, so we may apply the inductive hypothesis uniformly in λ to each of their corresponding one-parameter families of oscillatory integral operators, namely,

$$S_j^{\mu,\lambda}f_j := \int_{\mathbb{R}^{n_j}} e^{i\mu\phi_j^\lambda(x,\xi)}\psi_j^\lambda(x,\xi)f_j(x)dx \qquad \mu > 1.$$

We also need to define corresponding rescaled terms for the non-oscillatory parts, Let $B_j^\lambda(\xi) := \lambda^{1/2}B_j(\lambda^{-1/2}\xi)$ and $B_{Q_j}(\xi) := \nabla_x\phi_j^\lambda(x_{Q_j},\xi)$. We then have the following identities for all $\lambda > 1$ and $\xi \in \lambda^{1/2}U$:

$$S_j^\lambda f_j^U(\xi) = S_j^{\lambda^{1/2},\lambda}(e^{i\lambda\phi_j(x,\xi_U)} f_j^U)(\lambda^{1/2}\xi)$$
$$\rho_{\omega_j}^\lambda \circ B_{Q_j}(\xi) = \rho_{\omega_j}^\lambda(\lambda^{-1/2}(B_{Q_j}^\lambda(\lambda^{1/2}\xi) + \nabla_x\phi_j(x_{Q_j}, \xi_U))$$

We may substitute these identities into the left-hand side of (5.3.3) and rescale out the factors of $\lambda^{1/2}$ from the argument, bringing the integral up to unit scale.

$$\begin{aligned}\int_U \prod_{j\in J} |S_j^\lambda f_j^U|^{2p_j} F_J^U &= \int_U \prod_{j\in J} \left|S_j^{\lambda^{1/2},\lambda}(e^{i\lambda\phi_j(x,\xi_U)} f_j^U)(\lambda^{1/2}\xi)\right|^{2p_j} \prod_{j\notin J_0}\left(\sum_{B_j\in\mathbb{B}_j} g_{B_j}(\lambda^{-1/2}B_j^\lambda(\lambda^{1/2}\xi))\right)^{p_j} \\ &\quad\times \prod_{j\notin J, j\in J_0}\left(\sum_{Q_j,\omega_j} |a_{Q_j,\omega_j}|^2 \rho_{\omega_j}^\lambda(\lambda^{-1/2}(B_{Q_j}^\lambda(\lambda^{1/2}\xi) + \nabla_x\phi_j(x_{Q_j},\xi_U))^{2N}\right)^{p_j} d\xi \\ &= \lambda^{-n/2}\int_{\lambda^{1/2}U} \prod_{j\in J}\left|S_j^{\lambda^{1/2},\lambda}(e^{i\lambda\phi_j(x,\xi_U)} f_j^U)(\xi)\right|^{2p_j} \prod_{j\notin J_0}\left(\sum_{B_j\in\mathbb{B}_j} g_{B_j}(\lambda^{-1/2}B_j^\lambda(\xi))\right)^{p_j} \\ &\quad\times \prod_{j\notin J, j\in J_0}\left(\sum_{Q_j,\omega_j} |a_{Q_j,\omega_j}|^2 \rho_{\omega_j}^\lambda(\lambda^{-1/2}B_{Q_j}^\lambda(\xi) + \nabla_x\phi_j(x_{Q_j},\xi_U))^{2N}\right)^{p_j} d\xi\end{aligned}$$

Applying the inductive hypothesis (5.3.1) to $S_1^{\lambda^{1/2},\lambda}, ..., S_m^{\lambda^{1/2},\lambda}$, we then obtain the desired estimate.

$$
\begin{aligned}
&\int_U \prod_{j\in J} |S_j^\lambda f_j^U|^{2p_j} F_J^U \\
&\quad \leqslant \lambda^{-n/2} \int_{2\lambda^{1/2}U} \prod_{j\in J} \left| S_j^{\lambda^{1/2},\lambda}(e^{i\lambda\phi_j(x,\xi_U)} f_j^U)(\xi)\right|^{2p_j} \prod_{j\notin J_0} \left(\sum_{B_j\in\mathbb{B}_j} g_{B_j}(\lambda^{-1/2} B_j^\lambda(\xi))\right)^{p_j} \\
&\qquad\quad \times \prod_{j\notin J, j\in J_0} \left(\sum_{Q_j,\omega_j} |a_{Q_j,\omega_j}|^2 \rho_{\omega_j}^\lambda (\lambda^{-1/2} B_{Q_j}^\lambda(\xi) + \nabla_x \phi_j(x_{Q_j},\xi_U))^{2N})\right)^{p_j} d\xi \\
&\quad \lesssim_\alpha \lambda^{\alpha/2 - P_J/2 - n/2} \prod_{j\in J} \|f_j^U\|_{L^2}^{2p_j} \prod_{j\notin J_0} \left(\sum_{B_j\in\mathbb{B}_j} \int_{\mathbb{R}^{n_j}} g_{B_j}(\lambda^{-1/2} z) dz\right)^{p_j} \\
&\qquad\quad \times \prod_{j\notin J, j\in J_0} \left(\sum_{Q_j,\omega_j} |a_{Q_j,\omega_j}|^2 \int_{\mathbb{R}^{n_j}} \rho_{\omega_j}^\lambda (\lambda^{-1/2} z + \nabla_x \phi_j(x_{Q_j},\xi_U))^N dz\right)^{p_j} \\
&\quad = \lambda^{\alpha/2 - P_J/2 - n/2 + P_{J^c}/2} \prod_{j\in J} \|f_j^U\|_{L^2}^{2p_j} \prod_{j\notin J, j\in J_0} \left(\sum_{Q_j,\omega_j} |a_{Q_j,\omega_j}|^2 \int_{\mathbb{R}^{n_j}} (\rho_{\omega_j}^\lambda)^{2N}\right)^{p_j} \prod_{j\notin J_0} \left(\sum_{B_j\in\mathbb{B}_j} \int_{\mathbb{R}^{n_j}} g_{B_j}\right)^{p_j} \\
&\quad = \lambda^{\alpha/2 - P_J} \prod_{j\in J} \|f_j^U\|_{L^2}^{2p_j} \prod_{j\notin J, j\in J_0} \left(\sum_{Q_j,\omega_j} |a_{Q_j,\omega_j}|^2 \int_{\mathbb{R}^{n_j}} (\rho_{\omega_j}^\lambda)^{2N}\right)^{p_j} \prod_{j\notin J_0} \left(\sum_{B_j\in\mathbb{B}_j} \int_{\mathbb{R}^{n_j}} g_{B_j}\right)^{p_j}
\end{aligned}
$$

This establishes the inductive step, completing the proof. □

Chapter 6

Further Research

In this chapter, we shall discuss some of the questions that lead on from the results of this thesis and offer some speculative conjectures.

6.1 Exact Heat-flow Monotonicity

In this thesis, we do not achieve exact heat-flow monotonicity statements for nonlinear Brascamp–Lieb functionals, however such statements do hold in certain geometrically symmetric regimes, such as those considered by Carlen, Lieb, and Loss [30], later generalised by Bramati [23]. The following unpublished result of Hong Duong as we shall discuss is another such statement.

Theorem 6.1.1 *Let $n, n_1, ..., n_m \in \mathbb{N}$, $\boldsymbol{p} = (p_1, ..., p_m) \in [0,1]^m$ be an m-tuple of exponents such that $\sum_{j=1}^m p_j n_j = n$, and for each $j \in \{1, ..., m\}$, let $B_j : \mathbb{R}^n \to \mathbb{R}^{n_j}$ be a smooth submersion such that there exists a uniformly positive definite matrix valued function $A : \mathbb{R}^n \to \mathbb{R}^{n\times n}$, and for each $j \in \{1, ..., m\}$, a uniformly positive-definite matrix-valued* *function* $A_j : \mathbb{R}^n \to \mathbb{R}^{n_j \times n}$ *satisfying*

$$A(x) = \sum_{j=1}^{m} dB_j(x)^* A_j(x) dB_j(x)$$
$$A_j^{-1}(x) = dB_j(x) A^{-1}(x) dB_j(x)^*$$

Let $\mathcal{L} := \nabla \cdot (A^{-1}\nabla)$ *and* $f_j : \mathbb{R}^{n_j} \to \mathbb{R}$. *Let* $u_j : \mathbb{R}^{n_j} \times (0, \infty) \to \mathbb{R}$ *and suppose that there exists a smooth supersolution to* $\mathcal{L}$ *of the form* $u_j^{(t)} \circ B_j$, *i.e.*

$$\partial_t (u_j^{(t)} \circ B_j) \geqslant \mathcal{L}(u_j^{(t)} \circ B_j) \tag{6.1.2}$$

Then $\prod_{j=1}^{m} (u_j^{(t)} \circ B_j)^{p_j}$ *is also a supersolution to* $\mathcal{L}$.

The proof of this statement follows very similar reasoning to the proof of heat-flow monotonicity in the linear geometric case found in [14], and analogously to the linear case implies heat-flow monotonicity under certain reasonable integrability conditions.

Corollary 6.1.3 *Let* $u := \prod_{j=1}^{m} (u_j^{(t)} \circ B_j)$, *and assume that* $div(A\nabla u) \in L^1(\mathbb{R}^n)$ *is integrable,* $\nabla u \to 0$ *as* $|x| \to \infty$, *and* $\partial_t u$ *is uniformly bounded. Then, under the same assumptions as the above theorem, the quantity* $Q(t) := \int_{\mathbb{R}^n} u$ *is monotone increasing for all* $t > 0$.

Proof. By uniform boundedness of the time derivative we may exchange orders of differentiation and integration as follows

$$\begin{aligned} \frac{d}{dt} Q(t) = \partial_t \int_{\mathbb{R}^n} u^{(t)} = \int_{\mathbb{R}^n} \partial_t u^{(t)} \geqslant \int_{\mathbb{R}^n} div(A\nabla u^{(t)}) \\ = \lim_{R\to\infty} \int_{B(0,R)} div(A\nabla u^{(t)}) \\ = \lim_{R\to\infty} \int_{\partial B(0,R)} A\nabla u^{(t)} \cdot d\underline{n} = 0 \end{aligned}$$
□

This result poses a difficult question, this being that of when there exist supersolutions of the form $u_j^{(t)} \circ B_j$, i.e. supersolutions whose fibre structure coincides with that of the maps B_j, however the existing theory of supersolutions to diffusion equations cannot yet provide any immediate answers to this question. It is appealing to think that they may be constructed as solutions to some other variable coefficient diffusion equation, as then by our uniformity assumptions it is reasonable to suppose that these supersolutions may be written as $u_j^t \circ B_j(x) = \int_{\mathbb{R}^n} P_t(x, y_j) f_j \circ B_j(y_j) dy_j$ for some integral kernel P_t, which of course is the case if they may be constructed explicitly as solutions to some related diffusion equation. Using the co-area formula, we would then obtain a formal bound for the nonlinear Brascamp–Lieb inequality.

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \prod_{j=1}^m f_j \circ B_j(x)^{p_j} dx \leqslant \limsup_{t\to\infty} \int_{\mathbb{R}^n} \prod_{j=1}^m u_j^t \circ B_j(x)^{p_j} dx \\
&= \limsup_{t\to\infty} \int_{\mathbb{R}^n} \prod_{j=1}^m \left(\int_{\mathbb{R}^{n_j}} P_t(x, y_j) f_j \circ B_j(y_j) dy_j \right)^{p_j} dx \\
&= \limsup_{t\to\infty} \int_{\mathbb{R}^n} \prod_{j=1}^m \left(\int_{\mathbb{R}^{n_j}} \int_{B_j^{-1}(\{z_j\})} P_t(x, y_j) \frac{d\sigma(y_j)}{|\det(dB_j(y_j) dB_j(y_j)^*)|} f_j(z_j) dz_j \right)^{p_j} dx \\
&\leqslant \limsup_{t\to\infty} \left(\int_{\mathbb{R}^n} \prod_{j=1}^m \sup_{z_j} \left(\int_{B_j^{-1}(\{z_j\})} P_t(x, y_j) \frac{d\sigma(y_j)}{|\det(dB_j(y_j) dB_j(y_j)^*)|} \right)^{p_j} dx \right) \prod_{j=1}^m \left(\int_{\mathbb{R}^{n_j}} f_j \right)^{p_j}
\end{aligned}
$$

6.2 Sharp Nonlinear Brascamp–Lieb Inequalities

It is natural to ask the question of whether or not there is a version of Theorem 1.2.1 that does not include a $(1 + \varepsilon)$ error term in the constant on the right-hand side. This constant would then need to at least track the Brascamp–Lieb constant over the whole of the domain of integration on the left-hand side. This leads us to the following natural conjecture:

Conjecture 6.2.1 (Sharp Local Nonlinear Brascamp–Lieb) *Suppose that* $(\mathbf{B}, \mathbf{p})$ *is* *a C^2 nonlinear Brascamp–Lieb datum defined over some neighbourhood $\widetilde{U}$ of a point $x_0 \in \mathbb{R}^n$. There exists a neighbourhood $U \subset \widetilde{U}$ of x_0 such that the following inequality holds for all $f_j \in L^1(\mathbb{R}^{n_j})$:*

$$\int_U \prod_{j=1}^m f_j \circ B_j(x)^{p_j} dx \leqslant \sup_{x \in U} \mathrm{BL}(\mathbf{dB}(x), \mathbf{p}) \prod_{j=1}^m \left(\int_{\mathbb{R}^{n_j}} f_j \right)^{p_j} \tag{6.2.2}$$

One may also view this as a consequence of a related conjecture about the existence of localising extremising sequences.

Conjecture 6.2.3 *Given submersions $B_j : M \to M_j$ and $p_j \in [0,1]$, then there exists a collection of points $x_j \in M_j$ and an extremising sequence of inputs $\mathbf{f}^{(k)}$ such that $\mathrm{supp}(f_j^{(k)}) \subset U_{1/k}(x_j)$.*

We shall now give a heuristic for why Conjecture 6.2.1 would follow from Conjecture 6.2.3. Conjecture 6.2.3 suggests that in order to find the sharp constant for the local nonlinear Brascamp–Lieb inequality, it suffices to test on functions f_j with arbitrarily small support, i.e., given any non-extremal $f_j \in L^1(\mathbb{R}^{n_j})$ of unit mass, for $\delta > 0$ sufficiently small there exists an f_j^δ of unit mass and a δ-ball $V_j^\delta \subset U$ such that $\mathrm{supp}(f_j^\delta) \subset V_j^\delta$ and

$$\int_U \prod_{j=1}^m f_j \circ B_j(x)^{p_j} dx \leqslant \int_U \prod_{j=1}^m f_j^\delta \circ B_j(x)^{p_j} dx = \int_{U_\delta} \prod_{j=1}^m f_j^\delta \circ B_j(x)^{p_j} dx, \tag{6.2.4}$$

where $U_\delta := \bigcap_{j=1}^m B_j^{-1}(V_j^\delta)$. By transversality of the maps B_j, we may assume that U_δ is contained in some ball of radius $\lesssim_{\mathbf{B}} \delta$, and since δ can be taken to be arbitrarily small we may apply the local nonlinear Brascamp–Lieb inequality (Theorem 1.2.1) to find that, given any $\varepsilon > 0$, there exists some choice of δ such that, denoting the centre of U_δ by x_{U_δ},

$$\int_{U_\delta} \prod_{j=1}^m f_j^\delta \circ B_j(x)^{p_j} dx \leqslant (1+\varepsilon)\mathrm{BL}(\mathbf{dB}(x_{U_\delta}, \mathbf{p}) \leqslant (1+\varepsilon) \max_{x \in U} \mathrm{BL}(\mathbf{dB}(x, \mathbf{p}). \tag{6.2.5}$$

Combining this with (6.2.4) and taking the limit as $\varepsilon \to 0$ implies Conjecture 6.2.1. Of course, we may try to run a similar argument in a global setting, where now each $B_j : M \to M_j$ is a submersion between Riemannian manifolds of dimension n and n_j respectively, and we assume that $\mathrm{BL}(\mathbf{dB}(x), \mathbf{p})$, $|\mathbf{dB}(x)|$, and $|\mathbf{dB}(x)|^{-1}$ are uniformly bounded above, whence transversality yields that U_δ is contained in, not just one, but a union of balls with radius $\lesssim_{\mathbf{B}} \delta$, centred at the intersection points of the fibres associated to the centres $x_{V_j^\delta}$ of the balls V_j^δ, this being $\bigcap_{j=1}^m B_j^{-1}(x_{V_j^\delta})$. Nonetheless, by uniformity of Theorem 1.2.1, we may apply the local Brascamp–Lieb inequality to each of these balls simultaneously and take the limit as $\varepsilon \to 0$ as before, implying the following conjecture:

Conjecture 6.2.6 (Sharp Global Nonlinear Brascamp–Lieb) *Suppose that $M, M_1, ..., M_m$ are Riemannian manifolds and that $B_j : M \to M_j$ are submersions such that $|dB_j(x)|$ is uniformly bounded above and below for each $j \in \{1, ..., m\}$, then the following inequality holds,*

$$\int_M \prod_{j=1}^m f_j \circ B_j^{p_j} \leqslant \sup_{z_j \in M_j, j \in \{1,...,m\}} \left(\sum_{x \in M : B_j(x) = z_j} \mathrm{BL}(\mathbf{dB}(x), \mathbf{p}) \right) \prod_{j=1}^m \left(\int_{M_j} f_j \right)^{p_j} \tag{6.2.7}$$

6.3 Scale-Dependent Nonlinear Brascamp–Lieb Inequalities

Throughout this thesis, we have adhered to scaling conditions of the form $\sum_{j=1}^m p_j n_j = n$, since this usually imparts some essential scale-invariance required for the problems we are considering to be tractable, and as a result much of the existing literature carries an assumption of this form; there is however reason to believe that at least some of the inequalities of the type we consider may be feasible outside of this polytope. In the linear Brascamp–Lieb setting, the scaling condition is necessitated by the presence of certain trivial rescaling counterexamples, however they may be avoided if we truncate the domain of integration on the left-hand side and introduce some control to the level of constancy of the functions f_j, in which case we may then obtain Brascamp–Lieb type inequalities with a broader range of exponents.

Let $r, R > 0$, and let $L^1_r(\mathbb{R}^{n_j})$ denote the cone of non-negative functions $f_j \in L^1(\mathbb{R}^{n_j})$ such that $f(x) \simeq f(y)$ whenever $|x-y| \leqslant r$, we then define the *scale-dependent Brascamp–Lieb inequality* as

$$\int_{B_R(0)} \prod_{j=1}^m f_j \circ L_j(x)^{p_j} dx \leqslant C \prod_{j=1}^m \left(\int_{\mathbb{R}^{n_j}} f_j \right)^{p_j} \qquad \forall f_j \in L^1_r(\mathbb{R}^{n_j}). \tag{6.3.1}$$

and we define $\mathrm{BL}_{loc}(\mathbf{L}, \mathbf{p}; r, R)$ as the optimal constant $C \in (0, \infty]$ such that (6.3.1) holds. Maldague recently showed that $\mathrm{BL}_{loc}(\mathbf{L}, \mathbf{p}; r, R) \simeq r^\alpha R^\beta$, where $\alpha, \beta > 0$ are the following exponents [56]:

$$\alpha := \inf_{V \leqslant \mathbb{R}^n} \left(\mathrm{codim}(V) - \sum_{j=1}^m p_j \mathrm{codim}(L_j V) \right) \tag{6.3.2}$$

$$\beta := \sup_{V \leqslant \mathbb{R}^n} \left(\dim(V) - \sum_{j=1}^m p_j \dim(L_j V) \right) \tag{6.3.3}$$

This generalises the well-known finiteness characterisation for the classical Brascamp–Lieb inequality, which in this context is that $\alpha \geqslant 0$ and $\beta \leqslant 0$. It also more generally implies that $\mathrm{BL}_{loc}(\mathbf{L}, \mathbf{p}) := \lim_{r \to 0} \mathrm{BL}_{loc}(\mathbf{L}, \mathbf{p}; r, 1)$ is finite if and only if $\alpha \geqslant 0$. Given the local nature of the inequality (1.2.2) it would make sense to conjecture the following generalisation:

Conjecture 6.3.4 *Let $\varepsilon > 0$, and suppose that $(\mathbf{B}, \mathbf{p})$ is a C^2 nonlinear Brascamp–Lieb datum defined over some neighbourhood of a point $x_0 \in \mathbb{R}^n$. There exists a $\delta > 0$ such* *that the following inequality holds for all $f_j \in L^1(\mathbb{R}^{n_j})$:*

$$\int_{B_\delta(x_0)} \prod_{j=1}^{m} f_j \circ B_j(x)^{p_j} dx \leqslant (1+\varepsilon)\mathrm{BL}_{loc}(\mathbf{dB}(x_0), \mathbf{p}) \prod_{j=1}^{m} \left(\int_{\mathbb{R}^{n_j}} f_j \right)^{p_j}. \qquad (6.3.5)$$

Given that Theorem 1.2.3 is a consequence of Theorem 1.2.1, it is plausible that Conjecture 6.3.4 might, given some appropriate additional curvature assumptions to compensate for the lack of scale-invariance, imply similar multilinear convolution and restriction estimates, but for the broader range of exponents and geometries that the condition $\beta \leqslant 0$ allows.

6.4 Invariant Inequalities in Multilinear Harmonic Analysis

Currently the tools we have to prove invariant Brascamp–Lieb and Kakeya-type inequalities rely on the use of auxiliary algebraic varieties [27, 77]. The theorems that use such methods require that the underlying geometry interacts favourably with these varieties, hence we typically require that they have an algebraic structure of some kind. However, there is no evidence to suggest that these somewhat rigid hypotheses are necessary, since the role that the algebraic condition on the fibres plays appears heuristically to be purely combinatorial in nature, in that it is there simply so that Bézout's theorem prohibits unboundedly many intersections of fibres. If we are to follow this heuristic, then, it would suggest that a more general theorem might hold that does not require any algebraic assumptions on the underlying maps.

Conjecture 6.4.1 *Let $M, M_1, ..., M_m$ be Riemannian manifolds of dimensions $n, n_1, ..., n_m$ respectively, and take some exponents $p_j \in [0,1]$ satisfying $\sum_{j=1}^m p_j n_j = n$. Fix some $D > 0$, and for each $j \in \{1, ..., m\}$, let $B_j : M \to M_j$ be a C^1 map. Suppose that for all* *configurations* $(z_1, ..., z_m) \in M_1 \times ... \times M_m$,

$$\# \bigcap_{j=1}^{m} B_j^{-1}(\{z_j\}) \leqslant D.$$

Then, the following inequality holds for all nonnegative $f_j \in L^1(M_j)$,

$$\int_M \prod_{j=1}^{m} f_j \circ B_j(x)^{p_j} \frac{d\sigma_M(x)}{\mathrm{BL}(\mathbf{dB}(x), \mathbf{p})} \lesssim_D \prod_{j=1}^{m} \left(\int_{M_j} f_j \right)^{p_j}$$

One may devise some similar Kakeya analogue of the form (4.5.2) with a similar combinatorial assumption in place of an algebraic one.

Conjecture 6.4.2 *Take* $M, M_1, ..., M_m$ *as in Conjecture 6.4.1, and for each* $j \in \{1, ..., m\}$, *let* $\mathbb{B}_j$ *be a finite collection of almost everywhere* C^1 *maps* $B_j : M \to M_j$. *Suppose that there exists a* $D > 0$ *such that for any configuration of maps* $(B_1, ..., B_m) \in \mathbb{B}_1 \times ... \mathbb{B}_m$ *and points* $(z_1, ..., z_m) \in M_1 \times ... \times M_m$, *we have that*

$$\# \bigcap_{j=1}^{m} B_j^{-1}(\{z_j\}) \leqslant D.$$

Then, the following inequality holds for all $f_{B_j} \in L^1(M_j)$*:*

$$\int_M \left(\sum_{(B_1,...,B_m) \in \mathbb{B}_1 \times ... \times \mathbb{B}_m} \frac{\prod_{j=1}^m f_{B_j} \circ B_j(x)}{\mathrm{BL}(\mathbf{dB}(x), \mathbf{p})^{m-1}} \right)^{\frac{1}{m-1}} dx \lesssim_D \prod_{j=1}^{m} \left(\sum_{B_j \in \mathbb{B}_j} \int_{M_j} f_{B_j} \right)^{\frac{1}{m-1}}$$

We shall now discuss the issue of formulating invariant Kakeya–Brascamp–Lieb inequalities with general exponents. First, observe that if the exponents p_j are rationals written with numerator r_j and common denominator q, then, denoting an arbitrary r_j-tuple of maps in $\mathbb{B}_j$ by $\underline{B}_j := (B_{j,k_j})_{k=1}^{r_j} \in \mathbb{B}_j^{r_j}$, we may write the left-hand side of (4.5.4) as

$$\int_M \left(\sum_{(\underline{B}_1,...,\underline{B}_m)\in\mathbb{B}_1^{r_1}\times...\times\mathbb{B}_m^{r_m}} \prod_{j=1}^{m} \prod_{k_j=1}^{r_j} f_{B_{j,k_j}} \circ B_{j,k_j}(x) \right)^{\frac{1}{q}} dx$$

The upshot of this formulation is that we may then introduce an invariant weight factor as in (4.5.4). Given $\mathbf{k} = (k_j)_{j=1}^m \in \prod_{j=1}^m \{1, ..., r_j\}$, let $\mathbf{B_k} := (B_{j,k_j})_{j=1}^m$,

$$\int_M \left(\sum_{(\underline{B}_1,...,\underline{B}_m)\in\mathbb{B}_1^{r_1}\times...\times\mathbb{B}_m^{r_m}} \prod_{j=1}^{m} \prod_{k_j=1}^{r_j} \frac{f_{B_{j,k_j}} \circ B_{j,k_j}(x)}{\mathrm{BL}(\mathbf{dB_k}(x), \mathbf{p})^{\frac{q}{mr_j}}} \right)^{\frac{1}{q}} dx$$

Again, this integral satisfies diffeomorphism-invariance properties akin to Proposition 4.1.10. It is then natural to ask the question of what the appropriate form is for irrational exponents. One suggestion is that this could be formulated using the calculus of virtual integration, developed by Tao in [68], in order to make sense of summing over a cartesian product of non-integer powers of the sets $\mathbb{B}_j$, where we would then reinterpret the weight as a 'virtual function', however this avenue is as of yet unexplored.

Another possible route could be via the Fremlin tensor product technique of Zorin-Kranich as in [77], where we consider each term of the form $\frac{\prod_{j=1}^m f_j \circ B_j(x)^{p_j}}{\mathrm{BL}(\mathbf{dB}(x),\mathbf{p})}$ as a real-valued function on $\mathbb{B}_1 \times ... \times \mathbb{B}_m$, and therefore may be considered as an element of the Fremlin tensor product space $\overline{\bigotimes}_{j=1}^m L^{1/p_j}(\mathbb{B}_j)$ for each $x \in M$. We may then formulate the following nonlinear Kakeya–Brascamp–Lieb inequality:

$$\int_M \left\| \frac{\prod_{j=1}^m f_{B_j} \circ B_j(x)^{p_j}}{\mathrm{BL}(\mathbf{dB}(x), \mathbf{p})} \right\|_{\overline{\bigotimes}_{j=1}^m L^{1/p_j}(\mathbb{B}_j)} dx \lesssim \prod_{j=1}^{m} \left(\sum_{B_j\in\mathbb{B}_j} \int_{M_j} f_{B_j} \right)^{p_j} \tag{6.4.3}$$

While this formulation has the advantage of being able to access irrational exponents and is somewhat more succinct that (6.4), unfortunately, unlike (6.4), it does not coincide with (6.4.2) in the case when each $p_j = \frac{1}{m-1}$. It is therefore unclear, outside of the context of a given application, exactly which of these generalisations is the most elegant or natural.

Given Theorem 5.1.1, it would be reasonable to conjecture that if Conjecture 6.4.2 holds, then a corresponding oscillatory version might follow. We shall now formulate a candidate invariant version of the multilinear oscillatory integral estimate (5.1.3). For each $j \in \{1, ..., m\}$, let $\phi_j : \mathbb{R}^{n_j} \times \mathbb{R}^n \to \mathbb{R}$ be a C^2 phase function and $\psi : \mathbb{R}^{n_j} \times \mathbb{R}^n \to \mathbb{R}$ a smooth cutoff function that is compactly supported in the first variable. Setting $\mathbb{R}^N := \mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_m}$, Let $\phi : \mathbb{R}^N \times \mathbb{R}^n \to \mathbb{R}$ denote the direct sum of the phases ϕ_j, i.e. $\phi(x, \xi) := \sum_{j=1}^m \phi_j(x_j, \xi)$, and $\psi : \mathbb{R}^N \times \mathbb{R}^n \to \mathbb{R}$ the tensor product of the cut-off functions ψ_j, i.e. $\psi(x, \xi) := \prod_{j=1}^m \psi_j(x_j, \xi)$. We shall also define the collection of maps $B_{\phi_j} : \mathbb{R}^N \times \mathbb{R}^n \to \mathbb{R}^{n_j}$ as $B_{\phi_j}(x, \xi) := \nabla_{x_j} \phi_j(x_j, \xi)$, and denote their m-tuple as $\mathbf{B}_\phi := (B_{\phi_j})_{j=1}^m$. Consider the following one-parameter family of multilinear oscillatory integral operators:

$$T^\lambda_{\phi,\psi}(f_1, ..., f_m) := \int_{\mathbb{R}^N} e^{i\lambda\phi(x,\xi)} \psi(x, \xi) f_1 \otimes ... \otimes f_m(x) \frac{dx}{\mathrm{BL}(\mathbf{d}_\xi \mathbf{B}_\phi(x, \xi), \mathbf{p})^{\frac{m-1}{2}}} \tag{6.4.4}$$

Observe that if there exists $w : \mathbb{R}^n \to \mathbb{R}$ such that $\mathrm{BL}(\mathbf{d}_\xi \mathbf{B}_\phi(x, \xi), \mathbf{p}) = w(\xi)$ for all $x \in \mathbb{R}^N$, then $|T^\lambda_{\phi,\psi}(f_1, ..., f_m)| = \omega(\xi) \prod_{j=1}^m |S^\lambda_j f_j|$, where S^λ_j is defined as in Theorem 5.1.1. In this sense, we may view the following conjecture as an invariant version of Theorem 5.1.1.

Conjecture 6.4.5 (Invariant Multilinear L^2 Oscillatory Integral Estimate) *Suppose that (6.4.2) holds uniformly for all collections of maps $\mathbb{B}_j$ of the form $\mathbb{B}_j := \{B_j(y, \cdot) : y \in \mathcal{Y}\}$, where $\mathcal{Y} \subset \mathbb{R}^N$ is some finite set of points. Then, the following estimate holds for all $f_j \in L^2(\mathbb{R}^{n_j})$:*

$$\|T^\lambda_{\phi,\psi}(f_1, ..., f_m)\|_{L^{\frac{2}{m-1}}(\mathbb{R}^n)} \lesssim_\varepsilon \lambda^{\varepsilon - \frac{n(m-1)}{2}} \prod_{j=1}^m \|f_j\|_{L^2(\mathbb{R}^{n_j})} \tag{6.4.6}$$

Finally, we observe that the optimal constant in this inequality is invariant under the action of diffeomorphism on the underlying phase and amplitude.

Proposition 6.4.7 (Diffeomorphism-Invariance) *Let ϕ_j and ψ_j be phase and cut-off functions respectively for $j \in \{1, ..., m\}$ satisfying the conditions of Conjecture 6.4.5. Let $H : M \to \widetilde{M}$ and, for each $j \in \{1, ..., m\}$, $G_j : N_j \to \tilde{N}_j$ be diffeomorphisms, and define $\widetilde{\phi}_j(\xi) := \phi_j(G_j(x_j), H(\xi))$, $\widetilde{\psi}_j(x_j, \xi) := \psi_j(G_j(x_j), H(\xi))$, then,*

$$\|T^\lambda_{\phi,\psi}\|_{\bigotimes_{j=1}^m L^2(N_j)\to L^{\frac{2}{m-1}}(M)} = \|T^\lambda_{\tilde{\phi},\tilde{\psi}}\|_{\bigotimes_{j=1}^m L^2(\tilde{N}_j)\to L^{\frac{2}{m-1}}(\widetilde{M})}$$

Proof. For each $j \in \{1, ..., m\}$, let $f_j \in C_0^\infty(N_j)$ be a compactly supported smooth function, define $\widetilde{f}_j := \det(dG_j)^{1/2} f_j \circ G_j$. For brevity, write $N = N_1 \times ... \times N_m$ $F := f_1 \otimes ... \otimes f_m$, $\widetilde{F} := \widetilde{f}_1 \otimes ... \otimes \widetilde{f}_m$, and let $G : N \to N$ denote the diffeomorphism $G(x) := (G_1(x_1), ..., G_m(x_m))$. We first of all apply the change of variables $\xi \mapsto H(\xi)$ and $x_j \mapsto G_j(x_j)$.

$$\begin{aligned}
&\|T^\lambda_{\phi,\psi}(f_1, ..., f_m)\|^{\frac{2}{m-1}}_{L^{\frac{2}{m-1}}(M)} \\
&= \int_M \left| \int_N e^{i\lambda\phi(x,\xi)} \psi(x,\xi) F(x) \frac{dx}{\mathrm{BL}(\mathbf{d}_\xi \mathbf{B}_\phi(x,\xi), \mathbf{p})^{\frac{m-1}{2}}} \right|^{\frac{2}{m-1}} d\xi \\
&= \int_{\widetilde{M}} \left| \int_N e^{i\lambda\phi(x,H(\xi))} \psi(x, H(\xi)) F(x) \frac{dx}{\mathrm{BL}(\mathbf{d}_\xi \mathbf{B}_\phi(x,\xi), \mathbf{p})^{\frac{m-1}{2}}} \right|^{\frac{2}{m-1}} \det(dH(\xi)) d\xi \\
&= \int_{\widetilde{M}} \left| \int_{\tilde{N}} e^{i\lambda\phi(G(x),H(\xi))} \psi(G(x), H(\xi)) F \circ G(x) \frac{\det(dH(\xi))^{\frac{m-1}{2}} \prod_{j=1}^m \det(dG_j(x_j)) dx}{\mathrm{BL}(\mathbf{d}_\xi \mathbf{B}_\phi(x,\xi), \mathbf{p})^{\frac{m-1}{2}}} \right|^{\frac{2}{m-1}} d\xi \\
&= \int_{\widetilde{M}} \left| \int_{\tilde{N}} e^{i\lambda\tilde{\phi}(x,\xi)} \widetilde{\psi}(x,\xi) \widetilde{F}(x) \frac{\det(dH(\xi))^{\frac{m-1}{2}} \det(dG(x) dG(x)^*)^{\frac{1}{2}} dx}{\mathrm{BL}(\mathbf{d}_\xi \mathbf{B}_\phi(x,\xi), \mathbf{p})^{\frac{m-1}{2}}} \right|^{\frac{2}{m-1}} d\xi
\end{aligned}$$

By the chain rule, $d_\xi B_{\tilde{\phi}_j}(x_j, \xi) = d_\xi \nabla_{x_j} \widetilde{\phi}_j(x_j, \xi) = dG_j(x_j) d_\xi \nabla_{x_j} \widetilde{\phi}_j(G_j(x_j), H(\xi)) dH(\xi)$, so $(\mathbf{d}_\xi \mathbf{B}_\phi(x,\xi), \mathbf{p})$ and $(\mathbf{d}_\xi \mathbf{B}_{\tilde{\phi}}(x,\xi), \mathbf{p})$ are equivalent Brascamp–Lieb data, moreover by Lemma 3.3 of [14], $\mathrm{BL}(\mathbf{d}_\xi \mathbf{B}_{\tilde{\phi}}(x,\xi), \mathbf{p}) = \det(dH(x))^{-1} \prod_{j=1}^m \det(dG_j(x_j))^{\frac{-1}{m-1}} \mathrm{BL}(\mathbf{d}_\xi \mathbf{B}_\phi(x,\xi), \mathbf{p}) =$

$\det(dH(x))^{-1}\det(dG(x))^{\frac{-1}{m-1}}\mathrm{BL}(\mathbf{d}_\xi\mathbf{B}_\phi(x,\xi),\mathbf{p})$, hence,

$$\begin{aligned}
&\|T^\lambda_{\phi,\psi}(f_1,...,f_m)\|^{\frac{2}{m-1}}_{L^{\frac{2}{m-1}}(\mathbb{R}^n)}\\
&=\int_{\widetilde{M}}\left|\int_{\widetilde{N}}e^{i\lambda\widetilde{\phi}(x,\xi)}\widetilde{\psi}(x,\xi)\widetilde{F}(x)\frac{dx}{\mathrm{BL}(\mathbf{d}_\xi\mathbf{B}_{\widetilde{\phi}}(x,\xi),\mathbf{p})^{\frac{m-1}{2}}}\right|^{\frac{2}{m-1}}d\xi\\
&=\|T^\lambda_{\widetilde{\phi},\widetilde{\psi}}(\widetilde{f}_1,...,\widetilde{f}_m)\|^{\frac{2}{m-1}}_{L^{\frac{2}{m-1}}(\widetilde{M})}\\
&\leqslant\|T^\lambda_{\widetilde{\phi},\widetilde{\psi}}\|^{\frac{2}{m-1}}_{\bigotimes_{j=1}^m L^2(\tilde{N}_j)\to L^{\frac{2}{m-1}}(\widetilde{M})}\prod_{j=1}^m\|\widetilde{f}_j\|^{\frac{2}{m-1}}_{L^2(\tilde{N}_j)}
\end{aligned}$$

Note that $\|f_j\|_{L^2(N_j)}=\|\widetilde{f}_j\|_{L^2(\tilde{N}_j)}$, hence, by density of $C_0^\infty(N_j)$ in $L^2(N_j)$, $\|T^\lambda_{\widetilde{\phi},\widetilde{\psi}}\|_{op}\leqslant\|T^\lambda_{\phi,\psi}\|_{op}$, and so by symmetry the converse inequality also holds, proving the claim. □

Appendix

Chapter 3

Proof of Lemma 3.2.4 We should first clarify that, in this proof, double bars shall always denote an L^∞ norm. We first prove the case for derivatives of order 1. Let $p \in M$ and let $X, Y \in T_pM$, and $|X|, |Y| \leqslant 1$. We consider the following vector field $J(t) : (0, \infty) \to TM$ defined over the curve parametrised by $\gamma(t) := \exp(tX)$:

$$J(t) := \partial_s \exp_p(t(X + sY))|_{s=0}.$$

By definition of the exponential map, J is a Jacobi field with initial data $J(0) := 0$ and $J'(0) = Y$, hence it satisfies the Jacobi equation:

$$J'' + R(J, \gamma')\gamma' = 0 \tag{A.8}$$

Here R denotes the Riemannian curvature endomorphism. Now, define the following quantity $F(t) := |J(t)|^2 + |J'(t)|^2$. We shall aim to bound this quantity via bounding its derivative using (A.8) and the AM-GM inequality.

$$\begin{aligned} F' &= 2\langle J, J' \rangle + 2\langle J', J'' \rangle \\ &= 2(\langle J, J' \rangle + 2\langle J', R(J, \gamma')\gamma' \rangle) \\ &\leqslant 2(|J||J'| + |J'|\|R\||J||X|^2) \\ &\leqslant (1 + \|R\|)F \end{aligned}$$

Hence $F(t) \leqslant e^{t(1+\|R\|)}F(0)$, and so

$$|d\exp_p(X)Y| = J(1) \leqslant F(1)^{1/2} \leqslant e^{(1+\|R\|)/2}F(0)^{1/2} = e^{(1+\|R\|)/2}|Y|.$$

We then bootstrap to the second order case via a similar method. Let $Z \in T_pM$, $|Z| \leqslant 1$, and consider the following family of variations of J:

$$J_\varepsilon(t) := \partial_s \exp_p(t(X + sY + \varepsilon Z))|_{s=0}$$

Each such J_ε is a Jacobi field for all $\varepsilon > 0$, so we may then differentiate (A.8) in ε to find that

$$\partial_\varepsilon J_\varepsilon'' + \partial_J R(J_\varepsilon(t), \gamma')(\gamma', \partial_\varepsilon J_\varepsilon) = 0, \forall t, \varepsilon > 0.$$

Where $\partial_J R$ refers to the partial covariant derivative of the Riemannian curvature tensor in the first argument. We now consider the quantity $G(t) := |\partial_\varepsilon J_0(t)|^2 + |\partial_\varepsilon J_0'(t)|^2$, and apply a similar argument to last time

$$\begin{aligned} G' &= 2\langle \partial_\varepsilon J_0, \partial_\varepsilon J_0' \rangle + 2\langle \partial_\varepsilon J_0', \partial_\varepsilon J_0'' \rangle \\ &= 2(\langle \partial_\varepsilon J_0, \partial_\varepsilon J_0' \rangle + 2\langle J_0', \partial_J R(J_0, \gamma')(\gamma', \partial_\varepsilon J_0) \rangle \\ &\leqslant 2(|\partial_\varepsilon J_0||\partial_\varepsilon J_0'| + |\partial_\varepsilon J_0'|\|\partial_J R\||J_0||\partial_\varepsilon J_0||X|^2) \\ &\leqslant (1 + \|\partial_\varepsilon R\||J_0|^2)G \end{aligned}$$

Hence $G(t) \leqslant e^{t(1+\|\partial_J R\|(\sup_{0<l<t}|J_0|^2(l)))}G(0) \leqslant e^{t(1+\|\partial_J R\|e^{t(1+\|R\|)}|Y|)}G(0)$, therefore,

$$|d^2 \exp(X)(Y,Z)| = \partial_\varepsilon J_0(1) \leqslant G(1)^{1/2} \leqslant e^{t(1+\|\partial_J R\|e^{t(1+\|R\|)})/2}G(0)^{1/2} = e^{(1+\|\partial_J R\|e^{(1+\|R\|)/2})/2}|Z|$$

By symmetry, we also have that $|d^2 \exp(X)(Y,Z)| \leqslant e^{(1+\|\partial_J R\|e^{t(1+\|R\|)/2})/2}|Y|$, so we are done. □

Proof of Lemma 4.2.1 Let $w(x) := \det(dB(x)dB(x)^*)^{-\frac{1}{2}}$. The lemma follows from the co-area formula and the continuity of the quantity $\int_{B^{-1}(\{z\})} f(x)w(x)d\sigma(x)$ in $z \in N$, since we then have that

$$\begin{aligned} &\left| \int_A f(x)\chi_\delta \circ B(x)dx - \int_{B^{-1}(\{z_0\})} f(x)w(x)d\sigma(x) \right| \\ &= \delta^{d-n} \left| \int_{U_\delta(0)} \left(\int_{A\cap B^{-1}(\{z\})} f(x)w(x)d\sigma(x) - \int_{A\cap B^{-1}(\{z_0\})} f(x)w(x)d\sigma(x) \right) dz \right| \\ &\lesssim \left\| \int_{A\cap B^{-1}(\{z\})} f(x)w(x)d\sigma(x) - \int_{A\cap B^{-1}(\{z_0\})} f(x)w(x)d\sigma(x) \right\|_{L^\infty_z(U_\delta(z_0))} \\ &\xrightarrow[\delta\to 0]{} 0. \end{aligned}$$

□

Proof of Lemma 3.2.11 Let $\tau > 0$ be small. Since f_j is uniformly continuous, given $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $z, z' \in M$ such that $d(z,z') \leqslant \delta$, we have that $|f_j(z) - f_j(z')| \leqslant \varepsilon$. Therefore, provided $C\tau^\gamma \leqslant \delta$, we may bound $|f_j \circ B_j(x) - H_{x,\tau,j} f_j \circ B_j(x)|$ in the following way:

$$
\begin{aligned}
|f_j \circ B_j(x) - H_{x,\tau,j} f_j \circ B_j(x)| = &\left| f_j \circ B_j(x) - \int_{U_{\tau^\gamma,j}(0)} f_j(w) G_{x,\tau,j}(e^{-1}_{B_j(x)}(w)) dw \right| \\
\leqslant & f_j \circ B_j(x) \left(1 - \int_{U_{\tau^\gamma,j}(B_j(x))} G_{x,\tau,j} \circ e^{-1}_{B_j(x)} \right) \\
&+ \left| \int_{V_{\tau^\gamma,j}(0)} \left(f_j \circ B_j(x)\right) - f_j(w)\right) G_{x,\tau,j}(e^{-1}_{B_j(x)}(w)) dw \right|.
\end{aligned}
$$

By the uniform boundedness of the second derivative of the exponential map $e_{B_j(x)}$ established in Lemma 3.2.4, provided that $\tau > 0$ is sufficiently small, for all $x \in M$, $j \in \{1, ..., m\}$, and $v \in V_{\tau^\gamma,j}(0) \subset T_{B_j(x)} M_j$, we have

$$
(1 + \tau^\eta)^{-1} \leqslant \det(de_{B_j(x)})[v] \leqslant 1 + \tau^\eta. \tag{A.9}
$$

We may then apply Lemma 3.2.4 to bound the first term by a power of τ. For the second term, we apply the triangle inequality and bound the resulting gaussian integral similarly.

$$
\begin{aligned}
&|f_j \circ B_j(x) - H_{x,\tau,j} f_j \circ B_j(x)| \\
&\leqslant f_j \circ B_j(x) \left(1 - (1 + \tau^\eta)^{-1} \int_{V_{\tau^\gamma,j}(0)} G_{x,\tau,j} \right) \\
&\qquad + \int_{U_{\tau^\gamma,j}(B_j(x))} |f_j \circ B_j(x) - f_j(w)| \, G_{x,\tau,j} \circ e^{-1}_{B_j(x)}(w) dw \\
&\leqslant (1 + \tau^\eta)^{-1} (f_j \circ B_j(x) \tau^\eta + \varepsilon)
\end{aligned}
$$

This of implies the claim of the lemma. □

Chapter 4

Proof of Lemma 4.2.2 The fact that the scaling condition is satisfied is trivial. As for the second claim, we shall first prove that $\mathrm{BL}(\mathbf{L}, \mathbf{p}) \leqslant \det(L_{m+1} L_{m+1}^*)^{\frac{1}{2}} \mathrm{BL}(\widetilde{\mathbf{L}}, \widetilde{\mathbf{p}})$. If we let $\chi_\delta : \mathbb{R}^{n-d} \to \mathbb{R}$ be as in Lemma 4.2.1, with $z_0 = 0$, and we take arbitrary $f_j \in L^1(\mathbb{R}^{n_j})$, then by Lemma 4.2.1, we have that

$$
\begin{aligned}
\int_V \prod_{j=1}^m f_j \circ L_j(x)^{p_j} dx &= \det(L_{m+1} L_{m+1}^*)^{\frac{1}{2}} \lim_{\delta \to 0} \int_{\mathbb{R}^n} \left(\prod_{j=1}^m f_j \circ L_j(x)^{p_j} \right) \chi_\delta \circ L_{m+1}(x) dx \\
&\leqslant \det(L_{m+1} L_{m+1}^*)^{\frac{1}{2}} \mathrm{BL}(\widetilde{\mathbf{L}}, \widetilde{\mathbf{p}}) \prod_{j=1}^m \left(\int_{M_j} f_j \right)^{p_j},
\end{aligned}
$$

which establishes that $\mathrm{BL}(\mathbf{L},\mathbf{p}) \leqslant \det(L_{m+1}L_{m+1}^*)^{\frac{1}{2}}\mathrm{BL}(\widetilde{\mathbf{L}},\widetilde{\mathbf{p}})$. We shall now prove the converse inequality. For each $1 \leqslant j \leqslant m+1$, let $f_j \in C^\infty(\mathbb{R}^{n_j})$ be a smooth function with unit mass. The claim quickly follows upon decomposing $\mathbb{R}^n$ into $V \oplus V^\perp$ and applying the Brascamp-Lieb inequality associated to the datum $(\mathbf{L},\mathbf{p})$ to the integral over V:

$$\begin{aligned}
\int_{\mathbb{R}^n} \prod_{j=1}^{m+1} f_j \circ L_j(x)^{p_j}dx &= \int_{V^\perp}\left(\int_V \prod_{j=1}^{m} f_j(L_j(x)+L_j(y))^{p_j}dx\right) f_{m+1}\circ L_{m+1}(y)dy \\
&\leqslant \mathrm{BL}(\mathbf{L},\mathbf{p})\int_{V^\perp} f_{m+1}\circ L_{m+1}(y)\prod_{j=1}^{m}\left(\int_{M_j} f_j(z+L_j(y))dz\right)^{p_j} dy \\
&= \mathrm{BL}(\mathbf{L},\mathbf{p})\det(L_{m+1}L_{m+1}^*)^{-\frac{1}{2}}. \qquad \square
\end{aligned}$$

Proof of Lemma 4.3.11 Fix some $x \in \Omega$. Let $T := \exp_{B_j(x)} \circ (dB_j(x)dB_j(x)^*)^{-1/2} \circ \exp_{B_j(x)}^{-1}$, then for $\delta > 0$ smaller than the minimal injectivity radius among $z \in B_j(\Omega)$.

$$L_j^x(U_{\delta/2}(x)) \subset B_j(U_\delta(x)) \iff TL_j^x(U_{\delta/2}(x)) \subset TB_j(U_\delta(x))$$

Hence we may assume without loss of generality that $dB_j(x)$ is a projection, in the sense that $dB_j(x)dB_j(x)^* = I_{T_{B_j(x)}M_j}$, and thus we may also assume that $L_j(U_{\delta/2}(x)) = U_{\delta/2}(B_j(x))$. It then suffices to show that for $\delta > 0$ sufficiently small, $\partial B_j(U_\delta(x)) \cap U_{\delta/2}(B_j(x)) = \varnothing$, in other words that for all $z \in \partial B_j(U_\delta(x))$, $d(z, B_j(x)) > \delta/2$. First of all, $\partial B_j(U_\delta(x)) = B(\partial U_\delta(x))$, so for a given $z \in \partial B_j(U_\delta(x))$, there exists a $y \in \partial U_\delta(x)$ such that $\exp_{B_j(x)} \circ dB_j(x)(y) = z$. By Taylor's theorem, there exists a $c > 0$ depending on Ω such that, for $\delta > 0$ sufficiently small,

$$\begin{aligned}
d(z, B_j(x)) &= |dB_j(x)(y-x)| + \mathcal{O}(|y-x|^2) \\
&\geqslant \delta - c\delta^2 > \delta/2 \qquad \square
\end{aligned}$$

Proof of Corollary 4.1.8 By duality, (4.1.9) is equivalent to the bound

$$\int_G \phi(x)\left(*_{j=1}^m f_j\Delta^{\sum_{l=1}^{j-1}\frac{1}{p_l'}}\right)(x)d\mu(x) \lesssim \deg(G)\|\phi\|_{L^{r'}(G)}\prod_{j=1}^m \|f_j\|_{L^{p_j}(G)}. \qquad \text{(A.10)}$$

For $1 \leqslant j \leqslant m$, define the nonlinear maps $B_j : G^m \to G$, $B_j(x_1,...,x_m) := x_j$, and $B_{m+1} : G^m \to G$, $B_{m+1}(x_1,...,x_m) := \prod_{j=1}^m x_j$. Deleting the null set of singular points from their ranges, these maps are quasialgebraic of degree 1 for $1 \leqslant j \leqslant m$, and $\deg(B_{m+1}) \leqslant \deg(m_G)$, hence by Theorem 4.1.3 we know that

$$\int_{G^m} \phi\left(\prod_{j=1}^m x_j\right)\prod_{j=1}^m f_j(x_j)\frac{d\sigma_1(x_1)...d\sigma_m(x_m)}{\mathrm{BL}_{T_{\underline{x}}G^m}(\mathbf{dB}(\underline{x}),\mathbf{p})} \lesssim \deg(G)\deg(m_g)^\sigma\|\phi\|_{L^{r'}(G)}\prod_{j=1}^m \|f_j\|_{L^{p_j}(G)}$$

where $\underline{x} := (x_1, ..., x_m)$, which is equivalent to (A.10) provided we have the identity

$$\mathrm{BL}_{T_{\underline{x}}G^m}(\mathbf{dB}(\underline{x}), \mathbf{p}) = B_{\mathbf{p},n} \prod_{j=1}^{m} \omega(x_j)^{-1} \Delta(x_j)^{-\sum_{l=1}^{j-1} \frac{1}{p_j'}} \tag{A.11}$$

at all configurations of smooth points $x_1, ..., x_m$ of G, where $d\mu(x) = \omega(x)d\sigma(x)$, and $B_{\mathbf{p},n}$ is the best constant for the n-dimensional euclidean multilinear Young's inequality associated to the exponents $\mathbf{p} := (p_1, ..., p_m)$. Let $x_1, ..., x_m \in G$ be smooth points, the left-hand side of (A.11) is by definition the best constant $C > 0$ in the inequality

$$\int_{\prod_{j=1}^{m} T_{x_j}G} \phi\left(\sum_{j=1}^{m} x_1...x_{j-1}v_j x_{j+1}...x_m\right) \prod_{j=1}^{m} f_j(v_j)dv_j \leqslant C\|\phi\|_{L^{r'}(T_{x_1...x_m}G)} \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(T_{x_j}G)}, \tag{A.12}$$

where the Lebesgue measure on the left-hand side is induced by the Lebesgue measure on the ambient euclidean space, and the Lebesgue measures defining the norms on the right-hand side are induced by the left-invariant Riemannian metric on G.

First of all, we multiply the measure on the left by the constant $\prod_{j=1}^{m} \omega(x_j)$ for convenience. We then apply the linear transformation from the Lie algebra $\mathfrak{g}$ to $T_{x_j}G$ defined by the mapping $v_j \mapsto x_1...x_m(x_1...x_{j-1})^{-1}v_j(x_{j+1}...x_m)^{-1}$, this is to turn the left-hand side of (A.12) into an integral to which we may directly apply the euclidean Young's inequality:

$$\begin{aligned}
&\int_{\prod_{j=1}^{m} T_{x_j}G} \phi\left(\sum_{j=1}^{m} x_1...x_{j-1}v_j x_{j+1}...x_m\right) \prod_{j=1}^{m} f_j(x_j)\omega(x_j)dv_j \\
&= \int_{\mathfrak{g}^m} \phi\left(x_1...x_m \sum_{j=1}^{m} v_j\right) \prod_{j=1}^{m} f_j((x_1...x_{j-1})^{-1}x_1...x_m v_j(x_{j+1}...x_m)^{-1})dx_j \Delta(x_{j+1}...x_m)^{-1}dv_j \\
&\leqslant B_{\mathbf{p},n}\|\phi(x_1...x_m v)\|_{L_v^{r'}(\mathfrak{g})} \prod_{j=1}^{m} \Delta(x_{j+1}...x_m)^{-1}\|f_j((x_{j+1}...x_m)^{-1}x_1...x_m v_j(x_{j+1}...x_m)^{-1})\|_{L_{v_j}^{p_j}(\mathfrak{g})} \\
&= B_{\mathbf{p},n}\|\phi\|_{L^{r'}(T_{x_1...x_m}G)} \prod_{j=1}^{m} \Delta(x_{j+1}...x_m)^{\frac{1}{p_j}-1}\|f_j\|_{L^{p_j}(T_{x_j}G)} \\
&= B_{\mathbf{p},n}\|\phi\|_{L^{r'}(T_{x_1...x_m}G)} \prod_{j=1}^{m} \Delta(x_j)^{-\sum_{l=1}^{j-1} \frac{1}{p_l'}}\|f_j\|_{L^{p_j}(T_{x_j}G)}.
\end{aligned}$$

Since this inequality is sharp by definition of $B_{\mathbf{p},n}$, we have established (A.11), thus completing the proof. □

Chapter 5

Proof of Lemma 5.2.3. We proceed by induction on $|\alpha|$. The claim holds trivially for $|\alpha| = 0$, so suppose for inductive hypothesis that, for some $k \in \mathbb{N}\backslash\{0\}$, $|z|^{2k}\partial^\alpha W(z)$ is a homogenous polynomial of degree k for all $\alpha \in \mathbb{N}^d$ such that $|\alpha| = k - 1$. Take such an α and some $i \in \{1, ..., d\}$, denote the multi-index whose i^{th} entry is 1 and all others are 0 by e_i. Consider $\partial_{x_i}\partial^\alpha W(z)$. By the inductive hypothesis, for a given $j \in \{1, ..., d\}$, there exist coefficients $c_\beta \in \mathbb{R}$ such that,

$$\begin{aligned}
(\partial_{x_i}\partial^\alpha W(z))_j &:= \partial_{x_i}\left(|z|^{-2k}\sum_{|\beta|=k} c_\beta z^\beta\right) \\
&= -2kz_i|z|^{-2(k+1)}\sum_{|\beta|=k} c_\beta z^\beta + |z|^{-2k}\sum_{\substack{|\beta|=k \\ \beta_i>0}} \beta_i c_\beta z^{\beta-e_i} \\
&= |z|^{2(k+1)}\left(-2kz_i\sum_{|\beta|=k} c_\beta z^\beta + |z|^2\sum_{\substack{|\beta|=k \\ \beta_i>0}} \beta_i c_\beta z^{\beta-e_i}\right)
\end{aligned}$$

We then may observe that $|z|^{2(k+1)}\partial_{x_i}\partial^\alpha W(z)$ is a homogeneous polynomial of degree $k + 1$, closing the induction. The inequality (5.2.4) then follows from the fact that, given a vector-valued homogeneous polynomial $p : \mathbb{R}^d \to \mathbb{R}^d$, provided that $|c_\beta| \lesssim 1$ for each component c_β of p, we have $|p(z)| \lesssim |z|^{\deg(p)}$, hence $|\nabla^k W(z)| \lesssim |z|^{-2(k+1)}|z|^{k+1} = |z|^{-(k+1)}$. □

List of References

[1] Jong-Guk Bak, Daniel M Oberlin, and Andreas Seeger. Restriction of Fourier transforms to curves and related oscillatory integrals. *American journal of mathematics*, 131(2):277–311, 2009.

[2] Franck Barthe. On a reverse form of the Brascamp–Lieb inequality. *Inventiones mathematicae*, 134(2):335–361, 1998.

[3] Franck Barthe. Optimal Young's inequality and its converse: a simple proof. *Geometric & Functional Analysis GAFA*, 8(2):234–242, 1998.

[4] Ioan Bejenaru and Sebastian Herr. Convolutions of singular measures and applications to the Zakharov system. *Journal of Functional Analysis*, 261(2):478–506, 2011.

[5] Ioan Bejenaru, Sebastian Herr, and Daniel Tataru. A convolution estimate for two-dimensional hypersurfaces. *Revista Matemática Iberoamericana*, 26(2):707–728, 2010.

[6] Jonathan Bennett and Neal Bez. Some nonlinear Brascamp–Lieb inequalities and applications to harmonic analysis. *Journal of Functional Analysis*, 259(10):2520–2556, 2010.

[7] Jonathan Bennett and Neal Bez. Generating monotone quantities for the heat equation. *Journal für die reine und angewandte Mathematik (Crelle's Journal)*, 2015.

[8] Jonathan Bennett and Neal Bez. Higher order transversality in harmonic analysis. *Royal Institute for Mathematical Sciences*, (pending publication).

[9] Jonathan Bennett, Neal Bez, Stefan Buschenhenke, Michael G Cowling, Taryn C Flock, et al. On the nonlinear Brascamp–Lieb inequality. *Duke Mathematical Journal*, 2020.

[10] Jonathan Bennett, Neal Bez, Michael G Cowling, and Taryn C Flock. Behaviour of the Brascamp–Lieb constant. *Bulletin of the London Mathematical Society*, 49(3):512–518, 2017.

[11] Jonathan Bennett, Neal Bez, Taryn C Flock, and Sanghyuk Lee. Stability of the Brascamp–Lieb constant and applications. *American Journal of Mathematics*, 140(2):543–569, 2018.

[12] Jonathan Bennett, Neal Bez, Taryn C Flock, and Sanghyuk Lee. Stability of the Brascamp–Lieb constant and applications. *American Journal of Mathematics*, 140(2):543–569, 2018.

[13] Jonathan Bennett, Neal Bez, and Susana Gutiérrez. Global nonlinear Brascamp–Lieb inequalities. *Journal of Geometric Analysis*, 23(4):1806–1817, 2013.

[14] Jonathan Bennett, Anthony Carbery, Michael Christ, and Terence Tao. The Brascamp–Lieb inequalities: finiteness, structure and extremals. *Geometric and Functional Analysis*, 17(5):1343–1415, 2008.

[15] Jonathan Bennett, Anthony Carbery, and Terence Tao. On the multilinear restriction and Kakeya conjectures. *Acta mathematica*, 196(2):261–302, 2006.

[16] Jonathan Bennett, Anthony Carbery, and James Wright. A non-linear generalisation of the Loomis–Whitney inequality and applications. *Mathematical Research Letters*, 12(4):443, 2005.

[17] Jonathan Bennett, Anthony Carbery, and James Wright. A non-linear generalisation of the Loomis–Whitney inequality and applications. *Mathematical Research Letters*, 12(4):443–457, 2005.

[18] Jonathan Bennett and Marina Iliopoulou. A multilinear Fourier extension identity on $\mathbb{R}^n$. *arXiv preprint arXiv:1701.06099*, 2017.

[19] Thomas F Bloom. Quantitative inverse theory of Gowers uniformity norms. *arXiv preprint arXiv:2009.01774*, 2020.

[20] Jean Bourgain. L^p-estimates for oscillatory integrals in several variables. *Geometric & Functional Analysis GAFA*, 1(4):321–374, 1991.

[21] Jean Bourgain, Ciprian Demeter, and Larry Guth. Proof of the main conjecture in Vinogradov's mean value theorem for degrees higher than three. *Annals of Mathematics*, pages 633–682, 2016.

[22] Jean Bourgain and Larry Guth. Bounds on oscillatory integral operators based on multilinear estimates. *Geometric and Functional Analysis*, 21(6):1239–1295, 2011.

[23] Roberto Bramati. Brascamp–Lieb inequalities on compact homogeneous spaces. *Analysis and Geometry in Metric Spaces*, 7(1):130–157, 2019.

[24] Herm J Brascamp, Elliott H Lieb, and JM Luttinger. A general rearrangement inequality for multiple integrals. *Journal of functional analysis*, 17(2):227–237, 1974.

[25] Anthony Carbery, Timo S Hänninen, and Stefán Ingi Valdimarsson. Multilinear duality and factorisation for Brascamp–Lieb-type inequalities with applications. *to appear, Jour. Eur. Math. Soc*, 2018.

[26] Anthony Carbery, Carlos Kenig, and Sarah Ziesler. Restriction for flat surfaces of revolution in $\mathbb{R}^3$. *Proceedings of the American Mathematical Society*, 135(6):1905–1914, 2007.

[27] Anthony Carbery and Stefán Ingi Valdimarsson. The endpoint multilinear Kakeya theorem via the Borsuk–Ulam theorem. *Journal of Functional Analysis*, 264(7):1643–1663, 2013.

[28] Anthony Carbery and Sarah Ziesler. Restriction and decay for flat hypersurfaces. *Publicacions matematiques*, pages 405–434, 2002.

[29] Eric A Carlen. Brascamp–Lieb inequalities for non-commutative integration. *Documenta Mathematica*, 13:553–584, 2008.

[30] Eric A Carlen, Elliott H Lieb, and Michael Loss. A sharp analog of Young's inequality on $\mathbb{S}^n$ and related entropy inequalities. *The Journal of Geometric Analysis*, 14(3):487–520, 2004.

[31] Michael Christ, James Demmel, Nicholas Knight, Thomas Scanlon, and Katherine Yelick. On Holder–Brascamp–Lieb inequalities for torsion–free discrete abelian groups. *arXiv preprint arXiv:1510.04190*, 2015.

[32] Michael Christ, Spyridon Dendrinos, Betsy Stovall, and Brian Street. Endpoint Lebesgue estimates for weighted averages on polynomial curves. *American Journal of Mathematics*, 142(6):1661–1731, 2020.

[33] G Constantine and T Savits. A multivariate faa di bruno formula with applications. *Transactions of the American Mathematical Society*, 348(2):503–520, 1996.

[34] Michael G Cowling, Alessio Martini, Detlef Müller, and Javier Parcet. The Hausdorff–Young inequality on lie groups. *Mathematische Annalen*, 375(1-2):93–131, 2019.

[35] Spyridon Dendrinos, Magali Folch-Gabayet, and James Wright. An affine-invariant inequality for rational functions and applications in harmonic analysis. *Proceedings of the Edinburgh Mathematical Society*, 53(3):639–655, 2010.

[36] Spyridon Dendrinos, Norberto Laghi, and James Wright. Universal L^p-improving for averages along polynomial curves in low dimensions. *Journal of Functional Analysis*, 257(5):1355–1378, 2009.

[37] Spyridon Dendrinos and Eugen Zimmermann. On L^p-improving for averages associated to mixed homogeneous polynomial hypersurfaces in $\mathbb{R}^3$. *Journal d'Analyse Mathématique*, 138(2):563–595, 2019.

[38] Stephen W Drury and Bernard P Marshall. Fourier restriction theorems for curves with affine and euclidean arclengths. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 97, pages 111–125. Cambridge University Press, 1985.

[39] Jennifer Duncan. An algebraic Brascamp–Lieb inequality. *The Journal of Geometric Analysis*, pages 1–28, 2021.

[40] Jennifer Duncan. A nonlinear variant of Ball's inequality. *arXiv preprint arXiv:2101.07672*, 2021.

[41] Ankit Garg, Leonid Gurvits, Rafael Oliveira, and Avi Wigderson. Algorithmic and optimization aspects of Brascamp–Lieb inequalities, via operator scaling. *Geometric and Functional Analysis*, 28(1):100–145, 2018.

[42] Ben Green. Finite field models in additive combinatorics. *arXiv preprint math/0409420*, 2004.

[43] Philip T Gressman. L^p-improving estimates for Radon-like operators and the Kakeya–Brascamp–Lieb inequality. *arXiv preprint arXiv:2008.01886*, 2020.

[44] Philip T Gressman et al. On the Oberlin affine curvature condition. *Duke Mathematical Journal*, 168(11):2075–2126, 2019.

[45] Larry Guth. The endpoint case of the Bennett–Carbery–Tao multilinear Kakeya conjecture. *Acta mathematica*, 205(2):263–286, 2010.

[46] Larry Guth. Restriction estimates using polynomial partitioning ii. *Acta Mathematica*, 221(1):81–142, 2018.

[47] Larry Guth, Jonathan Hickman, and Marina Iliopoulou. Sharp estimates for oscillatory integral operators via polynomial partitioning. *Acta Mathematica*, 223(2):251–376, 2019.

[48] Larry Guth, Dominique Maldague, and Hong Wang. Improved decoupling for the parabola. *arXiv preprint arXiv:2009.07953*, 2020.

[49] Robin Hartshorne. Algebraic geometry. *Graduate texts in mathematics*, 52, 1977.

[50] Emmanuel Hebey. *Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities*, volume 5. American Mathematical Soc., 2000.

[51] Jonathan Hickman. An affine Fourier restriction theorem for conical surfaces. *Mathematika*, 60(2):374–390, 2014.

[52] Junko Inoue. L^p Fourier transforms on nilpotent Lie groups and solvable Lie groups acting on Siegel domains. *Pacific Journal of Mathematics*, 155(2):295–318, 1992.

[53] Herbert Koch and Stefan Steinerberger. Convolution estimates for singular measures and some global nonlinear Brascamp–Lieb inequalities. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, 145(6):1223–1237, 2015.

[54] Michel Ledoux. Heat flows, geometric and functional inequalities. In *Proceedings of the International Congress of Mathematicians*, volume 4, pages 117–135, 2014.

[55] Elliot H. Lieb. Gaussian kernels have only gaussian maximizers. *Inventiones mathematicae*, 102(1):179–208, 1990.

[56] Dominique Maldague. Regularized Brascamp–Lieb inequalities and an application. *arXiv preprint arXiv:1904.06450*, 2019.

[57] Daniel Oberlin. Convolution with affine arclength measures in the plane. In *Proc. Amer. Math. Soc*, volume 127, pages 3591–3592, 1999.

[58] Daniel M Oberlin. A uniform Fourier restriction theorem for surfaces in $\mathbb{R}^3$. *Proceedings of the American Mathematical Society*, pages 1195–1199, 2004.

[59] Tong S Quek and Leonard YH Yap. Sharpness of Young's inequality for convolution. *Mathematica Scandinavica*, 53(2):221–237, 1984.

[60] AR Schep. Factorization of positive multilinear maps. *Illinois Journal of Mathematics*, 28(4):579–591, 1984.

[61] Thomas Schick. Manifolds with boundary and of bounded geometry. *Mathematische Nachrichten*, 223(1):103–120, 2001.

[62] Elias M Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, volume 3. Princeton University Press, 1993.

[63] Elias M Stein and Rami Shakarchi. 8. Oscillatory integrals in Fourier analysis. In *Functional Analysis*, pages 321–408. Princeton University Press, 2011.

[64] Betsy Stovall. Endpoint $L^p \to L^q$ bounds for integration along certain polynomial curves. *Journal of Functional Analysis*, 259(12):3205–3229, 2010.

[65] Betsy Stovall. L^p improving multilinear radon-like transforms. *Revista Matemática Iberoamericana*, 27(3):1059–1085, 2011.

[66] Terence Tao. Some recent progress on the restriction conjecture. *Fourier analysis and convexity, Applied and Numerical Harmonic Analysis, Birkhäuser Boston*, pages 217–243, 2004.

[67] Terence Tao. *Higher order Fourier analysis, Graduate Studies in Mathematics*, volume 142. American Mathematical Soc., 2012.

[68] Terence Tao. Sharp bounds for multilinear curved Kakeya, restriction and oscillatory integral estimates away from the endpoint. *Mathematika*, 66(2):517–576, 2020.

[69] Terence Tao, Ana Vargas, and Luis Vega. A bilinear approach to the restriction and Kakeya conjectures. *Journal of the American Mathematical Society*, 11(4):967–1000, 1998.

[70] Terence Tao and James Wright. L^p improving bounds for averages along curves. *Journal of the American Mathematical Society*, 16(3):605–638, 2003.

[71] Marianne Terp. L^p Fourier transformation on non-unimodular locally compact groups. *Advances in Operator Theory*, 2(4):547–583, 2017.

[72] Stefán Ingi Valdimarsson. Optimisers for the Brascamp–Lieb inequality. *Israel Journal of Mathematics*, 168(1):253–274, 2008.

[73] Stefán Ingi Valdimarsson. Geometric Brascamp–Lieb has the optimal best constant. *J. Geom. Anal*, 21:1036–1043, 2011.

[74] Laura Jane Wisewell. Oscillatory integrals and curved Kakeya sets (PhD thesis). *The University of Edinburgh*, 2003.

[75] Laura Jane Wisewell. Kakeya sets of curves. *Geometric and Functional Analysis*, 15(6):1319–1362, 2005.

[76] Ruixiang Zhang. The endpoint perturbed Brascamp–Lieb inequalities with examples. *Analysis & PDE*, 11(3):555–581, 2017.

[77] Pavel Zorin-Kranich. Kakeya–Brascamp–Lieb inequalities. *Collectanea Mathematica*, pages 1–22, 2019.