Quadratic Estimates and Functional Calculi for Inhomogeneous First-Order Operators and Applications to Boundary Value Problems for Schrödinger Equations

by

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Abstract

We develop a holomorphic functional calculus for first-order operators DB to solve boundary value problems for Schrödinger equations $-\operatorname{div} A\nabla u + aVu = 0$ in the upper half-space $\mathbb{R}^{n+1}_+$ when $n \geq 3$. This relies on quadratic estimates for DB, which are proved for coefficients A, a, V that are independent of the transversal direction to the boundary, and comprised of a complex-elliptic pair A, a that are bounded and measurable, and a singular potential V in the reverse Hölder class $B^{\frac{n}{2}}(\mathbb{R}^n)$. The square function bounds are also shown to be equivalent to non-tangential maximal function bounds. This allows us to prove that the Dirichlet regularity and Neumann boundary value problems with $L^2(\mathbb{R}^n)$-data are well-posed if and only if certain boundary trace operators defined by the functional calculus are isomorphisms. We prove this property when the coefficient matrices A and a are either a Hermitian or block structure. More generally, the set of all complex-elliptic A for which the boundary value problems are well-posed is shown to be open in L^∞. We also prove these solutions coincide with those generated from the Lax–Milgram Theorem. Furthermore, we extend this theory to prove quadratic estimates for the magnetic Schrödinger operator $(\nabla + ib)^* A(\nabla + ib)$ when the magnetic field curl b is in the reverse Hölder class $B^{\frac{n}{2}}(\mathbb{R}^n)$.

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CONTENTS

CHAPTER 1

INTRODUCTION

In this thesis we will be concerned with the study of the purely electric Schrödinger operator $-\operatorname{div} A\nabla + aV$, and the purely magnetic Schrödinger operator $(\nabla+ib)^* A(\nabla+ib)$, and boundary value problems for the Schrödinger equation

$$-\operatorname{div}_{t,x} A(x)\nabla_{t,x} u(t,x) + a(x)V(x)u(t,x) = 0, \tag{1.0.1}$$

on the upper half-space $\mathbb{R}^{n+1}_+ := \{(t,x) \in \mathbb{R}\times\mathbb{R}^n : t > 0\}$, for integers $n \geq 3$, where $A \in L^\infty(\mathbb{R}^{n+1}; \mathcal{L}(\mathbb{C}^{1+n}))$ and $a \in L^\infty(\mathbb{R}^{n+1}; \mathcal{L}(\mathbb{C}))$ are complex, t-independent, and elliptic (which we will define in Section 2.6); the electric potential V is in the reverse Hölder class, $B^q(\mathbb{R}^n)$; and the magnetic field associated with the magnetic potential b is also in the reverse Hölder class. We will give definitions of these terms in Chapter 2.

By boundary value problem we mean (1.0.1) together with an additional constraint of functions at the boundary $\partial\mathbb{R}^{n+1}_+$ (which we identify naturally with $\mathbb{R}^n$). We will consider the Neumann an Dirichlet regularity problem in this thesis. We say u is a solution to the Neumann problem if u satisfies (1.0.1) and its normal derivative is equal to a prescribed function at the boundary. We say u is a solution to the Dirichlet regularity problem (henceforth simplified to the regularity problem) if u satisfies (1.0.1) and is equal to some prescribed function at the boundary where equality is in a particular space. We will give precise definitions of these in Chapter 2.

These boundary value problems arise from the study of the Schrödinger equation $\Delta w + Vw = 0$ above a Lipschitz graph $\Omega = \{(t,x) \in \mathbb{R}^{n+1} : t > g(x)\}$ where $g\colon \mathbb{R}^n \to \mathbb{R}$

is a Lipschitz function. The change of variables $u(t,x) := w(t+g(x),x)$ then gives $-\operatorname{div} A\nabla u + Vu = 0$ on $\mathbb{R}^{n+1}_+$ where A is real and symmetric and independent of t. Work in this setting started with Shen in [48]. Here Shen studied the equation $-\Delta u + Vu = 0$ above a Lipschitz curve with $V \in B^\infty(\mathbb{R}^{n+1})$. It was shown there exists a unique solution of the Neumann problem with L^p boundary data for $p \in (1,2]$. Later, in [53], Tao and Wang extended these results to include solving the Neumann problem where $V \in B^n(\mathbb{R}^{n+1})$ and with L^p boundary data for $p \in (1,2]$ or in the Hardy space H^p for $p \in (1-\varepsilon, 1]$ where $\varepsilon \in (0, \frac{1}{n})$. Tao also proved, in [52], the solvability of the corresponding regularity problem for boundary data in a Hardy space adapted to the potential $V \in B^n(\mathbb{R}^{n+1})$. In [55], Yang proved that for a bounded Lipschitz domain and $V \in B^n(\mathbb{R}^n)$, the Neumann and regularity problems are well-posed in L^p, for $p > 2$, if the non-tangential maximal function of the gradient of solutions satisfies an $L^p \to L^2$ weak reverse Hölder estimate. We note that these works build on the theory of boundary value problems for the Laplacian $\Delta u = 0$ on similar domains which have been extensively studied in [20, 21, 22, 26, 35, 54].

More recently, boundary value problems for equations of the form

$$-\operatorname{div}(A\nabla u + b\cdot u) + c\cdot\nabla u + du \tag{1.0.2}$$

have been studied. In [38], Kim and Sakellaris study Green's functions for this equation when b, c, d are in certain Lebesgue spaces, without any smallness condition, but with additional conditions that $d \geq \operatorname{div} b$ or $d \geq \operatorname{div} c$. Here A is assumed to be real and uniformly elliptic. In [43], Sakellaris, considered boundary value problems for (1.0.2) on bounded domains with Dirichlet and regularity boundary data, and the additional condition that A is Hölder continuous. Sakellaris, then extended this to arbitrary domains in [44], where the estimates on the Green functions are in Lorentz spaces and are scale invariant. Also, in [42], Mourgoglou proves well-posedness for the Dirichlet problem in unbounded domains, with coefficients in a local Stummel–Kato class. He also constructs Green functions and proves scale invariance for them. In [23], Davey, Hill, and Mayboroda construct Green's matrices for complex bounded coefficients under particular conditions on the solutions of (1.0.1). Exponential decay of the fundamental solution to $-(\nabla - ia)^T A(\nabla - ia)u + Vu = 0$ was proven by Mayboroda and Poggi in [40]. Also, recent work by Bortz, Luna Garcia, Hofmann, Mayboroda and Poggi, in [19] treats well-posedness of these equations when the coefficients have sufficiently small L^p-norm .

In this thesis we will focus on boundary value problems for (1.0.1) when A and a are complex-valued and elliptic. This includes the situation when $\Delta u + Vu = 0$ on Lipschitz domains. Unlike in [45, 52, 53], we restrict to the case when V is t-independent, however by doing this we will be allowed to have the potential in $B^{\frac{n}{2}}(\mathbb{R}^n)$ which is a more general reverse Hölder class as $B^n(\mathbb{R}^n) \subset B^{\frac{n}{2}}(\mathbb{R}^n)$.

To this end, we develop methods introduced by Auscher, Axelsson, and McIntosh in [4] for boundary value problems for the equation

$$\operatorname{div}_{t,x} A(x)\nabla_{t,x} u(t,x) = 0, \tag{1.0.3}$$

and adapt these methods to include the 0^{th} order term aV in (1.0.1). These methods rely on the bounded holomorphic functional calculus, which we will define later, of a first-order operator DB, where D is a self-adjoint, first-order differential operator and B is a bounded matrix-valued multiplication operator, that is elliptic on $\overline{\mathsf{R}(D)}$. In the case of (1.0.3), this was proved when A has a certain block-type structure, is self-adjoint, or has constant coefficients by Auscher, Axelsson, McIntosh in [4] building on the work on the bounded holomorphic functional calculus for Dirac-type operators by Axelsson, Keith, and McIntosh in [13], which expanded on methods developed for the solution of the Kato square root problem obtained by Auscher, Hofmann, Lacey, McIntosh, and Tchamitchian in [12]. The first-order approach developed in [4] and later in [2] shows an equivalence between the second-order elliptic equation with a first-order Cauchy–Riemann-type system

$$\partial_t F + DBF = 0. \tag{1.0.4}$$

If DB were to be a sectorial operator with angle less than $\frac{\pi}{2}$, then it would generate an analytic semigroup which would solve (1.0.4). However, DB is in fact a bisectorial operator. In the case $V \equiv 0$, the boundedness of the holomorphic functional calculus of DB was proved in [13] and this was used to prove that DB is sectorial on a subspace of $\overline{\mathsf{R}(D)}$. Then, in [4], this was used along with analytic semigroup theory to generate solutions to (1.0.4). The solvability of the boundary value problems were then reduced to showing these solutions come from the correct spaces of boundary data, by constructing a mapping between the initial value of (1.0.4) with the correct boundary data for (1.0.1) and showing that this mapping is invertible. This first-order method has already been adapted to the degenerate elliptic case by Auscher, Rosen, and Rule in [10] and to the parabolic case by Auscher, Egert, and Nyström in [6]. The first-order method was also used recently to prove solvability for elliptic systems with block triangular coefficient matrices A by Auscher, Mourgoglou, and McIntosh in [7].

Thus, the operator DB having a bounded holomorphic functional calculus is pivotal to proving solvability. Therefore, a large part of this thesis is dedicated to establishing the functional calculus results. To do this we develop ideas from [13] and [3] by reducing this to proving quadratic estimates. To overcome the lack of coercivity ((H8) in [13]) from the presence of the potential, we exploit the structure of the operator DB using some ideas introduced by Bailey in [16]. In [16], it is proved that the operator $-\operatorname{div} A\nabla + V$ satisfies a Kato square-root-type estimate for a large class of potentials V using the Axelsson-Keith-McIntosh framework of [13]. However, in [16], the bounded holomorphic functional calculus is proved for Dirac-type operators of the form $\Gamma + \Gamma^* B$ which does not directly imply the same results for operators of the form DB. This is because in our setting we need our perturbation B to have a more general structure, whereas in [16] the perturbation B is of a certain block-type. Therefore, we adapt the methods for such operators, by considering projections onto different components of $\overline{\mathsf{R}(D)}$, and in doing so we restrict the class of potentials we are interested in, namely the reverse Hölder class, $B^{\frac{n}{2}}(\mathbb{R}^n)$ and $L^{\frac{n}{2}}(\mathbb{R}^n)$. To establish bounds on the holomorphic functional calculus for DB, we divide the usual dyadic cube structure of $\mathbb{R}^n$ into certain 'big' and 'small' cubes, depending on a property of the potential V. We shall see that heuristically on 'small' cubes DB behaves similarly to the case when $V \equiv 0$, whereas, on 'big' cubes DB will be treated differently by using the Fefferman–Phong inequality from [5].

Extending the concept of well-posedness, we discuss the notion of compatible well-posedness. That is we compare the solution we generate from the first-order method with the energy solutions (also known as variational solutions) generated from the Lax–Milgram theorem, when the boundary data is in the intersection of the respective boundary spaces. To do this we develop the theory of trace spaces for the Schrödinger equation and in doing so we discuss the notion of fractional Sobolev spaces adapted to the potential V. We also use the fractional Sobolev spaces associated with an operator introduced by Auscher, McIntosh, and Nahmod in [8], and their relationship with interpolation theory and quadratic estimates. We use both of these different types of fractional Sobolev spaces to give a semigroup representation of energy solutions for the Schrödinger equation. We then use this and the bounded holomorphic functional calculus to prove that the solutions from the first-order method coincide with those derived from the Lax–Milgram theorem.

We also consider the magnetic Schrödinger operator $(\nabla + ib)^* A(\nabla + ib)$ and prove quadratic estimates (and therefore the existence of a bounded holomorphic functional calculus) for the associated first-order operator. The difficulty here, compared with the purely electric case, is the conditions are imposed on the magnetic field $\operatorname{curl} b$ and not the magnetic potential b. Therefore, we take advantage of a gauge transform, introduced by Iwatsuka in [34], which allows us to introduce a new magnetic potential, corresponding to the same magnetic field, which can be controlled by $\operatorname{curl} b$. The drawback of this is that the new magnetic potential is only defined locally and so we must localise the first-order operator to attain local quadratic estimates. We do this by using a localised version of the usual dyadic decomposition of $\mathbb{R}^n$, where we perform a stopping time argument to obtain a collection of cubes on which we can introduce the new local magnetic potentials. These methods have evolved from the theory of Riesz transform estimates for the magnetic Schödinger equation developed by Ben Ali in [17] and [18] which builds on Shen's work in [47].

This thesis consists of two broad parts. The first is establishing the quadratic estimates and the second is the applications of these quadratic estimates to proving solvability of boundary value problems. The thesis is structured as follows: Chapter 2 is the preliminary chapter with known results which we will use repeatedly throughout; Chapters 3 and 4 are dedicated to the proof of the quadratic estimates associated with the Schrödinger and magnetic Schrödinger operators respectively; Chapter 5 is concerned with direct consequences of the quadratic estimates including Kato square-root-type estimates and analytic dependence on the coefficients; in Chapter 6 the connection between equations (1.0.1) and (1.0.4) is explored; the non-tangential maximal function estimates are the focus of Chapter 7; and finally Chapter 8 finishes the mathematical content of the thesis with proofs of well-posedness and compatible well-posedness for the Schrödinger equation.

Much of the work is contained in the preprint [41], in particular, forming the majority of Chapters 3, 5, 6, and 7. I also acknowledge the contribution of my collaborator Andrew Morris.

CHAPTER 2

PRELIMINARIES

This chapter is dedicated to giving some of the important standard results that we will be using throughout.

2.1 Functional Calculus of Bisectorial Operators

This section gives the definitions and some important results regarding the functional calculus of bisectorial operators and the relationship between functional calculus and quadratic estimates. For the proofs and more details see [1] or [31]. We start by defining the closed and open sectors as

$$S_{\omega+} := \{z \in \mathbb{C} : |\arg(z)| \leq \omega\} \cup \{0\},$$
$$S^o_{\mu+} := \{z \in \mathbb{C} : z \neq 0, |\arg(z)| < \mu\},$$

where $0 \leq \omega < \mu < \pi$. Then we define the open bisector $S^o_\mu := S^o_{\mu+} \cup (-S^o_{\mu+})$ and closed bisector $S_\mu := S_{\mu+} \cup (-S_{\mu+})$ for $0 \leq \mu < \frac{\pi}{2}$. For a closed operator T we denote $\sigma(T)$ as the spectrum of T.

Definition 2.1.1. Let $\mathcal{X}$ be a Banach space. Let $0 \leq \omega < \pi$. Then a closed operator T on $\mathcal{X}$ is sectorial of type $S_{\omega+}$ (or ω-sectorial) if $\sigma(T) \subseteq S_{\omega+}$ and, for each $\mu > \omega$, there exists $C_\mu > 0$ such that

$$\|(T - zI)^{-1}\| \leq C_\mu |z|^{-1}, \qquad \forall z \in \mathbb{C} \setminus S_{\mu+}.$$

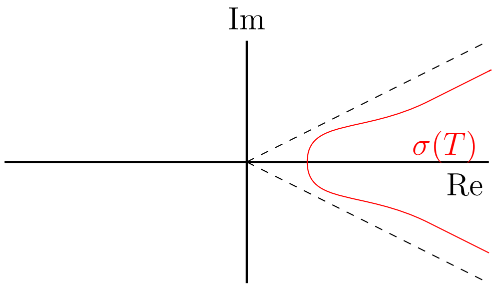

Figure 2.1: The sector with spectrum of a sectorial operator

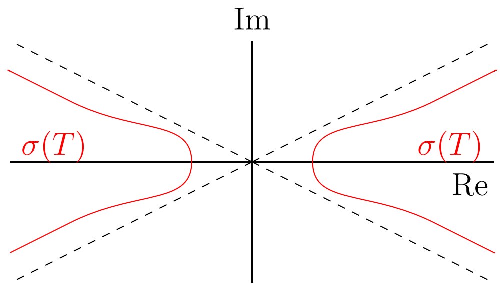

Figure 2.2: The bisector with spectrum of a bisetorial operator

We have a similar definition for bisectorial.

Definition 2.1.2. Let $\mathcal{X}$ be a Banach space. Let $0 \le \omega < \frac{\pi}{2}$. Then a closed operator T on $\mathcal{X}$ is is bisectorial of type S_ω (or ω-bisectorial) if $\sigma(T) \subseteq S_\omega$ and, for each $\mu > \omega$, there exists $C_\mu > 0$ such that

$$\|(T - zI)^{-1}\| \le C_\mu |z|^{-1}, \qquad \forall z \in \mathbb{C} \setminus S_\mu.$$

Let $H(S_\mu^o)$ be the set of all holomorphic functions on S_μ^o. We define the following

$$H^\infty(S_\mu^o) = \{f \in H(S_\mu^o) : \|f\|_\infty < \infty\},$$
$$\Psi(S_\mu^o) = \{\psi \in H^\infty(S_\mu^o) : \exists s, C > 0,\ |\psi(z)| \le C|z|^s(1 + |z|^{2s})^{-1}\},$$
$$\mathcal{F}(S_\mu^o) = \{f \in H(S_\mu^o) : \exists s, C > 0,\ |f(z)| \le C(|z|^s + |z|^{-s})\}.$$

Note that $\Psi(S_\mu^o) \subseteq H^\infty(S_\mu^o) \subseteq \mathcal{F}(S_\mu^o) \subseteq H(S_\mu^o)$.

Now for $\psi \in \Psi(S_{\mu+}^o)$ and γ, the unbounded contour $\gamma = \{z = re^{\pm i\theta} : r \ge 0\}$ parametrised clockwise around $S_{\omega+}$ such that $0 \le \omega < \theta < \mu < \frac{\pi}{2}$. Then for an injective, sectorial of type $S_{\omega+}$ operator T we define $\psi(T)$ as

$$\psi(T) = \frac{1}{2\pi i} \int_\gamma \psi(z)(T - zI)^{-1} dz.$$

Similarly, let $\psi \in \Psi(S_\mu^o)$ and the unbounded contour $\gamma = \{z = \pm re^{\pm i\theta} : r \ge 0\}$ parametrised clockwise around S_ω such that $0 \le \omega < \theta < \mu < \frac{\pi}{2}$. Then for an injective, bisectorial of type S_ω operator T we define $\psi(T)$ as

$$\psi(T) = \frac{1}{2\pi i} \int_\gamma \psi(z)(T - zI)^{-1} dz.$$

Note as $\psi \in \Psi(S^o_{\mu+})$ (or $\Psi(S^o_\mu)$) then the integral converges and so the resolvent operators are bounded we have $\psi(T) \in \mathcal{L}(\mathcal{X})$. Then we define the $\mathcal{F}$-functional calculus for $f \in \mathcal{F}(S^o_{\mu+})$ in the following way: let $\psi \in \Psi(S^o_{\mu+})$ be defined as

$$\psi(z) = \left(\frac{z}{(1+z)^2}\right)^{s+1},$$

where $s > 0$ is such that $|f(z)| \leq C(|z|^s + |z|^{-s})$ for some $C > 0$. Therefore $\psi f \in \Psi(S^o_{\mu+})$. Then for an injective, bisectorial of type S_ω operator T we define the closed operator $f(T)$ as

$$f(T) = (\psi(T))^{-1}(f\psi)(T). \tag{2.1.1}$$

One of the most important questions related to the functional calculus is whether the operator's functional calculus is bounded.

Definition 2.1.3. Let $\mathcal{X}$ be a Banach space. Let T be an injective, bisectorial operator of type S_ω in $\mathcal{X}$, and $0 \leq \omega < \mu < \frac{\pi}{2}$. Then we say that T has bounded holomorphic (or H^∞) functional calculus if, for all $f \in H^\infty(S^o_\mu)$, then $f(T) \in \mathcal{L}(\mathcal{X})$ and there exists $c_\mu > 0$ such that

$$\|f(T)u\| \leq c_\mu \|f\|_\infty \|u\|, \quad \forall u \in \mathcal{X}.$$

In general it is hard to prove whether an operator has a bounded H^∞ functional calculus or not. The following theorem gives an equivalent (and somewhat easier to prove) condition to an operator having a bounded holomorphic functional calculus.

Theorem 2.1.4. Let $\mathcal{H}$ be a Hilbert space. Let T be an injective operator of type S_ω in $\mathcal{H}$. Then the following are equivalent:

1. T has a bounded S^o_μ holomorphic functional calculus for all $\mu \in (\omega, \frac{\pi}{2})$;

2. there exists $c_\mu > 0$ such that $\|\psi(T)u\| \leq c_\mu \|\psi\|_\infty \|u\|$ for all $u \in \mathcal{H}$ and for all $\psi \in \Psi(S^o_\mu)$ for some $\mu \in (\omega, \frac{\pi}{2})$;

3. the following estimate holds,

$$\int_0^\infty \|tT(I + t^2T^2)^{-1}u\|_2^2 \frac{\mathrm{d}t}{t} \approx \|u\|_2^2, \quad \text{for all } u \in \mathcal{H}.$$

We note that in the definition of bounded holomorphic functional calculus we present the operator T as being injective. In general we will not be dealing with injective operators, however, we will later restrict our operators to a subspace where they are injective (see Proposition 2.5.1), and so these result will be applicable after such a restriction.

2.2 Reverse Hölder Weights

This section give the definitions and some of the important properties associated with reverse Hölder weights. First, for a cube, $Q \subset \mathbb{R}^n$, we denote the sidelength as $l(Q)$ and the Lebesgue measure as $|Q|$. we define $⨍_Q f := |Q|^{-1} \int_Q f$ for all $f \in L^1_{\text{loc}}(\mathbb{R}^n)$.

Definition 2.2.1. Let $q \in (1, \infty)$. Then a non-negative locally L^1-function, V, is in the reverse Hölder class, $B^q(\mathbb{R}^n)$, if there exists $C > 0$ such that the reverse Hölder inequality

$$\left(⨍_Q V^q \,\mathrm{d}x \right)^{\frac{1}{q}} \leq C \left(⨍_Q V \,\mathrm{d}x \right)$$

for all cubes Q.

We define the best constant in the reverse Hölder inequality as

$$[\![V]\!]_q := \sup_{Q \subset \mathbb{R}^n} \frac{\left(⨍_Q V^q\right)^{\frac{1}{q}}}{⨍_Q V} < \infty.$$

We first we note for all $q > 1$ we have $B^q(\mathbb{R}^n) \subset A_\infty$, the set of all Muckenhoupt weights, see [50, Chapter V Theorem 3]. We say $\omega \in A_\infty$ if there exists constants

$\gamma, \delta \in (0,1)$ such that for all cubes Q and all subsets $E \subseteq Q$ we have, if

$$|E| \leq \gamma |Q|, \text{ then } \int_E \omega(x)\,\mathrm{d}x \leq \delta \int_Q \omega(x)\,\mathrm{d}x. \tag{2.2.1}$$

We call γ and δ the A_∞ constants of ω. In [28] the important property that: if $V \in B^q(\mathbb{R}^n)$ then there exists $\varepsilon > 0$, depending only on n and $[\![V]\!]_q$, such that $V \in B^{q+\varepsilon}(\mathbb{R}^n)$. Another key property of $B^q(\mathbb{R}^n)$ is that $V(x)\,\mathrm{d}x$ is a doubling measure, that is

$$\int_{2Q} V(x)\,\mathrm{d}x \leq c_d \int_Q V(x)\,\mathrm{d}x,$$

where $c_d > 1$ is the doubling constant for V.

For $V \in B^q(\mathbb{R}^n)$ we define $m(x,V)$, the Shen maximal function, first introduced by Shen in [45], as

$$m(x,V) \coloneqq \sup \left\{ r > 0 \colon r^2 ⨍_{B_r(x)} V(y)\,\mathrm{d}y \leq 1 \right\}, \tag{2.2.2}$$

where $B_r(x)$ is the ball of radius r and centre x. We will also use a discrete version of $m(x,V)$ which we will incorporate into the dyadic mesh of $\mathbb{R}^n$ at each scale. Therefore, as in [5], we will consider the following two types of cubes:

$$l(Q)^2 ⨍_Q V(x)\,\mathrm{d}x \leq 1 \quad \text{or} \quad l(Q)^2 ⨍_Q V(x)\,\mathrm{d}x > 1,$$

we call the former type as small and the latter type as big. To see why we use the term small let $x \in \mathbb{R}^n$ be a Lebesgue point of V and let $\{Q_k\}_{k\in\mathbb{N}}$ be the collection of cubes such that $\lim_{k\to\infty} l(Q_k) = 0$ and $x \in Q_k$ for all $k \in \mathbb{N}$. Then by the Lebesgue differentiation theorem (see [51, Corollary I.1]) we have

$$\lim_{k\to\infty} \left[l(Q_k)^2 ⨍_{Q_k} V \right] = 0 \cdot V(x) = 0. \tag{2.2.3}$$

Note that, as $V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ then, again using [51, Corollary I.1], that almost every $x \in \mathbb{R}^n$ is a Lebesgue point of V.

A natural question to ask is: Given a small cube Q are there conditions on a subcube

$R \subset Q$ which can guarantee that R is also small? We give the following general lemma which answers this question. The lemma expands on [45, Lemma 1.2].

Lemma 2.2.2. Let $V \in B^q(\mathbb{R}^n)$, for $q > \frac{n}{2}$. Let $C \geq c > 0$ Suppose that cubes Q and $R \subset Q$ we have

$$l(Q)^2 ⨍_Q V \leq C, \quad \text{and} \quad l(R)^2 ⨍_R V \geq c, \tag{2.2.4}$$

Then, there exists $\tilde{c} > 0$ such that

$$\tilde{c} l(Q) \leq l(R),$$

where $\tilde{c}$ depends only on n, q, $[\![V]\!]_q$, C, and c.

Proof. Then, using Jensen's inequality and the reverse Hölder property of V, we have

$$⨍_R V \leq \left(⨍_R V^q \right)^{\frac{1}{q}} \leq \left(\frac{|Q|}{|R|} ⨍_Q V^q \right)^{\frac{1}{q}} \leq \left(\frac{l(Q)}{l(R)} \right)^{\frac{n}{q}} [\![V]\!]_q ⨍_Q V.$$

Thus, by (2.2.4) we have

$$c < l(R)^2 ⨍_R V \leq \left(\frac{l(Q)}{l(R)} \right)^{\frac{n}{q}-2} [\![V]\!]_q l(Q)^2 ⨍_Q V \leq \left(\frac{l(Q)}{l(R)} \right)^{\frac{n}{q}-2} [\![V]\!]_q C.$$

Therefore, as $\frac{n}{q} - 2 < 0$ we have

$$l(Q) \leq \left(\frac{[\![V]\!]_q C}{c} \right)^{\frac{1}{2-\frac{n}{q}}} l(R).$$

Then letting $\tilde{c} = \left(\frac{[\![V]\!]_q C}{c} \right)^{\frac{-1}{2-\frac{n}{q}}}$ completes the proof. $\square$

The contrapositive of Lemma 2.2.2 will also be very useful and so we present it as well.

Lemma 2.2.3. Let $V \in B^q(\mathbb{R}^n)$, with $q > \frac{n}{2}$. Let $C \geq c > 0$ Let Q be a cube such that

$$l(Q)^2 ⨍_Q V \leq C.$$

Then, if R is a subcube of Q with

$$l(R) < \left(\frac{[\![V]\!]_q C}{c}\right)^{\frac{-1}{2-\frac{n}{q}}} l(Q),$$

then

$$l(R)^2 \fint_R V < c.$$

Remark 2.2.4. If $V \in B^{\frac{n}{2}}(\mathbb{R}^n)$ then by the self improvement property of reverse Hölder weights we have there exists $q > \frac{n}{2}$ such that $V \in B^q(\mathbb{R}^n)$. Therefore, Lemmas 2.2.2 and 2.2.3 apply to all $V \in B^{\frac{n}{2}}(\mathbb{R}^n)$ with constants depending on the exponent q.

We also include a version of the Fefferman–Phong inequality on cubes as in [5]. This inequality is used to bound the local L^p-norm with a local Sobolev norm adapted to the weight V, where the constant depends explicitly on the behaviour of V on Q.

Proposition 2.2.5 (Improved Fefferman–Phong Inequality)**.** Let $p \in [1, \infty)$ Let $w \in A_\infty$. Then there are constants $C > 0$ and $\beta \in (0, 1)$ depending only on the A_∞ constants of w (as in (2.2.1)) and n, such that for all cubes Q with side-length $l(Q)$ and $u \in \mathcal{C}^1(\mathbb{R}^n)$ we have

$$\frac{C m_\beta(l(Q)^p \operatorname{av}_Q w)}{l(Q)^p} \int_Q |u|^p \leq \int_Q |\nabla u|^p + w|u|^p,$$

where $\operatorname{av}_Q w = \fint_Q w$, and $m_\beta(x) = x$ for $x \leq 1$ and $m_\beta(x) = x^\beta$ for $x > 1$.

2.3 Sobolev Spaces Adapted to Singular Potentials

Throughout this section, suppose that $n > 2$ is an integer and that Ω is an open subset of $\mathbb{R}^n$. The potential $V : \mathbb{R}^n \to \mathbb{C}$ always denotes a locally integrable function. In most results, it will be either complex-valued in $L^p(\mathbb{R}^n)$ for some $p \in (1, \infty)$, or nonnegative-valued in the reverse Hölder class $B^q(\mathbb{R}^n)$ for some $q \in (1, \infty)$ so that either

$$\|V\|_p := \left(\int_{\mathbb{R}^n} |V|^p\right)^{\frac{1}{p}} < \infty \quad \text{or} \quad [\![V]\!]_q := \sup_{Q \subset \mathbb{R}^n} \frac{\left(\fint_Q V^q\right)^{\frac{1}{q}}}{\fint_Q V} < \infty.$$

We will need to adapt the usual Sobolev spaces to account for the potential V. The following notation will be convenient for this purpose. If $f \in L^2_{\text{loc}}(\Omega)$, then $\nabla_\mu f \in \mathcal{D}'(\Omega)$ denotes the distribution

$$\nabla_\mu f := \begin{pmatrix} \nabla f \\ |V|^{\frac{1}{2}} f \end{pmatrix}, \tag{2.3.1}$$

where $\nabla f = (\partial_1 f, \dots, \partial_n f)^T$ is the standard distributional gradient of f, whilst the product $(|V|^{\frac{1}{2}} f)(x) = |V(x)|^{\frac{1}{2}} f(x)$ is defined pointwise almost everywhere on Ω and belongs to $L^1_{\text{loc}}(\Omega)$ (and thus can be interpreted as a distribution) because $f \in L^2_{\text{loc}}(\Omega)$ and $V \in L^1_{\text{loc}}(\mathbb{R}^n)$. We note that ∇_μ depends on the dimension of the domain of the function. In particular, we will use ∇_μ for function on both $\mathbb{R}^n$ and $\mathbb{R}^{n+1}_+$ and ∇_μ will have different dimensions in these two cases, n and $n+1$ dimensions respectively. This should be clear from the context.

Our starting point is a minor variant of the standard Sobolev inequality (see, for instance, Section 2 in Chapter V of [51]): If $f \in L^p(\mathbb{R}^n)$ for some $p \in [1, \infty)$ and $\nabla f \in L^2(\mathbb{R}^n)$, then

$$\|f\|_{2^*} \lesssim_n \|\nabla f\|_2. \tag{2.3.2}$$

where $2^* := \frac{2n}{n-2}$ is the Sobolev exponent for $\mathbb{R}^n$. We will consider potentials that can be controlled by this inequality as follows: If $V \in L^{\frac{n}{2}}(\mathbb{R}^n)$, then Hölder's inequality implies that

$$\||V|^{\frac{1}{2}} f\|_2^2 \leq \|V\|_{\frac{n}{2}} \|f\|_{2^*}^2. \tag{2.3.3}$$

If $V \in B^{\frac{n}{2}}(\mathbb{R}^n)$, then Hölder's inequality implies the local variant

$$\||V|^{\frac{1}{2}} f\|_{L^2(Q)}^2 \leq \|V\|_{L^{\frac{n}{2}}(Q)} \|f\|_{L^{2^*}(Q)}^2 \leq [\![V]\!]_{\frac{n}{2}} \left(l(Q)^2 \fint_Q |V| \right) \|f\|_{2^*}^2 \tag{2.3.4}$$

for all cubes $Q \subset \mathbb{R}^n$. The following technical lemma provides the basis for the definition of our adapted Sobolev spaces.

Lemma 2.3.1. Let $p \in [1, \infty)$ and suppose that $V \in L^{\frac{n}{2}}_{\text{loc}}(\mathbb{R}^n)$. If $\{f_m\}$ is a sequence in $L^p(\Omega)$ that converges to some f in $L^p(\Omega)$, and $\{\nabla_\mu f_m\}$ is a Cauchy sequence in

$L^2(\Omega, \mathbb{C}^{n+1})$, then $\{\nabla_\mu f_m\}$ converges to $\nabla_\mu f$ in $L^2(\Omega, \mathbb{C}^{n+1})$.

Proof. Suppose that $\{f_m\}$ and f satisfy the hypotheses of the lemma, in which case $\{\nabla f_m\}$ converges to some $(F_1, \ldots, F_n)^T$ in $L^2(\Omega, \mathbb{C}^n)$ and $\{|V|^{\frac{1}{2}} f_m\}$ converges to some F_{n+1} in $L^2(\Omega)$. It suffices to prove that $F_j = \partial_j f$ when $j \in \{1, \ldots, n\}$ whilst $F_{n+1} = |V|^{\frac{1}{2}} f$. Fix $\varepsilon > 0$ and $\varphi \in \mathcal{C}_c^\infty(\Omega)$ arbitrarily. Let $N \in \mathbb{N}$ be such that

$$\|F_j - \partial_j f_m\|_2 \|\varphi\|_2 < \frac{\varepsilon}{2} \quad \text{and} \quad \|f_m - f\|_p \|\partial_j \varphi\|_{p'} < \frac{\varepsilon}{2}$$

for all $j \in \{1, \ldots, n\}$ whenever $m > N$. Now, if $j \in \{1, \ldots, n\}$, then by the definition of the distributional derivative and Hölder's inequality imply the simple estimate

$$\left| \left(\int F_j \varphi \right) - \left(- \int f \partial_j \varphi \right) \right| \leq \|F_j - \partial_j f_m\|_2 \|\varphi\|_2 + \|f_m - f\|_p \|\partial_j \varphi\|_{p'} < \varepsilon.$$

Therefore, as $\varepsilon > 0$ and φ were arbitrary then $\int F_j \varphi = - \int f \partial_j \varphi$ and thus $F_j = \partial_j f$. In particular, this shows that ∇f_m converges to ∇f in $L^2(\Omega, \mathbb{C}^n)$, which we shall now rely on to complete the proof. Suppose that Q is a cube contained in Ω. Again fix $\varepsilon > 0$ and $\varphi \in \mathcal{C}_c^\infty(\Omega)$ supported in Q arbitrarily. Let $N \in \mathbb{N}$ be such that

$$\|F_{n+1} - |V|^{\frac{1}{2}} f_m\|_2 \|\varphi\|_2 < \frac{\varepsilon}{2} \quad \text{and} \quad \|\nabla(f_m - f)\|_2 \|\varphi\|_2 < \frac{\varepsilon}{2}$$

Then, by Hölder's Inequality and (2.3.2), we obtain

$$\begin{aligned} \left| \int (F_{n+1} - |V|^{\frac{1}{2}} f) \varphi \right| &\leq \|F_{n+1} - |V|^{\frac{1}{2}} f_m\|_2 \|\varphi\|_2 + \||V|^{\frac{1}{2}} (f_m - f)\|_{L^2(Q)} \|\varphi\|_2 \\ &\lesssim_{n,Q} \|F_{n+1} - |V|^{\frac{1}{2}} f_m\|_2 \|\varphi\|_2 + \|V\|_{L^{\frac{n}{2}}(Q)} \|\nabla f_m - \nabla f\|_2 \|\varphi\|_2 \\ &\lesssim_{n,V,Q} \varepsilon. \end{aligned}$$

Similar to before we have $F_{n+1} = |V|^{\frac{1}{2}} f_m$ almost everywhere on Q, and thus also almost everywhere on Ω. $\square$

We now define the adapted Sobolev space $\mathcal{V}^{1,2}(\Omega)$ to be the inner-product space consisting of the set

$$\mathcal{V}^{1,2}(\Omega) := \{f \in L^2(\Omega) : \nabla_\mu f \in L^2(\Omega, \mathbb{C}^{n+1})\}$$

with the (complex) inner-product

$$\langle f, g\rangle_{\mathcal{V}^{1,2}(\Omega)} := \int_\Omega f\overline{g} + \int_\Omega \nabla_\mu f \cdot \overline{\nabla_\mu g}$$

and the associated norm

$$\|f\|_{\mathcal{V}^{1,2}(\Omega)} := \left(\|f\|^2_{L^2(\Omega)} + \|\nabla_\mu f\|^2_{L^2(\Omega,\mathbb{C}^{n+1})}\right)^{\frac{1}{2}}.$$

If either $V \in L^{\frac{n}{2}}(\mathbb{R}^n)$ or $V \in B^{\frac{n}{2}}(\mathbb{R}^n)$, then Lemma 2.3.1 shows that $\mathcal{V}^{1,2}(\Omega)$ is a Hilbert space. We then define $\mathcal{V}^{1,2}_c(\Omega)$ to be the closure of $\mathcal{C}^\infty_c(\Omega)$ in $\mathcal{V}^{1,2}(\Omega)$. In the case $\Omega = \mathbb{R}^n$, it holds that $\mathcal{V}^{1,2}(\mathbb{R}^n) = \mathcal{V}^{1,2}_c(\mathbb{R}^n)$, since $\mathcal{C}^\infty_c(\mathbb{R}^n)$ is a dense subspace of $\mathcal{V}^{1,2}(\mathbb{R}^n)$. The density of $\mathcal{C}^\infty_c(\mathbb{R}^n)$ in the case $V \in B^{\frac{n}{2}}(\mathbb{R}^n)$ is proved in [24, Theorem 1.8.1], which only requires that V is nonnegative and locally integrable. In the case $V \in L^{\frac{n}{2}}(\mathbb{R}^n)$, the Sobolev inequality in (2.3.3) implies that $\|f\|^2_{\mathcal{V}^{1,2}} \eqsim \|f\|^2_2 + \|\nabla f\|^2_2 =: \|f\|^2_{W^{1,2}}$, so in fact $\mathcal{V}^{1,2}(\mathbb{R}^n)$ is then just an equivalent normed space to the usual Sobolev space $W^{1,2}(\mathbb{R}^n)$, for which density of $\mathcal{C}^\infty_c(\mathbb{R}^n)$ is well-known. We also define $\mathcal{V}^{1,2}_{\text{loc}}(\Omega)$ to be the set of all $f \in L^2_{\text{loc}}(\Omega)$ such that $f \in \mathcal{V}^{1,2}(\Omega')$ for all open sets Ω' with compact closure $\overline{\Omega'} \subset \Omega$ (henceforth denoted $\Omega' \subset\subset \Omega$).

We also define the homogeneous space $\dot{\mathcal{V}}^{1,2}(\mathbb{R}^n)$ to be the completion of the normed space consisting of the set $\mathcal{C}^\infty_c(\mathbb{R}^n)$ with the norm

$$\|f\|_{\dot{\mathcal{V}}^{1,2}(\mathbb{R}^n)} := \|\nabla_\mu f\|_{L^2(\mathbb{R}^n,\mathbb{C}^{n+1})}.$$

The precompleted space is a genuine normed space, since if $f \in \mathcal{D}'(\mathbb{R}^n)$ and $\|\nabla f\|_{L^2(\mathbb{R}^n)} = 0$, then f is a constant function, so when f is also in $\mathcal{C}^\infty_c(\mathbb{R}^n)$, it must hold that $f = 0$. Moreover, the Sobolev inequality (2.3.2), and Lemma 2.3.1 in the case $p = 2^*$, show that there is an injective embedding from the completion into $L^{2^*}(\mathbb{R}^n)$, allowing us to henceforth identify it as the set

$$\dot{\mathcal{V}}^{1,2}(\mathbb{R}^n) = \{f \in L^{2^*}(\mathbb{R}^n) : \nabla_\mu f \in L^2(\mathbb{R}^n)\}$$

with the norm equivalence

$$\|f\|_{\dot{\mathcal{V}}^{1,2}(\mathbb{R}^n)} = \|\nabla_\mu f\|_{L^2(\mathbb{R}^n,\mathbb{C}^{n+1})} \eqsim \left(\|f\|^2_{L^{2^*}(\mathbb{R}^n)} + \|\nabla_\mu f\|_{L^2(\mathbb{R}^n,\mathbb{C}^{n+1})}\right)^{\frac{1}{2}}.$$

In particular, the set inclusion $\dot{\mathcal{V}}^{1,2}(\mathbb{R}^n) \supseteq \{f \in L^{2^*}(\mathbb{R}^n) : \nabla_\mu f \in L^2(\mathbb{R}^n)\}$ requires the density of $\mathcal{C}_c^\infty(\mathbb{R}^n)$, with respect to the norm $\|\nabla_\mu f\|_{L^2(\mathbb{R}^n,\mathbb{C}^{n+1})}$, in the latter set. This density can be proved using the arguments in Theorem 1.8.1 of [24], as discussed above for the space $\mathcal{V}^{1,2}(\mathbb{R}^n)$. In the case $V \in L^{\frac{n}{2}}(\mathbb{R}^n)$, we also have the equivalence $\|f\|^2_{\dot{\mathcal{V}}^{1,2}} \eqsim \|f\|^2_{2^*} + \|\nabla f\|^2_2 \eqsim \|\nabla f\|^2_2$, so $\dot{\mathcal{V}}^{1,2}(\mathbb{R}^n)$ is then just the realisation of the usual homogeneous Sobolev space $\dot{W}^{1,2}(\mathbb{R}^n)$ in which each equivalence class of locally integrable functions modulo constant functions $[f] \in L^1_{\text{loc}}(\mathbb{R}^n)/\mathbb{C}$ is identified with a unique function $g \in L^{2^*}(\mathbb{R}^n)$ such that $g \in [f]$.

Now we define $\dot{\mathcal{V}}^{1,2}(\mathbb{R}^{n+1}_+)$ to be the Banach Space consisting of the set

$$\dot{\mathcal{V}}^{1,2}(\mathbb{R}^{n+1}_+) := \{f \in L^{2^*}(\mathbb{R}^{n+1}_+) : \nabla_\mu f \in L^2(\mathbb{R}^{n+1}_+, \mathbb{C}^{n+2})\}$$

with the norm

$$\|f\|_{\dot{\mathcal{V}}^{1,2}(\mathbb{R}^{n+1}_+)} := \|\nabla_\mu f\|_{L^2(\mathbb{R}^{n+1}_+,\mathbb{C}^{n+2})},$$

where $2^* := \frac{2(n+1)}{(n+1)-2} = \frac{2(n+1)}{n-1}$ denotes the Sobolev exponent on $\mathbb{R}^{n+1}$, and since $\mathbb{R}^{n+1}_+ \subset \mathbb{R}^{n+1}$ and so following (2.3.1) then ∇_μ is understood on $\mathbb{R}^{n+1}_+$ as

$$\nabla_\mu f(t,x) := \begin{bmatrix} \partial_t f(t,x) \\ \nabla_\| f(t,x) \\ |V|^{\frac{1}{2}}(x) f(t,x) \end{bmatrix}$$

for all $t > 0$ and $x \in \mathbb{R}^n$. For a function defined on $\mathbb{R}^{n+1}_+$ we use $\nabla_\|$ to denote the derivatives in the transversal directions. That is

$$\nabla_{\|} f = \begin{bmatrix} \partial_1 f \\ \vdots \\ \partial_n f \end{bmatrix},$$

where $\partial_k = \frac{\partial}{\partial x_k}$ (we use $\partial_0 = \partial_t$).

Following [30], we define $\mathcal{C}(\overline{\mathbb{R}^{n+1}_+})$ to be the set of continuous functions on $\overline{\mathbb{R}^{n+1}_+}$ and $\mathcal{C}^k(\overline{\mathbb{R}^{n+1}_+})$ to be the set of functions in $\mathcal{C}^k(\mathbb{R}^{n+1}_+)$ all of whose derivatives of order less than k have a continuous extension to $\overline{\mathbb{R}^{n+1}_+}$. We also denote $\mathcal{C}^k_c(\overline{\mathbb{R}^{n+1}_+})$ to be the set of function in $\mathcal{C}^k(\overline{\mathbb{R}^{n+1}_+})$ with compact support. Note that, the support of functions in $\mathcal{C}^k_c(\overline{\mathbb{R}^{n+1}_+})$ can include the boundary of $\overline{\mathbb{R}^{n+1}_+}$. We have the following proposition about the density of $\mathcal{C}^k_c(\overline{\mathbb{R}^{n+1}_+})$ in $\dot{\mathcal{V}}(\mathbb{R}^{n+1}_+)$.

Proposition 2.3.2. If $V \in B^{\frac{n}{2}}(\mathbb{R}^n)$, then $C^\infty_c(\overline{\mathbb{R}^{n+1}_+})$ is dense in $\dot{\mathcal{V}}^{1,2}(\mathbb{R}^{n+1}_+)$

Proof. If $f \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^{n+1}_+)$, then $f \in \dot{H}^1(\mathbb{R}^{n+1}_+)$, so by [7, Lemma 3.1] the extension $\tilde{f}$ of f defined by reflection across the boundary of $\mathbb{R}^{n+1}_+$ is in $\dot{H}^1(\mathbb{R}^{n+1})$, further details can be found in [30, Theorem 7.25]. The main point is that $f \in L^2_{\mathrm{loc}}(\overline{\mathbb{R}^{n+1}_+})$ because $f \in L^{2^*}(\mathbb{R}^{n+1}_+)$. In particular, this property ensures that the "translated mollifiers" used in the proof of [30, Theorem 7.25] converge in $\dot{H}^1(\mathbb{R}^{n+1}_+)$. Meanwhile, the extension $\tilde{f}$ is also in $L^{2^*}(\mathbb{R}^{n+1})$ with $\|\tilde{f}\|_{L^{2^*}(\mathbb{R}^{n+1})} = 2\|f\|_{L^{2^*}(\mathbb{R}^{n+1}_+)}$ and $\||V|^{\frac{1}{2}}\tilde{f}\|_{L^2(\mathbb{R}^{n+1})} = 2\||V|^{\frac{1}{2}}f\|_{L^2(\mathbb{R}^{n+1}_+)}$, hence $\tilde{f}$ is in $\dot{\mathcal{V}}^{1,2}(\mathbb{R}^{n+1})$.

Next, since $\mathcal{C}^\infty_c(\mathbb{R}^{n+1})$ is dense in $\dot{\mathcal{V}}^{1,2}(\mathbb{R}^{n+1})$, there exists a sequence $(f_k)_{k\in\mathbb{N}}$ in $\mathcal{C}^\infty_c(\mathbb{R}^{n+1})$ that converges to $\tilde{f}$ in $\dot{\mathcal{V}}^{1,2}(\mathbb{R}^{n+1})$. This means that $\nabla_\mu f_k$ converges to $\nabla_\mu \tilde{f}$ in $L^2(\mathbb{R}^{n+1}, \mathbb{C}^{n+2})$, hence the restrictions to the upper half-space $(\nabla_\mu f_k)|_{\mathbb{R}^{n+1}_+} = \nabla_\mu (f_k|_{\mathbb{R}^{n+1}_+})$ converge to $(\nabla_\mu \tilde{f})|_{\mathbb{R}^{n+1}_+} = \nabla_\mu f$ in $L^2(\mathbb{R}^{n+1}_+, \mathbb{C}^{n+2})$. The required density follows since $f_k|_{\mathbb{R}^{n+1}_+}$ is in $C^\infty_c(\overline{\mathbb{R}^{n+1}_+})$. $\square$

Remark 2.3.3. We define $\dot{\mathcal{V}}^{1,2}$ on the upper half-space $\mathbb{R}^{n+1}_+$ in a different way to how we define it on the full space $\mathbb{R}^{n+1}$. The density of $\mathcal{C}^\infty_c$ in the full space definition was instrumental to the result above. In fact, the above proof actually shows that

$$\dot{\mathcal{V}}^{1,2}(\mathbb{R}^{n+1}_+) = \{f|_{\mathbb{R}^{n+1}_+} : f \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^{n+1})\},$$

since the set inclusion $\supseteq$ is immediate.

We will translate these Sobolev spaces into the language of first-order operators. First, We introduce notation to represent vectors $v \in \mathbb{C}^{n+2}$ as follows

$$v = \begin{bmatrix} v_\perp \\ v_\parallel \\ v_\mu \end{bmatrix} = \begin{bmatrix} v_\perp \\ v_r \end{bmatrix}, \text{ that is } v_r = \begin{bmatrix} v_\parallel \\ v_\mu \end{bmatrix}$$

where $v_\perp \in \mathbb{C}$ represents the normal part, $v_\parallel \in \mathbb{C}^n$ represents the tangential part, $v_\mu \in \mathbb{C}$ represents the potential adapted part and $v_r \in \mathbb{C}^{n+1}$ represents the combination of $v_\parallel$ and v_μ.

Now, we define the self-adjoint, hence closed, operator $D\colon \mathscr{D}(D) \subset L^2(\mathbb{R}^n;\mathbb{C}^{n+2}) \to L^2(\mathbb{R}^n;\mathbb{C}^{n+2})$ by

$$D := \begin{bmatrix} 0 & -\nabla_\mu^* \\ -\nabla_\mu & 0 \end{bmatrix} = \begin{bmatrix} 0 & \operatorname{div} & -|V|^{\frac{1}{2}} \\ -\nabla & 0 & 0 \\ -|V|^{\frac{1}{2}} & 0 & 0 \end{bmatrix}$$

with its maximal domain in $L^2(\mathbb{R}^n,\mathbb{C}^{n+2})$ being given by

$$\mathscr{D}(D) := \{f \in L^2(\mathbb{R}^n,\mathbb{C}^{n+2}) : f_\perp \in \mathcal{V}^{1,2}(\mathbb{R}^n) \text{ and } (\operatorname{div} f_\parallel - |V|^{1/2} f_\mu) \in L^2(\mathbb{R}^n)\}.$$

This domain is maximal since Df is well-defined as a distribution for all f in $L^2(\mathbb{R}^n,\mathbb{C}^{n+2})$. Indeed, recalling the requirement that $f_\perp \in \mathcal{V}^{1,2}(\mathbb{R}^n)$ is just that $\nabla_\mu f_\perp \in L^2(\mathbb{R}^n)$, whilst $\operatorname{div} f_\parallel$ denotes the distributional divergence of $f_\parallel$, and $|V|^{\frac{1}{2}} f_\mu \in L^1_{\text{loc}}(\mathbb{R}^n)$ when $f_\mu \in L^2(\mathbb{R}^n)$ because $V \in L^1_{\text{loc}}(\mathbb{R}^n)$. Also, ∇_μ is interpreted as the unbounded operator from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n,\mathbb{C}^{n+1})$ defined by (2.3.1) on the domain $\mathscr{D}(\nabla_\mu) := \mathcal{V}^{1,2}(\mathbb{R}^n)$. In particular, the operator ∇_μ is closed by Lemma 2.3.1, and since $\mathcal{C}_c^\infty(\mathbb{R}^n)$ is dense in

$\mathcal{V}^{1,2}(\mathbb{R}^n)$, its adjoint satisfies $\nabla_\mu^* = (-\operatorname{div}, |V|^{\frac{1}{2}})$ on its domain

$$\mathscr{D}(\nabla_\mu^*) = \{(f_\parallel, f_\mu) \in L^2(\mathbb{R}^n, \mathbb{C}^{n+1}) : (-\operatorname{div} f_\parallel + |V|^{\frac{1}{2}} f_\mu) \in L^2(\mathbb{R}^n)\}, \tag{2.3.5}$$

where $(-\operatorname{div} f_\parallel + |V|^{\frac{1}{2}} f_\mu) \in L^2(\mathbb{R}^n)$ means there exists $F \in L^2(\mathbb{R}^n)$, and $\nabla_\mu^*((f_\parallel, f_\mu)) \coloneqq F$, such that $\int (f_\parallel \cdot \nabla\varphi + f_\mu |V|^{\frac{1}{2}} \varphi) = \int F\varphi$ for all $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$. We also note that we have the following product rule for ∇_μ^*.

Lemma 2.3.4. Let $u \in \mathscr{D}(\nabla_\mu^*)$ and $\eta \in \mathcal{C}_c^\infty(\mathbb{R}^n)$. Then $\eta u \in \mathscr{D}(\nabla_\mu^*)$ and

$$\nabla_\mu^*(\eta u) = \eta \nabla_\mu^* u - \nabla\eta \cdot u_\parallel.$$

Proof. Note as $u \in L^2(\mathbb{R}^n; \mathbb{C}^{n+1})$ and $\nabla_\mu^* u \in L^2(\mathbb{R}^n)$ we have $\nabla\eta \cdot u + \eta\nabla_\mu^* u \in L^2(\mathbb{R}^n)$. Now, let $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, then

$$\begin{aligned}
\langle \eta\nabla_\mu^* u - \nabla\eta \cdot u_\parallel, \varphi\rangle &= \langle \eta\nabla_\mu^* u, \varphi\rangle - \langle \nabla\eta \cdot u_\parallel, \varphi\rangle \\
&= \langle u, \nabla_\mu(\eta\varphi)\rangle - \langle u_\parallel, \varphi\nabla\eta\rangle \\
&= \langle u_\parallel, \eta\nabla\varphi + \varphi\nabla\eta\rangle + \langle u_\mu, |V|^{\frac{1}{2}}\eta\varphi\rangle - \langle u_\parallel, \varphi\nabla\eta\rangle \\
&= \langle \eta u, \nabla_\mu\varphi\rangle.
\end{aligned}$$

Thus, $\nabla_\mu^*(\eta u) = \eta\nabla_\mu^* u - \nabla\eta \cdot u_\parallel$. Therefore, $\nabla_\mu^*(\eta u) \in L^2(\mathbb{R}^n; \mathbb{C}^{n+1})$ and $\eta u \in \mathscr{D}(\nabla_\mu^*)$. $\square$

We introduce the definition of a topological splitting of a Banach space

Definition 2.3.5. Let $\mathcal{X}$ be a Banach space. Let $\mathcal{Y}, \mathcal{Z} \subseteq \mathcal{X}$. Then we write $\mathcal{X} = \mathcal{Y} \oplus \mathcal{Z}$ if:

1. $\mathcal{Y}, \mathcal{Z}$ are linear subspaces of $\mathcal{X}$;

2. $\mathcal{Y} \cap \mathcal{Z} = \{0\}$;

3. for all $x \in \mathcal{X}$ there exists $y \in \mathcal{Y}$, $z \in \mathcal{Z}$, and $C > 0$ such that $x = y + z$ and

$$\|y\| + \|z\| \leq C\|x\|$$

If $\mathcal{H}$ is a Hilbert space and $\mathcal{Y}, \mathcal{Z} \subseteq \mathcal{H}$, then we write $\mathcal{H} = \mathcal{Y} \overset{\perp}{\oplus} \mathcal{Z}$ if $\mathcal{H} = \mathcal{Y} \oplus \mathcal{Z}$ and $\mathcal{Y} = \mathcal{Z}^{\perp}$.

The self-adjointness of D provides the orthogonal Hodge decomposition

$$L^2(\mathbb{R}^n, \mathbb{C}^{n+2}) = \mathsf{N}(D) \overset{\perp}{\oplus} \overline{\mathsf{R}(D)}. \tag{2.3.6}$$

Moreover, the null space of D is the set

$$\mathsf{N}(D) = \{f \in L^2(\mathbb{R}^n, \mathbb{C}^{n+2}) : f_\perp = 0 \text{ and } \operatorname{div} f_\parallel = |V|^{\frac{1}{2}} f_\mu\},$$

whilst the closure of the range of D is characterised in the following lemma.

Lemma 2.3.6. The closure of the range of D in $L^2(\mathbb{R}^n, \mathbb{C}^{n+2})$ is the set

$$\overline{\mathsf{R}(D)} = \{f \in L^2(\mathbb{R}^n, \mathbb{C}^{n+2}) : f_\perp \in L^2(\mathbb{R}^n) \text{ and } (f_\parallel, f_\mu)^T = \nabla_\mu g \text{ for some } g \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^n)\}.$$

Proof. First, suppose that $f \in \overline{\mathsf{R}(D)}$, so then $f_\perp \in L^2(\mathbb{R}^n)$ and there exists a sequence $\{g_m\}$ in $\mathcal{V}^{1,2}(\mathbb{R}^n)$ such that $\{\nabla_\mu g_m\}$ converges to $(f_\parallel, f_\mu)^T$ in $L^2(\mathbb{R}^n, \mathbb{C}^{n+1})$. The Sobolev inequality (2.3.2) then implies that $\{g_m\}$ is Cauchy and hence convergent to some function g in $L^{2^*}(\mathbb{R}^n)$. Therefore, by Lemma 2.3.1 in the case $p = 2^*$, the sequence $\{\nabla_\mu g_m\}$ must converge to $\nabla_\mu g$ in $L^2(\mathbb{R}^n, \mathbb{C}^{n+1})$, hence $(f_\parallel, f_\mu)^T = \nabla_\mu g$ and $g \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^n)$, as required.

For the converse, suppose that $f_\perp \in L^2(\mathbb{R}^n)$ and that $(f_\parallel, f_\mu)^T = \nabla_\mu g$ for some $g \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^n)$. If $h \in \mathsf{N}(D)$, then $h_\perp = 0$ and $\operatorname{div} h_\parallel = |V|^{\frac{1}{2}} h_\mu$, hence

$$\langle f, h \rangle_{L^2(\mathbb{R}^n, \mathbb{C}^{n+2})} = \langle \nabla g, h_\parallel \rangle_{L^2(\mathbb{R}^n, \mathbb{C}^n)} + \langle |V|^{\frac{1}{2}} g, h_\mu \rangle_{L^2(\mathbb{R}^n)} = \langle g, |V|^{\frac{1}{2}} h_\mu - \operatorname{div} h_\parallel \rangle_{L^2(\mathbb{R}^n)} = 0,$$

where the last equality, which is immediate when $g \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, relies on the density of $\mathcal{C}_c^\infty(\mathbb{R}^n)$ in $\dot{\mathcal{V}}^{1,2}(\mathbb{R}^n)$. The orthogonal Hodge decomposition in (2.3.6) then allows us to conclude that $f \in [\mathsf{N}(D)]^\perp = \overline{\mathsf{R}(D)}$. $\square$

2.4 Sobolev Spaces Adapted to Vector Potentials

In this section we aim to replicate some of the results in Section 2.3 where we replace the operator ∇_μ with $\nabla + ib$ for some fixed $b \in L^2_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$. Here we call b the magnetic potential and we define the magnetic field generated by b as $\mathbf{B} := \mathrm{curl}(b)$, where

$$\mathbf{B}_{jk} = \mathrm{curl}(b)_{jk} := \frac{\partial b_j}{\partial x_k} - \frac{\partial b_k}{\partial x_j}.$$

Note that as $b_j \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ then $b_j f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ for all $f \in L^2_{\mathrm{loc}}\mathbb{R}^n)$. Therefore, we can make sense of $(\partial_j + ib_j)f$ in the following distributional sense

$$[(\partial_j + ib_j)f](\varphi) := \int f(ib_j - \partial)\varphi.$$

We then define the $(\nabla + ib)f := ((\partial_1 + ib_1)f, \ldots, (\partial_n + ib_n)f)^T \in \mathcal{D}'(\mathbb{R}^n)$. Then we define the inner-product space

$$W^{1,2}_b := \{f \in L^2(\mathbb{R}^n) : (\nabla + ib)f \in L^2(\mathbb{R}^n)\},$$

with (complex) inner-product

$$\langle f, g\rangle_{W^{1,2}_b(\Omega)} := \int_\Omega f\overline{g} + (\nabla + ib)f \cdot \overline{(\nabla + ib)g},$$

and norm

$$\|f\|_{W^{1,2}_b(\Omega)} := \left(\|f\|^2_{L^2(\Omega)} + \|(\nabla + ib)f\|^2_{L^2(\Omega;\mathbb{C}^n)}\right)^{\frac{1}{2}}.$$

In the definition of $W^{1,2}_b(\Omega)$ we do not assume that ∇f or bf are in $L^2(\mathbb{R}^n;\mathbb{C}^n)$ separately. That is, if $f \in W^{1,2}_b(\mathbb{R}^n)$ then it is not necessarily true that $f \in W^{1,2}(\mathbb{R}^n)$, however the following theorem shows that if $f \in W^{1,2}_b(\mathbb{R}^n)$ then $|f| \in W^{1,2}(\mathbb{R}^n)$, see [39, Theorem 6.17] for the definition of $\nabla|f|$. We give a pointwise bound below.

Proposition 2.4.1 (Diamagnetic Inequality)**.** Let $b \in L^2_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$. Then, for all $f \in$

$W_b^{1,2}(\mathbb{R}^n)$, we have

$$|\nabla|f|(x)| \leq |(\nabla + ib)f(x)|,$$

for almost every $x \in \mathbb{R}^n$.

Proof. See [39, Theorem 7.21]. $\square$

We are able to combine the Diamagnetic Inequality with the Sobolev Inequality (2.3.2) to obtain the following magnetic version

$$\|f\|_{2^*} = \||f|\|_{2^*} \lesssim \|\nabla|f|\|_2 \leq \|(\nabla + ib)f\|_2. \tag{2.4.1}$$

The following lemma is the analogous to Lemma 2.3.1.

Lemma 2.4.2. Let $p \in [1,\infty]$ and suppose that $b \in L^2_{\text{loc}}(\mathbb{R}^n)$. If $\{f_m\}$ is a sequence in $L^p(\Omega)$ that converges to some f in $L^p(\Omega)$, and $\{(\nabla + ib)f_m\}$ is a Cauchy sequence in $L^2(\Omega, \mathbb{C}^{n+1})$, then $\{(\nabla + ib)f_m\}$ converges to $(\nabla + ib)f$ in $L^2(\Omega, \mathbb{C}^n)$.

Proof. Suppose that $\{f_m\}$ and f satisfy the hypotheses of the lemma, in which case $\{(\nabla + ib)f_m\}$ converges to some $(F_1, \ldots, F_n)^T$ in $L^2(\Omega, \mathbb{C}^{n+1})$. It suffices to prove that $F_j = (\partial_j + ib_j)f$ for $j \in \{1, \ldots, n\}$. Fix $\varepsilon > 0$ and $\varphi \in \mathcal{C}_c^\infty(\Omega)$ arbitrarily. Let $N \in \mathbb{N}$ be such that

$$\|F_j - (\partial_j + ib_j)f_m\|_2\|\varphi\|_2 < \frac{\varepsilon}{2} \quad \text{and} \quad \|f_m - f\|_p\|(\partial_j + ib_j)\varphi\|_{p'} < \frac{\varepsilon}{2}$$

for all $j \in \{1, \ldots, \}$ whenever $m > N$. Now, if $j \in \{1, \ldots, n\}$, then by the definition of the distributional derivative and Hölder's inequality imply the simple estimate

$$\begin{aligned}\left|\left(\int F_j\varphi\right) - \left(-\int f(ib_j - \partial_j)\varphi\right)\right| &\leq \|F_j - (\partial_j + ib_j)f_m\|_2\|\varphi\|_2 \\ &\qquad + \|f_m - f\|_p\|(\partial_j + ib_j)\varphi\|_{p'} \\ &< \varepsilon.\end{aligned}$$

Therefore, as $\varepsilon > 0$ and φ were arbitrary then $\int F_j\varphi = -\int f(ib_j - \partial_j)\varphi$ and thus $F_j = (\partial_j + ib_j)f$. $\square$

A corollary to Lemma 2.4.2 is that $W_b^{1,2}(\Omega)$ is a Hilbert space. Following the notation in Section 2.3, we define $W_{b,c}^{1,2}(\Omega)$ to be the closure of $\mathcal{C}_c^\infty(\Omega)$ in $W_b^{1,2}(\Omega)$. By [39, Theorem 7.22] we have $\mathcal{C}_c^\infty(\mathbb{R}^n)$ is dense in $W_b^{1,2}(\mathbb{R}^n)$, and so $W_b^{1,2}(\mathbb{R}^n) = W_{b,c}^{1,2}(\mathbb{R}^n)$. We also define $W_{b,\mathrm{loc}}^{1,2}(\Omega)$ to be the set of all $f \in L^2_{\mathrm{loc}}(\Omega)$ such that $f \in W_b^{1,2}(\Omega')$ for all open sets $\Omega' \subset\subset \Omega$.

We define the homogeneous $\dot{W}_b^{1,2}(\mathbb{R}^n)$ to be the completion of $\mathcal{C}_c^\infty(\mathbb{R}^n)$ with the norm

$$\|f\|_{\dot{W}_b^{1,2}(\mathbb{R}^n)} \coloneqq \|(\nabla + ib)f\|_{L^2(\mathbb{R}^n;\mathbb{C}^n)}.$$

The precompleted space is again a genuine normed space as if $f \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ is such that $\|(\nabla + ib)f\|_{L^2(\mathbb{R}^n;\mathbb{C}^n)} = 0$, then by (2.4.1) we have $\|f\|_{L^{2^*}(\mathbb{R}^n)} \lesssim \|(\nabla + ib)f\|_{L^2(\mathbb{R}^n;\mathbb{C}^n))} = 0$. Hence $f = 0$. Now by (2.4.1) and Lemma 2.4.2, we may use a similar approach to Section 2.3 to get

$$\dot{W}_b^{1,2}(\mathbb{R}^n) = \{f \in L^{2^*}(\mathbb{R}^n) : (\nabla + ib)f \in L^2(\mathbb{R}^n;\mathbb{C}^n)\},$$

with the norm equivalence

$$\|f\|_{\dot{W}_b^{1,2}(\mathbb{R}^n)} = \|(\nabla + ib)f\|_{L^2(\mathbb{R}^n;\mathbb{C}^n)} \eqsim \left(\|f\|^2_{L^{2^*}(\mathbb{R}^n)} + \|(\nabla + ib)f\|_{L^2(\mathbb{R}^n;\mathbb{C}^n)}\right)^{\frac{1}{2}}.$$

It is important to note that b does not uniquely determine $\mathbf{B}$, since $\operatorname{curl}(b + \nabla\varphi) = \mathbf{B}$ for any $\varphi \in W^{1,2}(\mathbb{R}^n)$. This is known as the gauge invariance and is why we will impose our conditions on $\mathbf{B}$ instead of b itself. We will however, be able to take advantage of this through the following Proposition in [34, Proposition 3.2].

Proposition 2.4.3 (Iwatsuka Gauge Transform)**.** Let $b \in L^2_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$ and Q a cube in $\mathbb{R}^n$. We assume $\operatorname{curl} b = \mathbf{B} \in L^{\frac{n}{2}}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^{n\times n})$. Then, there exists $h \in L^n(Q;\mathbb{R}^n)$ and $\varphi \in W^{1,2}(\mathbb{R}^n;\mathbb{R})$ such that $\operatorname{curl} h = \mathbf{B}$ and

$$h = b - \nabla\varphi \quad \text{a.e. } x \in Q,$$

with

$$\left(\fint_Q |h|^n\right)^{\frac{1}{n}} \lesssim l(Q)\left(\fint_Q |\mathbf{B}|^{\frac{n}{2}}\right)^{\frac{2}{n}}, \tag{2.4.2}$$

where $c > 0$ depends only on n.

The conditions we will impose on the magnetic field $\mathbf{B}$ are as follows:

$$\begin{cases} |\mathbf{B}| \in B^{\frac{n}{2}}(\mathbb{R}^n) \\ |\nabla \mathbf{B}(x)| \leq cm(x, |\mathbf{B}|)^3, \end{cases} \tag{2.4.3}$$

for some $c > 0$, where $m(\cdot, V)$ is the Shen maximal function in (2.2.2).

In [46] the following Fefferman–Phong inequality was proven.

Lemma 2.4.4 (Global Fefferman–Phong)**.** Let $b \in L^2_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^n)$. Also, assume that $\mathbf{B}$ satisfies (2.4.3). Then

$$\|m(\cdot, |\mathbf{B}|)u\|_2 \lesssim \|(\nabla + ib)u\|_2,$$

for all $u \in \mathcal{C}^1(\mathbb{R}^n)$.

We recall the magnetic version of the Fefferman–Phong inequality (Proposition 2.2.5) in [17].

Proposition 2.4.5. Let $\omega \in A^\infty$ and $p \in [1, \infty)$. Then there exists constants $c > 0$ and $\beta \in (0, 1)$ depending only on p, n, and the A^∞ constant of ω, such that for all cubes Q and $u \in \mathcal{C}^1(\mathbb{R}^n)$, then

$$\int_Q |(\nabla + ib)u|^p + \omega|u|^p \geq \frac{cm_\beta(l(Q)^p \fint_Q \omega)}{l(Q)^p} \int_Q |u|^p.$$

where $m_\beta(x) = x$ if $x \leq 1$ and $m_\beta(x) = x^\beta$ if $x \geq 1$.

As with Section 2.3 we wish to use the language of first-order operators. First, We introduce notation to represent vectors $v \in \mathbb{C}^{n+1}$ as follows

$$v = \begin{bmatrix} v_\perp \\ v_\parallel \end{bmatrix},$$

where $v_\perp \in \mathbb{C}$ represents the normal part, $v_\parallel \in \mathbb{C}^n$ represents the tangential part. Now, define the self-adjoint, hence closed, operator $D\colon \mathscr{D}(D) \subset L^2(\mathbb{R}^n; \mathbb{C}^{n+1}) \to L^2(\mathbb{R}^n; \mathbb{C}^{n+1})$

by

$$D := \begin{bmatrix} 0 & -(\nabla + ib)^* \\ -(\nabla + ib) & 0 \end{bmatrix}$$

with its maximal domain in $L^2(\mathbb{R}^n, \mathbb{C}^{n+1})$ being given by

$$\mathscr{D}(D) := \{f \in L^2(\mathbb{R}^n, \mathbb{C}^{n+1}) : f_\perp \in W_b^{1,2}(\mathbb{R}^n) \text{ and } (\nabla + ib)^* f_\parallel \in L^2(\mathbb{R}^n)\}.$$

This domain is maximal since Df is well-defined as a distribution for all f in $L^2(\mathbb{R}^n, \mathbb{C}^{n+1})$. Indeed, recalling the requirement that $f_\perp \in W_b^{1,2}(\mathbb{R}^n)$ is just that $(\nabla + ib) f_\perp \in L^2(\mathbb{R}^n)$, whilst $(\nabla + ib)^* f_\parallel$ denotes the adjoint of $(\nabla + ib)$. Here $(\nabla + ib)$ is interpreted as the unbounded operator from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n, \mathbb{C}^n)$ on the domain $\mathscr{D}(\nabla + ib) := W_b^{1,2}(\mathbb{R}^n)$. In particular, the operator $(\nabla + ib)$ is closed by Lemma 2.4.2, and since $\mathcal{C}_c^\infty(\mathbb{R}^n)$ is dense in $W_b^{1,2}(\mathbb{R}^n)$, its adjoint satisfies $(\nabla + ib)^* = -(\operatorname{div} + ib)$ on its domain

$$\mathscr{D}((\nabla + ib)^*) = \{f \in L^2(\mathbb{R}^n, \mathbb{C}^n) : -(\operatorname{div} f + ibf) \in L^2(\mathbb{R}^n)\}, \tag{2.4.4}$$

where $-(\operatorname{div} f + ibf) \in L^2(\mathbb{R}^n)$ means there exists $F \in L^2(\mathbb{R}^n)$, and $(\nabla + ib)^* f := F$, such that $\int f \cdot (\nabla + ib)\varphi = \int F \varphi$ for all $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$.

As before the self-adjointness of D provides the orthogonal Hodge decomposition

$$L^2(\mathbb{R}^n, \mathbb{C}^{n+1}) = \mathsf{N}(D) \overset{\perp}{\oplus} \overline{\mathsf{R}(D)}. \tag{2.4.5}$$

Moreover, the null space of D is the set

$$\mathsf{N}(D) = \{f \in L^2(\mathbb{R}^n, \mathbb{C}^{n+1}) : f_\perp = 0 \text{ and } -\operatorname{div} f_\parallel = ib f_\parallel\},$$

whilst the closure of the range of D is characterised in the following lemma which is analogous to Lemma 2.3.1.

Lemma 2.4.6. The closure of the range of D in $L^2(\mathbb{R}^n, \mathbb{C}^{n+1})$ is the set

$$\overline{\mathsf{R}(D)} = \{f \in L^2(\mathbb{R}^n, \mathbb{C}^{n+1}) : f_\perp \in L^2(\mathbb{R}^n) \text{ and } f_\parallel = (\nabla + ib)g \text{ for some } g \in \dot{W}_b^{1,2}(\mathbb{R}^n)\}.$$

Proof. First, suppose that $f \in \overline{\mathsf{R}(D)}$, so then $f_\perp \in L^2(\mathbb{R}^n)$ and there exists a sequence $\{g_m\}$ in $\dot{W}_b^{1,2}(\mathbb{R}^n)$ such that $\{(\nabla + ib)g_m\}$ converges to $f_\|$ in $L^2(\mathbb{R}^n, \mathbb{C}^{n+1})$. The Sobolev inequality (2.4.1) then implies that $\{g_m\}$ is Cauchy and hence convergent to some function g in $L^{2^*}(\mathbb{R}^n)$. Therefore, by Lemma 2.4.2 in the case $p = 2^*$, the sequence $\{(\nabla + ib)g_m\}$ must converge to $(\nabla + ib)g$ in $L^2(\mathbb{R}^n, \mathbb{C}^{n+1})$, hence $f_\| = (\nabla + ib)g$ and $g \in \dot{W}_b^{1,2}(\mathbb{R}^n)$, as required.

For the converse, suppose that $f_\perp \in L^2(\mathbb{R}^n)$ and that $f_\| = (\nabla + ib)g$ for some $g \in \dot{W}_b^{1,2}(\mathbb{R}^n)$. If $h \in \mathsf{N}(D)$, then $h_\perp = 0$ and $-\operatorname{div} h_\| = ibh_\|$, hence

$$\langle f, h\rangle_{L^2(\mathbb{R}^n,\mathbb{C}^{n+2})} = \langle (\nabla + ib)g, h_\|\rangle_{L^2(\mathbb{R}^n,\mathbb{C}^n)} = \langle g, -(\operatorname{div} + ib)h_\|\rangle_{L^2(\mathbb{R}^n)} = 0,$$

where the last equality, which is immediate when $g \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, relies on the density of $\mathcal{C}_c^\infty(\mathbb{R}^n)$ in $\dot{W}_b^{1,2}(\mathbb{R}^n)$. The orthogonal Hodge decomposition in (2.4.5) then allows us to conclude that $f \in [\mathsf{N}(D)]^\perp = \overline{\mathsf{R}(D)}$. $\square$

2.5 The Theory of Perturbed Self-adjoint operators

In this section we discuss operators of the form DB or BD where D is a self-adjoint operator on a Hilbert space $\mathcal{H}$, and B is bounded on $\mathcal{H}$ and elliptic on $\overline{\mathsf{R}(D)}$, in the sense that there exists $\kappa > 0$ such that

$$\operatorname{Re}\langle Bu, u\rangle \geq \kappa\|u\|_2^2, \quad \text{for all } u \in \overline{\mathsf{R}(D)}. \tag{2.5.1}$$

These are the same as (H2),(H4),(H5) from [3]. We also define the angle of ellipticity of B to be

$$\omega := \sup_{u \in \mathsf{R}(D)\setminus\{0\}} |\arg\langle Bu, u\rangle| < \frac{\pi}{2}.$$

We have that following proposition about the behaviour of DB in [4, Proposition 3.3].

Proposition 2.5.1. Let $\mathcal{H}$ be a Hilbert space. Let D be a self-adjoint operator and let B be a bounded operator in $\mathcal{H}$ satisfying (2.5.1). Then the operator DB is a closed, densely defined ω–bisectorial operator with resolvent bounds $\|(\lambda I - DB)^{-1}u\|_{\mathcal{H}} \lesssim \frac{\|u\|_{\mathcal{H}}}{\operatorname{dist}(\lambda, S_\omega)}$ when $\lambda \notin S_\omega$. Also

1. The operator DB has range $\mathsf{R}(DB) = \mathsf{R}(D)$ and null space $\mathsf{N}(DB) = B^{-1}\mathsf{N}(D)$ with $\mathcal{H} = \overline{\mathsf{R}(DB)} \oplus \mathsf{N}(DB)$.

2. The restriction of DB to $\overline{\mathsf{R}(D)}$ is a closed and injective operator with dense range in $\overline{\mathsf{R}(D)}$ with spectrum and resolvent bounds as above.

Proof. As D is closed and densely defined and B is bounded then DB and BD is closed and densely defined. Also, $(DB)^* = B^*D$, and $(BD)^* = DB^*$. Therefore, we have

$$\mathcal{H} = \overline{\mathsf{R}(DB)} \oplus \mathsf{N}(B^*D) = \overline{\mathsf{R}(B^*D)} \oplus \mathsf{N}(DB).$$

Then $\overline{\mathsf{R}(DB)} = \mathsf{N}(B^*D)^\perp = \mathsf{N}(D)^\perp = \overline{\mathsf{R}(D)}$. Now, let $v \in \mathscr{D}(DB)$ and $w \in \mathsf{N}(DB)$. Then, using ellipticity, Cauchy-Schwarz, and the boundedness of B^*, we have

$$\begin{aligned}
\|DBv\|_{\mathcal{H}}^2 &\leq \kappa^{-1}|\langle BDBv, DBv\rangle| \\
&= \kappa^{-1}|\langle DBv, B^*DBv\rangle + \langle B^*DBw, v\rangle| \\
&= \kappa^{-1}|\langle DBv + w, B^*DBv\rangle| \\
&= \kappa^{-1}|\langle B(DBv + w), DBv\rangle| \\
&\leq \kappa^{-1}\|B(DBv + w)\|_{\mathcal{H}}\|DBv\|_{\mathcal{H}} \\
&\leq \kappa^{-1}\|B\|_{\mathrm{op}}\|DBv + w\|_{\mathcal{H}}\|DBv\|_{\mathcal{H}},
\end{aligned}$$

where $\|B\|_{\mathrm{op}}$ is the operator norm of B. So $\|DBv\|_{\mathcal{H}} \lesssim \|DBv + w\|_{\mathcal{H}}$ with constant depending only on κ and $\|B\|_{\mathrm{op}}$. Thus, we have $\|DBv\|_{\mathcal{H}} + \|w\|_{\mathcal{H}} \eqsim \|DBv + w\|_{\mathcal{H}}$. Then, by a density argument we have $\|u\|_{\mathcal{H}} + \|w\|_{\mathcal{H}} \eqsim \|u + w\|_{\mathcal{H}}$ for all $u \in \overline{\mathsf{R}(DB)}$ and $w \in \mathsf{N}(DB)$. A similar argument gives that $\|u+w\|_{\mathcal{H}} \eqsim \|u\|_{\mathcal{H}} + \|w\|_{\mathcal{H}}$ for all $u \in \overline{\mathsf{R}(B^*D)}$ and $w \in \mathsf{N}(B^*D)$. Now let $u \in \overline{\mathsf{R}(DB)} \cap \mathsf{N}(DB)$. Then, we have $\|u\|_{\mathcal{H}} \eqsim \|u\|_{\mathcal{H}} + \|-u\|_{\mathcal{H}} \eqsim \|u - u\|_{\mathcal{H}} = 0$. That is, $u = 0$. Thus, $\overline{\mathsf{R}(DB)} \cap \mathsf{N}(DB) = \{0\}$. Again, a similar argument gives $\overline{\mathsf{R}(B^*D)} \cap \mathsf{N}(B^*D) = \{0\}$. Now

$$\mathcal{H} = \overline{\mathsf{R}(DB)} \oplus \mathsf{N}(DB) \oplus \left(\overline{\mathsf{R}(DB)} \oplus \mathsf{N}(DB)\right)^{\perp}.$$

Let $u \in \left(\overline{\mathsf{R}(DB)} \oplus \mathsf{N}(DB)\right)^{\perp}$. Then, $\langle u, v+w\rangle = 0$ for all $v \in \overline{\mathsf{R}(DB)}$ and all $w \in \mathsf{N}(DB)$. Therefore, $\langle u, v\rangle = 0$ and $\langle u, w\rangle = 0$ for all $v \in \overline{\mathsf{R}(DB)}$ and all $w \in \mathsf{N}(DB)$. That is, $u \in \overline{\mathsf{R}(DB)}^{\perp} \cap \mathsf{N}(DB)^{\perp}$. Since $\overline{\mathsf{R}(DB)}^{\perp} = \mathsf{N}(B^*D)$ and $\mathsf{N}(DB)^{\perp} = \overline{\mathsf{R}(B^*D)}$ we have $u \in \overline{\mathsf{R}(B^*D)} \cap \mathsf{N}(B^*D) = \{0\}$. That is, $u = 0$. Thus, $\left(\overline{\mathsf{R}(DB)} \oplus \mathsf{N}(DB)\right)^{\perp} = \{0\}$. Hence,

$$\mathcal{H} = \overline{\mathsf{R}(DB)} \oplus \mathsf{N}(DB).$$

Again, the same argument yields

$$\mathcal{H} = \overline{\mathsf{R}(B^*D)} \oplus \mathsf{N}(B^*D).$$

We now prove DB is bisectorial. Let $\mu \in (\omega, \frac{\pi}{2})$. Let $\lambda \in \mathbb{C} \setminus S_\mu$. First, note that if $u \in \mathsf{N}(DB)$ then $\|(\lambda I - DB)u\|_{\mathcal{H}} = |\lambda|\|u\|_{\mathcal{H}} \geq \operatorname{dist}(\lambda, S_\mu)\|u\|_{\mathcal{H}}$. Now, assume $u \in \overline{\mathsf{R}(D)}$. As D is self-adjoint then $\langle Bu, DBu\rangle \in \mathbb{R}$. Therefore, $\operatorname{Im}\langle Bu, (\lambda I - DB)u\rangle = \operatorname{Im}\langle Bu, \lambda u\rangle$. Now, using the boundedness of B, we have $|\operatorname{Im}(\overline{\lambda}\langle Bu, u\rangle)| = |\operatorname{Im}\langle Bu, (\lambda I - DB)u\rangle| \lesssim \|u\|_{\mathcal{H}}\|(\lambda I - DB)u\|_{\mathcal{H}}$. Since $u \in \overline{\mathsf{R}(D)}$ using the ellipticity of B we have

$$|\operatorname{Im}(\overline{\lambda}\langle Bu, u\rangle)| \geq |\lambda||\langle Bu, u\rangle| \sin(\mu - \omega) \gtrsim \|u\|_{\mathcal{H}}^2 \operatorname{dist}(\lambda, S_\omega).$$

Thus

$$\|u\|_{\mathcal{H}} \lesssim \frac{1}{\operatorname{dist}(\lambda, S_\omega)}\|(\lambda I - DB)u\|_{\mathcal{H}},$$

which implies bisectoriality. □

And for BD we have a similar result

Proposition 2.5.2. Let D be a self-adjoint operator and let B be a bounded operator in $\mathcal{H}$ satisfying (2.5.1). Then the operator BD is a closed, densely defined ω–bisectorial operator with resolvent bounds $\|(\lambda I - BD)^{-1}u\|_{\mathcal{H}} \lesssim \frac{\|u\|_{\mathcal{H}}}{\operatorname{dist}(\lambda, S_\omega)}$ when $\lambda \notin S_\omega$. Also

1. The operator BD has range $\mathsf{R}(BD) = B\mathsf{R}(D)$ and null space $\mathsf{N}(BD) = \mathsf{N}(D)$ with $\mathcal{H} = \overline{\mathsf{R}(BD)} \oplus \mathsf{N}(BD)$.
2. The restriction of BD to $\overline{\mathsf{R}(BD)}$ is a closed and injective operator with dense range in $\overline{\mathsf{R}(BD)}$ with spectrum and resolvent bounds as above.

Proof. Proved similarly to the DB case. $\square$

For $t \in \mathbb{R}$ with $t \neq 0$, we define the following operators

$$\begin{aligned} R_t^B &:= (I + itDB)^{-1} \\ P_t^B &:= (I + (tDB)^2)^{-1} = \frac{1}{2}(R_t^B + R_{-t}^B) = R_t^B R_{-t}^B, \\ Q_t^B &:= tDB(I + (tDB)^2)^{-1} = \frac{1}{2i}(-R_t^B + R_{-t}^B). \end{aligned}$$

We let $P_t = P_t^I$ and $Q_t = Q_t^I$ to be the unperturbed versions. Now using Proposition 2.5.1 we have

$$\|R_t^B u\|_{\mathcal{H}} = \|(I + itDB)^{-1}u\|_{\mathcal{H}} = \frac{\|((-it)^{-1} - DB)^{-1}u\|_{\mathcal{H}}}{|t|} \lesssim \frac{1}{|t|}\frac{\|u\|_{\mathcal{H}}}{\operatorname{dist}((-it)^{-1}, S_\omega)} \lesssim \|u\|_{\mathcal{H}},$$

uniformly in t where the implicit constants depend only on dimension and the properties of B. Therefore, $\|P_t^B u\|_{\mathcal{H}} \lesssim \|u\|_{\mathcal{H}}$ and $\|Q_t^B u\|_{\mathcal{H}} \lesssim \|u\|_{\mathcal{H}}$ uniformly in t where the implicit constants depend only on dimension and the properties of B. Similar statements hold when DB is replaced by BD. An key observation is that as D is self-adjoint then we have the quadratic estimate

$$\int_0^\infty \|Q_t u\|_{\mathcal{H}}^2 \frac{\mathrm{d}t}{t} \approx \|u\|_{\mathcal{H}}^2, \quad \text{for all } u \in \overline{\mathsf{R}(D)}.$$

Another important property is the Calderón reproducing formula

$$\int_0^\infty (Q_t^B)^2 u \frac{\mathrm{d}t}{t} = \frac{u}{2}$$

for all $u \in \overline{\mathsf{R}(D)}$, see [13, eqn. (17)] and [1, Section (E)] for details.

Now let

$$D = \begin{bmatrix} 0 & -\nabla_\mu^* \\ -\nabla_\mu & 0 \end{bmatrix}, \quad \text{or} \quad D = \begin{bmatrix} 0 & -(\nabla + ib)^* \\ -(\nabla + ib) & 0 \end{bmatrix}.$$

We define the bounded operator $B \colon \mathcal{H} \to \mathcal{H}$ to be multiplication by a matrix valued function with the following structure

$$B := \begin{bmatrix} B_{\perp\perp} & B_{\perp\parallel} & 0 \\ B_{\parallel\perp} & B_{\parallel\parallel} & 0 \\ 0 & 0 & b \end{bmatrix} \in L^\infty(\mathbb{R}^n; \mathcal{L}(\mathbb{C}^{n+2})), \text{ or } B := \begin{bmatrix} B_{\perp\perp} & B_{\perp\parallel} \\ B_{\parallel\perp} & B_{\parallel\parallel} \end{bmatrix} \in L^\infty(\mathbb{R}^n; \mathcal{L}(\mathbb{C}^{n+1})),$$

respectively. Here $\mathcal{H} = L^2(\mathbb{R}^n; \mathbb{C}^{n+1})$ and $\mathcal{H} = L^2(\mathbb{R}^n; \mathbb{C}^{n+1})$ for the electric and magnetic operators respectively. Then, the aims of Chapters 3 and 4 are to proof the quadratic estimates

$$\int_0^\infty \|Q_t^B u\|_2^2 \frac{\mathrm{d}t}{t} = \int_0^\infty \|tDB(I + t^2 DBDB)^{-1} u\|_2^2 \frac{\mathrm{d}t}{t} \eqsim \|u\|_2^2, \quad \forall u \in \overline{\mathsf{R}(D)}, \tag{2.5.2}$$

for the first-order operator DB as defined in Sections 2.3 and 2.4 respectively. To prove (2.5.2) we will first prove

$$\int_0^\infty \|tDB(I + t^2 DBDB)^{-1} u\|_2^2 \frac{\mathrm{d}t}{t} \lesssim \|u\|_2^2, \quad \forall u \in \mathcal{H}. \tag{2.5.3}$$

To see that (2.5.3) implies (2.5.2) we present a duality argument. Since $Q_t^B = tDBP_t^B = P_t^B tDB$, then for $u \in \mathsf{N}(DB)$ we have $Q_t^B u = 0$. Then, by Proposition 2.5.1 we have $\mathcal{H} = \overline{\mathsf{R}(DB)} \oplus \mathsf{N}(DB)$ and so now we are left to prove the quadratic estimate on $\overline{\mathsf{R}(D)} = \overline{\mathsf{R}(DB)}$. Now, assume DB satisfies the estimate (2.5.3). As

$$(I + t^2 BDBD)u = (BB^{-1} + t^2 BDBDBB^{-1})u = B(I + t^2 DBDB)B^{-1}u,$$

for all $u \in \overline{\mathsf{R}(D)}$, then we have

$$(I + t^2 BDBD)^{-1}u = B(I + t^2 DBDB)^{-1}B^{-1}u,$$

for all $u \in \overline{\mathsf{R}(D)}$. Then using above, (2.5.3), and the boundedness of B and (2.5.1), we have

$$\int_0^\infty \|tBD(I+t^2BDBD)^{-1}u\|_2^2 \frac{\mathrm{d}t}{t} = \int_0^\infty \|tBDB(I+t^2DBDB)^{-1}B^{-1}u\|_2^2 \frac{\mathrm{d}t}{t} \lesssim \|B^{-1}u\|_2^2, \tag{2.5.4}$$

for all $u \in \mathcal{H}$. Note that we have $\operatorname{Re}\langle B^*v, v\rangle \geq \kappa\|v\|$ for all $v \in \overline{\mathsf{R}(D)}$. Therefore, by (2.5.4), where B is replaced by B^*, we have the estimate of the dual

$$\left(\int_0^\infty \|(Q_t^B)^*v\|_2^2 \frac{\mathrm{d}t}{t}\right)^{\frac{1}{2}} = \left(\int_0^\infty \|B^*D(I + t^2B^*DB^*D)^{-1}v\|_2^2 \frac{\mathrm{d}t}{t}\right)^{\frac{1}{2}} \lesssim \|v\|_2^2$$

for all $v \in \mathcal{H}$. For the lower estimate we now follow the proof of [13, Proposition 4.8 (iii)]. Therefore, using the Calderón reproducing formula, Cauchy–Schwarz, and above, we have

$$\begin{aligned}
\|u\|_2 &= \sup_{\|v\|_2=1} |\langle u, v\rangle| \\
&\eqsim \sup_{\|v\|_2=1} \left|\left\langle \int_0^\infty (Q_t^B)^2 u \frac{\mathrm{d}t}{t}, v\right\rangle\right| \\
&= \sup_{\|v\|_2=1} \left|\int_0^\infty \left\langle Q_t^B u, (Q_t^B)^* v\right\rangle \frac{\mathrm{d}t}{t}\right| \\
&\leq \left(\int_0^\infty \|Q_t^B u\|_2^2 \frac{\mathrm{d}t}{t}\right)^{\frac{1}{2}} \sup_{\|v\|_2=1} \left(\int_0^\infty \|(Q_t^B)^* v\|_2^2 \frac{\mathrm{d}t}{t}\right)^{\frac{1}{2}} \\
&\lesssim \left(\int_0^\infty \|Q_t^B u\|_2^2 \frac{\mathrm{d}t}{t}\right)^{\frac{1}{2}} \sup_{\|v\|_2=1} \|v\|_2 \\
&= \left(\int_0^\infty \|tDB(I + t^2DBDB)^{-1}u\|_2^2 \frac{\mathrm{d}t}{t}\right)^{\frac{1}{2}}.
\end{aligned}$$

for all $u \in \overline{\mathsf{R}(D)}$. Therefore, to prove (2.5.2) it suffices to prove (2.5.3).

2.6 Weak Solutions to Elliptic Equations

In Chapters 6, 7 and 8 we will discuss the solvability of the Schrödinger equation

$$H_{A,a,V}\, u = -\operatorname{div} A\nabla u + aVu = 0, \tag{2.6.1}$$

on the upper-half space $\mathbb{R}^{n+1}_+$ where A and a are elliptic (to be defined later in (2.6.2)) and t-independent and $V \in B^{\frac{n}{2}}(\mathbb{R}^n)$. In this section we will give the basic definitions and concepts associated with weak solutions and the Schrödinger equation (2.6.1). We note that, as we only have an electric potential in (2.6.1), the spaces and operators we consider in this section are those from Section 2.3.

We start with the coefficient matrices. We define the following two matrices $a \in L^\infty(\mathbb{R}^{n+1}; \mathcal{L}(\mathbb{C}))$ and $A \in L^\infty(\mathbb{R}^{n+1}; \mathcal{L}(\mathbb{C}^{n+1}))$ as t-independent 1×1 and $(n+1)\times(n+1)$-dimensional matrices respectively, with complex components. We split the coefficients of A to obtain the following

$$\mathcal{A}^{\mathcal{V}} := \begin{bmatrix} A & 0 \\ 0 & ae^{i\arg V} \end{bmatrix} = \begin{bmatrix} A_{\perp\perp} & A_{\perp\|} & 0 \\ A_{\|\perp} & A_{\|\|} & 0 \\ 0 & 0 & ae^{i\arg V} \end{bmatrix}.$$

where $A_{\perp\perp}(x) \in \mathcal{L}(\mathbb{C})$, $A_{\perp\|}(x) \in \mathcal{L}(\mathbb{C}^n;\mathbb{C})$, $A_{\|\perp}(x) \in \mathcal{L}(\mathbb{C};\mathbb{C}^n)$, and $A_{\|\|}(x) \in \mathcal{L}(\mathbb{C}^n)$. In the case when $V(x) \geq 0$ for all $x \in \mathbb{R}^n$ we let $\mathcal{A} := \mathcal{A}^{\mathcal{V}}$ to simplify notation. We will assume that $\mathcal{A}^{\mathcal{V}}$ satisfies the following ellipticity condition: there exists $\kappa > 0$ such that

$$\sum_{l=0}^{n+1}\sum_{k=0}^{n+1} \operatorname{Re} \int_{\mathbb{R}^n} \mathcal{A}^{\mathcal{V}}_{k,l}(x) f_k(x) \overline{f_l(x)}\, \mathrm{d}x \geq \kappa \sum_{k=0}^{n} \int_{\mathbb{R}^n} |f_k(x)|^2\, \mathrm{d}x, \tag{2.6.2}$$

for all $f \in \{g \in L^2(\mathbb{R}^n, \mathbb{C}^{n+2}) : g_\perp \in L^2(\mathbb{R}^n) \text{ and } (g_\|, g_\mu)^T = \nabla_\mu h \text{ some } h \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^n)\}$. We note that this is similar to the ellipticity condition in [4] adapted to the potential V.

We note that (2.6.2) implies the following

$$\begin{cases} \operatorname{Re}((A_{\perp\perp}(x)\xi)\cdot\overline{\xi}) \geq \kappa|\xi|^2, & \forall \xi \in \mathbb{C},\ \text{a.e. } x \in \mathbb{R}^n, \\ \operatorname{Re}\int_{\mathbb{R}^n}\left[(A_{\|\|}\nabla f)\cdot\nabla f + aV|f|^2\right]\mathrm{d}x \geq \kappa\int_{\mathbb{R}^n}|\nabla_\mu f|^2\,\mathrm{d}x, & \forall f \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^n). \end{cases}$$

To see the second inequality above set $f_\perp = 0$ in (2.6.2). For the first inequality set $f = (u, 0, 0)^T$ in (2.6.2) to get that

$$\operatorname{Re}\int_{\mathbb{R}^n} A_{\perp\perp}(x)u(x)\overline{u(x)}\,\mathrm{d}x \geq \kappa\int_{\mathbb{R}^n}|u(x)|^2\,\mathrm{d}x, \quad \forall u \in L^2(\mathbb{R}^n).$$

Now let E be any measurable set and then for $\xi \in \mathbb{C}$ choose $u(x) = \xi\mathbb{1}_E(x)$. Therefore we have

$$\int_E(\operatorname{Re} A_{\perp\perp}(x)\xi\overline{\xi} - \kappa|\xi|^2)\,\mathrm{d}x \geq 0$$

Thus, as E was arbitrary we have

$$\operatorname{Re}((A_{\perp\perp}(x)\xi)\cdot\overline{\xi}) \geq \kappa|\xi|^2, \quad \forall \xi \in \mathbb{C},\ \text{a.e. } x \in \mathbb{R}^n.$$

We note that the ellipticity condition (2.6.2) is between the pointwise ellipticity condition

$$\operatorname{Re}(A^{\mathcal{V}}(x)\xi\cdot\overline{\xi}) \geq \kappa|\xi|^2, \quad \forall \xi \in \mathbb{C}^{n+1},\ \text{a.e. } x \in \mathbb{R}^n,$$

and the following Gårding-type inequality adapted to the potential V

$$\operatorname{Re}\iint_{\mathbb{R}^{n+1}_+}(\mathcal{A}^{\mathcal{V}}(x)\nabla_\mu f(t,x))\cdot\nabla_\mu f(t,x)\,\mathrm{d}t\,\mathrm{d}x \geq \kappa\iint_{\mathbb{R}^{n+1}_+}|\nabla_\mu f(t,x)|^2\,\mathrm{d}t\,\mathrm{d}x,$$

for all $f \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^{n+1}_+)$. There is an explicit connection between (2.6.2) and the operator D from Section 2.3 in that using Lemma 2.3.6 we can write (2.6.2) as

$$\operatorname{Re}\langle\mathcal{A}^{\mathcal{V}}u, u\rangle \geq \kappa\|u\|_2^2, \text{ for all } u \in \overline{\mathsf{R}(D)}.$$

This will be the way in which we interpret the notion of ellipticity of A and a.

We now turn our attention to weak solutions of $H_{A,a,V}$.

Definition 2.6.1. We shall write that u is a weak solution of $-\operatorname{div} A\nabla u + aVu = 0$ in Ω, or simply that $H_{A,a,V}\, u = 0$ in Ω, if $u \in \mathcal{V}^{1,2}_{\mathrm{loc}}(\Omega)$ and

$$\int_\Omega A\nabla u \cdot \overline{\nabla v} + aVu\overline{v} = 0,$$

for all $v \in \mathcal{C}^\infty_c(\Omega)$.

We require some control at the boundary $\partial\mathbb{R}^{n+1}_+$. For this we introduce the non-tangential maximal function. For $F \in L^2_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$ and $x \in \mathbb{R}^n$, we define the non-tangential maximal operator as

$$(\widetilde{N}_* F)(x) := \sup_{t>0} \left(\fint\!\!\fint_{W(t,x)} |F(s,y)|^2 \,\mathrm{d}y \,\mathrm{d}s \right)^{\frac{1}{2}},$$

where $W(t,x) := [t, 2t] \times Q(t,x)$ is the Whitney box of scale $t > 0$, centred at $x \in \mathbb{R}^n$ where $Q(t,x)$ is the cube of side-length $l(Q(t,x)) = t$, centred at x. Also for $F \in L^2_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$ and $f \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$, we say that F converges to f pointwise on Whitney averages if

$$\lim_{t\to 0} \fint\!\!\fint_{W(t,x)} |F(s,y) - f(x)|^2 \,\mathrm{d}y \,\mathrm{d}s = 0, \quad \text{for almost all } x \in \mathbb{R}^n. \tag{2.6.3}$$

We will impose one of the following conditions on u on the boundary: Neumann,

$$(\mathcal{N})^{\mathcal{A}}_{L^2} \begin{cases} -\operatorname{div} A\nabla u + aVu = 0 \text{ in } \mathbb{R}^{n+1}_+, \\ \widetilde{N}_*(\nabla_\mu u) \in L^2(\mathbb{R}^n), \\ \lim_{t\to 0} \partial_{\nu_A} u(t,\cdot) = \varphi,\ \varphi \in L^2(\mathbb{R}^n) \end{cases} \tag{2.6.4}$$

or Dirichlet regularity,

$$(\mathcal{R})^{\mathcal{A}}_{L^2} \begin{cases} -\operatorname{div} A\nabla u + aVu = 0 \text{ in } \mathbb{R}^{n+1}_+, \\ \widetilde{N}_*(\nabla_\mu u) \in L^2(\mathbb{R}^n), \\ \lim_{t\to 0} \nabla^{\|}_\mu u(t,\cdot) = \nabla_\mu \varphi,\ \varphi \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^n) \end{cases} \tag{2.6.5}$$

where the limits are taken to be in L^2 and pointwise on Whitney averages and $\partial_{\nu_A} u = (A\nabla u)_\perp$ is the conormal derivative and $\nabla_\mu^\parallel \colon \dot{\mathcal{V}}^{1,2}(\mathbb{R}^{n+1}_+) \to L^2(\mathbb{R}^n;\mathbb{C}^n)$ defined by

$$\nabla_\mu^\parallel u = \begin{bmatrix} \nabla_\parallel u \\ |V|^{\frac{1}{2}} u \end{bmatrix} = \begin{bmatrix} \partial_1 u \\ \vdots \\ \partial_n u \\ |V|^{\frac{1}{2}} u \end{bmatrix}.$$

That is $\nabla_\mu^\parallel u = (\nabla_\mu u)_r$ the gradient without the first component, which is the derivative in t. We say $(\mathcal{N})^{\mathcal{A}}_{L^2}$ and $(\mathcal{R})^{\mathcal{A}}_{L^2}$ are well-posed if, for each boundary data, there exists a unique u satisfying the above boundary value problems. We also define the sets $WP(\mathcal{N})$ and $WP(\mathcal{R})$ to be the set of all $\mathcal{A}$ such that $(\mathcal{N})^{\mathcal{A}}_{L^2}$ and $(\mathcal{R})^{\mathcal{A}}_{L^2}$ are well-posed respectively.

CHAPTER 3

QUADRATIC ESTIMATES FOR THE PURELY ELECTRIC SCHRÖDINGER OPERATOR

The focus of this chapter is to prove quadratic estimates for first-order systems of the DB-type for the electric Schrödinger operator $-\operatorname{div} A\nabla + V$. We follow the methods developed [3] which were first introduced in [8] We adapt these methods to incorporate a zeroth-order term as a singular potential. We use the framework from Section 2.3 with $\Omega = \mathbb{R}^n$, since ultimately we will solve boundary value problems in the upper half-space $\mathbb{R}^{n+1}_+$ by applying the quadratic estimates obtained here on the domain boundary $\partial\mathbb{R}^{n+1}_+$, this will be done in Chapters 6, 7 and 8. Therefore we define the operators

$$D = \begin{bmatrix} 0 & -(\nabla_\mu)^* \\ -\nabla_\mu & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} B_{\perp\perp} & B_{\perp\|} & 0 \\ B_{\|\perp} & B_{\|\|} & 0 \\ 0 & 0 & b \end{bmatrix}$$

as defined in Section 2.3. Then, the aim of this chapter is to prove the following theorem.

Theorem 3.0.1. Let $n \geq 3$. If $V \in B^{\frac{n}{2}}(\mathbb{R}^n)$, then we have the quadratic estimate

$$\int_0^\infty \|tDB(I + t^2 DBDB)^{-1}u\|_2^2 \frac{\mathrm{d}t}{t} \eqsim \|u\|_2^2, \quad \forall u \in \overline{\mathsf{R}(D)},$$

where the implicit constants depends only on V, n, κ, and $\|B\|_\infty$.

We also have the case when V is integrable.

Theorem 3.0.2. Let $n > 4$. If $V \in L^{\frac{n}{2}}(\mathbb{R}^n)$ with sufficiently small norm, then we have the quadratic estimate

$$\int_0^\infty \|tDB(I + t^2 DBDB)^{-1}u\|_2^2 \frac{\mathrm{d}t}{t} \eqsim \|u\|_2^2, \quad \forall u \in \overline{\mathsf{R}(D)},$$

where the implicit constants depends only on V, n, κ, and $\|B\|_\infty$.

We specialise to the case when $n \geq 3$ and $V \in B^{\frac{n}{2}}(\mathbb{R}^n)$, as this is in fact more difficult. However, we will summarise the differences between the case when $n > 4$ and $V \in L^{\frac{n}{2}}(\mathbb{R}^n)$ with sufficiently small norm as we move forward.

We note that by the discussion after (2.5.3) to prove Theorem 3.0.1 it suffice to prove (2.5.3) itself. That is, we are left to prove

$$\int_0^\infty \|tDB(I + t^2 DBDB)^{-1}u\|_2^2 \frac{\mathrm{d}t}{t} \lesssim \|u\|_2^2, \quad \forall u \in L^2(\mathbb{R}^n; \mathbb{C}^{n+2}).$$

To do this we will reduce to a Carleson measure estimate and then use a stopping time argument to prove the Carleson measure estimate.

3.1 Initial Estimates

We start by giving some estimates which are key for proving the quadratic estimate. First note that we can decompose $L^2(\mathbb{R}^n; \mathbb{C}^{n+2})$ as follows $L^2(\mathbb{R}^n; \mathbb{C}^{n+2}) = L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n; \mathbb{C}^n) \oplus L^2(\mathbb{R}^n)$. Now define the projections on $L^2(\mathbb{R}^n; \mathbb{C}^{n+2})$ onto each of these spaces as

$$\mathbb{P}_\perp f := \begin{bmatrix} f_\perp \\ 0 \\ 0 \end{bmatrix}, \quad \mathbb{P}_\| f := \begin{bmatrix} 0 \\ f_\| \\ 0 \end{bmatrix}, \text{ and } \mathbb{P}_\mu f := \begin{bmatrix} 0 \\ 0 \\ f_\mu \end{bmatrix}.$$

Moreover, define $\tilde{\mathbb{P}} := (\mathbb{P}_\perp + \mathbb{P}_\|)$ and $\mathbb{P}_r = (\mathbb{P}_\| + \mathbb{P}_\mu)$. We give the following Riesz transform type bounds which will be important to replacing the coercivity in [13] which our operators do not have.

Lemma 3.1.1. If $V \in B^{\frac{n}{2}}(\mathbb{R}^n)$, then we have the following estimates

$$\|DB\mathbb{P}_\mu u\|_2 \lesssim \|Du\|_2, \quad \|\nabla\tilde{\mathbb{P}}u\|_2 \lesssim \|Du\|_2, \quad \||V|^{\frac{1}{2}}\tilde{\mathbb{P}}u\|_2 \lesssim \|Du\|_2,$$

for all $u \in \overline{\mathsf{R}(D)}$, where the constants depend only on V, n and $\|B\|_\infty$.

Proof. First note as $u \in \overline{\mathsf{R}(D)}$ then

$$u = \begin{bmatrix} u_\perp \\ \nabla_\mu f \end{bmatrix}$$

for some $f \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^n)$. Therefore,

$$Du = \begin{bmatrix} \operatorname{div}\nabla f + Vf \\ -\nabla u_\perp \\ -|V|^{\frac{1}{2}}u_\perp \end{bmatrix}.$$

Then by direct computations we have that

$$DB\mathbb{P}_\mu u = \begin{bmatrix} bVf \\ 0 \\ 0 \end{bmatrix}, \quad \nabla\tilde{\mathbb{P}}u = \begin{bmatrix} \nabla u_\perp \\ -\nabla^2 f \\ 0 \end{bmatrix}, \quad \text{and } |V|^{\frac{1}{2}}\tilde{\mathbb{P}}u = \begin{bmatrix} |V|^{\frac{1}{2}}u_\perp \\ -|V|^{\frac{1}{2}}\nabla f \\ 0 \end{bmatrix}.$$

Now by the boundedness of B and Riesz transform bounds, from [45] and [5], we have that

$$\|bVf\|_2 \leq \|B\|_\infty \|Vf\|_2 \lesssim \|(-\operatorname{div}\nabla + V)f\|_2.$$

Thus $\|DB\mathbb{P}_\mu u\|_2 \lesssim \|Du\|_2$. For the second two inequalities we use $\|\nabla^2 f\|_2 \lesssim \|(-\operatorname{div}\nabla + V)f\|_2$ and $\||V|^{\frac{1}{2}}\nabla f\|_2 \lesssim \|(-\operatorname{div}\nabla + V)f\|_2$ from [45] and [5], to obtain $\|\nabla\tilde{\mathbb{P}}u\|_2 \lesssim \|Du\|_2$ and $\||V|^{\frac{1}{2}}\tilde{\mathbb{P}}u\|_2 \lesssim \|Du\|_2$. This completes the proof. $\square$

We have a similar result for $V \in L^{\frac{n}{2}}(\mathbb{R}^n)$ with small norm. We note that the smallness of V is needed here when using the Riesz transform estimates.

Lemma 3.1.2. Let $n > 4$. If $V \in L^{\frac{n}{2}}(\mathbb{R}^n)$ with sufficiently small norm, then

$$\|DB\mathbb{P}_\mu u\|_2 \lesssim \|Du\|_2, \quad \|\nabla\tilde{\mathbb{P}}u\|_2 \lesssim \|Du\|_2$$

for all $u \in \mathsf{R}(D)$, where the constants depend only on V, n and $\|B\|_\infty$.

Proof. The proof will follow as in Lemma 3.1.1 once we have established

$$\|\nabla^2 f\|_2 \lesssim \|(\Delta + V)f\|_2 \quad \text{and} \quad \|Vf\|_2 \leq \|(\Delta + V)f\|_2.$$

To this end we use Hölder's inequality, and the Sobolev inequality to obtain

$$\|Vf\|_2 \leq \|V\|_{\frac{n}{2}}\|f\|_{2^{**}} \leq C^2\|V\|_{\frac{n}{2}}\|\nabla^2 f\|_2, \tag{3.1.1}$$

where $C > 0$ is the constant associated with the Sobolev inequality. Then by the classical Reisz transform estimates and (3.1.1), we get

$$\|\nabla^2 f\|_2 \lesssim \|\Delta f\|_2 \leq \|(\Delta + V)f\|_2 + \|Vf\|_2 \leq \|(\Delta + V)f\|_2 + C^2\|V\|_{\frac{n}{2}}\|\nabla^2 f\|_2,$$

Therefore, as $\|V\|_{\frac{n}{2}}$ is sufficiently small we may hide the last term above as

$$(1 - C^2\|V\|_{\frac{n}{2}})\|\nabla^2 f\|_2 \lesssim \|(\Delta + V)f\|_2.$$

We also have

$$\|Vf\|_2 \leq C^2\|V\|_{\frac{n}{2}}\|\nabla^2 f\|_2 \lesssim \|(\Delta + V)f\|_2.$$

Then, the proof follows that of Lemma 3.1.1 verbatim. □

Denote $\langle x\rangle := 1+|x|$ and $\operatorname{dist}(E, F) := \inf\{|x-y| : x \in E, y \in F\}$ for every $E, F \subseteq \mathbb{R}^n$. We will now state the off-diagonal estimates for the operators R_t^B, P_t^B, and Q_t^B. These estimates are important for later sections as they relate how the operators R_t^B, P_t^B and Q_t^B are bounded on cubes.

Proposition 3.1.3 (Off-Diagonal Estimates)**.** Let U_t be given by either R_t^B for every nonzero $t \in \mathbb{R}^n$, or P_t^B or Q_t^B for every $t > 0$. Then for any $M \in \mathbb{N}$ there exists $C_M > 0$, which depends only on V, n, κ, and $\|B\|_\infty$, such that

$$\|U_t u\|_{L^2(E)} \leq C_M \left\langle \frac{\operatorname{dist}(E, F)}{t} \right\rangle^{-M} \|u\|_{L^2(F)},$$

for every $E, F \subseteq \mathbb{R}^n$ Borel sets, and $u \in L^2(\mathbb{R}^n; \mathbb{C}^{n+2})$ with $\operatorname{supp} u \subset F$.

Proof. Let $u \in \mathscr{D}(D)$ and $\eta \in \mathcal{C}_c^\infty(\mathbb{R}^n)$. Using the product rule and Lemma 2.3.4 we have $\eta u \in \mathscr{D}(D)$ and the following commutator bound

$$|[\eta I, D]u| = |\eta D u - D(\eta u)| = \left| \begin{bmatrix} \eta \nabla_\mu^* u_r - \nabla_\mu^*(\eta u_r) \\ \nabla(\eta u_\perp) - \eta \nabla u_\perp \\ 0 \end{bmatrix} \right| = \left| \begin{bmatrix} -u_\parallel \cdot \nabla \eta \\ u_\perp \nabla \eta \\ 0 \end{bmatrix} \right| \leq |\nabla \eta||u|,$$

The proof then follows in the same manner as in [13, Proposition 5.2]. □

We define the standard dyadic decomposition of $\mathbb{R}^n$ as $\Delta := \bigcup_{k=-\infty}^{\infty} \Delta_{2^k}$ where $\Delta_t := \{2^k(m + (0,1]^n) : m \in \mathbb{Z}^n\}$ if $2^{k-1} < t \leq 2^k$. Now we introduce the follow collection of dyadic cubes depending on the potential V.

Definition 3.1.4. Let $V \in B^{\frac{n}{2}}(\mathbb{R}^n)$. Then define Δ_t^V to be all the dyadic cubes, $Q \in \Delta_t$, with

$$l(Q)^2 \fint_Q V \leq 1. \tag{3.1.2}$$

And define $\Delta^V := \bigcup_{t>0} \Delta_t^V$.

We refer to cubes in Δ^V as 'small' since for almost all $x \in \mathbb{R}^n$ then by (2.2.3) there exists $\varepsilon > 0$ such that for all $t < \varepsilon$ then the unique dyadic cubes containing x of scale t will be in Δ_t^V. These cubes were first introduced in [5]. We introduce the small cube so that in the following Lemma we obtain the homogeneous estimate (3.1.3) so long as we are on a small cube by using (2.3.4). We also note that if $V \in L^{\frac{n}{2}}(\mathbb{R}^n)$ then we will consider all cubes to be 'small'. The proof of the lemma is adapted from the proofs of [13, Lemma 5.6] and [12, Lemma 5.15] to incorporate the inhomogeneity of the operator D.

To do this we use the fact that the Sobolev exponent $2^* = 2\left(\frac{n}{2}\right)'$, the Hölder conjugate of the regularity of V.

Lemma 3.1.5. We have the estimate

$$\left|\fint_Q Du\right|^2 \lesssim \frac{1}{l(Q)}\left(1+\left(\int_Q |V|^{\frac{n}{2}}\right)^{\frac{2}{n}}\right)\left(\fint_Q |u|^2\right)^{\frac{1}{2}}\left(\fint_Q |Du|^2\right)^{\frac{1}{2}},$$

for all $Q \in \Delta$ and $u \in \mathscr{D}(D)$. Moreover, if $V \in B^{\frac{n}{2}}(\mathbb{R}^n)$ and $Q \in \Delta^V$ or $V \in L^{\frac{n}{2}}(\mathbb{R}^n)$, then we have

$$\left|\fint_Q Du\right|^2 \lesssim \frac{1}{l(Q)}\left(\fint_Q |u|^2\right)^{\frac{1}{2}}\left(\fint_Q |Du|^2\right)^{\frac{1}{2}}, \tag{3.1.3}$$

for all $u \in \mathscr{D}(D)$.

Proof. Let

$$t = \left(\int_Q |u|^2\right)^{\frac{1}{2}}\left(\int_Q |Du|^2\right)^{-\frac{1}{2}}.$$

If $t \geq \frac{1}{4}l(Q)$ then proceed as in [13, Lemma 5.6]. By the Cauchy-Schwarz Inequality we have

$$\begin{aligned}\left|\int_Q Du\right|^2 &\leq \left(\int_Q 1\right)\left(\int_Q |Du|^2\right)\\ &= |Q|\left(\int_Q |Du|^2\right)^{\frac{1}{2}}\left(\int_Q |Du|^2\right)^{\frac{1}{2}}\\ &\leq \frac{4|Q|}{l(Q)}\left(\int_Q |Du|^2\right)^{\frac{1}{2}}\left(\int_Q |u|^2\right)^{\frac{1}{2}}.\end{aligned}$$

Then dividing by $|Q|^2$ gives

$$\begin{aligned}\left|\fint_Q Du\right|^2 &\lesssim \frac{1}{l(Q)}\left(\fint_Q |u|^2\right)^{\frac{1}{2}}\left(\fint_Q |Du|^2\right)^{\frac{1}{2}}\\ &\leq \frac{1}{l(Q)}\left(1+l(Q)^2\fint_Q V\right)\left(\fint_Q |u|^2\right)^{\frac{1}{2}}\left(\fint_Q |Du|^2\right)^{\frac{1}{2}}.\end{aligned}$$

Now suppose $t \leq \frac{1}{4}l(Q)$. Let $\eta \in \mathcal{C}_c^\infty(Q)$ such that $\eta(x) = 1$ when $\operatorname{dist}(x, \mathbb{R}^n \setminus Q) > t$ and $\|\nabla\eta\|_\infty \lesssim \frac{1}{t}$. Note that $\eta u \in \mathscr{D}(D)$ with compact support. We will bound each component of Du separately. First, we have

$$\left|\int_Q (Du)_\perp\right| = \left|\int_Q \eta(Du)_\perp + (1-\eta)(Du)_\perp\right| \leq \left|\int_Q \eta\nabla_\mu^* u_r\right| + \int_Q |1-\eta||Du|.$$

Then, as η has compact support and by the definition of ∇_μ^*, we have

$$\left|\int_Q \eta\nabla_\mu^* u_r\right| = \left|\int_Q \nabla_\mu\eta\cdot u_r\right| \leq \int_Q |\nabla_\mu\eta||u| = \int_Q |\nabla\eta||u| + \int_Q |V|^{\frac{1}{2}}\eta|u|$$

For the second component we have

$$\left|\int_Q (Du)_\parallel\right| = \left|\int_Q \eta\cdot(Du)_\parallel + (1-\eta)\cdot(Du)_\parallel\right| \leq \left|\int_Q \eta\cdot\nabla u_\perp\right| + \int_Q |1-\eta||Du|$$

Since $\eta u_\perp$ has compact support in Q, then by the Fundamental Theorem of Calculus we have

$$\int_Q \nabla(\eta u_\perp) = 0.$$

Using this and the product rule, we have

$$\left|\int_Q \eta\cdot\nabla u_\perp\right| = \left|\int_Q \eta\cdot\nabla u_\perp - \nabla(\eta u_\perp)\right| = \left|\int_Q u_\perp\nabla\eta\right| \leq \int_Q |u||\nabla\eta|.$$

Also,

$$\left|\int_Q (Du)_\mu\right| = \left|\int_Q \eta(Du)_\mu + (1-\eta)(Du)_\mu\right| \leq \int_Q \eta|V|^{\frac{1}{2}}|u| + \int_Q |1-\eta||Du|.$$

Thus, using the Cauchy-Schwarz Inequality, $\|\nabla\eta\|_\infty \lesssim \frac{1}{t}$, and $|\operatorname{supp}(\nabla\eta)| = l(Q)^{n-1}t$, we have

$$\begin{aligned}\int_Q |\nabla\eta||u| &\leq \left(\int_Q |\nabla\eta|^2\right)^{\frac{1}{2}}\left(\int_Q |u|^2\right)^{\frac{1}{2}}\\
&\leq \|\nabla\eta\|_\infty|\operatorname{supp}(\nabla\eta)|^{\frac{1}{2}}\left(\int_Q |u|^2\right)^{\frac{1}{2}}\\
&\lesssim l(Q)^{\frac{n-1}{2}}t^{-\frac{1}{2}}\left(\int_Q |u|\right)^{\frac{1}{2}}.\end{aligned}$$

Now using the Cauchy–Schwarz inequality, Hölder's inequality, Sobolev Embedding, where $\left(\frac{n}{2}\right)' = \frac{2^*}{2}$, and the same argument as above we have

$$\begin{aligned}
\int_Q |V|^{\frac{1}{2}}\eta|u| &\leq \left(\int_Q |V||\eta|^2\right)^{\frac{1}{2}} \left(\int_Q |u|^2\right)^{\frac{1}{2}} \\
&\leq \left(\int_Q |V|^{\frac{n}{2}}\right)^{\frac{1}{n}} \left(\int_Q |\eta|^{2^*}\right)^{\frac{1}{2^*}} \left(\int_Q |u|^2\right)^{\frac{1}{2}} \\
&\lesssim \left(\int_Q |V|^{\frac{n}{2}}\right)^{\frac{1}{n}} \left(\int_Q |\nabla\eta|^2\right)^{\frac{1}{2}} \left(\int_Q |u|^2\right)^{\frac{1}{2}} \\
&\lesssim l(Q)^{\frac{n-1}{2}} t^{-\frac{1}{2}} \left(\int_Q |V|^{\frac{n}{2}}\right)^{\frac{1}{n}} \left(\int_Q |u|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Now, by the Cauchy–Schwarz inequality, also $|Q \cap \operatorname{supp}(1-\eta)| = l(Q)^{n-1}t$ and $|1-\eta| \leq 1$ by the definition of η, we have

$$\int_Q |1-\eta||Du| \leq \left(\int_Q |1-\eta|^2\right)^{\frac{1}{2}} \left(\int_Q |Du|^2\right)^{\frac{1}{2}} \leq l(Q)^{\frac{n-1}{2}} t^{\frac{1}{2}} \left(\int_Q |Du|^2\right)^{\frac{1}{2}}.$$

Thus, recalling the definition of t, we obtain

$$\begin{aligned}
\left|\int_Q Du\right| &\lesssim l(Q)^{\frac{n-1}{2}} t^{-\frac{1}{2}} \left(\int_Q |u|\right)^{\frac{1}{2}} \left(1 + \left(\int_Q |V|^{\frac{n}{2}}\right)^{\frac{1}{n}}\right) + (l(Q)^{n-1}t)^{\frac{1}{2}} \left(\int_Q |Du|\right)^{\frac{1}{2}}, \\
&\lesssim \left(\frac{|Q|}{l(Q)}\right)^{\frac{1}{2}} \left(\int_Q |u|\right)^{\frac{1}{4}} \left(\int_Q |Du|\right)^{\frac{1}{4}} \left(1 + \left(\int_Q |V|^{\frac{n}{2}}\right)^{\frac{1}{n}}\right).
\end{aligned}$$

Thus dividing by $|Q|$ and then squaring gives

$$\left|\fint_Q Du\right|^2 \lesssim \frac{1}{l(Q)} \left(1 + \left(\int_Q |V|^{\frac{n}{2}}\right)^{\frac{2}{n}}\right) \left(\fint_Q |u|^2\right)^{\frac{1}{2}} \left(\fint_Q |Du|^2\right)^{\frac{1}{2}}.$$

Now if $V \in L^{\frac{n}{2}}(\mathbb{R}^n)$ then the inequality (3.1.3) holds for all cubes. If $V \in B^{\frac{n}{2}}(\mathbb{R}^n)$, and $Q \in \Delta^V$, we have

$$\left(\int_Q V^{\frac{n}{2}}\right)^{\frac{2}{n}} = |Q|^{\frac{2}{n}} \left(\fint_Q V^{\frac{n}{2}}\right)^{\frac{2}{n}} \lesssim l(Q)^2 \fint_Q V \leq 1.$$

Then inequality (3.1.3) holds. This completes the proof. $\square$

3.2 Reduction to Carleson measure estimate

We start by with the approach, as developed in [13], of reducing the quadratic estimate to proving a Carleson measure estimate. Our approach differs in that the Carleson measure will have to be adapted to the potential V, in the sense that the measure is only a Carleson measure on small cubes. The reason we treat the big and small cubes differently is that on the small cubes we have inequality (3.1.3) which is an important step in proving Lemmas 3.2.7 and 3.3.4. In Lemma 3.2.1 we will prove that Q_t^B extends to an operator from $L^\infty(\mathbb{R}^n;\mathbb{C}^{n+1})$ to $L^2_{\text{loc}}(\mathbb{R}^n;\mathbb{C}^{n+2})$ so, as in [13], we define can $\gamma_t(x)w := (Q_t^B w)(x)$ for every $w \in \mathbb{C}^n$. Here we view w on the right-hand side of the above equation as the constant function defined on $\mathbb{R}^n$ by $w(x) := w$. We additionally define $\tilde{\gamma}_t := \gamma_t \tilde{\mathbb{P}} = \gamma_t(\mathbb{P}_\perp + \mathbb{P}_\parallel)$, similar to as in [16]. For fixed $t > 0$ we define the mapping $\tilde{\gamma}_t \colon \mathbb{R}^n \to \mathcal{L}(\mathbb{C}^{n+2})$ by $\tilde{\gamma}_t \colon x \mapsto \tilde{\gamma}_t(x)$. We also define the dyadic averaging operator $A_t \colon L^2(\mathbb{R}^n;\mathbb{C}^{n+2}) \to L^2(\mathbb{R}^n;\mathbb{C}^{n+2})$ given by

$$A_t u(x) := \mathrm{av}_Q\, u := \fint_Q u(y)\,\mathrm{d}y,$$

for every $x \in \mathbb{R}^n$ and $t > 0$, where $Q \in \Delta_t$ is the unique dyadic cube such that $x \in Q$. We have the following properties for $\tilde{\gamma}_t$ and A_t.

Lemma 3.2.1. We have the following:

1. The operator $\tilde{Q}_t^B$ extends to a bounded operator from $L^\infty(\mathbb{R}^n;\mathbb{C}^{n+2})$ into the space $L^2_{\text{loc}}(\mathbb{R}^n;\mathbb{C}^{n+2})$. In particular, for $t > 0$, we have $\tilde{\gamma}_t \in L^2_{\text{loc}}(\mathbb{R}^n;\mathcal{L}(\mathbb{C}^{n+2}))$ with

$$\fint_Q |\tilde{\gamma}_t(x)|^2_{\mathcal{L}(\mathbb{C}^{n+2})}\,\mathrm{d}x \lesssim 1,$$

   for all $Q \in \Delta_t$

2. $\sup_{t>0} \|\tilde{\gamma}_t A_t\|_{op} \lesssim 1$.

Proof. The proof follows Lemma 3.5 in [4]. Let $f \in L^\infty(\mathbb{R}^n;\mathbb{C}^{n+2})$. Now define $C_0(Q) := Q$ and for $k > 0$ define $C_k(Q) := (2^k Q \setminus 2^{k-1} Q)$. Then, using the off-diagonal estimates in Proposition 3.1.3 and the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
\|\mathbb{1}_Q \tilde{Q}_t^B f\|_2^2 &\lesssim \left(\sum_{k=0}^{\infty} \|\mathbb{1}_Q \tilde{Q}_t^B \mathbb{1}_{C_k(Q)}\| \|\mathbb{1}_{C_k(Q)} f\|_2 \right)^2 \\
&\lesssim \left(\sum_{k=0}^{\infty} \left\langle \frac{\operatorname{dist}(Q, C_k(Q))}{t} \right\rangle^{-M} \|\mathbb{1}_{C_k(Q)} f\|_2 \right)^2 \\
&\lesssim \left(\sum_{k=0}^{\infty} 2^{-kM} \|\mathbb{1}_{C_k(Q)} f\|_2 \right)^2 \\
&\leq \left(\sum_{k=0}^{\infty} 2^{-kM} \right) \left(\sum_{k=0}^{\infty} 2^{-kM} \|\mathbb{1}_{C_k(Q)} f\|_2^2 \right).
\end{aligned}$$

Now, choosing $M > n$, we have

$$\|\mathbb{1}_Q \tilde{Q}_t^B f\|_2^2 \lesssim \sum_{k=0}^{\infty} 2^{-kM} \|\mathbb{1}_{2^k Q} f\|_2^2 \lesssim \|f\|_\infty^2 |Q| \sum_{k=0}^{\infty} 2^{-k(M-n)} \lesssim \|f\|_\infty^2 |Q|$$

Therefore, we have $\tilde{Q}_t^B \colon L^\infty(\mathbb{R}^n; \mathbb{C}^{n+2}) \to L^2_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{C}^{n+2})$ is a bounded operator. Now, let $\{e_1, \ldots, e_{n+2}\}$ be an orthonormal basis for $\mathbb{C}^{n+2}$. First note that for any $w \in \mathbb{C}^{n+2}$, using the Cauchy–Schwarz inequality, we have

$$
\begin{aligned}
|[\tilde{\gamma}_t(x)](w)|^2 &= \left|[\tilde{\gamma}_t(x)]\left(\sum_{k=1}^{n+2} w_k e_k\right)\right|^2 \\
&= \left|\sum_{k=1}^{n+2} w_k [\tilde{\gamma}_t(x)](e_k)\right|^2 \\
&= \left|\sum_{k=1}^{n+2} w_k \left(\sum_{l=1}^{n+2} \langle [\tilde{\gamma}_t(x)](e_k), e_l\rangle e_l\right)\right|^2 \\
&= \left|\sum_{l=1}^{n+2} \left(\sum_{k=1}^{n+2} w_k \langle [\tilde{\gamma}_t(x)](e_k), e_l\rangle\right) e_l\right|^2 \\
&= \sum_{l=1}^{n+2} \left(\sum_{k=1}^{n+2} w_k \langle [\tilde{\gamma}_t(x)](e_k), e_l\rangle\right)^2 \\
&\leq \sum_{l=1}^{n+2} \left(\sum_{k=1}^{n+2} w_k^2\right)\left(\sum_{k=1}^{n+2} \langle [\tilde{\gamma}_t(x)](e_k), e_l\rangle^2\right) \\
&\leq |w|^2 \sum_{l=1}^{n+2}\sum_{k=1}^{n+2} |[\tilde{\gamma}_t(x)](e_k)|^2 |e_l|^2 \\
&= (n+2)|w|^2 \sum_{k=1}^{n+2} |[\tilde{\gamma}_t(x)](e_k)|^2.
\end{aligned}
$$

Then, by the definition of operator norm, the above two calculations, and the definition of $\tilde{\gamma}_t$, we have

$$
\begin{aligned}
⨍_Q |\tilde{\gamma}_t(x)|^2_{\mathcal{L}(\mathbb{C}^{n+2})}\,\mathrm{d}x &= ⨍_Q \sup_{w=1} |[\tilde{\gamma}_t(x)](w)|^2\,\mathrm{d}x \\
&\lesssim ⨍_Q \sup_{w=1} |w|^2 \sum_{k=1}^{n+2} |[\tilde{\gamma}_t(x)](e_k)|^2\,\mathrm{d}x \\
&= \sum_{k=1}^{n+2} ⨍_Q |[\tilde{Q}_t^B(\tilde{e}_k)](x)|^2\,\mathrm{d}x \\
&\lesssim \sum_{k=1}^{n+2} \|\tilde{e}_k\|_\infty \\
&\lesssim 1,
\end{aligned}
$$

where $\tilde{e}_k(x) = e_k$ is the constant function. This completes the proof of part (1).

Let $u \in L^2(\mathbb{R}^n; \mathbb{C}^{n+2})$. For part (2) we use part (1), the definition of A_t, and Jensen's inequality, to get

$$
\begin{aligned}
\|\tilde{\gamma}_t A_t u\|_2^2 &= \sum_{Q\in\Delta_t} \int_Q |[\tilde{\gamma}_t(x)](A_t u)(x)|^2 \,\mathrm{d}x \\
&\lesssim \sum_{Q\in\Delta_t} \int_Q |[\tilde{\gamma}_t(x)]|^2_{\mathcal{L}(\mathbb{C}^{n+2})} \left| \fint_Q u(y)\,\mathrm{d}y \right|^2 \mathrm{d}x \\
&= \sum_{Q\in\Delta_t} \left(\int_Q |[\tilde{\gamma}_t(x)]|^2_{\mathcal{L}(\mathbb{C}^{n+2})} \right) \left| \fint_Q u(y)\,\mathrm{d}y \right|^2 \\
&\lesssim \sum_{Q\in\Delta_t} |Q| \fint_Q |u(y)|^2 \,\mathrm{d}y \\
&= \|u\|_2^2.
\end{aligned}
$$

Taking supremum in $t > 0$ completes the proof. □

Then the main result of this section is the following.

Proposition 3.2.2. Let $V \in B^{\frac{n}{2}}(\mathbb{R}^n)$. If

$$
\int_0^\infty \sum_{Q\in\Delta_t^V} \|\mathbb{1}_Q \tilde{\gamma}_t A_t u\|_2^2 \frac{\mathrm{d}t}{t} \lesssim \|u\|_2^2, \quad \forall u \in \overline{\mathsf{R}(D)}, \tag{3.2.1}
$$

then we have

$$
\int_0^\infty \|Q_t^B u\|_2^2 \frac{\mathrm{d}t}{t} \lesssim \|u\|_2^2, \quad \forall u \in L^2(\mathbb{R}^2).
$$

We proceed in proving Proposition 3.2.2 by introducing and proving the required lemmas. We will then assemble the lemmas to prove Proposition 3.2.2

Lemma 3.2.3. If $V \in L^{\frac{n}{2}}_{\mathrm{loc}}(\mathbb{R}^n)$, then we have

$$
\int_0^\infty \|Q_t^B (I - P_t) u\|_2^2 \frac{\mathrm{d}t}{t} \lesssim \|u\|_2^2,
$$

for all $u \in \overline{\mathsf{R}(D)}$.

Proof. Let $u \in \overline{\mathsf{R}(D)}$. We begin by proving $tQ_t^B D$ is uniformly bounded in t. As ellipticity gives that B^{-1} exists on $\overline{\mathsf{R}(D)}$, then using $tQ_t^B DB = (I - P_t^B)$ and ellipticity again gives

$$\|tQ_t^B Du\|_2^2 = \|tQ_t^B DBB^{-1}u\|_2^2 = \|(I - P_t^B)B^{-1}u\|_2^2 \lesssim \|B^{-1}u\|_2^2 \lesssim \|u\|_2^2.$$

Then, as $I - P_t = t^2D^2P_t$ and the quadratic estimates for the self-adjoint operator D, we have

$$\begin{aligned}
\int_0^\infty \|Q_t^B(I - P_t)u\|_2^2 \frac{\mathrm{d}t}{t} &= \int_0^\infty \|Q_t^B(t^2D^2P_t)u\|_2^2 \frac{\mathrm{d}t}{t} \\
&= \int_0^\infty \|(tQ_t^B D)(tDP_t)u\|_2^2 \frac{\mathrm{d}t}{t^3} \\
&\lesssim \int_0^\infty \|Q_t u\|_2^2 \frac{\mathrm{d}t}{t} \\
&\lesssim \|u\|_2^2,
\end{aligned}$$

as required. □

We will now exploit the structure of D to bound the third component. This follows a similar approach to [16], however, because our perturbation B is not block-diagonal and as $B_{\perp\perp}$ is not necessarily 1 we cannot bound the first component in the same way. Therefore, unlike in [16] we do not reduce to a homogeneous differential operator and so we do not get Lemma 3.1.5 on all cubes, and this is why we need to introduce the big and small cubes. We note an important consequence of Lemmas 3.1.1 and 3.1.2 is that the projection $\mathbb{P}_\mu$ maps $\overline{\mathsf{R}(D)}$ into $\mathscr{D}(DB)$.

Lemma 3.2.4. If $V \in B^{\frac{n}{2}}(\mathbb{R}^n)$, then we have

$$\int_0^\infty \|Q_t^B \mathbb{P}_\mu P_t u\|_2^2 \frac{\mathrm{d}t}{t} \lesssim \|u\|_2^2,$$

for all $u \in \overline{\mathsf{R}(D)}$.

Proof. Let $u \in \overline{\mathsf{R}(D)}$. Thus by the uniform boundedness of P_t^B, Lemma 3.1.1, and D

being self-adjoint, we have

$$\begin{aligned}\int_0^\infty \|Q_t^B \mathbb{P}_\mu P_t u\|_2^2 \frac{\mathrm{d}t}{t} &= \int_0^\infty \|tP_t^B DB\mathbb{P}_\mu P_t u\|_2^2 \frac{\mathrm{d}t}{t} \\ &\lesssim \int_0^\infty \|tDB\mathbb{P}_\mu P_t u\|_2^2 \frac{\mathrm{d}t}{t} \\ &\lesssim \int_0^\infty \|tDP_t u\|_2^2 \frac{\mathrm{d}t}{t} \\ &= \int_0^\infty \|Q_t u\|_2^2 \frac{\mathrm{d}t}{t} \\ &\lesssim \|u\|_2^2\end{aligned}$$

As required. □

We will need to be able to use the inequality (3.1.3). However, this is only available to us when we are on small cubes. Therefore, we need a bound on all large cubes. We do this by using the off-diagonal estimates, the Fefferman–Phong Inequality, and Lemma 3.1.1.

Lemma 3.2.5. If $V \in B^{\frac{n}{2}}(\mathbb{R}^n)$, then we have

$$\int_0^\infty \sum_{Q \in \Delta_t \setminus \Delta_t^V} \|\mathbb{1}_Q Q_t^B \tilde{\mathbb{P}} P_t u\|_2^2 \frac{\mathrm{d}t}{t} \lesssim \|u\|_2^2,$$

for all $u \in \overline{\mathsf{R}(D)}$.

Proof. Let $u \in \overline{\mathsf{R}(D)}$. Define $f := \tilde{\mathbb{P}} P_t u$. Let $M \in \mathbb{N}$ to be chosen later. Define $C_0(Q) := Q$ and for $k > 0$ define $C_k(Q) := (2^k Q \setminus 2^{k-1} Q)$. Then, using off-diagonal estimates, and the Cauchy–Schwarz inequality we have

$$\begin{aligned}\sum_{Q\in\Delta_t\setminus\Delta_t^V}\|\mathbb{1}_Q Q_t^B\tilde{\mathbb{P}}P_t u\|_2^2 &\lesssim \sum_{Q\in\Delta_t\setminus\Delta_t^V}\left(\sum_{k=0}^{\infty}\|\mathbb{1}_Q Q_t^B\mathbb{1}_{C_k(Q)}\|\|\mathbb{1}_{C_k(Q)}f\|_2\right)^2\\
&\lesssim \sum_{Q\in\Delta_t\setminus\Delta_t^V}\left(\sum_{k=0}^{\infty}\left\langle\frac{\operatorname{dist}(Q,C_k(Q))}{t}\right\rangle^{-M}\|\mathbb{1}_{C_k(Q)}f\|_2\right)^2\\
&\lesssim \sum_{Q\in\Delta_t\setminus\Delta_t^V}\left(\sum_{k=0}^{\infty}2^{-kM}\|\mathbb{1}_{C_k(Q)}f\|_2\right)^2\\
&\leq \sum_{Q\in\Delta_t\setminus\Delta_t^V}\left(\sum_{k=0}^{\infty}2^{-kM}\right)\left(\sum_{k=0}^{\infty}2^{-kM}\|\mathbb{1}_{C_k(Q)}f\|_2^2\right)\\
&\lesssim \sum_{Q\in\Delta_t\setminus\Delta_t^V}\sum_{k=0}^{\infty}2^{-kM}\|\mathbb{1}_{2^kQ}f\|_2^2.\end{aligned}$$

First, suppose

$$l(2^kQ)^2\fint_{2^kQ}V>1.$$

Then using the Fefferman–Phong inequality in Proposition 2.2.5, we have

$$\|\mathbb{1}_{2^kQ}f\|_2^2<\left(l(2^kQ)^2\fint_{2^kQ}V\right)^{\beta}\|\mathbb{1}_{2^kQ}f\|_2^2\lesssim l(2^kQ)^2\|\mathbb{1}_{2^kQ}\nabla_\mu f\|_2^2=2^{2k}l(Q)^2\|\mathbb{1}_{2^kQ}\nabla_\mu f\|_2^2.$$

Now, suppose

$$l(2^kQ)^2\fint_{2^kQ}V\leq 1,$$

then using $Q\in\Delta_t\setminus\Delta_t^V$ and the Fefferman–Phong inequality in Proposition 2.2.5 again, we have

$$\begin{aligned}\|\mathbb{1}_{2^kQ}f\|_2^2&<\left(l(Q)^2\fint_Q V\right)\|\mathbb{1}_{2^kQ}f\|_2^2\\
&\lesssim 2^{(n-2)(k-2)}\left(l(2^kQ)^2\fint_{2^kQ}V\right)|\mathbb{1}_{2^kQ}\nabla_\mu f\|_2^2\\
&\lesssim 2^{(n-2)(k-2)}l(2^kQ)^2\|\mathbb{1}_{2^kQ}\nabla_\mu f\|_2^2\\
&\lesssim 2^{nk}l(Q)^2\|\mathbb{1}_{2^kQ}\nabla_\mu f\|_2^2\end{aligned}$$

Noting that $2^{nk}\leq 2^{2nk}$ and $2^{2k}\leq 2^{2nk}$, then using $l(Q)\approx t$, the covering inequality

$\sum_{Q\in\Delta_t} \mathbb{1}_{2^kQ}(x) \lesssim 2^{kn}$ for all $x \in \mathbb{R}^n$, and choosing $M > 3n$, we obtain

$$\begin{aligned}\sum_{Q\in\Delta_t\setminus\Delta_t^V} \sum_{k=0}^{\infty} 2^{-kM} \|\mathbb{1}_{2^kQ} f\|_2^2 &\lesssim \sum_{Q\in\Delta_t\setminus\Delta_t^V} \sum_{k=0}^{\infty} 2^{-k(M-2n)} l(Q)^2 \|\mathbb{1}_{2^kQ} \nabla_\mu f\|_2^2 \\ &\lesssim t^2 \sum_{k=0}^{\infty} 2^{-k(M-3n)} \|\nabla_\mu f\|_2^2 \\ &\lesssim t^2 \|\nabla_\mu f\|_2^2.\end{aligned}$$

Recall that $f = \tilde{\mathbb{P}} P_t u$, and that if $u \in \overline{\mathsf{R}(D)}$ then $P_t u \in \overline{\mathsf{R}(D)}$. Therefore, using Lemma 3.1.1 we have

$$\begin{aligned}\int_0^\infty \sum_{Q\in\Delta_t\setminus\Delta_t^V} \|\mathbb{1}_Q Q_t^B \tilde{\mathbb{P}} P_t u\|_2^2 \frac{\mathrm{d}t}{t} &\lesssim \int_0^\infty t^2 \|\nabla_\mu \tilde{\mathbb{P}} P_t u\|_2^2 \frac{\mathrm{d}t}{t} \\ &\lesssim \int_0^\infty \|tDP_t u\|_2^2 \frac{\mathrm{d}t}{t} \\ &\lesssim \int_0^\infty \|Q_t u\|_2^2 \frac{\mathrm{d}t}{t} \\ &\lesssim \|u\|_2^2.\end{aligned}$$

This completes the proof. □

In the following lemma is where having the projection $\tilde{\mathbb{P}}$ is needed, otherwise, we would have $\nabla|V|^{\frac{1}{2}}$ in the last component and this would force us to assume some differentiability on V. In [14] the coercivity ((H8) in [14]) of the operators is used, however, in the inhomogeneous case we do not have coercivity of the operator but we do have coercivity in the sense of Lemma 3.1.1.

Lemma 3.2.6. If $V \in B^{\frac{n}{2}}(\mathbb{R}^n)$, then we have that

$$\int_0^\infty \sum_{Q\in\Delta_t^V} \|\mathbb{1}_Q (Q_t^B - \gamma_t A_t) \tilde{\mathbb{P}} P_t u\|_2^2 \frac{\mathrm{d}t}{t} \lesssim \|u\|_2^2,$$

for all $u \in \overline{\mathsf{R}(D)}$.

Proof. Let $u \in \overline{\mathsf{R}(D)}$. Define $f := \tilde{\mathbb{P}} P_t u$. Recall the definition of $C_k(Q)$ from Lemma 3.2.5. Then, by the off-diagonal estimates in Proposition 3.1.3, the Cauchy–Schwarz inequality, the Poincare inequality, Lemma 3.1.1 and choosing $M > n+2$, we have

$$\begin{aligned}
\sum_{Q\in\Delta_t^V} \|\mathbb{1}_Q(Q_t^B - \gamma_t A_t)\tilde{\mathbb{P}}P_t u\|_2^2 &\lesssim \sum_{Q\in\Delta_t^V} \left(\sum_{k=0}^{\infty} \|\mathbb{1}_Q Q_t^B \mathbb{1}_{C_k(Q)}\| \|\mathbb{1}_{C_k(Q)}(f - f_Q)\|_2 \right)^2 \\
&\lesssim \sum_{Q\in\Delta_t^V} \sum_{k=0}^{\infty} 2^{-kM} \|\mathbb{1}_{2^k Q}(f - f_Q)\|_2^2 \\
&\lesssim \sum_{Q\in\Delta_t^V} \sum_{k=0}^{\infty} 2^{-kM} l(2^k Q)^2 \|\mathbb{1}_{2^k Q}\nabla f\|_2^2 \\
&\lesssim t^2 \sum_{k=0}^{\infty} 2^{-k(M-(n+2))} \|\nabla f\|_2^2 \\
&\lesssim t^2 \|\nabla \tilde{\mathbb{P}} P_t u\|_2^2 \\
&\lesssim t^2 \|D P_t u\|_2^2.
\end{aligned}$$

Thus, we have

$$\int_0^\infty \sum_{Q\in\Delta_t^V} \|\mathbb{1}_Q(Q_t^B - \gamma_t A_t)\tilde{\mathbb{P}}P_t u\|_2^2 \frac{\mathrm{d}t}{t} \lesssim \int_0^\infty \|Q_t u\|_2^2 \frac{\mathrm{d}t}{t} \lesssim \|u\|_2^2.$$

This completes the proof. □

The following Lemma is analogous to [14, lemma 5.6]. This is where Lemma 3.1.5 is used. Therefore, it is important that we have already reduced to proving the estimate on small cubes.

Lemma 3.2.7. If $V \in B^{\frac{n}{2}}(\mathbb{R}^n)$, then we have

$$\int_0^\infty \sum_{Q\in\Delta_t^V} \|\mathbb{1}_Q \gamma_t A_t \tilde{\mathbb{P}}(P_t - I)u\|_2^2 \frac{\mathrm{d}t}{t} \lesssim \|u\|_2^2,$$

for all $u \in \overline{\mathsf{R}(D)}$.

Proof. We will perform a Schur-type estimate after we have established the bound

$$\|\mathbb{1}_{\Omega_t^V} A_t \tilde{\mathbb{P}}(P_t - I)Q_s v\|_2^2 = \sum_{Q\in\Delta_t^V} \|\mathbb{1}_Q A_t \tilde{\mathbb{P}}(P_t - I)v\|_2^2 \lesssim \min\left\{\frac{s}{t}, \frac{t}{s}\right\} \|v\|_2^2 \tag{3.2.2}$$

for all $v \in L^2(\mathbb{R}^n; \mathbb{C}^{n+2})$, where $\Omega_t^V := \bigcup_{Q \in \Delta_t^V} Q$. Now $(P_t - I)Q_s = \frac{t}{s} Q_t (I - P_s)$ and $P_t Q_s = \frac{s}{t} Q_t P_s$. If $t \leq s$ we use the uniform boundedness of Q_t and $I - P_t$ to obtain

$$\sum_{Q \in \Delta_t^V} \|\mathbb{1}_Q A_t \tilde{\mathbb{P}}(P_t - I) Q_s v\|_2^2 \lesssim \|\frac{t}{s} Q_s (I - P_t) v\|_2^2 \lesssim \left(\frac{t}{s}\right)^2 \|v\|_2^2 \leq \frac{t}{s} \|v\|_2^2.$$

If $s \leq t$ then using the boundedness of P_t and Q_t we have

$$\begin{aligned}\sum_{Q \in \Delta_t^V} \|\mathbb{1}_Q A_t \tilde{\mathbb{P}}(P_t - I) Q_s v\|_2^2 &\lesssim \|P_t Q_s v\|_2^2 + \sum_{Q \in \Delta_t^V} \|\mathbb{1}_Q A_t \tilde{\mathbb{P}} Q_s v\|_2^2 \\ &\lesssim \frac{s}{t} \|v\|_2^2 + \sum_{Q \in \Delta_t^V} \|\mathbb{1}_Q A_t \tilde{\mathbb{P}} Q_s v\|_2^2.\end{aligned}$$

Then, using Lemma 3.1.5 for cubes in Δ^V, the Cauchy–Schwarz inequality, and the uniform boundedness of P_t, and Q_s, we have

$$\begin{aligned}\sum_{Q \in \Delta_t^V} \|\mathbb{1}_Q A_t \tilde{\mathbb{P}} Q_s v\|_2^2 &= \sum_{Q \in \Delta_t^V} \int_Q \left| \fint_Q \tilde{\mathbb{P}} Q_s v \right|^2 \\ &\leq \sum_{Q \in \Delta_t^V} s^2 |Q| \left| \fint_Q D P_s v \right|^2 \\ &\lesssim \sum_{Q \in \Delta_t^V} s^2 \frac{|Q|}{l(Q)} \left(\fint_Q |D P_s v|^2 \right)^{\frac{1}{2}} \left(\fint_Q |P_s v|^2 \right)^{\frac{1}{2}} \\ &\lesssim \frac{s}{t} \left(\sum_{Q \in \Delta_t^V} \int_Q |Q_s v|^2 \right)^{\frac{1}{2}} \left(\sum_{Q \in \Delta_t^V} \int_Q |P_s v|^2 \right)^{\frac{1}{2}} \\ &\leq \frac{s}{t} \|Q_s u\|_2 \|P_t v\|_2 \\ &\lesssim \frac{s}{t} \|v\|_2^2.\end{aligned}$$

This proves (3.2.2).

We now use a Schur-type estimate to complete the proof. In particular, since A_t is defined component-wise on $L^2(\mathbb{R}^n; \mathbb{C}^{n+2})$, whilst both γ_t and $\tilde{\gamma}_t$ are defined pointwise as multiplication operators on $\mathbb{R}^n$, and $A_t^2 = A_t$ maps locally to constants in $\mathbb{C}^n$ on each dyadic cube in Δ_t, observe that

$$\mathbb{1}_Q\gamma_t(A_t\tilde{\mathbb{P}}v) = \mathbb{1}_Q\gamma_t(\tilde{\mathbb{P}}A_t\tilde{\mathbb{P}}v) = \mathbb{1}_Q\tilde{\gamma}_t(A_t\tilde{\mathbb{P}}v) = \mathbb{1}_Q\tilde{\gamma}_t(\mathbb{1}_Q A_t\tilde{\mathbb{P}}v) = \mathbb{1}_Q\tilde{\gamma}_t(A_t\mathbb{1}_Q A_t\tilde{\mathbb{P}}v)$$

whenever $Q \in \Delta_t$, $t > 0$ and $v \in L^2(\mathbb{R}^n;\mathbb{C}^{n+2})$. We now let $m(s,t) := \min\left\{\frac{s}{t},\frac{t}{s}\right\}^{\frac{1}{2}}$, and combine the above observation with Lemma 3.2.1, the reproducing formula, Minkowski's inequality and Tonnelli's Theorem to obtain

$$\begin{aligned}
\int_0^\infty \sum_{Q\in\Delta_t^V} \|\mathbb{1}_Q\gamma_t A_t\tilde{\mathbb{P}}(P_t-I)u\|_2^2\frac{\mathrm{d}t}{t} &= \int_0^\infty \sum_{Q\in\Delta_t^V} \|\mathbb{1}_Q\tilde{\gamma}_t A_t\mathbb{1}_Q A_t\tilde{\mathbb{P}}(P_t-I)u\|_2^2\frac{\mathrm{d}t}{t} \\
&\lesssim \int_0^\infty \sum_{Q\in\Delta_t^V} \|\mathbb{1}_Q A_t\tilde{\mathbb{P}}(P_t-I)u\|_2^2\frac{\mathrm{d}t}{t} \\
&= \int_0^\infty \|\mathbb{1}_{\Omega_t^V} A_t\tilde{\mathbb{P}}(P_t-I)u\|_2^2\frac{\mathrm{d}t}{t} \\
&\lesssim \int_0^\infty \left\|\mathbb{1}_{\Omega_t^V} A_t\tilde{\mathbb{P}}(P_t-I)\left(\int_0^\infty Q_s^2 u\frac{\mathrm{d}s}{s}\right)\right\|_2^2\frac{\mathrm{d}t}{t} \\
&\lesssim \int_0^\infty \left(\int_0^\infty \|\mathbb{1}_{\Omega_t^V} A_t\tilde{\mathbb{P}}(P_t-I)Q_s(Q_s u)\|_2\frac{\mathrm{d}s}{s}\right)^2\frac{\mathrm{d}t}{t} \\
&\leq \int_0^\infty \left(\int_0^\infty m(s,t)\|Q_s u\|_2\frac{\mathrm{d}s}{s}\right)^2\frac{\mathrm{d}t}{t} \\
&\lesssim \int_0^\infty \sup_{t>0}\left(\int_0^\infty m(s,t)\frac{\mathrm{d}s}{s}\right)\left(\int_0^\infty m(s,t)\|Q_s u\|_2^2\frac{\mathrm{d}s}{s}\right)\frac{\mathrm{d}t}{t} \\
&\lesssim \sup_{s>0}\left(\int_0^\infty m(s,t)\frac{\mathrm{d}t}{t}\right)\int_0^\infty \|Q_s u\|_2^2\frac{\mathrm{d}s}{s} \\
&\lesssim \|u\|_2^2.
\end{aligned}$$

This completes the proof. □

Combining all the previous lemmas we can now prove Proposition 3.2.2.

Proof of Proposition 3.2.2. First note that if $u \in \mathsf{N}(DB)$ then we have that

$$\int_0^\infty \|Q_t^B u\|_2^2\frac{\mathrm{d}t}{t} = \int_0^\infty \|tP_t^B DBu\|_2^2\frac{\mathrm{d}t}{t} = 0.$$

Therefore, assume that $u \in \overline{\mathsf{R}(D)}$, we have

$$\int_0^\infty \|Q_t^B u\|_2^2 \frac{\mathrm{d}t}{t} \lesssim \int_0^\infty \|Q_t^B P_t u\|_2^2 \frac{\mathrm{d}t}{t} + \int_0^\infty \|Q_t^B (I - P_t) u\|_2^2 \frac{\mathrm{d}t}{t}.$$

Then, by Lemma 3.2.3, the second term above is bounded by $\|u\|_2^2$. Now as $I = \tilde{\mathbb{P}} + \mathbb{P}_\mu$

$$\int_0^\infty \|Q_t^B P_t u\|_2^2 \frac{\mathrm{d}t}{t} \lesssim \int_0^\infty \|Q_t^B \tilde{\mathbb{P}} P_t u\|_2^2 \frac{\mathrm{d}t}{t} + \int_0^\infty \|Q_t^B \mathbb{P}_\mu P_t u\|_2^2 \frac{\mathrm{d}t}{t}.$$

Then, by Lemma 3.2.4, the second term above is bounded by $\|u\|_2^2$. Now

$$\int_0^\infty \|Q_t^B \tilde{\mathbb{P}} P_t u\|_2^2 \frac{\mathrm{d}t}{t} \lesssim \int_0^\infty \sum_{Q \in \Delta_t^V} \|\mathbb{1}_Q Q_t^B \tilde{\mathbb{P}} P_t u\|_2^2 \frac{\mathrm{d}t}{t} + \int_0^\infty \sum_{Q \in \Delta_t \setminus \Delta_t^V} \|\mathbb{1}_Q Q_t^B \tilde{\mathbb{P}} P_t u\|_2^2 \frac{\mathrm{d}t}{t}.$$

Then, by Lemma 3.2.5, the second term is bounded by $\|u\|_2^2$. Now

$$\begin{aligned}\int_0^\infty \sum_{Q \in \Delta_t^V} \|\mathbb{1}_Q Q_t^B \tilde{\mathbb{P}} P_t u\|_2^2 \frac{\mathrm{d}t}{t} &\lesssim \int_0^\infty \sum_{Q \in \Delta_t^V} \|\mathbb{1}_Q \tilde{\gamma}_t A_t P_t u\|_2^2 \frac{\mathrm{d}t}{t} \\ &\quad + \int_0^\infty \sum_{Q \in \Delta_t^V} \|\mathbb{1}_Q (Q_t^B \tilde{\mathbb{P}} - \tilde{\gamma}_t A_t) P_t u\|_2^2 \frac{\mathrm{d}t}{t}.\end{aligned}$$

Then, by Lemma 3.2.6 and as $\tilde{\gamma}_t A_t = \gamma_t \tilde{\mathbb{P}} A_t = \gamma_t A_t \tilde{\mathbb{P}}$ the second term above is bounded by $\|u\|_2^2$. Now, again using $\tilde{\gamma}_t A_t = \gamma_t A_t \tilde{\mathbb{P}}$, Lemma 3.2.7, and the hypothesis, we have

$$\begin{aligned}\int_0^\infty \sum_{Q \in \Delta_t^V} \|\mathbb{1}_Q \tilde{\gamma}_t A_t P_t u\|_2^2 \frac{\mathrm{d}t}{t} &\lesssim \int_0^\infty \sum_{Q \in \Delta_t^V} \|\mathbb{1}_Q \tilde{\gamma}_t A_t u\|_2^2 \frac{\mathrm{d}t}{t} \\ &\quad + \int_0^\infty \sum_{Q \in \Delta_t^V} \|\mathbb{1}_Q \gamma_t A_t \tilde{\mathbb{P}} (P_t - I) u\|_2^2 \frac{\mathrm{d}t}{t} \\ &\lesssim \|u\|_2^2.\end{aligned}$$

This completes the proof. $\square$

We note that the only part that depends on V being in the reverse Hölder class is Lemma 3.2.5, but if $V \in L^{\frac{n}{2}}(\mathbb{R}^n)$ then we say all cubes are small. Therefore, the key is to note that the second inequality in Lemma 3.1.5 holds for all cubes in this case. We also use the smallness of the norm to obtain the Riesz Transform estimates in Lemma 3.1.2. Hence, we have the following proposition.

Proposition 3.2.8. Let $n > 4$. Let $V \in L^{\frac{n}{2}}(\mathbb{R}^n)$ with small norm. If

$$\int_0^\infty \|\tilde{\gamma}_t A_t u\|_2^2 \lesssim \|u\|_2^2,$$

for all $u \in \overline{\mathsf{R}(D)}$, then we have

$$\int_0^\infty \|Q_t^B u\|_2^2 \lesssim \|u\|_2^2,$$

for all $u \in L^2(\mathbb{R}^2)$.

3.3 Carleson Measure Estimate

To prove the quadratic estimate we are left to prove the estimate (3.2.1). We will do this by reducing to a Carleson measure type estimate adapted to the potential V. This will be done in a similar manner as in [13, Section 5.3] by constructing tests functions and using a stopping time argument.

Definition 3.3.1. Let μ be a measure on $\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times \mathbb{R}_+$. Then we will say that μ is V-Carleson if

$$\|\mu\|_{\mathcal{C}_V} := \sup_{Q \in \Delta^V} \frac{1}{|Q|} \mu(\mathcal{C}(Q)) < \infty,$$

where $\mathcal{C}(Q) := Q \times (0, l(Q)]$, is the Carleson box of the cube Q.

That is μ is V-Carleson if μ is Carleson when restricted to small cubes. The following proposition is adapted to our case from [50, p. 59], and, states that if $|\tilde{\gamma}_t(x)|^2 \frac{\mathrm{d}t\,\mathrm{d}x}{t}$ is a V-Carleson measure then (3.2.1) is bounded above by $\|u\|_2^2$. We note the following proposition, like the definition of V-Carleson, only considers small cubes.

Proposition 3.3.2. If μ is a V-Carleson measure, then we have

$$\int_0^\infty \sum_{Q\in\Delta_t^V} \int_Q |A_t u(x)|^2 \, \mathrm{d}\mu(x,t) \lesssim \|\mu\|_{\mathcal{C}_V} \|u\|_2^2,$$

for all $u \in L^2(\mathbb{R}^n; \mathbb{C}^{n+2})$.

Proof. First, using the fact that $\Delta_t^V = \Delta_{2^k}^V$ for $k \in \mathbb{Z}$ and $2^{k-1} < t \leq 2^k$ and Tonelli's Theorem, we have

$$\begin{aligned}
\int_0^\infty \sum_{Q\in\Delta_t^V} \int_Q |A_t u(x)|^2 \, \mathrm{d}\mu(x,t) &= \sum_{k=-\infty}^{\infty} \int_{2^{k-1}}^{2^k} \sum_{Q\in\Delta_t^V} \int_Q \left| \fint_Q u(y) \, \mathrm{d}y \right|^2 \mathrm{d}\mu(x,t) \\
&= \sum_{k=-\infty}^{\infty} \int_{2^{k-1}}^{2^k} \sum_{Q\in\Delta_{2^k}^V} \int_Q \left| \fint_Q u(y) \, \mathrm{d}y \right|^2 \mathrm{d}\mu(x,t) \\
&\leq \sum_{k=-\infty}^{\infty} \sum_{Q\in\Delta_{2^k}^V} \left(\fint_Q |u(y)| \, \mathrm{d}y \right)^2 \int_{2^{k-1}}^{2^k} \int_Q \mathrm{d}\mu(x,t).
\end{aligned}$$

Now let $I_k^V \subseteq \mathbb{N}$ be an indexing set such that $\Delta_{2^k}^V = \{Q_\alpha^k : \alpha \in I_k^V\}$. We also introduce the notation

$$u_{\alpha,k} = \fint_{Q_\alpha^k} |u(y)| \, \mathrm{d}y, \quad \text{and} \quad \mu_{\alpha,k} = \mu(Q_\alpha^k \times (2^{k-1}, 2^k]).$$

Therefore, rearranging and using Tonelli's Theorem, we have

$$\begin{aligned}
\int_0^\infty \sum_{Q\in\Delta_{2^k}^V} \int_Q |A_t u(x)|^2 \, \mathrm{d}\mu(x,t) &\leq \sum_{k=-\infty}^{\infty} \sum_{\alpha\in I_k^V} u_{\alpha,k}^2 \mu_{\alpha,k} \\
&= \sum_{k=-\infty}^{\infty} \sum_{\alpha\in I_k^V} \mu_{\alpha,k} \int_0^{u_{\alpha,k}} 2r \, \mathrm{d}r \\
&= \int_0^\infty \sum_{k=-\infty}^{\infty} \sum_{\alpha\in I_k^V} \mu_{\alpha,k} \mathbb{1}_{\{|u_{\alpha,k}|>r\}}(r) 2r \, \mathrm{d}r,
\end{aligned}$$

where dr denotes the Lebesgue measure on $(0,\infty)$. For each $r > 0$ let $\{R_j(r)\}_{j\in\mathbb{N}}$ be an enumeration of the collection of maximal dyadic cubes $Q_\alpha^k \in \Delta^V$ such that $u_{\alpha,k} > r$.

Note as $u \in L^2(\mathbb{R}^n; \mathbb{C}^{n+2})$, then by Jensen's inequality

$$\fint_Q |u(y)|\,\mathrm{d}y \leq \left(\fint_Q |u(y)|^2\,\mathrm{d}y\right)^{\frac{1}{2}} \leq \frac{\|u\|_2}{|Q|^{\frac{1}{2}}} \to 0 \text{ as } l(Q) \to \infty.$$

Therefore, for all $Q \in \Delta^V$ with

$$\fint_Q |u| > r$$

there exists a maximal $Q_\alpha^k \in \Delta^V$ for which $u_{\alpha,k} > r$ and $Q \subseteq Q_\alpha^k$. Define

$$\mathcal{M}_{\Delta^V} u(x) \coloneqq \sup\left\{\fint_Q u : Q \in \Delta^V, \text{ with } x \in Q\right\}.$$

We claim

$$\bigcup_{j=1}^{\infty} R_j(r) = \{x \in \mathbb{R}^n : (\mathcal{M}_{\Delta^V}|u|)(x) > r\}.$$

Let $x \in \bigcup_{j=1}^{\infty} R_j(r)$. Therefore $x \in Q$ such that $Q = R_j(r)$ for some $j \in \mathbb{N}$, then

$$r < \fint_Q |u| \leq (\mathcal{M}_{\Delta^V}|u|)(x).$$

Now if $x \in \mathbb{R}^n$ such that $(\mathcal{M}_{\Delta^V}|u|)(x) > r$, then there exists $Q' \in \Delta^V$ with $x \in Q'$ such that

$$r < \fint_{Q'} |u|.$$

Then either $Q' = R_j(r)$ for some $j \in \mathbb{N}$ or, as the cubes in $\{R_j(r)\}_{j\in\mathbb{N}}$ are maximal, there exists a cube $Q = R_j(r)$ for some $j \in \mathbb{N}$ with $Q' \subseteq Q$. Therefore, $x \in \bigcup_{j=1}^{\infty} R_j(r)$. This proves the claim.

Now suppose $Q_\alpha^k \in \Delta^V$ is such that $u_{\alpha,k} > r$. Then, as the cubes in $\{R_j(r)\}_{j\in\mathbb{N}}$ are maximal, either $Q_\alpha^k = R_j(r)$ or $Q_\alpha^k \subseteq R_j(r)$ for some $j \in \mathbb{N}$. Therefore, using the definition of a V-Carleson measure, the above claim, standard results for maximal functions (see [51, p. 7]), and the fact that $\|\mathcal{M}_{\Delta^V}\|_2^2 \leq \|\mathcal{M}\|_2^2$, where $\mathcal{M}$ is the Hardy–Littlewood maximal function, we have

$$
\begin{aligned}
\int_0^\infty \sum_{k=-\infty}^{\infty} \sum_{\alpha \in I_k^V} \mu_{\alpha,k} \mathbb{1}_{\{|u_{\alpha,k,}|>r\}}(r) 2r \,\mathrm{d}r &\leq \int_0^\infty \sum_{j=1}^{\infty} \sum_{\substack{R \in \Delta^V \\ R \subseteq R_j(r)}} \mu(R \times (\tfrac{l(R)}{2}, l(R)]) 2r \,\mathrm{d}r \\
&\leq \int_0^\infty 2r \sum_{j=1}^{\infty} \mu(\mathcal{C}(R_j(r))) \,\mathrm{d}r \\
&\lesssim \|\mu\|_{\mathcal{C}_V} \int_0^\infty 2r \sum_{j=1}^{\infty} |R_j(r)| \,\mathrm{d}r \\
&= \|\mu\|_{\mathcal{C}_V} \int_0^\infty 2r \left| \bigcup_{j=1}^{\infty} R_j(r) \right| \mathrm{d}r \\
&= \|\mu\|_{\mathcal{C}_V} \int_0^\infty 2r \, |\{x \in \mathbb{R}^n : (\mathcal{M}_{\Delta^V} |u|)(x) > r\}| \,\mathrm{d}r \\
&= \|\mu\|_{\mathcal{C}_V} \|\mathcal{M}_{\Delta^V} |u|\|_2^2 \\
&\lesssim \|\mu\|_{\mathcal{C}_V} \|u\|_2^2,
\end{aligned}
$$

where $\mathcal{C}(R_j(r))$ is the Carleson box of $R_j(r)$. This completes the proof. $\square$

Adapting the work of Bailey in [16, Section 4.1], which in turn is based on [13], we define the space

$$\tilde{\mathcal{L}} := \{\nu \in \mathcal{L}(\mathbb{C}^{n+2}) \setminus \{0\} : \nu\tilde{\mathbb{P}} = \nu\},$$

equipped with the operator norm $|\nu| := |\nu|_{\mathcal{L}(\mathbb{C}^{n+1})}$ for all $\nu \in \tilde{\mathcal{L}}$. We give a small technical lemma about some of the properties of $\tilde{\mathcal{L}}$

Lemma 3.3.3. Let $\nu \in \tilde{\mathcal{L}}$ with $|\nu| = 1$. Then there exists $\xi, \zeta \in \mathbb{C}^{n+2}$ such that $|\xi| = |\zeta| = 1$, $\xi = \nu^*(\zeta)$, and $\tilde{\mathbb{P}}\xi = \xi$.

Proof. As $|\nu| = 1$ there exists $\eta \in \mathbb{C}^{n+2}$ such that $|\eta| = 1$ and $|\nu(\eta)| = 1$. Then define $\xi := \nu^*(\nu(\eta))$ and $\zeta := \nu(\eta)$. Then $\nu^*(\zeta) = \nu^*(\nu(\eta)) = \xi$ by definition. Now $|\zeta| = |\nu(\eta)| = 1$. And $|\xi| \leq |\nu^*||\nu(\eta)| = 1$. Also, $1 = |\nu(\eta)|^2 = \langle \nu(\eta), \nu(\eta) \rangle = \langle \eta, \nu^*(\nu(\eta)) \rangle \leq |\eta||\nu^*(\nu(\eta))| = |\xi|$. Thus $|\xi| = 1$.

Let $z \in \mathbb{C}^{n+2}$. Then, since $\nu \in \tilde{\mathcal{L}}$, we have $\langle \xi, z \rangle = \langle \nu(\eta), \nu(z) \rangle = \langle \nu(\eta), \nu\tilde{\mathbb{P}}(z) \rangle = \langle \xi, \tilde{\mathbb{P}}z \rangle = \langle \tilde{\mathbb{P}}\xi, z \rangle$. As $z \in \mathbb{C}^{n+2}$ was arbitrary we have that $\xi = \tilde{\mathbb{P}}\xi$. As required. $\square$

Let $\sigma > 0$ be a constant to be chosen later. Let $\mathcal{V}_\sigma \subset \tilde{\mathcal{L}}$ be a finite set of matrices $\nu \in \tilde{\mathcal{L}}$ with $|\nu| = 1$, such that $\bigcup_{\nu \in \mathcal{V}_\sigma} K_{\nu,\sigma} = \tilde{\mathcal{L}}$, where

$$K_{\nu,\sigma} := \left\{ \mu \in \tilde{\mathcal{L}} : \left| \frac{\mu}{|\mu|} - \nu \right| \leq \sigma \right\}.$$

It suffices to prove the Carleson measure estimate on each cone $K_{\nu,\sigma}$. That is, we need to prove

$$\iint_{\substack{(x,t)\in \mathcal{C}(Q) \\ \tilde{\gamma}_t(x) \in K_{\nu,\sigma}}} |\tilde{\gamma}_t(x)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t} \lesssim |Q|, \tag{3.3.1}$$

for every $v \in \mathcal{V}_\sigma$. Using Lemma 3.3.3 we choose $\xi, \zeta \in \mathbb{C}^{n+2}$ with $|\xi| = |\zeta| = 1$ such that $\xi = \nu^*(\zeta)$, and $\tilde{\mathbb{P}}\xi = \xi$. Let $\eta_Q : \mathbb{R}^{n+2} \to [0,1]$ be a smooth function equal to 1 on $2Q$, with support in $4Q$, and $\|\nabla \eta_Q\|_\infty \lesssim \frac{1}{l}$, where $l = l(Q)$. Define $\xi_Q = \eta_Q \xi$. We define the test functions, in a similar way to those used in [10, Section 3.6], as follows, for $\varepsilon > 0$, define the test functions as

$$f^\xi_{Q,\varepsilon} := (I + (\varepsilon l)^2 (DB)^2)^{-1}(\xi_Q) = P^B_{\varepsilon l}\xi_Q.$$

We now present some useful properties of the test functions $f^\xi_{Q,\varepsilon}$. The following lemma is adapted to accommodate the potential V from [14, Lemma 5.3] and [10, Lemma 3.16].

Lemma 3.3.4. We have the following estimates:

1. $\|f^\xi_{Q,\varepsilon}\|_2 \lesssim |Q|^{\frac{1}{2}}$,

2. $\|\varepsilon l DB f^\xi_{Q,\varepsilon}\|_2 \lesssim |Q|^{\frac{1}{2}}$,

for all $Q \in \Delta$. Also

3. $\left| \fint_Q f^\xi_{Q,\varepsilon} - \xi \right| \lesssim \varepsilon^{\frac{1}{2}}$ for all $Q \in \Delta^V$ if $V \in B^{\frac{n}{2}}(\mathbb{R}^n)$.

Proof. By definition, and uniform boundedness of P^B_t, we have

$$\|f^\xi_{Q,\varepsilon}\|_2^2 = \|P^B_{\varepsilon l}\xi_Q\|_2^2 \lesssim \|\xi_Q\|_2^2 \lesssim |Q|.$$

Similarly we have

$$\|\varepsilon l D B f^{\xi}_{Q,\varepsilon}\|_2^2 = \|Q^B_{\varepsilon l}\xi_Q\|_2^2 \lesssim \|\xi_Q\|_2^2 \lesssim |Q|.$$

For (3), we use the definition of $f^{\xi}_{Q,\varepsilon}$, Lemma 3.1.5, and the uniform boundedness of $P^B_t - I$ and Q^B_t to obtain

$$\begin{aligned}
\left|\fint_Q f^{\xi}_{Q,\varepsilon} - \xi\right|^2 &= \left|\fint_Q (f^{\xi}_{Q,\varepsilon} - \eta_Q \xi)\right|^2 \\
&= \left|\fint_Q (\varepsilon l)^2 (DB)^2 P^B_{\varepsilon l}\xi_Q\right|^2 \\
&\lesssim \frac{(\varepsilon l)^4}{l}\left(\fint_Q |(DB)^2 P^B_{\varepsilon l}\xi_Q|^2\right)^{\frac{1}{2}}\left(\fint_Q |BDBP^B_{\varepsilon l}\xi_Q|^2\right)^{\frac{1}{2}} \\
&\leq \frac{\varepsilon\|B\|_\infty}{|Q|}\left(\int_Q |(P^B_{\varepsilon l} - I)\xi_Q|^2\right)^{\frac{1}{2}}\left(\int_Q |Q^B_{\varepsilon l}\xi_Q|^2\right)^{\frac{1}{2}} \\
&\lesssim \frac{\varepsilon}{|Q|}\|\xi_Q\|_2^2 \\
&\lesssim \varepsilon.
\end{aligned}$$

This completes the proof. $\square$

For each $Q \in \Delta^V$ we consider a sub-collection of disjoint subcubes which give us the following reduction of (3.3.1).

Proposition 3.3.5. There exists $\tau \in (0,1)$ such that for all cubes $Q \in \Delta^V$ and for all $\nu \in \tilde{\mathcal{L}}$ with $|\nu| = 1$, there is a collection $\{Q_k\}_{k\in I_Q} \subset \Delta^V$ of disjoint subcubes of Q, where I_Q is the indexing set of the collection, such that $|E_{Q,\nu}| > \tau|Q|$ where $E_{Q,\nu} = Q\setminus\bigcup_{k\in I_Q} Q_k$ and with

$$\iint_{\substack{(x,t)\in E^*_{Q,\nu}\\ \tilde{\gamma}_t(x)\in K_{\nu,\sigma}}} |\tilde{\gamma}_t(x)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t} \lesssim |Q|,$$

where $E^*_{Q,\nu} \coloneqq \mathcal{C}(Q) \setminus \bigcup_{k\in I_Q} \mathcal{C}(Q_k)$.

We will use a stopping-time argument to give a suitable collection of dyadic subcubes for Proposition 3.3.5. We note that unlike in [13] we need all the bad cubes in the stopping time argument to be small, therefore, we need to use Lemma 2.2.3 which gives a uniform bound on the number of times we need to subdivide a small dyadic cube until we can guarantee that the subcubes at that scale are also small.

Define $f_Q^\xi := f_{Q,\varepsilon_0}^\xi$ where $\varepsilon_0 > 0$ is such that when we apply Lemma 3.3.4 part (3) we obtain

$$\left| \fint_Q f_Q^\xi - \xi \right| \leq \frac{1}{2},$$

for all $Q \in \Delta^V$. Then the polarisation identity gives

$$\begin{aligned} \operatorname{Re}\left\langle \xi, \fint_Q f_Q^\xi \right\rangle &= \frac{1}{4}\left(\left| \xi + \fint_Q f_Q^\xi \right|^2 - \left| \fint_Q f_Q^\xi - \xi \right|^2 \right) \\ &\geq \frac{1}{4}\left(2\operatorname{Re}\left\langle \xi, \fint_Q f_Q^\xi \right\rangle + \left| \fint_Q f_Q^\xi \right|^2 + \frac{3}{4} \right). \end{aligned}$$

Therefore, using Lemma 3.3.4, we have that

$$\operatorname{Re}\left\langle \xi, \fint_Q f_Q^\xi \right\rangle \geq \frac{1}{2}\left(\left| \fint_Q f_Q^\xi \right|^2 + \frac{3}{4} \right) \geq \frac{1}{2}\left(\frac{1}{4} + \frac{3}{4} \right) = \frac{1}{2}. \tag{3.3.2}$$

We now describe the bad cubes which we will use in Proposition 3.3.5, using the above lemma so that we can make sure there are no big bad cubes.

Lemma 3.3.6. Let $Q \in \Delta^V$ if $V \in B^{\frac{n}{2}}(\mathbb{R}^n)$. Then, there are constants $c_1, c_2 > 0$, $\tau \in (0,1)$ and a disjoint collection $\{Q_k\}_{k \in I_Q} \subset \Delta^V$ such that

- $Q_k \subseteq Q$
- $|E_{Q,\nu}| > \tau |Q|$
- $l(Q_k) < [\![V]\!]_q^{\frac{-1}{2-\frac{n}{q}}} l(Q)$, where $q > \frac{n}{2}$ is as in Remark 2.2.4,

satisfying

$$\fint_S |f_Q^\xi| \leq c_1, \quad \text{and} \quad \operatorname{Re}\left\langle \xi, \fint_S f_Q^\xi \right\rangle \geq c_2, \tag{3.3.3}$$

for all dyadic subcubes S of Q with $l(S) < [\![V]\!]_q^{\frac{-1}{2-\frac{n}{q}}} l(Q)$ for which $\mathcal{C}(S) \cap \widetilde{E}^*_{Q,\nu} \neq \emptyset$, where $\widetilde{E}^*_{Q,\nu} := (Q \times [0, [\![V]\!]_q^{\frac{-1}{2-\frac{n}{q}}} l(Q))) \setminus \bigcup_k \mathcal{C}(Q_k)$.

Proof. Let $\alpha \in (0,1)$. Let $\mathcal{B}_1$ be the collection of maximal dyadic subcubes of Q, with

$l(Q_k) < [\![V]\!]_q^{\frac{-1}{2-\frac{n}{q}}} l(Q)$, for which

$$⨍_{Q_k} |f_Q^\xi| > \frac{1}{\alpha}. \tag{3.3.4}$$

Since $Q \in \Delta^V$ then by Lemma 2.2.3 with $C = c = 1$ we have $Q_k \in \Delta^V$ for all $k \in I_Q$. Then, using the Cauchy–Schwarz inequality and (1) from Lemma 3.3.4, we have

$$\left|\bigcup \mathcal{B}_1\right| = \sum_{Q_k \in \mathcal{B}_1} |Q_k| \leq \alpha \sum_{Q_k \in \mathcal{B}_1} \int_{Q_k} |f_Q^\xi| \leq \alpha \int_Q |f_Q^\xi| \leq \alpha |Q|^{\frac{1}{2}} \left(\int_Q |f_Q^\xi|^2\right)^{\frac{1}{2}} \leq C\alpha|Q|,$$

where $C > 0$ is the implicit constant in (1) from Lemma 3.3.4. Now let $\mathcal{B}_2$ be the collection of maximal dyadic subcubes of Q, with $l(Q_k) < [\![V]\!]_q^{\frac{-1}{2-\frac{n}{q}}} l(Q)$, such that

$$\operatorname{Re}\left\langle \xi, ⨍_{Q_k} f_Q^\xi \right\rangle < \alpha. \tag{3.3.5}$$

Then, using (3.3.2), the properties of $\mathcal{B}_2$, the Cauchy–Schwarz inequality, and Lemma 3.3.4 part (1), we have that

$$\begin{aligned}
\frac{1}{2} &\leq \operatorname{Re}\left\langle \xi, ⨍_Q f_Q^\xi \right\rangle \\
&= \sum_{Q_k \in \mathcal{B}_2} \frac{|Q_k|}{|Q|} \operatorname{Re}\left\langle \xi, ⨍_{Q_k} f_Q^\xi \right\rangle + \operatorname{Re}\left\langle \xi, \frac{1}{|Q|} \int_{Q \setminus \bigcup \mathcal{B}_2} f_Q^\xi \right\rangle \\
&\leq \alpha + \frac{1}{|Q|} \int_{Q \setminus \bigcup \mathcal{B}_2} |f_Q^\xi| \\
&\leq \alpha + C\left(\frac{|Q \setminus \bigcup \mathcal{B}_2|}{|Q|}\right)^{\frac{1}{2}}.
\end{aligned}$$

Making the restriction $\alpha \in (0, \frac{1}{2})$ we have

$$\left(\frac{\frac{1}{2} - \alpha}{C}\right)^2 |Q| \leq |Q \setminus \bigcup \mathcal{B}_2|.$$

Therefore, letting $\{Q_k\}_{k \in I_Q} := \mathcal{B}_1 \cup \mathcal{B}_2$ where I_Q is an enumeration of the cubes in $\mathcal{B}_1 \cup \mathcal{B}_2$

$$|E_{Q,\nu}| \geq |Q \setminus \bigcup \mathcal{B}_2| - |\bigcup \mathcal{B}_1| \geq \left[\left(\frac{\frac{1}{2} - \alpha}{C}\right)^2 - C\alpha\right] |Q|.$$

Now, choosing $\alpha \in (0, \frac{1}{2})$ sufficiently small, gives $\tau := \left[\left(\frac{\frac{1}{2}-\alpha}{C}\right)^2 - C\alpha\right] > 0$. Now, let R be a dyadic subcube of Q with $l(R) < [\![V]\!]_q^{\frac{-1}{2-\frac{n}{q}}} l(Q)$ and $\mathcal{C}(R) \cap \tilde{E}^*_{Q,\nu} \neq \emptyset$. Then, by definition $R \cap (Q \times [0, [\![V]\!]_q^{\frac{-1}{2-\frac{n}{q}}} l(Q))) \setminus \bigcup_k \mathcal{C}(Q_k) \neq \emptyset$ and so we have $R \not\subseteq Q_k$ for all $Q_k \in \mathcal{B}_1 \cup \mathcal{B}_2$. Since, the cubes in $\mathcal{B}_1$ and $\mathcal{B}_2$ are maximal for conditions (3.3.4) and (3.3.5) respectively and $R \not\subseteq Q_k$ for all $Q_k \in \mathcal{B}_1 \cup \mathcal{B}_2$, then R cannot satisfy either (3.3.4) or (3.3.5). Thus, R satisfies (3.3.3). This completes the proof. $\square$

Now we choose $\sigma = \frac{c_2}{2c_1}$. The following lemma will allow us to introduce the test functions into our argument.

Lemma 3.3.7. Let $Q \in \Delta^V$ if $V \in B^{\frac{n}{2}}(\mathbb{R}^n)$. If $(x,t) \in \widetilde{E}^*_{Q,\nu}$ and $\gamma_t(x) \in K_{v,\sigma}$, then

$$|\tilde{\gamma}_t(x)(A_t f^{\xi}_Q(x))| \geq \frac{1}{2} c_2 |\tilde{\gamma}_t(x)|.$$

Proof. As $(x,t) \in \widetilde{E}^*_{Q,\nu}$ there exists $S \in \Delta^V_t$ such that $S \subseteq Q$, $x \in S$, $l(S) \leq \tilde{c}l(Q)$ and $\mathcal{C}(S) \cap \widetilde{E}^*_{Q,\nu} \neq \emptyset$. Then by Lemmas 3.3.6, and the definitions of ξ and ζ, we have

$$|\nu(A_t f^{\xi}_Q(x))| \geq \operatorname{Re}\left\langle \zeta, \nu(A_t f^{\xi}_Q(x))\right\rangle = \operatorname{Re}\left\langle \xi, \fint_S f^{\xi}_Q \right\rangle \geq c_2.$$

Then, by above and Lemma 3.3.6, we have

$$\left|\frac{\tilde{\gamma}_t(x)}{|\tilde{\gamma}_t(x)|}(A_t f^{\xi}_Q(x))\right| \geq |\nu(A_t f^{\xi}_Q(x))| - \left|\left(\frac{\tilde{\gamma}_t(x)}{|\tilde{\gamma}_t(x)|} - \nu\right)(A_t f^{\xi}_Q(x))\right| \geq c_2 - \sigma c_1 \geq \frac{1}{2} c_2.$$

As required. $\square$

We are now ready to give the proof of Proposition 3.3.5 This is adapted from [13, Proof of Proposition 5.9] to the set $\widetilde{E}^*_{Q,v}$ and to the presence of the potential V.

Proof of Proposition 3.3.5. First we break up the integral as follows

$$\iint_{\substack{(x,t)\in E^*_{Q,\nu} \\ \tilde{\gamma}_t(x)\in K_{\nu,\sigma}}} |\tilde{\gamma}_t(x)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t} = \iint_{\substack{(x,t)\in \widetilde{E}^*_{Q,\nu} \\ \tilde{\gamma}_t(x)\in K_{\nu,\sigma}}} |\tilde{\gamma}_t(x)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t} + \iint_{\substack{(x,t)\in Q\times[\tilde{c}l(Q),l(Q)] \\ \tilde{\gamma}_t(x)\in K_{\nu,\sigma}}} |\tilde{\gamma}_t(x)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t}.$$

Then, using the lower bound $t > \tilde{c}l(Q)$ and the local boundedness of $\tilde{\gamma}$ as in (1) in Lemma 3.2.1, we have

$$
\begin{aligned}
\iint_{\substack{(x,t)\in Q\times[\tilde{c}l(Q),l(Q)] \\ \tilde{\gamma}_t(x)\in K_{\nu,\sigma}}} |\tilde{\gamma}_t(x)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t} &\leq \int_{\tilde{c}l(Q)}^{l(Q)} \int_Q |\tilde{\gamma}_t(x)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t} \\
&\leq \frac{1}{\tilde{c}l(Q)} \int_{\tilde{c}l(Q)}^{l(Q)} \left(\int_Q |\tilde{\gamma}_t(x)|^2 \,\mathrm{d}x \right) \mathrm{d}t \\
&= \frac{1}{\tilde{c}l(Q)} \int_{\tilde{c}l(Q)}^{l(Q)} \left(\sum_{\substack{R\in\Delta_t \\ R\subseteq Q}} \int_R |\tilde{\gamma}_t(x)|^2 \,\mathrm{d}x \right) \mathrm{d}t \\
&\lesssim \frac{1}{\tilde{c}l(Q)} \int_{\tilde{c}l(Q)}^{l(Q)} \left(\sum_{\substack{R\in\Delta_t \\ R\subseteq Q}} |R| \right) \mathrm{d}t \\
&\lesssim \frac{|Q|}{\tilde{c}l(Q)} \int_{\tilde{c}l(Q)}^{l(Q)} \mathrm{d}t \\
&\eqsim |Q|.
\end{aligned}
$$

Therefore, by Lemma 3.3.7 we introduce the test functions to obtain

$$
\begin{aligned}
\iint_{\substack{(x,t)\in \widetilde{E}^*_{Q,\nu} \\ \tilde{\gamma}_t(x)\in K_{\nu,\sigma}}} |\tilde{\gamma}_t(x)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t} &\lesssim \iint_{\substack{(x,t)\in \widetilde{E}^*_{Q,\nu} \\ \tilde{\gamma}_t(x)\in K_{\nu,\sigma}}} |\tilde{\gamma}_t(x)(A_t f_Q^\xi(x))|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t} \\
&\lesssim \iint_{\substack{(x,t)\in \widetilde{E}^*_{Q,\nu} \\ \tilde{\gamma}_t(x)\in K_{\nu,\sigma}}} |(Q_t^B - \tilde{\gamma}_t(x)A_t) f_Q^\xi|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t} + \iint_{\mathcal{C}(Q)} |Q_t^B f_Q^\xi|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t}.
\end{aligned}
$$

Now, by the uniform boundedness of P_t^B and Lemma 3.3.4, we have

$$
\iint_{\mathcal{C}(Q)} |Q_t^B f_Q^\xi|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t} \leq \int_0^l \left(\frac{t}{\varepsilon_0 l}\right)^2 \|P_t^B \varepsilon_0 l DB f^\xi_{Q,\varepsilon_0}\|_2^2 \frac{\mathrm{d}t}{t} \lesssim \int_0^l \frac{1}{\varepsilon_0^2 l} \|\varepsilon_0 l DB f^\xi_{Q,\varepsilon_0}\|_2^2 \,\mathrm{d}t \lesssim \frac{|Q|}{\varepsilon_0^2}.
$$

Also,

$$
\begin{aligned}
\iint_{\substack{(x,t)\in \widetilde{E}^*_{Q,\nu} \\ \tilde{\gamma}_t(x)\in K_{\nu,\sigma}}} |(Q_t^B - \tilde{\gamma}_t(x)A_t) f_Q^\xi|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t} &\lesssim \iint_{\substack{(x,t)\in \widetilde{E}^*_{Q,\nu} \\ \tilde{\gamma}_t(x)\in K_{\nu,\sigma}}} |(Q_t^B - \tilde{\gamma}_t(x)A_t)(f_Q^\xi - \xi_Q)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t} \\
&\quad + \iint_{\mathcal{C}(Q)} |(Q_t^B - \tilde{\gamma}_t(x)A_t)\xi_Q|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t}
\end{aligned} \tag{3.3.6}
$$

Now, for the first term in (3.3.6) we have

$$\begin{aligned}
&\iint_{\substack{(x,t)\in \widetilde{E}^*_{Q,\nu}\\ \tilde{\gamma}_t(x)\in K_{\nu,\sigma}}} |(Q_t^B - \tilde{\gamma}_t(x)A_t)(f_Q^\xi - \xi_Q)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t}\\
&\qquad\lesssim \iint_{\substack{(x,t)\in \widetilde{E}^*_{Q,\nu}\\ \tilde{\gamma}_t(x)\in K_{\nu,\sigma}}} |(Q_t^B P_t - \tilde{\gamma}_t(x)A_t)(f_Q^\xi - \xi_Q)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t}\\
&\qquad\quad + \iint_{\mathcal{C}(Q)} |Q_t^B(I-P_t)(f_Q^\xi - \xi_Q)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t}.
\end{aligned}\tag{3.3.7}$$

As $f_Q^\xi - \xi_Q = (P^B_{\varepsilon_0 l} - I)\xi_Q = -(\varepsilon_0 lDB)^2 P^B_{\varepsilon_0 l}\xi_Q \in \mathsf{R}(D)$, by Lemma 3.2.3 we have the second term in (3.3.7) is bounded by $\|f_Q^\xi - \xi_Q\|_2^2$. Then, by definitions of $\tilde{\mathbb{P}}$ and $\mathbb{P}_\mu$, we obtain

$$\begin{aligned}
&\iint_{\substack{(x,t)\in \widetilde{E}^*_{Q,\nu}\\ \tilde{\gamma}_t(x)\in K_{\nu,\sigma}}} |(Q_t^B P_t - \tilde{\gamma}_t(x)A_t)(f_Q^\xi - \xi_Q)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t}\\
&\qquad\lesssim \iint_{\substack{(x,t)\in \widetilde{E}^*_{Q,\nu}\\ \tilde{\gamma}_t(x)\in K_{\nu,\sigma}}} |(Q_t^B \tilde{\mathbb{P}} P_t - \tilde{\gamma}_t(x)A_t)(f_Q^\xi - \xi_Q)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t}\\
&\qquad\quad + \iint_{\mathcal{C}(Q)} |Q_t^B \mathbb{P}_\mu P_t(f_Q^\xi - \xi_Q)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t}.
\end{aligned}\tag{3.3.8}$$

Then by Lemma 3.2.4 we have that the second term in (3.3.8) is bounded by $\|f_Q^\xi - \xi_Q\|_2^2$. Now using Lemmas 3.2.6 and 3.2.7 we obtain

$$\begin{aligned}
&\iint_{\substack{(x,t)\in \widetilde{E}^*_{Q,\nu}\\ \tilde{\gamma}_t(x)\in K_{\nu,\sigma}}} |(Q_t^B \tilde{\mathbb{P}} P_t - \tilde{\gamma}_t(x)A_t)(f_Q^\xi - \xi_Q)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t}\\
&\qquad\lesssim \iint_{\substack{(x,t)\in \widetilde{E}^*_{Q,\nu}\\ \gamma_t(x)\in K_{\nu,\sigma}}} |(Q_t^B \tilde{\mathbb{P}} - \gamma_t(x)\tilde{\mathbb{P}}A_t)P_t(f_Q^\xi - \xi_Q)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t}\\
&\qquad\quad + \iint_{\substack{(x,t)\in \widetilde{E}^*_{Q,\nu}\\ \tilde{\gamma}_t(x)\in K_{\nu,\sigma}}} |\gamma_t(x)\tilde{\mathbb{P}}A_t(I-P_t)(f_Q^\xi - \xi_Q)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t}\\
&\qquad\lesssim \int_0^\infty \sum_{R\in\Delta_t^V} \|\mathbb{1}_R(Q_t^B - \gamma_t(x)A_t)\tilde{\mathbb{P}}P_t(f_Q^\xi - \xi_Q)\|_2^2 \frac{\mathrm{d}t}{t}\\
&\qquad\quad + \int_0^\infty \sum_{R\in\Delta_t^V} \|\mathbb{1}_R\gamma_t(x)A_t\tilde{\mathbb{P}}(I-P_t)(f_Q^\xi - \xi_Q)\|_2^2 \frac{\mathrm{d}t}{t}\\
&\qquad\lesssim \|f_Q^\xi - \xi_Q\|_2^2.
\end{aligned}$$

Therefore, by the uniform boundedness of $P_t^B - I$ we have

$$\iint_{\substack{(x,t)\in \widetilde{E}^*_{Q,\nu} \\ \tilde{\gamma}_t(x)\in K_{\nu,\sigma}}} |(Q_t^B - \tilde{\gamma}_t(x)A_t)(f_Q^\xi - \xi_Q)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t} \lesssim \|f_Q^\xi - \xi_Q\|_2^2 = \|(P^B_{\varepsilon_0 l} - I)\xi_Q\|_2^2 \lesssim |Q|$$

We now start to bound the second term in (3.3.6). Now, using that $\eta_Q = 1$ on $2Q$ and so $A_t(\eta_Q\xi) = \xi$ on $2Q$, and $\xi = \tilde{\mathbb{P}}\xi$, we have

$$(Q_t^B - \tilde{\gamma}_t A_t)\xi_Q = Q_t^B(\eta_Q\xi - \gamma_t\tilde{\mathbb{P}}A_t(\eta_Q\xi)) = Q_t^B(\eta_Q\xi - \tilde{\mathbb{P}}\xi) = Q_t^B((\eta_Q - I)\xi),$$

for all $(x,t) \in \mathcal{C}(Q)$, and so in particular $x \in Q \subset 2Q$. Therefore, since $\operatorname{supp}((\eta_Q - 1)\xi) \cap 2Q = \emptyset$, using the off-diagonal estimates in Proposition 3.1.3 with $M > n$, we have

$$\begin{aligned}
\|\mathbb{1}_Q Q_t^B((\eta_Q - 1)\xi)\|_2^2 &\leq \left(\sum_{j=1}^{\infty} \|\mathbb{1}_Q Q_t^B \mathbb{1}_{C_j(Q)}((\eta_Q - 1)\xi)\|_2\right)^2 \\
&\lesssim \left(\frac{t}{l(Q)}\right)^M \sum_{j=1}^{\infty} 2^{-jM} \|\mathbb{1}_{C_j(Q)}((\eta_Q - 1)\xi)\|_2^2 \\
&\lesssim \left(\frac{t}{l(Q)}\right)^M \sum_{j=1}^{\infty} 2^{-j(M-n)} |Q| \\
&\lesssim \left(\frac{t}{l(Q)}\right)^M |Q|.
\end{aligned}$$

Thus, integrating in t gives

$$\iint_{\mathcal{C}(Q)} |(Q_t^B f_Q^\xi - \tilde{\gamma}_t(x)A_t)\xi_Q|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t} \lesssim \int_0^l \frac{t^{M-1}|Q|}{l^M}\,\mathrm{d}t \lesssim |Q|$$

Therefore,

$$\iint_{\substack{(x,t)\in E^*_{Q,\nu} \\ \tilde{\gamma}_t(x)\in K_{\nu,\sigma}}} |\tilde{\gamma}_t(x)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t} \lesssim |Q|.$$

This completes the proof. □

Now we are finally ready to prove Theorem 3.0.1.

Proof of Theorem 3.0.1. We start by showing that Proposition 3.3.5 proves (3.3.1). Consider an arbitrary $Q \in \Delta^V$ and fix $\nu \in \mathcal{V}_\sigma$. Then, for all $\delta \in (0,1)$, we have

$$\begin{aligned}\iint_{\mathcal{C}(Q)} \mathbb{1}_{\tilde{\gamma}_t(x)\in K_{\nu,\sigma}}(x,t)\mathbb{1}_{(\delta,\delta^{-1})}(t)|\tilde{\gamma}_t(x)|^2\frac{\mathrm{d}x\,\mathrm{d}t}{t} &\leq \frac{1}{\delta}\int_0^{l(Q)} \mathbb{1}_{(\delta,\delta^{-1})}(t)\left(\int_Q |\tilde{\gamma}_t(x)|^2\,\mathrm{d}x\right)\mathrm{d}t \\ &\leq \frac{1}{\delta}\int_0^{l(Q)} \mathbb{1}_{(\delta,\delta^{-1})}(t)\left(\sum_{\substack{R\in\Delta_t\\ R\subseteq Q}}\int_R |\tilde{\gamma}_t(x)|^2\,\mathrm{d}x\right)\mathrm{d}t \\ &\lesssim \frac{1}{\delta}\int_0^{l(Q)} \mathbb{1}_{(\delta,\delta^{-1})}(t)\left(\sum_{\substack{R\in\Delta_t\\ R\subseteq Q}}|R|\right)\mathrm{d}t \\ &\lesssim \frac{|Q|}{\delta^2},\end{aligned}$$

where the penultimate inequality comes from Lemma 3.2.1 part (1). Therefore, we have that the measure $\mu_{\delta,\nu} := \mathbb{1}_{\tilde{\gamma}_t(x)\in K_{\nu,\sigma}}(x,t)\mathbb{1}_{(\delta,\delta^{-1})}(t)|\tilde{\gamma}_t(x)|^2\frac{\mathrm{d}x\,\mathrm{d}t}{t}$ is V-Carleson. We now show that $\|\mu_{\delta,\nu}\|_{\mathcal{C}}$ does not depend on $\delta \in (0,1)$. Then, as each $Q_k \in \Delta^V$, $\mu_{\delta,\nu}$ being V-Carleson and by Proposition 3.3.5, we have

$$\begin{aligned}\iint_{\mathcal{C}(Q)} \mathbb{1}_{\tilde{\gamma}_t(x)\in K_{\nu,\sigma}}(x,t)\mathbb{1}_{(\delta,\delta^{-1})}(t)|\tilde{\gamma}_t(x)|^2\frac{\mathrm{d}x\,\mathrm{d}t}{t} &= \iint_{\substack{(x,t)\in E^*_{Q,\nu}\\ \tilde{\gamma}_t(x)\in K_{\nu,\sigma}}} |\tilde{\gamma}_t(x)|^2\frac{\mathrm{d}x\,\mathrm{d}t}{t} + \sum_{k\in I_Q}\mu_{\delta,\nu}(\mathcal{C}(Q_k)) \\ &\leq\ C_0|Q| + \|\mu_{\delta,\nu}\|_{\mathcal{C}}\sum_{k\in I_Q}|Q_k| \\ &\leq C_0|Q| + \|\mu_{\delta,\nu}\|_{\mathcal{C}}|Q\setminus E_{Q,\nu}|,\end{aligned}$$

Then using the fact that $|E_{Q,\nu}| > \tau|Q|$, dividing by $|Q|$ and taking the supremum over all cubes $Q \in \Delta^V$, we have that

$$\|\mu_{\delta,\nu}\|_{\mathcal{C}_V} = \sup_{Q\in\Delta^V}\frac{1}{|Q|}\iint_{\mathcal{C}(Q)} \mathbb{1}_{\tilde{\gamma}_t(x)\in K_{\nu,\sigma}}(x,t)\mathbb{1}_{(\delta,\delta^{-1})}(t)|\tilde{\gamma}_t(x)|^2\frac{\mathrm{d}x\,\mathrm{d}t}{t} < C + \|\mu_{\delta,\nu}\|_{\mathcal{C}_V}(1-\tau).$$

Rearranging then gives us

$$\|\mu_{\delta,\nu}\|_{\mathcal{C}_V} \lesssim \frac{1}{\tau}.$$

That is $\mu_{\delta,\nu}$ is a Carleson measure with Carleson norm independent of δ. Now, note that $\mathbb{1}_{(\delta,\delta^{-1})}(t)$ is a pointwise increasing function. Thus, by the Monotone Convergence

Theorem

$$\iint_{\substack{(x,t)\in\mathcal{C}(Q)\\ \tilde{\gamma}_t(x)\in K_{\nu,\sigma}}} |\tilde{\gamma}_t(x)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t} = \iint_{\substack{(x,t)\in\mathcal{C}(Q)\\ \tilde{\gamma}_t(x)\in K_{\nu,\sigma}}} \lim_{\delta\to 0} \mathbb{1}_{(\delta,\delta^{-1})}(t)|\tilde{\gamma}_t(x)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t}$$
$$= \lim_{\delta\to 0} \iint_{\substack{(x,t)\in\mathcal{C}(Q)\\ \tilde{\gamma}_t(x)\in K_{\nu,\sigma}}} \mathbb{1}_{(\delta,\delta^{-1})}(t)|\tilde{\gamma}_t(x)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t}$$
$$\lesssim |Q|.$$

Thus, we have proved (3.3.1). Now, since $\mathcal{V}_\sigma$ is a finite set and the size of $\mathcal{V}_\sigma$ is independent of Q, we have

$$\iint_{(x,t)\in\mathcal{C}(Q)} |\tilde{\gamma}_t(x)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t} \leq \sum_{\nu\in\mathcal{V}_\sigma} \iint_{\substack{(x,t)\in\mathcal{C}(Q)\\ \tilde{\gamma}_t(x)\in K_{\nu,\sigma}}} |\tilde{\gamma}_t(x)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t} \lesssim |Q|.$$

Thus, $|\tilde{\gamma}_t(x)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t}$ is a V-Carleson measure. Therefore, by Proposition 3.3.2 we have proven (3.2.1); finally, applying Proposition 3.2.2 completes the proof. □

For $V \in L^{\frac{n}{2}}(\mathbb{R}^n)$ with sufficiently small norm we note that as all dyadic cubes are small then the bounds on the length of the cubes in Lemma 3.3.6 are not needed to remove all the big bad cubes. Therefore, the proof is similar but easier.

Proposition 3.3.8. Let $V \in L^{\frac{n}{2}}(\mathbb{R}^n)$ with sufficiently small norm. Then we have the square function estimate

$$\int_0^\infty \|tDB(I+t^2DBDB)^{-1}u\|_2^2 \frac{\mathrm{d}t}{t} \lesssim \|u\|_2^2,$$

for all $u \in L^2(\mathbb{R}^n;\mathbb{C}^{n+2})$.

CHAPTER 4

QUADRATIC ESTIMATES FOR THE PURELY MAGNETIC SCHRÖDINGER OPERATOR

Like Chapter 3 the focus of this chapter is to prove quadratic estimates for a first-order systems of the DB-type, which we now adapt to incorporate a first-order term as a potential. We again follow the methods in [14] and so this chapter follows in a similar structure to Chapter 3. Again, the goal would be to prove well-posedness results for the magnetic Schrödinger equation on the upper half-space; however, this is outside the scope of this thesis. We use the framework from Section 2.4 and define the operators

$$D = \begin{bmatrix} 0 & -(\nabla + ib)^* \\ -(\nabla + ib) & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} B_{\perp\perp} & B_{\perp\|} \\ B_{\|\perp} & B_{\|\|} \end{bmatrix}$$

as defined in Section 2.4. Then, the aim of this chapter is to prove the following theorem. We consider the purely magnetic Schrödinger oprator

$$H_{b,A}u = (\nabla + ib)^* A(\nabla + ib)u, \tag{4.0.1}$$

for $n > 2$, where $b \in L^2(\mathbb{R}^n; \mathbb{R}^n)$ is the magnetic potential, $A \in L^\infty(\mathbb{R}^{n+1}; \mathcal{L}(\mathbb{C}^{n+1}))$ is complex and elliptic operator. Recall the definition of the magnetic field generated by b as

$$\mathbf{B} := \operatorname{curl}(b),$$

We will use the following notation from now on

$$Lu := (\nabla + ib)u.$$

Throughout this chapter we will assume the magnetic field satisfies conditions (2.4.3), that is:

$$\begin{cases} |\mathbf{B}| \in B^{\frac{n}{2}}(\mathbb{R}^n) \\ |\nabla \mathbf{B}(x)| \leq cm(x, |\mathbf{B}|)^3, \end{cases}$$

for some $c > 0$, where $m(\cdot, |\mathbf{B}|)$ is the Shen maximal function in (2.2.2).

Remark 4.0.1. Originally in [46] the additional condition $|\mathbf{B}(x)| \lesssim m(x, |\mathbf{B}|)^2$ was included but in [47, Remark 1.8] it was observed that this is a consequence of the condition $|\nabla \mathbf{B}(x)| \leq cm(x, |\mathbf{B}|)^3$.

The main theorem of the chapter is the following.

Theorem 4.0.2. If $\mathbf{B}$ satisfies (2.4.3), then we have the following quadratic estimate

$$\int_0^\infty \|Q_t^B u\|_2^2 \frac{\mathrm{d}t}{t} \approx \|u\|_2^2, \quad \forall u \in \overline{\mathsf{R}(D)},$$

where the implicit constants depend on $\mathbf{B}$, n, κ, and $\|B\|_\infty$.

To do this we will prove a localised quadratic estimates by reducing to a localised Carleson measure estimate and then use a stopping time argument to prove the Carleson measure estimate.

4.1 Initial Estimates

We start by giving some estimates which are key for proving the quadratic estimate. The key part being the definition of the maximal dyadic mesh we will be using and the finite overlap property. This will be the main difference between this chapter and Chapter 3 as we use the fact that we impose the reverse Hölder condition on the magnetic field $\mathbf{B}$

instead of the magnetic potential b. And because our method will involve localising the quadratic estimate we discuss the estimates that are globally on $\mathbb{R}^n$ and those which are on the maximal dyadic mesh in separately.

4.1.1 Global Estimates

We begin with the global estimates. We have the following proposition (see [39, Theorem 7.21] for details) states that the absolute value of a function with finite magnetic gradient is a $W^{1,2}(\mathbb{R}^n)$. We give a pointwise bound below.

Proposition 4.1.1 (Diamagnetic Inequality)**.** For all $u \in L^2(\mathbb{R}^n)$ with $Lu \in L^2(\mathbb{R}^n; \mathbb{C}^n)$, we have

$$|\nabla |u|| \leq |Lu|.$$

In [46] the following Fefferman–Phong inequality was proven.

Lemma 4.1.2 (Global Fefferman–Phong)**.** Let $b \in L^2_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^n)$. Also, assume that $\mathbf{B}$ satisfies (2.4.3). Then

$$\|m(\cdot, |\mathbf{B}|)u\|_2 \lesssim \|Lu\|_2,$$

for all $u \in \mathcal{C}^1(\mathbb{R}^n)$.

Another important property is that the first-order operator satisfies a commutator bound. To see this we use the product rule and the product rule for divergence to get

$$[\eta I, D]u = \eta Du - D(\eta u) = \begin{bmatrix} \eta L^* u_{\|} \\ \eta L u_{\perp} \end{bmatrix} - \begin{bmatrix} L^*(\eta u_{\|}) \\ L(\eta u_{\perp}) \end{bmatrix} = \begin{bmatrix} u_{\|} \cdot \nabla \eta \\ u_{\perp} \nabla \eta \end{bmatrix}$$

Therefore, we have

$$|[\eta I, D]u(x)| \lesssim |\nabla \eta||u(x)|, \tag{4.1.1}$$

for a.e. $x \in \mathbb{R}^n$. The commutator bounds allows us to prove the off-diagonal estimates for D.

Proposition 4.1.3. Let U_t be $\tilde{R}_t^B$ for $t \in \mathbb{R} \setminus \{0\}$ or $\tilde{P}_t^B$ or $\tilde{Q}_t^B$ for every $t > 0$. Then for any $M \in \mathbb{N}$ there exists $C_M > 0$, which depends only on $\mathbf{B}$, n, κ, and $\|B\|_\infty$, such that

$$\|U_t u\|_{L^2(E)} \leq C_M \left\langle \frac{\operatorname{dist}(E,F)}{t} \right\rangle^{-M} \|u\|_{L^2(F)},$$

for every $E, F \subseteq \mathbb{R}^n$ Borel sets, and $u \in L^2(\mathbb{R}^n; \mathbb{C}^{n+1})$ with $\operatorname{supp}(u) \subseteq F$

Proof. The proof follows as in [13, Proposition 5.2] using (4.1.1). □

We will need the Riesz transform bounds first proven in [47] and later improved in [17].

Theorem 4.1.4. Let $b \in L^2_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^n)$. Also, Assume $\mathbf{B}$ satisfies (2.4.3). Then $L^2 H_b^{-1}$ is L^p-bounded for any $p \in (1, \infty)$.

We adapt Theorem 4.1.4 so that we may use them in the context of our first-order operator D. Here we bound the magnetic gradient with D.

Proposition 4.1.5. Let $b \in L^2_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^n)$. Also, Assume $\mathbf{B}$ satisfies (2.4.3). Then

$$\|Lu\|_2 \lesssim \|Du\|_2,$$

for all $u \in \overline{\mathsf{R}(D)}$.

Proof. Let $u \in \overline{\mathsf{R}(D)}$ By Lemma 2.4.6 we have there exists $f \in L^2(\mathbb{R}^n)$ and $g \in \dot{W}_b^{1,2}(\mathbb{R}^n)$ such that

$$u = \begin{bmatrix} f \\ Lg \end{bmatrix}.$$

Therefore, we have

$$Lu = \begin{bmatrix} L^2 g \\ Lf \end{bmatrix} \quad \text{and} \quad Du = \begin{bmatrix} L^* L g \\ Lf \end{bmatrix}.$$

Thus using Theorem 4.1.4 and $H = L^* L$, we have

$$\|Lu\|_2 \leq \|L^2 g\|_2^2 + \|Lf\|_2 \lesssim \|Hf\|_2 + \|Lf\|_2 = \left\| \begin{bmatrix} L^* L f \\ Lf \end{bmatrix} \right\| = \|Du\|_2.$$

This completes the proof. $\square$

Combining Proposition 4.1.5 with the global Fefferman–Phong inequality, Lemma 4.1.2, and the observation in Remark 4.0.1, we get the following corollary.

Corollary 4.1.6. Let $b \in L^2_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^n)$. Also, Assume $\mathbf{B}$ satisfies (2.4.3). Then

$$\|Lu\|_2 + \||\mathbf{B}|^{\frac{1}{2}}u\|_2 \lesssim \|Du\|_2$$

for all $u \in \mathsf{R}(D)$.

Proof. Using Remark 4.0.1 and Lemma 4.1.2, we have

$$\||\mathbf{B}|^{\frac{1}{2}}u\|_2 \lesssim \|m(x, |\mathbf{B}|)P_t u\|_2 \lesssim \|Lu\|_2.$$

Then, using Proposition 4.1.5, we get

$$\|Lu\|_2 + \||\mathbf{B}|^{\frac{1}{2}}u\|_2 \lesssim \|Lu\|_2 \lesssim \|Du\|_2.$$

As required $\square$

4.1.2 Maximal Dyadic Mesh Adapted to the Magnetic Field

We start by defining the notion of a dyadic decomposition of an arbitrary (not necessarily dyadic) cube. To this end fix a cube Q. For $t > l(Q)$ we define $\Delta_t(Q) \coloneqq \emptyset$. For $t \leq l(Q)$, there exists $k \in \mathbb{N}$ such that $2^{-(k-1)}l(Q) < t \leq 2^{-k}l(Q)$. Then, we define $\Delta_t(Q)$ to be the set of dyadic subcubes of Q of side-length $2^k l(Q)$. Note $Q = \bigcup_{R \in \Delta_t(Q)} R$.

We define, for each $T > 0$, the collection of dyadic cubes, $\tilde{\Delta}^{\mathbf{B}}_T$, as follows: for each $Q \in \Delta_T$ if

$$l(4Q)^2 \fint_{4Q} |\mathbf{B}| \leq 1 \tag{4.1.2}$$

then add $Q \in \tilde{\Delta}^{\mathbf{B}}_T$, if not then sub-divide Q dyadically, and stop when (4.1.2) is satisfied. To see that the subdivision stops, fix $x \in Q$ and consider the collection $\{Q_n\}_{n \in \mathbb{N}}$ obtained from dyaically sub-dividing Q and taking Q_n to be the cube of side-length $2^{-n}l(Q)$ such that $x \in Q_n$. Then, by the Lebesgue differentiation theorem we have

$$\lim_{n\to\infty}\left[l(4Q_n)^2 \fint_{4Q_n} |\mathbf{B}|\right] = 0 \cdot |\mathbf{B}(x)| = 0.$$

Therefore, there exists $N \in \mathbb{N}$ such that for all $n > N$ (4.1.2) holds. Also, note that by construction $\tilde{\Delta}_T^{\mathbf{B}}$ is maximal in the sense that for every $Q \in \tilde{\Delta}_T^{\mathbf{B}}$ then either $Q \in \Delta_T$ or the parent, $\widehat{Q}$, of Q is such that

$$l(4\widehat{Q})^2 \fint_{4\widehat{Q}} |\mathbf{B}| > 1.$$

Thus $\tilde{\Delta}_T^{\mathbf{B}}$ is a maximal collection of dyadic cubes satisfying (4.1.2) and we have $\tilde{\Delta}_T^{\mathbf{B}}$ is a covering of $\mathbb{R}^n$. An important property of the maximal dyadic mesh $\tilde{\Delta}_T^{\mathbf{B}}$ is that it has a finite overlap property which we present in the following proposition.

Proposition 4.1.7. Let $T > 0$. Then there exists a constant $c > 0$ such that

$$\sum_{Q\in\tilde{\Delta}_T^{\mathbf{B}}} \mathbb{1}_{2^kQ}(x) \le c2^{kl_{\mathbf{B}}}, \quad \forall x \in \mathbb{R}^n, \tag{4.1.3}$$

where c and $l_{\mathbf{B}}$ depend only on dimension and the properties of $|\mathbf{B}|$.

Proof. Fix $x \in \mathbb{R}^n$. Then there is a unique $Q \in \tilde{\Delta}_T^{\mathbf{B}}$ such that $x \in Q$. We will give a bound on the number of $\tilde{Q} \in \tilde{\Delta}_T^{\mathbf{B}}$ such that $Q \cap 2^k\tilde{Q} \neq \emptyset$. First we calculate a lower bound on $l(\tilde{Q})$ so we assume that $l(\tilde{Q}) < l(Q)$. Now let $R \in \Delta_{l(Q)}$ be the unique cube such that $\tilde{Q} \subset R$. As $\emptyset \neq 2^k\tilde{Q} \cap Q \subset 2^kR \cap Q$. Thus, as $l(Q) = l(R)$ we have $R \subset 2^{k+1}Q$. Therefore, using the doubling property of $|\mathbf{B}|$ we obtain

$$l(R)^2 \fint_R |\mathbf{B}| \le \frac{4^{-2}l(4R)^2}{4^{-n}|4R|}\int_{2^{k+1}Q} |\mathbf{B}| \le c_d^{k-1}4^{n-2}\frac{l(4Q)^2}{|4Q|}\int_{4Q}|\mathbf{B}| \le c_d^{k-1}4^{n-2}, \tag{4.1.4}$$

where $c_d > 0$ is the doubling constant for $|\mathbf{B}|$. Now assume for contradiction, that

$$l(\tilde{Q}) < \left(\frac{[\![|\mathbf{B}|]\!]_p c_d^{k-1}4^{n-2}}{2^{-6}c_d^{-4}}\right)^{\frac{-1}{2-\frac{n}{p}}} l(R).$$

Then by Lemma 2.2.3 with $C = c_d^{k-1}4^{n-2}$ and $c = 2^{-6}c_d^{-4}$, and (4.1.4), we have

$$l(\tilde{Q})^2 \fint_{\tilde{Q}} |\mathbf{B}| \leq 2^{-6}c_d^{-4}.$$

Let $\widehat{\tilde{Q}}$ be the dyadic parent of $\tilde{Q}$, that is $\tilde{Q} \subset \widehat{\tilde{Q}}$ and $\widehat{\tilde{Q}} \in \Delta_{2l(\tilde{Q})}$. Then

$$l(4\widehat{\tilde{Q}})^2 \fint_{4\widehat{\tilde{Q}}} |\mathbf{B}| \leq l(8\tilde{Q})^2 \fint_{16\tilde{Q}} |\mathbf{B}| \leq 2^6 c_d^4 l(\tilde{Q})^2 \fint_{\tilde{Q}} |\mathbf{B}| \leq 1.$$

Thus, $\widehat{\tilde{Q}}, \tilde{Q} \in \tilde{\Delta}_T^{\mathbf{B}}$ and $\tilde{Q} \subset \widehat{\tilde{Q}}$. Also, as $l(\tilde{Q}) < l(Q)$ then $\tilde{Q} \notin \Delta_T$. Thus, by the maximality of $\tilde{\Delta}_T^{\mathbf{B}}$ we also have

$$l(4\widehat{\tilde{Q}})^2 \fint_{4\widehat{\tilde{Q}}} |\mathbf{B}| > 1$$

This is a contradiction. Hence

$$l(\tilde{Q}) \geq \left(\frac{[\![|\mathbf{B}|]\!]_p c_d^{k-1} 4^{n-2}}{2^{-6}c_d^{-4}} \right)^{\frac{-1}{2-\frac{n}{p}}} l(R) = \left([\![|\mathbf{B}|]\!]_p c_d^{k+3} 2^{2(n+1)} \right)^{\frac{-1}{2-\frac{n}{p}}} l(Q).$$

Now we find an upper bound for $l(\tilde{Q})$. First suppose $Q \in \tilde{\Delta}_T^{\mathbf{B}} \cap \Delta_T$. Then for $\tilde{Q} \in \tilde{\Delta}_T^{\mathbf{B}}$ and $l(\tilde{Q}) \geq l(Q)$ then we must have $\tilde{Q} \in \Delta_T$. That is $l(\tilde{Q}) = l(Q)$. Now suppose $Q \in \tilde{\Delta}_T^{\mathbf{B}} \setminus \Delta_T$ Let $\widehat{Q}$ be the dyadic parent of Q. As $Q \subset \widehat{Q}$, we have $2^k\tilde{Q} \cap \widehat{Q} \neq \emptyset$. Then, by the maximality of $\tilde{\Delta}_T^{\mathbf{B}}$ and since $Q \in \tilde{\Delta}_T^{\mathbf{B}} \setminus \Delta_T$, we have

$$l(4\widehat{Q})^2 \fint_{4\widehat{Q}} |\mathbf{B}| > 1.$$

We also have $4\widehat{Q} \subseteq 2^{k+3}\tilde{Q}$ and

$$l(2^{k+3}\tilde{Q})^2 \fint_{2^{k+3}\tilde{Q}} |\mathbf{B}| \leq \frac{c_d^{k+1}4^{2(k+1)} l(4\tilde{Q})^2}{2^{n(k+1)}|4\tilde{Q}|} \int_{4\tilde{Q}} |\mathbf{B}| \leq c_d^{k+1} 2^{(k+1)(2-n)}.$$

Then, by Lemma 2.2.2 we have $\left([\![|\mathbf{B}|]\!]_p c_d^{k+1} 2^{(k+1)(2-n)}\right)^{\frac{-1}{2-\frac{n}{p}}} l(2^{k+2}\tilde{Q}) \leq l(4\widehat{Q})$. Therefore, if $\tilde{Q} \in \tilde{\Delta}_T^{\mathbf{B}}$ and $2^k\tilde{Q} \cap Q \neq \emptyset$, and as $l(4\widehat{Q}) = 8l(Q)$, we have

$$\begin{aligned} l(\tilde{Q}) &\leq 2^{k+4} \left([\![|\mathbf{B}|]\!]_p c_d^{k+1} 2^{(k+1)(2-n)} \right)^{\frac{1}{2-\frac{n}{p}}} l(Q) \\ &= 2^4 \left([\![|\mathbf{B}|]\!]_p c_d 2^{2-n} \right)^{\frac{1}{2-\frac{n}{p}}} \left(2^{k+\frac{k(2-n)}{2-\frac{n}{p}}} c_d^{\frac{k}{2-\frac{n}{p}}} \right) l(Q). \end{aligned}$$

Define $d \coloneqq \log_2 c_d$. To recap, we have

$$c_1 2^{\frac{-kd}{2-\frac{n}{p}}} l(Q) \leq l(\tilde{Q}) \leq c_2 2^{k\left(1+\frac{d+2-n}{2-\frac{n}{p}}\right)} l(Q),$$

where $c_1, c_2 > 0$ depend only on dimension and the properties of $|\mathbf{B}|$. Now suppose $l(\tilde{Q}) = 2^j l(Q)$ for some $j \in \mathbb{R}$, then

$$2^j \in \left[c_1 2^{\frac{-kd}{2-\frac{n}{p}}}, c_2 2^{k\left(1+\frac{d+2-n}{2-\frac{n}{p}}\right)} \right]$$

As $2^k\tilde{Q} \cap Q \neq \emptyset$, then $\mathrm{dist}(\tilde{Q}, Q) \leq 2^k l(\tilde{Q}) = 2^{k+j} l(Q)$. There are at most 2^{2kn} many dyadic cubes at scale $2^j l(Q)$ which are a distance of at most $2^{k+j} l(Q)$ from Q. Therefore an upper bound on the number of possible cubes $\tilde{Q} \in \tilde{\Delta}_T^{\mathbf{B}}$ for which $2^k\tilde{Q} \cap Q \neq \emptyset$ is

$$\begin{aligned} \sum_{j=\frac{-kd}{2-\frac{n}{p}}+\tilde{c}_1}^{k\left(1+\frac{d+2-n}{2-\frac{n}{p}}\right)+\tilde{c}_2} 2^{2kn} &\leq 2^{2kn} \left(k + \frac{k(d+2-n)}{2-\frac{n}{p}} + \frac{kd}{2-\frac{n}{p}} + \tilde{c}_2 - \tilde{c}_1 \right) \\ &\leq 2^{\tilde{c}_2 - \tilde{c}_1} 2^{k\left(1+2n+\frac{2d+2-n}{2-\frac{n}{p}}\right)}, \end{aligned}$$

where $\tilde{c}_1 \coloneqq \log_2 c_1$ and $\tilde{c}_2 \coloneqq \log_2 c_2$. Thus, letting $l_{\mathbf{B}} \coloneqq 1 + 2n + \frac{(2d+2-n)}{2-\frac{n}{p}}$ we have

$$\sum_{Q \in \tilde{\Delta}_T^{\mathbf{B}}} \mathbb{1}_{2^k Q}(x) \lesssim 2^{kl}.$$

This completes the proof. $\square$

4.1.3 Local Estimates

One of the most important properties of the magnetic gradient is the gauge transform in Proposition 2.4.3. This will allow us to introduce the magnetic field $\mathbf{B}$ and take advantage of the reverse Hölder properties that $\mathbf{B}$ satisfies. To do this we will to localise onto the dyadic mesh $\tilde{\Delta}_T^{\mathbf{B}}$. To this end we introduce the notation $L^2_Q(\mathbb{R}^n;\Omega) := \{u \in L^2(\mathbb{R}^n;\Omega) : \operatorname{supp}(u) \subseteq 4Q\}$. If $\Omega = \mathbb{C}$ then we define $L^2_Q(\mathbb{R}^n) := L^2_Q(\mathbb{R}^n;\mathbb{C})$. We now define the gauge invariant magnetic gradient. Fix a cube Q. Let $u \in L^2_Q(\mathbb{R}^n)$. Then, define

$$\tilde{L}_Q u := (\nabla + ih_Q)u,$$

where h_Q is as in Proposition 2.4.3 for the cube $4Q$. Let $u \in L^2_Q(\mathbb{R}^n)$. To see why the gauge transform is so useful we present the following property which is called the gauge invariance. By the product rule and the definitions of h_Q an φ_Q from Proposition 2.4.3, we have

$$\begin{aligned}
\tilde{L}_Q(e^{i\varphi_Q}u) &= \nabla(e^{i\varphi_Q}u) + ih_Q(e^{i\varphi_Q}u) \\
&= e^{i\varphi_Q}\nabla u + u\nabla e^{i\varphi_Q} + ih_Q^{i\varphi_Q}u \\
&= e^{i\varphi_Q}\nabla u + uie^{i\varphi_Q}\nabla\varphi_Q + ih_Q e^{i\varphi_Q}u \\
&= e^{i\varphi_Q}\nabla u + ie^{i\varphi_Q}(\nabla\varphi_Q + h_Q)u \\
&= e^{i\varphi_Q}(\nabla u + ibu) \\
&= e^{i\varphi_Q}Lu
\end{aligned}$$

From above and the properties of adjoints, we also have

$$e^{i\varphi_Q}L^*u = (Le^{-i\varphi_Q})^*u = (e^{-i\varphi_Q}e^{i\varphi_Q}Le^{-i\varphi_Q})^*u = (e^{-i\varphi_Q}\tilde{L}_Q)^*u = \tilde{L}_Q^*(e^{i\varphi_Q}u).$$

We can also define the gauge invariant first-order operator defined on the cube Q as

$\tilde{D}^Q\colon \mathscr{D}(\tilde{D}^Q) \subset L^2_Q(\mathbb{R}^n;\mathbb{C}^{n+1}) \to L^2_Q(\mathbb{R}^n;\mathbb{C}^{n+1})$, given by

$$\tilde{D}^Q := \begin{bmatrix} 0 & \tilde{L}^*_Q \\ \tilde{L}_Q & 0 \end{bmatrix}.$$

Let $u \in L^2_Q(\mathbb{R}^n)$. Then, considering $e^{i\varphi_Q}$ to be multiplication by the scalar field $e^{i\varphi_Q}$ to each component, we have

$$e^{i\varphi_Q} Du = \begin{bmatrix} e^{i\varphi_Q} Lu_\perp \\ e^{i\varphi_Q} L^* u_\| \end{bmatrix} = \begin{bmatrix} \tilde{L}_Q(e^{i\varphi_Q} u_\perp) \\ \tilde{L}^*_Q(e^{i\varphi_Q} u_\|) \end{bmatrix} = \tilde{D}^Q(e^{i\varphi_Q} u) \tag{4.1.5}$$

The operator B also retains the elliptic on $\overline{\mathsf{R}(\tilde{D}^Q)}$. To see this let $u \in \mathsf{R}(\tilde{D}^Q)$. Then using (4.1.5), and the ellipticity of B on $\overline{\mathsf{R}(D)}$, we have

$$\begin{aligned} \kappa\|\tilde{D}^Q u\|_2^2 &= \kappa\|e^{i\varphi_Q}\tilde{D}^Q u\|_2^2 \\ &= \kappa\|De^{i\varphi_Q} u\|_2^2 \\ &\leq \mathrm{Re}\langle BDe^{i\varphi_Q} u, De^{i\varphi_Q} u\rangle \\ &= \mathrm{Re}\langle e^{i\varphi_Q} B\tilde{D}^Q u, e^{i\varphi_Q}\tilde{D}^Q u\rangle \\ &= \mathrm{Re}\langle e^{-i\varphi_Q} e^{i\varphi_Q} B\tilde{D}^Q u, \tilde{D}^Q u\rangle \\ &= \mathrm{Re}\langle B\tilde{D}^Q u, \tilde{D}^Q u\rangle \end{aligned}$$

Then let $u \in \overline{\mathsf{R}(\tilde{D}^Q)}$ and let $\{u_u\}_{n\in\mathbb{N}} \subset \mathsf{R}(\tilde{D}^Q)$ converging to u. Fix $\varepsilon > 0$ arbitrarily.

Then, by above and the Cauchy–Schwarz inequality, we have

$$
\begin{aligned}
\kappa\|u\|_2^2 &\lesssim \kappa\|u-u_n\|_2^2 + \kappa\|u_n\|_2^2 \\
&\leq \kappa\|u-u_n\|_2^2 + \mathrm{Re}\langle Bu_n, u_n\rangle \\
&= \kappa\|u-u_n\|_2^2 + \mathrm{Re}\langle Bu_n - u, u_n\rangle + \mathrm{Re}\langle Bu, u_n\rangle \\
&= \kappa\|u-u_n\|_2^2 + \mathrm{Re}\langle Bu_n - u, u_n - u\rangle + Re\langle B(u_n-u), u\rangle \\
&\qquad + \mathrm{Re}\langle Bu, u_n - u\rangle + \mathrm{Re}\langle Bu, u\rangle \\
&\leq \kappa\|u-u_n\|_2^2 + \|B\|_\infty\|u-u_n\|_2^2 + 2\|B\|_\infty\|u-u_n\|_2\|u\|_2 + \mathrm{Re}\langle Bu, u\rangle.
\end{aligned}
$$

As u_n converges to u, there exists $N \in \mathbb{N}$ such that

$$
\|u-u_n\|_2 < \max\left\{\sqrt{\frac{\varepsilon}{3\kappa}}, \sqrt{\frac{\varepsilon}{3\|B\|_\infty}}, \frac{\varepsilon}{6\|B\|_\infty\|u\|_2}\right\},
$$

for all $n > N$. That is

$$
\kappa\|u\|_2^2 \lesssim \varepsilon + \mathrm{Re}\langle Bu, u\rangle.
$$

Thus, as $\varepsilon > 0$ was arbitrary we have

$$
\kappa\|u\|_2^2 \lesssim \mathrm{Re}\langle Bu, u\rangle, \quad \forall u \in \overline{R(\tilde{D}^Q)}. \tag{4.1.6}
$$

For $t \in \mathbb{R}\setminus\{0\}$, we can then define

$$
\begin{aligned}
\tilde{R}_t^{B,Q} &:= (I - it\tilde{D}^Q B)^{-1} \\
\tilde{P}_t^{B,Q} &:= (I + t^2\tilde{D}^Q B\tilde{D}^Q B)^{-1} = \frac{1}{2}\left(\tilde{R}_t^{B,Q} + \tilde{R}_{-t}^{B,Q}\right) = \tilde{R}_t^{B,Q}\tilde{R}_{-t}^{B,Q} \\
\tilde{Q}_t^{B,Q} &:= t\tilde{D}^Q B(I + t^2\tilde{D}^Q B\tilde{D}^Q B)^{-1} = t\tilde{D}^Q BP_t^B = \frac{1}{2i}\left(-\tilde{R}_t^{B,Q} + \tilde{R}_t^{B,Q}\right).
\end{aligned}
$$

We will now compare the resolvents of DB with the resolvents of $\tilde{D}^Q B$.

Lemma 4.1.8. Let Q be a cube. Let $t \in \mathbb{R}$. Then we have the identity

$$
R_t^B(\eta_Q v) = e^{-i\varphi_Q}\eta_Q\tilde{R}_t^{B,Q}e^{i\varphi_Q}v + itR_t^B e^{-i\varphi_Q}[\eta_Q, \tilde{D}^Q]B\tilde{R}_t^{B,Q}e^{i\varphi_Q}v,
$$

for all $v \in L^2(\mathbb{R}^n;\mathbb{C}^{n+1})$, where $\eta_Q \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ with $\operatorname{supp}(\eta_Q) \subset 4Q$.

Proof. First note $\eta_Q v \in L^2_Q(\mathbb{R}^n;\mathbb{C}^{n+1})$. Then, we have

$$\begin{aligned} R_t^B(\eta_Q v) &= R_t^B(e^{-i\varphi_Q}e^{i\varphi_Q}\eta_Q v) \\ &= R_t^B(e^{-\varphi_Q}\eta_Q e^{i\varphi_Q}v) - e^{-i\varphi_Q}\eta_Q\tilde{R}_t^{B,Q}(e^{i\varphi_Q}v) + e^{-i\varphi_Q}\eta_Q\tilde{R}_t^{B,Q}(e^{i\varphi_Q}v). \end{aligned}$$

Then, the error term is

$$\begin{aligned} R_t^B(e^{-i\varphi_Q}\eta_Q e^{i\varphi_Q}v) - e^{-i\varphi_Q}\eta_Q\tilde{R}_t^{B,Q}e^{i\varphi_Q}v &= R_t^B e^{-i\varphi_Q}\eta_Q(I+it\tilde{D}^Q B)\tilde{R}_t^{B,Q}e^{i\varphi_Q}v \\ &\quad - R_t^B(I+itDB)e^{-i\varphi_Q}\eta_Q\tilde{R}_t^{B,Q}e^{i\varphi_Q}v \\ &= R_t^B\big[e^{-i\varphi_Q}\eta_Q(I+it\tilde{D}^Q B) \\ &\quad - (I+itDB)e^{-i\varphi_Q}\eta_Q\big]\tilde{R}_t^{B,Q}e^{i\varphi_Q}v \\ &= itR_t^B e^{-i\varphi_Q}\big[\eta_Q\tilde{D}^Q B \\ &\quad - e^{i\varphi_Q}DBe^{-i\varphi_Q}\eta_Q\big]\tilde{R}_t^{B,Q}e^{i\varphi_Q}v \\ &= itR_t e^{-i\varphi_Q}\eta_Q\tilde{D}^Q B - \tilde{D}^Q B\eta_Q\tilde{R}_t^{B,Q}e^{i\varphi_Q}v \\ &= itR_t^B e^{-i\varphi_Q}[\eta_Q,\tilde{D}^Q]B\tilde{R}_t^{B,Q}e^{i\varphi_Q}v \end{aligned}$$

This completes the proof. □

Using the product rule and the product rule for divergence we have the following identity

$$[\eta I,\tilde{D}^Q]u = \eta\tilde{D}^Q u - \tilde{D}^Q(\eta u) = \begin{bmatrix} \eta\tilde{L}_Q^* u_\parallel \\ \eta\tilde{L}_Q u_\perp \end{bmatrix} - \begin{bmatrix} \tilde{L}_Q^*(\eta u_\parallel) \\ \tilde{L}_Q(\eta u_\perp) \end{bmatrix} = \begin{bmatrix} u_\parallel\cdot\nabla\eta \\ u_\perp\nabla\eta \end{bmatrix}$$

for all $u \in L^2_Q(\mathbb{R}^n;\mathbb{C}^{n+1}) \cap \mathscr{D}(D)$. Thus, we have

$$|[\eta I,\tilde{D}^Q]u(x)| \lesssim |\nabla\eta||u(x)|, \tag{4.1.7}$$

for all $x \in 4Q$, and all $\eta \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ with $\operatorname{supp}(\eta) \subset 4Q$.

Proposition 4.1.9. Let U_t be $\tilde{R}_t^{B,Q}$ for $t \in \mathbb{R} \setminus \{0\}$ or $\tilde{P}_t^{B,Q}$ or $\tilde{Q}_t^{B,Q}$ for every $t > 0$. Then for any $M \in \mathbb{N}$ there exists $C_M > 0$, which depends only on $\mathbf{B}$, n, κ, and $\|B\|_\infty$, such that

$$\|U_t u\|_{L^2_Q(E)} \leq C_M \left\langle \frac{\operatorname{dist}(E,F)}{t} \right\rangle^{-M} \|u\|_{L^2_Q(F)},$$

for every $E, F \subseteq 4Q$ Borel sets, and $u \in L^2_Q(\mathbb{R}^n; \mathbb{C}^{n+1})$ with $\operatorname{supp}(u) \subseteq F$

Proof. The proof follows as in [13, Proposition 5.2] using (4.1.7). □

We now give an analogous result to that of Lemma 3.1.5 which is based on [13, Lemma 5.6].

Lemma 4.1.10. Let Q be a cube. Then we have the estimate

$$\left| \fint_R \tilde{D}^Q f \right|^2 \lesssim \frac{1}{l(R)} \left(1 + l(4Q)^2 \fint_{4Q} |\mathbf{B}|\right) \left(\fint_R |\tilde{D}^Q f|^2\right)^{\frac{1}{2}} \left(\fint_R |f|^2\right)^{\frac{1}{2}},$$

for all subcubes $R \subseteq 4Q$ and $f \in \mathscr{D}(\tilde{D}^Q)$.

Proof. Let

$$t = \left(\int_R |f|^2\right)^{\frac{1}{2}} \left(\int_R |\tilde{D}^Q f|^2\right)^{-\frac{1}{2}}.$$

If $t \geq \frac{1}{4} l(R)$ then proceed as in Lemma 3.1.5 or [13, Lemma 5.6]. Now suppose $t \leq \frac{1}{4} l(R)$. Let $\eta \in \mathcal{C}_c^\infty(R)$ such that $\eta(x) = 1$ when $\operatorname{dist}(x, \mathbb{R}^n \setminus R) > t$ and $\|\nabla \eta\|_\infty \lesssim \frac{1}{t}$. We now bound both components separately. First, using the definition of $\tilde{L}_Q^*$ as the adjoint of $\tilde{L}_Q$ and η having compact support gives

$$\begin{aligned}
\left| \int_R (\tilde{D}^Q f)_\perp \right| &= \left| \int_R \eta (\tilde{D}^Q f)_\perp + (1-\eta)(\tilde{D}^Q f)_\perp \right| \\
&\leq \left| \int_R \eta \tilde{L}_Q^* f_\parallel \right| + \int_R |1-\eta| |(\tilde{D}^Q f)_\perp| \\
&= \left| \int_R \tilde{L}_Q \eta \cdot f_\parallel \right| + \int_R |1-\eta| |\tilde{D}^Q f| \\
&\leq \int_R |\nabla \eta| |f| + \int_R |h_Q \eta| |f| + \int_R |1-\eta| |\tilde{D}^Q f|.
\end{aligned}$$

And for he second component we have

$$\left|\int_R (\tilde{D}^Q f)_\parallel\right| = \left|\int_R \eta(\tilde{D}^Q f)_\parallel + (1-\eta)(\tilde{D}^Q f)_\parallel\right| \leq \left|\int_R \eta L f_\perp\right| + \int_R |1-\eta||\tilde{D}^Q f|$$

Then, by the compact support of $\eta f_\perp$, the Fundamental Theorem of Calculus and the product rule we have

$$\begin{aligned}\left|\int_R \eta L f_\perp - \nabla(\eta f_\perp)\right| &\leq \int_R |\eta \nabla f_\perp - \nabla(\eta f_\perp)| + \int_R |h_Q \eta||f_\perp| \\ &= \int_R |f_\perp \nabla \eta| + \int_R |h_Q\eta||f| \\ &\leq \int_R |f||\nabla\eta| + \int_R |h_Q\eta||f|\end{aligned}$$

Thus, using the Cauchy-Schwarz Inequality, $\|\nabla\eta\|_\infty \lesssim \frac{1}{t}$, and $|\operatorname{supp}(\nabla\eta)| = l(R)^{n-1}t$, we have

$$\begin{aligned}\int_R |\nabla\eta||f| &\leq \left(\int_R |\nabla\eta|^2\right)^{\frac{1}{2}}\left(\int_R |f|^2\right)^{\frac{1}{2}} \\ &\leq \|\nabla\eta\|_\infty |\operatorname{supp}(\nabla\eta)|^{\frac{1}{2}}\left(\int_R |f|^2\right)^{\frac{1}{2}} \\ &\lesssim l(R)^{\frac{n-1}{2}} t^{-\frac{1}{2}}\left(\int_R |f|\right)^{\frac{1}{2}}.\end{aligned}$$

Now, by the Cauchy–Schwarz inequality, also $|R \cap \operatorname{supp}(1-\eta)| = l(R)^{n-1}t$ and $|1-\eta| \leq 1$ by the definition of η, we have

$$\int_R |1-\eta||\tilde{D}^Q f| \leq \left(\int_R |1-\eta|^2\right)^{\frac{1}{2}}\left(\int_R |\tilde{D}^Q f|^2\right)^{\frac{1}{2}} \leq l(R)^{\frac{n-1}{2}} t^{\frac{1}{2}}\left(\int_R |\tilde{D}^Q f|^2\right)^{\frac{1}{2}}.$$

We now bound the last term above by using the Cauchy–Schwarz inequity, Hölder's inequality, $R \subseteq 4Q$, the Sobolev inequality, the gauge transform inequality (2.4.2), the reverse Hölder inequality for $|\mathbf{B}|$, and the definition of η, to get

$$\begin{aligned}
\int_R |\eta h_Q||f| &\leq \left(\int_R |\eta h_Q|^2\right)^{\frac{1}{2}} \left(\int_R |f|^2\right)^{\frac{1}{2}} \\
&\leq \left(\int_{4Q} |h_Q|^n\right)^{\frac{1}{n}} \left(\int_R |\eta|^{2^*}\right)^{\frac{1}{2^*}} \left(\int_R |f|^2\right)^{\frac{1}{2}} \\
&\lesssim l(4Q)^2 \left(\fint_{4Q} |h_Q|^n\right)^{\frac{1}{n}} \left(\int_R |\nabla\eta|^2\right)^{\frac{1}{2}} \left(\int_R |f|^2\right)^{\frac{1}{2}} \\
&\lesssim l(4Q)^2 \left(\fint_{4Q} |\mathbf{B}|^{\frac{n}{2}}\right)^{\frac{2}{n}} \|\nabla\eta\|_\infty |\operatorname{supp}(\nabla\eta)|^{\frac{1}{2}} \left(\int_R |f|^2\right)^{\frac{1}{2}} \\
&\lesssim \left(l(4Q)^2 \fint_{4Q} |\mathbf{B}|\right) (l(R)^{n-1}t^{-1})^{\frac{1}{2}} \left(\int_R |f|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Therefore, by the definition of t, we have

$$\begin{aligned}
\left(\int_R |f|^2\right)^{\frac{1}{2}} &\lesssim \left(l(4Q)^2 \fint_{4Q} |\mathbf{B}|\right) (l(R)^{n-1}t^{-1})^{\frac{1}{2}} \left(\int_R |f|^2\right)^{\frac{1}{2}} \\
&\leq l(R)^{\frac{n-1}{2}} \left(l(4Q)^2 \fint_{4Q} |\mathbf{B}|\right) \left(\int_R |\tilde{D}^Q f|^2\right)^{\frac{1}{4}} \left(\int_R |f|^2\right)^{\frac{1}{4}}
\end{aligned}$$

Finally, combining the above, squaring, and dividing by $|R|$, we have

$$\left|\fint_R \tilde{D}^Q f\right|^2 \lesssim \frac{1}{l(R)} \left(1 + l(4Q)^2 \fint_{4Q} |\mathbf{B}|\right) \left(\fint_R |\tilde{D}^Q f|^2\right)^{\frac{1}{2}} \left(\fint_R |f|^2\right)^{\frac{1}{2}},$$

as required. $\square$

4.2 Localisation

The aim of this section is to localise the quadratic estimate in Theorem 4.0.2. This is where we will take advantage of the gauge transform in Proposition 2.4.3 to introduce $\mathbf{B}$, and then exploit the local properties of $\mathbf{B}$.

Now, for all $t \leq T$ define $E_t^T := \{R \in \Delta_t \colon \text{there exists } Q \in \tilde{\Delta}_T^{\mathbf{B}} \text{ such that } Q \subset R\}$, the set of dyadic cubes of scale t which contain a cube in the maximal dyadic mesh $\tilde{\Delta}_T^{\mathbf{B}}$. We note that the set inclusion in E_t^T is strict and so $Q \notin E_t^T$ for all $Q \in \tilde{\Delta}_T^{\mathbf{B}}$. Then, by maximality of $\tilde{\Delta}_T^{\mathbf{B}}$ we have if $Q \in E_t^T$ then

$$l(4Q)^2 \fint_{4Q} |\mathbf{B}| > 1.$$

The collection E_t^T corresponds to the big cubes at scale t in the electric case in Chapter 3 and the proof for bounds follows in a similar fashion.

Lemma 4.2.1. Fix $T > 0$. Then we have

$$\int_0^T \sum_{Q \in E_t^T} \|\mathbb{1}_Q Q_t^B u\|_2^2 \frac{\mathrm{d}t}{t} \lesssim \|u\|_2^2,$$

for all $u \in \overline{\mathsf{R}(D)}$ independently of T.

Proof. First consider

$$\int_0^T \sum_{R \in E_t^T} \|\mathbb{1}_R Q_t^B u\|_2^2 \frac{\mathrm{d}t}{t} \lesssim \int_0^T \sum_{R \in E_t^T} \|\mathbb{1}_R Q_t^B P_t u\|_2^2 \frac{\mathrm{d}t}{t} + \int_0^T \sum_{R \in E_t^T} \|\mathbb{1}_R Q_t^B (I - P_t) u\|_2^2 \frac{\mathrm{d}t}{t}.$$

Then, as $I - P_t = t^2 D^2 P_t$, the boundedness of $tQ_t^B D$ and the self-adjointness of D, gives

$$\begin{aligned}
\int_0^T \sum_{R \in E_t^T} \|\mathbb{1}_R Q_t^B (I - P_t) u\|_2^2 \frac{\mathrm{d}t}{t} &\leq \int_0^\infty \|Q_t^B t^2 D^2 P_t u\|_2^2 \frac{\mathrm{d}t}{t} \\
&= \int_0^\infty \|(tQ_t^B D)(tDP_t) u\|_2^2 \frac{\mathrm{d}t}{t} \\
&\lesssim \int_0^\infty \|Q_t u\|_2^2 \frac{\mathrm{d}t}{t} \\
&\lesssim \|u\|_2^2.
\end{aligned}$$

Now, using off diagonal estimates, $\mathrm{dist}(Q, C_k(Q)) \eqsim 2^k t$, and the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
\sum_{R\in E_t^T} \|\mathbb{1}_R Q_t^B P_t u\|_2^2 &\leq \sum_{R\in E_t^T} \left[\sum_{k=0}^{\infty} \|\mathbb{1}_R Q_t^B \mathbb{1}_{C_k(R)}\| \|\mathbb{1}_{C_k(R)} P_t u\|_2\right]^2 \\
&\lesssim \sum_{R\in E_t^T} \left[\sum_{k=0}^{\infty} 2^{-kM} \|\mathbb{1}_{2^k R} P_t u\|_2\right]^2 \\
&\lesssim \sum_{R\in E_t^T} \sum_{k=0}^{\infty} 2^{-kM} \|\mathbb{1}_{2^k R} P_t u\|_2^2.
\end{aligned}$$

Suppose $2^k R$ is such that

$$l(2^k R)^2 \fint_{2^k R} |\mathbf{B}| \geq 1.$$

Then, using the Fefferman–Phong inequality, we have

$$\begin{aligned}
\|\mathbb{1}_{2^k R} P_t u\|_2^2 &\leq \left(l(2^k R)^2 \fint_{2^k R} |\mathbf{B}|\right)^{\beta} \|\mathbb{1}_{2^k R} P_t u\|_2^2 \\
&\lesssim l(2^k R)^2 \left(\|\mathbb{1}_{2^k R} L P_t u\|_2^2 + \|\mathbb{1}_{2^k R} |\mathbf{B}|^{\frac{1}{2}} P_t u\|_2^2\right)
\end{aligned}$$

Now suppose $2^k R$ is such that

$$l(2^k R)^2 \fint_{2^k R} |\mathbf{B}| \leq 1.$$

Firstly, if $k \in \{0, 1, 2\}$ then using the doubling property of $|\mathbf{B}|$, $k \leq 2$, and $c_d \geq 1$, we get

$$1 \leq l(4R)^2 \fint_{4R} |\mathbf{B}| = 2^{2(2-k)} l(2^k R)^2 \frac{c_d^{2-k} |2^k R|}{|4R|} \fint_{2^k R} |\mathbf{B}| \leq 2^4 c_d^2 l(2^k R)^2 \fint_{2^k R} |\mathbf{B}|.$$

That is, for $k \in \{0, 1, 2\}$ we have

$$\begin{aligned}
\|\mathbb{1}_{2^k R} P_t u\|_2^2 &\leq m_\beta \left(2^4 c_d^2 l(2^k R)^2 \fint_{2^k R} |\mathbf{B}|\right) \|\mathbb{1}_{2^k R} P_t u\|_2^2 \\
&\lesssim 2^4 c_d^2 l(2^k R)^2 \left(\|\mathbb{1}_{2^k R} L P_t u\|_2^2 + \|\mathbb{1}_{2^k R} |\mathbf{B}|^{\frac{1}{2}} P_t u\|_2^2\right).
\end{aligned}$$

Now let $k > 2$. Then, as $R \in E_t^T$ there exists $Q \subset R$ such that $Q \in \tilde{\Delta}_T^{\mathbf{B}}$. Therefore, by the maximality of $\tilde{\Delta}_T^{\mathbf{B}}$ we have

$$l(4R)^2 \fint_{4R} |\mathbf{B}| \geq 1.$$

As $k > 2$, and so $4R \subset 2^k R$, and using the Fefferman–Phong inequality, we have

$$\begin{aligned}
\|\mathbb{1}_{2^k R} P_t u\|_2^2 &\leq \left(l(4R)^2 \fint_{4R} |\mathbf{B}| \right) \|\mathbb{1}_{2^k R} P_t u\|_2^2 \\
&= 2^{(n-2)(k-2)} \left(l(2^k R)^2 \fint_{2^k R} |\mathbf{B}| \right) \|\mathbb{1}_{2^k R} P_t u\|_2^2 \\
&\lesssim 2^{(n-2)(k-2)} l(2^k R)^2 \left(\|\mathbb{1}_{2^k R} L P_t u\|_2^2 + \|\mathbb{1}_{2^k R} |\mathbf{B}|^{\frac{1}{2}} P_t u\|_2^2 \right) \\
&\lesssim 2^{kn} l(R)^2 \left(\|\mathbb{1}_{2^k R} L P_t u\|_2^2 + \|\mathbb{1}_{2^k R} |\mathbf{B}|^{\frac{1}{2}} P_t u\|_2^2 \right).
\end{aligned}$$

Therefore, as $2^k \leq 2^{kn}$ and $l(R) \eqsim t$, we have

$$\|\mathbb{1}_{2^k R} P_t u\|_2^2 \lesssim 2^{kn} t^2 \left(\|\mathbb{1}_{2^k R} L P_t u\|_2^2 + \|\mathbb{1}_{2^k R} |\mathbf{B}|^{\frac{1}{2}} P_t u\|_2^2 \right),$$

for all $R \in E_t^T$. Thus, by using the covering inequality $\sum_{R \in \Delta_t} \mathbb{1}_{2^k R}(x) \lesssim 2^{kn}$, the global Fefferman–Phong inequality in Proposition 4.1.2, choosing $M > 2n$, and Corollary 4.1.6, we have

$$\begin{aligned}
\sum_{R \in E_t^T} \sum_{k=0}^{\infty} 2^{-kM} \|\mathbb{1}_{2^k R} P_t u\|_2^2 &\lesssim \sum_{R \in E_t^T} \sum_{k=0}^{\infty} 2^{-k(M-n)} t^2 \left(\|\mathbb{1}_{2^k R} L P_t u\|_2^2 + \|\mathbb{1}_{2^k R} |\mathbf{B}|^{\frac{1}{2}} P_t u\|_2^2 \right) \\
&\lesssim \sum_{k=0}^{\infty} 2^{-k(M-2n)} t^2 \left(\|L P_t u\|_2^2 + \||\mathbf{B}|^{\frac{1}{2}} P_t u\|_2^2 \right) \\
&\lesssim t^2 \|D P_t u\|_2^2.
\end{aligned}$$

Finally, we have

$$\int_0^T \sum_{R \in E_t^T} \|\mathbb{1}_R Q_t^B P_t u\|_2^2 \frac{\mathrm{d}t}{t} \lesssim \int_0^T \|t D P_t u\|_2^2 \frac{\mathrm{d}t}{t} \leq \int_0^\infty \|Q_t u\|_2^2 \frac{\mathrm{d}t}{t} \lesssim \|u\|_2^2.$$

As required. $\square$

We now begin to localise the problem. Firstly we have

$$\int_0^T \|Q_t^B u\|_2^2 \frac{\mathrm{d}t}{t} \lesssim \int_0^T \sum_{R\in E_t^T} \|\mathbb{1}_R Q_t^B u\|_2^2 \frac{\mathrm{d}t}{t} + \int_0^T \sum_{R\in \Delta_t\setminus E_t^T} \|\mathbb{1}_R Q_t^B u\|_2^2 \frac{\mathrm{d}t}{t}, \tag{4.2.1}$$

and so by Lemma 4.2.1 we have the first term above is bounded by $\|u\|_2^2$. If $Q \in \tilde{\Delta}_T^{\mathbf{B}}$ then $\Delta_t(Q)\cap E_t^T = \emptyset$ by the definition of E_t^T. Therefore $\bigcup_{Q\in\tilde{\Delta}_T^{\mathbf{B}}} \Delta_t(Q) \subseteq \Delta_t \setminus E_t^T$. Also, by definition of E_t^T we have for every $R \in \Delta_t \setminus E_t^T$ then there exists $Q \in \tilde{\Delta}_T^{\mathbf{B}}$ such that $R \subseteq Q$. In particular, as $\in \Delta_t$ we have $R \in \Delta_t(Q)$. Thus, $\bigcup_{Q\in\tilde{\Delta}_T^{\mathbf{B}}} \Delta_t(Q) = \Delta_t \setminus E_t^T$. Therefore

$$\begin{aligned}
\int_0^T \sum_{R\in \Delta_t\setminus E_t^T} \|\mathbb{1}_R Q_t^B u\|_2^2 \frac{\mathrm{d}t}{t} &= \int_0^T \sum_{Q\in\tilde{\Delta}_T^{\mathbf{B}}} \sum_{R\in\Delta_t(Q)} \|\mathbb{1}_R Q_t^B u\|_2^2 \frac{\mathrm{d}t}{t} \\
&= \sum_{Q\in\tilde{\Delta}_T^{\mathbf{B}}} \int_0^T \mathbb{1}_{(0,l(Q))}(t) \sum_{R\in\Delta_t(Q)} \|\mathbb{1}_R Q_t^B u\|_2^2 \frac{\mathrm{d}t}{t} \\
&= \sum_{Q\in\tilde{\Delta}_T^{\mathbf{B}}} \int_0^{l(Q)} \|\mathbb{1}_Q Q_t^B u\|_2^2 \frac{\mathrm{d}t}{t}.
\end{aligned} \tag{4.2.2}$$

Now we define the test function $\eta_Q \in C_0^\infty(\mathbb{R}^n)$, such that $\eta_Q(x) \in [0,1]$ for all $x \in \mathbb{R}^n$, $\eta_Q \equiv 1$ on $2Q$, $\mathrm{supp}(\eta_Q) \subseteq 4Q$, and $\|\nabla \eta_Q\|_\infty \lesssim \frac{1}{l(Q)}$. Now, we introduce the test function η_Q as follows

$$\begin{aligned}
\sum_{Q\in\tilde{\Delta}_T^{\mathbf{B}}} \int_0^{l(Q)} \|\mathbb{1}_Q Q_t^B u\|_2^2 \frac{\mathrm{d}t}{t} \lesssim{}& \sum_{Q\in\tilde{\Delta}_T^{\mathbf{B}}} \int_0^{l(Q)} \|\mathbb{1}_Q Q_t^B (\eta_Q u)\|_2^2 \frac{\mathrm{d}t}{t} \\
&+ \sum_{Q\in\tilde{\Delta}_T^{\mathbf{B}}} \int_0^{l(Q)} \|\mathbb{1}_Q Q_t^B \left((1-\eta_Q)u\right)\|_2^2 \frac{\mathrm{d}t}{t}.
\end{aligned} \tag{4.2.3}$$

The following lemma allows us to localise the quadratic estimate so that we can break up $\mathbb{R}^n$ and work on each $Q \in \tilde{\Delta}_T^{\mathbf{B}}$ separately.

Lemma 4.2.2. Fix $T > 0$. Let η_Q be as defined above. Then we have the following estimate

$$\sum_{Q\in\tilde{\Delta}_T^{\mathbf{B}}} \int_0^{l(Q)} \|\mathbb{1}_Q Q_t^B((1-\eta_Q)u)\|_2^2 \frac{\mathrm{d}t}{t} \lesssim \|u\|_2^2$$

for all $u \in \overline{\mathsf{R}(D)}$.

Proof. To begin with we use the fact that $\mathrm{supp}(1-\eta_Q) \subset \mathbb{R}^n \setminus 2Q$, and so $\mathrm{dist}(\mathrm{supp}(1-\eta_Q), Q) \geq l(Q)$, off diagonal estimates, and then the Cauchy–Schwarz inequality, to get

$$\begin{aligned}
\|\mathbb{1}_Q Q_t^B((1-\eta_Q)u)\|_2^2 &\leq \left(\sum_{k=1}^{\infty} \|\mathbb{1}_Q Q_t^B \mathbb{1}_{C_k(Q)}\| \|\mathbb{1}_{C_k(Q)} u\|_2\right)^2 \\
&\lesssim \left(\sum_{k=1}^{\infty} \left(\frac{t}{\mathrm{dist}(Q, C_k(Q))}\right)^M \|\mathbb{1}_{2^k Q} u\|_2\right)^2 \\
&\lesssim \left(\sum_{k=1}^{\infty} \left(\frac{t}{2^k l(Q)}\right)^M \|\mathbb{1}_{2^k Q} u\|_2\right)^2 \\
&\lesssim \left(\frac{t}{l(Q)}\right)^M \left(\sum_{k=1}^{\infty} 2^{-kM} \|\mathbb{1}_{2^k Q} u\|_2^2\right)\left(\sum_{k=1}^{\infty} 2^{-kM}\right) \\
&\lesssim \left(\frac{t}{l(Q)}\right)^M \sum_{k=1}^{\infty} 2^{-kM} \|\mathbb{1}_{2^k Q} u\|_2^2.
\end{aligned}$$

Thus, integrating and using Proposition 4.1.7 we have

$$\begin{aligned}
\sum_{Q \in \tilde{\Delta}_T^{\mathbf{B}}} \int_0^{l(Q)} \|\mathbb{1}_Q Q_t^B(1-\eta_Q)u\|_2^2 \frac{\mathrm{d}t}{t} &\lesssim \sum_{Q \in \tilde{\Delta}_T^{\mathbf{B}}} \sum_{k=1}^{\infty} 2^{-kM} \frac{\|\mathbb{1}_{2^k Q} u\|_2^2}{l(Q)^M} \int_0^{l(Q)} t^{M-1}\,\mathrm{d}t \\
&= \sum_{Q \in \tilde{\Delta}_T^{\mathbf{B}}} \sum_{k=1}^{\infty} 2^{-kM} \frac{\|\mathbb{1}_{2^k Q} u\|_2^2}{l(Q)^M} \frac{l(Q)^M}{M} \\
&\lesssim \sum_{k=1}^{\infty} 2^{-k(M-l)} \|u\|_2^2 \\
&\lesssim \|u\|_2^2,
\end{aligned}$$

where $l_{\mathbf{B}}$ is as in (4.1.3). This completes the proof. $\square$

The reason for localising is so that we may take advantage of the gauge invariance of the magnetic gradient and we may only use the gauge invariance on cubes and not the whole of $\mathbb{R}^n$. Now we want to replace Q_t^B with $\tilde{Q}_t^{B,Q}$ and then control the error terms that appear from Lemma 4.1.8 using the commutator bounds (4.1.7).

Proposition 4.2.3. If

$$\sum_{Q\in\tilde{\Delta}_T^{\mathbf{B}}}\int_0^{l(Q)}\|\mathbb{1}_{4Q}e^{-i\varphi_Q}\tilde{Q}_t^{B,Q}e^{i\varphi_Q}\eta_Q u\|_2^2\frac{\mathrm{d}t}{t}\lesssim\|u\|_2^2 \tag{4.2.4}$$

for all $u\in L^2(\mathbb{R}^n;\mathbb{C}^{n+1})$, independently of T, then

$$\int_0^\infty\|Q_t^B u\|_2^2\frac{\mathrm{d}t}{t}\lesssim\|u\|_2^2,$$

for all $u\in L^2(\mathbb{R}^n;\mathbb{C}^{n+1})$.

Proof. First, using the monotone convergence theorem, we have

$$\int_0^\infty\|Q_t^B u\|_2^2\frac{\mathrm{d}t}{t}=\int_0^\infty\lim_{T\to\infty}\mathbb{1}_{(0,T)}(t)\|Q_t^B u\|_2^2\frac{\mathrm{d}t}{t}=\lim_{T\to\infty}\int_0^T\|Q_t^B u\|_2^2\frac{\mathrm{d}t}{t}.$$

Now fix $T>0$. Then, by (4.2.1) and (4.2.2), Lemma 4.2.1, (4.2.3), Lemma 4.2.2, and $\operatorname{supp}\eta_Q\subset 4Q$, we have

$$\begin{aligned}\int_0^T\|Q_t^B u\|_2^2\frac{\mathrm{d}t}{t}&\lesssim\sum_{Q\in\tilde{\Delta}_T^{\mathbf{B}}}\int_0^{l(Q)}\|\mathbb{1}_Q Q_t^B u\|_2^2\frac{\mathrm{d}t}{t}+\int_0^T\sum_{R\in E_t^T}\|\mathbb{1}_R Q_t^B u\|_2^2\frac{\mathrm{d}t}{t}\\&\lesssim\sum_{Q\in\tilde{\Delta}_T^{\mathbf{B}}}\int_0^{l(Q)}\left(\|\mathbb{1}_Q Q_t^B\eta_Q u\|_2+\|\mathbb{1}_Q Q_t^B((1-\eta_Q)u)\|_2\right)^2\frac{\mathrm{d}t}{t}+\|u\|_2^2\\&\lesssim\sum_{Q\in\tilde{\Delta}_T^{\mathbf{B}}}\int_0^T\|\mathbb{1}_Q Q_t^B\eta_Q\mathbb{1}_{4Q}u\|_2^2\frac{\mathrm{d}t}{t}+2\|u\|_2^2.\end{aligned}$$

Now, by Lemma 4.1.8, the uniform boundedness of resolvents and the properties of η_Q, we have

$$
\begin{aligned}
\|\mathbb{1}_Q Q_t^B \eta_Q u\|_2^2 &\lesssim \|\mathbb{1}_Q (R_{-t}^B - R_t^B)\eta_Q u\|_2^2 \\
&\lesssim \|e^{-i\varphi_Q}\eta_Q(\tilde{R}_{-t}^{B,Q} - \tilde{R}_t^{B,Q})e^{i\varphi_Q}\eta_Q u\|_2^2 \\
&\qquad + t^2\|R_t^B e^{-i\varphi_Q}[\eta_Q, \tilde{D}^Q]B\tilde{R}_t^{B,Q}e^{i\varphi_Q}\mathbb{1}_{4Q}u\|_2^2 \\
&\qquad + t^2\|R_{-t}^B e^{-i\varphi_Q}[\eta_Q, \tilde{D}^Q]B\tilde{R}_{-t}^{B,Q}e^{i\varphi_Q}\mathbb{1}_{4Q}u\|_2^2 \\
&\lesssim \|e^{-i\varphi_Q}\eta_Q\tilde{Q}_t^{B,Q}e^{i\varphi_Q}\eta_Q u\|_2^2 \\
&\qquad + t^2(\|R_t\|^2 + \|R_{-t}\|^2)\|[\eta_Q, \tilde{D}]\|^2\|B\|_\infty(\|\tilde{R}_t\|^2 + \|\tilde{R}_{-t}\|^2)\|\mathbb{1}_{4Q}u\|_2^2 \\
&\lesssim \|e^{-i\varphi_Q}\eta_Q\tilde{Q}_t^{B,Q}e^{i\varphi_Q}\eta_Q u\|_2^2 + \frac{t^2}{l(Q)^2}\|\mathbb{1}_{4Q}u\|_2^2.
\end{aligned}
$$

Therefore, using the above, (4.2.4), and Proposition 4.1.7, we have

$$
\begin{aligned}
\sum_{Q\in\tilde{\Delta}_T^{\mathbf{B}}}\int_0^{l(Q)}\|\mathbb{1}_Q Q_t^B\eta_Q u\|_2^2\frac{\mathrm{d}t}{t} &\lesssim \sum_{Q\in\tilde{\Delta}_T^{\mathbf{B}}}\int_0^{l(Q)}\|e^{-i\varphi_Q}\mathbb{1}_{4Q}\tilde{Q}_t^{B,Q}e^{i\varphi_Q}\eta_Q u\|_2^2\frac{\mathrm{d}t}{t} \\
&\qquad + \sum_{Q\in\tilde{\Delta}_T^{\mathbf{B}}}\int_0^{l(Q)}\frac{t^2}{l(Q)^2}\|\mathbb{1}_{4Q}u\|_2^2\frac{\mathrm{d}t}{t} \\
&\lesssim \|u\|_2^2 + \sum_{Q\in\tilde{\Delta}_T^{\mathbf{B}}}\frac{\|\mathbb{1}_{4Q}u\|_2^2}{l(Q)^2}\int_0^{l(Q)} t\,\mathrm{d}t \\
&\lesssim \|u\|_2^2 + \sum_{Q\in\tilde{\Delta}_T^{\mathbf{B}}}\frac{\|\mathbb{1}_{4Q}u\|_2^2}{l(Q)^2}\frac{l(Q)^2}{2} \\
&\lesssim \|u\|_2^2
\end{aligned}
$$

Thus, combining the above four calculations gives

$$
\int_0^\infty \|Q_t^B u\|_2^2\frac{\mathrm{d}t}{t} = \lim_{T\to\infty}\int_0^T\|Q_t^B u\|_2^2\frac{\mathrm{d}t}{t} \lesssim \lim_{T\to\infty}\|u\|_2^2 = \|u\|_2^2.
$$

As required. $\square$

Thus to prove the quadratic estimate Theorem 4.0.2 it suffices to prove (4.2.4).

4.3 Reduction to Carleson measure estimate

Define the following averaging operator

$$A_t^Q u(x) := \fint_R u(y)\,\mathrm{d}y$$

where $R \in \Delta_t(4Q)$ is the unique cube such that $x \in R$. Also, for each $(t,x) \in (0, 4l(Q)) \times 4Q$ we define the following multiplication operator $\tilde{\gamma}_t^Q(x) \in \mathcal{L}(\mathbb{C}^n)$

$$\tilde{\gamma}_t^Q(x)w := \mathbb{1}_{4Q}(x)[\tilde{Q}_t^{B,Q}(\mathbb{1}_{4Q}w^*)](x)$$

for all $w \in \mathbb{C}^{n+1}$, where $w^*(x) := w$ for all $x \in \mathbb{R}^n$. For fixed $t > 0$ we also define the mapping $\tilde{\gamma}_t^Q : \mathbb{R}^n \to \mathcal{L}(\mathbb{C}^{n+1})$ by $\tilde{\gamma}_t^Q : x \mapsto \mathbb{1}_{4Q}(x)\tilde{\gamma}_t^Q(x)$.

Lemma 4.3.1. Let Q be a cube. Then we have the following:

1. The operator $\tilde{Q}_t^{B,Q}$ extends to a bounded operator from $L_Q^\infty(\mathbb{R}^n;\mathbb{C}^{n+1})$ into the space $L^2_{Q,\mathrm{loc}}(\mathbb{R}^n;\mathbb{C}^{n+1})$. In particular for $t > 0$, then $\tilde{\gamma}_t^Q \in L^2_{Q,\mathrm{loc}}(\mathbb{R}^n;\mathcal{L}(\mathbb{C}^{n+1}))$ with

$$\fint_R |\tilde{\gamma}_t^Q(x)|^2_{\mathcal{L}(\mathbb{C}^{n+1})}\,\mathrm{d}x \lesssim 1,$$

   for all $R \in \Delta_t(4Q)$.

2. $\sup_{t\in(0,4l(Q))} \|\tilde{\gamma}_t^Q A_t^Q\| \lesssim 1.$

Proof. Let $f \in L_Q^\infty(\mathbb{R}^n;\mathbb{C}^{n+1})$. Let $M > n$. Define $\tilde{C}_0^Q(R) := R$ and for $k > 0$ define $\tilde{C}_k^Q(R) := (2^kR \setminus 2^{k-1}R) \cap 4Q$. Therefore, $\tilde{C}_k^Q(R) \cap \tilde{C}_j^Q(R) = \emptyset$ if $k \neq j$ and there exists $\tilde{K}_Q$ such that $\bigcup_{k=0}^{\tilde{K}_Q} \tilde{C}_k^Q(R) = 4Q$. Also, as $\mathrm{dist}(R,(2^kR \setminus 2^{k-1}R)) = 2^kl(R)$ then $\mathrm{dist}(R,\tilde{C}_k^Q(R)) \geq 2^kl(R)$. Then, by the off-diagonal estimates in Proposition 4.1.9,

$\operatorname{dist}(R, \tilde{C}_k^Q(R)) \geq 2^k l(R) \eqsim 2^k t$, the Cauchy–Schwarz inequality, and $M > n$, gives

$$
\begin{aligned}
\|\mathbb{1}_R \tilde{Q}_t^{B,Q} f\|_2^2 &\leq \left(\sum_{j=0}^{\tilde{K}_Q} \|\mathbb{1}_R \tilde{Q}_t^{B,Q}(\mathbb{1}_{\tilde{C}_j^Q(R)} f)\|_2 \right)^2 \\
&\leq \left(\sum_{j=0}^{\tilde{K}_Q} \|\mathbb{1}_R \tilde{Q}_t^{B,Q} \mathbb{1}_{\tilde{C}_j^Q(R)}\| \|\mathbb{1}_{\tilde{C}_j^Q(R)} f\|_2 \right)^2 \\
&\lesssim \left(\sum_{j=0}^{\tilde{K}_Q} \left\langle \frac{\operatorname{dist}(R, \tilde{C}_k^Q(R))}{t} \right\rangle^{-M} \|\mathbb{1}_{\tilde{C}_j^Q(R)} f\|_2 \right)^2 \\
&\lesssim \left(\sum_{j=0}^{\infty} 2^{-jM} \right) \left(\sum_{j=0}^{\tilde{K}_Q} 2^{-jM} \|\mathbb{1}_{\tilde{C}_j^Q(R)} f\|_2^2 \right) \\
&\lesssim \sum_{j=0}^{\infty} 2^{-jM} \|f\|_\infty^2 \|\mathbb{1}_{2^j R}\|_2^2 \\
&\lesssim \|f\|_\infty^2 |R| \sum_{j=0}^{\infty} 2^{-j(M-n)} \\
&\lesssim \|f\|_\infty^2 |R|.
\end{aligned}
$$

Then a similar argument to the one in Lemma 3.2.1 gives

$$
|[\tilde{\gamma}_t(x)](w)|^2 = \left| [\tilde{\gamma}_t(x)] \left(\sum_{k=1}^{n+1} w_k e_k \right) \right|^2 \lesssim |w|^2 \sum_{k=1}^{n+1} |[\tilde{\gamma}_t(x)](e_k)|^2,
$$

for all $w \in \mathbb{C}^{n+1}$, where $\{e_1, \dots, e_{n+1}\}$ is an orthonormal basis for $\mathbb{C}^{n+1}$. Then combining the above gives

$$
\fint_R |\tilde{\gamma}_t^Q(x)|^2_{\mathcal{L}(\mathbb{C}^{n+1})} \, dx \lesssim \sum_{k=1}^{n+1} \fint_R |[\tilde{\gamma}_t(x)](e_k)|^2 \, dx = \sum_{k=1}^{n+1} \fint_R |[\tilde{Q}_t^{B,Q}(\tilde{e}_k)](x)|^2 \, dx \lesssim \sum_{k=1}^{n+1} \|\tilde{e}_k\|_\infty \lesssim 1,
$$

where $\tilde{e}_k(x) = e_k$ is the constant function. Thus, $\tilde{\gamma}_t^Q \in L^2_{Q,\mathrm{loc}}(\mathbb{R}^n; \mathcal{L}(\mathbb{C}^{n+1}))$, which complete the proof od part (1)

For part (2) we use part (1), the definition of A_t^Q, and Jensen's inequality, to obtain

$$\begin{aligned}
\|\tilde{\gamma}_t^Q A_t^Q u\|_2^2 &= \sum_{R\in\Delta_t(4Q)} \int_R |[\tilde{\gamma}_t^Q(x)](A_t^Q u)(x)|^2 \,\mathrm{d}x \\
&\lesssim \sum_{R\in\Delta_t(4Q)} \int_R |\tilde{\gamma}_t^Q(x)|^2_{\mathcal{L}(\mathbb{C}^{n+1})} \left| \fint_R u(y)\,\mathrm{d}y \right|^2 \mathrm{d}x \\
&= \sum_{R\in\Delta_t(4Q)} \left(\int_R |\tilde{\gamma}_t^Q(x)|^2_{\mathcal{L}(\mathbb{C}^{n+1})} \,\mathrm{d}x \right) \left| \fint_R u(y)\,\mathrm{d}y \right|^2 \\
&\lesssim \sum_{R\in\Delta_t(4Q)} |R| \fint_R |u|^2 \\
&= \|\mathbb{1}_{4Q} u\|_2^2.
\end{aligned}$$

Now taking supremum in $t \in (0, 4l(Q))$ completes the proof. $\square$

Fix $Q \in \tilde{\Delta}_T^{\mathbf{B}}$ and $t > 0$. Then, we have

$$\tilde{Q}_t^{B,Q} e^{i\varphi_Q} \eta_Q u = (\tilde{Q}_t^{B,Q} - \tilde{\gamma}_t^Q A_t) e^{i\varphi_Q} \eta_Q u + \tilde{\gamma}_t^Q A_t e^{i\varphi_Q} \eta_Q u, \tag{4.3.1}$$

on $4Q$. In this section we will bound the first term on the right-hand side above, giving the following proposition

Proposition 4.3.2. If $\mathbf{B}$ satisfies (2.4.3), then we have the following estimate

$$\sum_{Q\in\tilde{\Delta}_T^{\mathbf{B}}} \int_0^{l(Q)} \|\mathbb{1}_{4Q}(\tilde{Q}_t^{B,Q} - \tilde{\gamma}_t^Q A_t) e^{i\varphi_Q} \eta_Q u\|_2^2 \frac{\mathrm{d}t}{t} \lesssim \|u\|_2^2, \quad \forall u \in \overline{\mathsf{R}(D)},$$

where the implicit constants depend on $\mathbf{B}$, n, κ, and $\|B\|_\infty$.

To this end we consider the following terms to bound

$$\begin{aligned}
(\tilde{Q}_t^{B,Q} - \tilde{\gamma}_t^Q A_t) e^{i\varphi_Q} \eta_Q u = {} & \tilde{Q}_t^{B,Q} e^{i\varphi_Q} \eta_Q (I - P_t) u \\
& + (\tilde{Q}_t^{B,Q} - \tilde{\gamma}_t^Q A_t) e^{i\varphi_Q} \eta_Q P_t u \\
& + \tilde{\gamma}_t^Q A_t e^{i\varphi_Q} \eta_Q (P_t - I) u.
\end{aligned} \tag{4.3.2}$$

We begin by estimating the first term in (4.3.2), we give a lemma we will use here and later in the Carleson measure estimate.

Lemma 4.3.3. Let $Q \in \tilde{\Delta}_T^{\mathbf{B}}$ and R a cube with $R \subseteq Q$. Define $\eta_R \in \mathcal{C}_0^\infty(4R)$ such that $\eta_R(x) \in [0,1]$ for all $x \in 4R$, and $\eta_R \equiv 1$ on $2R$. Then

$$\|\mathbb{1}_{4R}\tilde{Q}_t^{B,Q}e^{i\varphi_Q}\eta_R(I-P_t)u\|_2^2 \lesssim \left(\frac{t^2}{l(R)^2}+1\right)\|\mathbb{1}_{4R}Q_tu\|_2^2,$$

for all $t>0$ and all $u \in \overline{\mathsf{R}(D)}$.

Proof. Firstly, using the splitting in Proposition 2.5.1, by the ellipicity of B, as in (4.1.6), we have the existence of B^{-1} on $\overline{\mathsf{R}(\tilde{D}^Q)}$, the uniform bounds for $t\tilde{Q}_t^{B,Q}\tilde{D}^QB = I - \tilde{P}_t^{B,Q}$, and again by (4.1.6), we have

$$\begin{aligned}
\|t\tilde{Q}_t^{B,Q}\tilde{D}^Qu\|_2^2 &= \|t\tilde{Q}_t^{B,Q}\tilde{D}^Q\left(\mathbb{P}_{\overline{\mathsf{R}(\tilde{D}^Q)}} + \mathbb{P}_{N(\tilde{D}^Q)}\right)u\|_2^2 \\
&= \|t\tilde{Q}_t^{B,Q}\tilde{D}^QBB^{-1}\mathbb{P}_{\overline{\mathsf{R}(\tilde{D}^Q)}}u\|_2^2 \\
&\lesssim \|B^{-1}\mathbb{P}_{\overline{\mathsf{R}(\tilde{D}^Q)}}u\|_2^2 \\
&\lesssim \|u\|_2^2,
\end{aligned} \tag{4.3.3}$$

for all $t>0$. Now, using the identity $I-P_t = t^2D^2P_t$, then the uniform bounds on $\tilde{Q}_t^{B,Q}$, (4.3.3), the gauge invariance (4.1.5), the commutator bounds (4.1.1), and that $\operatorname{supp}([\eta_R, D]) \subset 4R$ to obtain

$$\begin{aligned}
\|\mathbb{1}_{4R}\tilde{Q}_t^{B,Q}e^{i\varphi_Q}\eta_R(I-P_t)u\|_2^2 &= \|\mathbb{1}_{4R}\tilde{Q}_t^{B,Q}e^{i\varphi_Q}\eta_Rt^2D^2P_tu\|_2^2 \\
&= \|t\mathbb{1}_{4R}\tilde{Q}_t^{B,Q}e^{i\varphi_Q}(\eta_RD - D\eta_R + D\eta_R)tDP_tu\|_2^2 \\
&\lesssim t^2\|\mathbb{1}_{4R}\tilde{Q}_t^{B,Q}e^{i\varphi_Q}[\eta_R, D]Q_tu\|_2^2 \\
&\qquad + \|\mathbb{1}_{4R}(t\tilde{Q}_t^{B,Q}\tilde{D}^Q)e^{i\varphi_Q}\eta_RQ_tu\|_2^2 \\
&\lesssim t^2\|\nabla\eta_R\|_\infty\|\mathbb{1}_{4R}Q_tu\|_2^2 + \|\eta_RQ_tu\|_2^2 \\
&\lesssim \left(\frac{t^2}{l(R)^2}+1\right)\|\mathbb{1}_{4R}Q_tu\|_2^2.
\end{aligned}$$

As required. □

We now bound the first term in (4.3.2).

Lemma 4.3.4. we have the estimate

$$\sum_{Q\in\tilde{\Delta}_T^{\mathbf{B}}}\int_0^{l(Q)}\|\mathbb{1}_{4Q}\tilde{Q}_t^{B,Q}e^{i\varphi_Q}\eta_Q(I-P_t)u\|_2^2\frac{\mathrm{d}t}{t}\lesssim\|u\|_2^2$$

for all $u\in\overline{\mathsf{R}(D)}$ independently of T.

Proof. Using Lemma 4.3.3 (with $R=Q$), $t\le l(Q)$, Proposition 4.1.7, and the quadratic estimate for self-adjoint operators, we have

$$\begin{aligned}\sum_{Q\in\tilde{\Delta}_T^{\mathbf{B}}}\int_0^{l(Q)}\|\mathbb{1}_{4Q}\tilde{Q}_t^{B,Q}e^{i\varphi_Q}\eta_Q(I-P_t)u\|_2^2\frac{\mathrm{d}t}{t}&\lesssim\sum_{Q\in\tilde{\Delta}_T^{\mathbf{B}}}\int_0^{l(Q)}\left(\frac{t^2}{l(Q)^2}+1\right)\|\mathbb{1}_{4Q}Q_tu\|_2^2\frac{\mathrm{d}t}{t}\\&\lesssim\int_0^{\infty}\sum_{Q\in\tilde{\Delta}_T^{\mathbf{B}}}\|\mathbb{1}_{4Q}Q_tu\|_2^2\frac{\mathrm{d}t}{t}\\&\lesssim\int_0^{\infty}\|Q_tu\|_2^2\frac{\mathrm{d}t}{t}\\&\lesssim\|u\|_2^2.\end{aligned}$$

As required. □

We now begin to estimate the second term in (4.3.2), but first we prove an important lemma. This is where we see the need for the localisation so that we can use the gauge invariance.

Lemma 4.3.5. Let $Q\in\tilde{\Delta}_T^{\mathbf{B}}$ and R a dyadic cube with $R\subseteq Q$. Then

$$\|\mathbb{1}_{4R}(\tilde{Q}_t^{B,Q}-\tilde{\gamma}_t^QA_t)e^{i\varphi_Q}\eta_Rf\|_2^2\lesssim\|\mathbb{1}_{4R}L\eta_Rf\|_2^2+\|\mathbb{1}_{4R}|\mathbf{B}|^{\frac{1}{2}}\eta_Rf\|_2^2,$$

for all $t\in(0,l(Q))$ and all $f\in\overline{\mathsf{R}(D)}$, independently of T and Q.

Proof. Let $g=e^{i\varphi_Q}\eta_Rf$. For cubes R,S with $S\subset R$, define $\tilde{C}_k^R(S)$ as in Lemma 4.3.1. Then, as $\tilde{\gamma}_t^QA_t^Qu=\tilde{Q}_t^{B,Q}A_t^Qu$ for all $S\in\Delta_t(4R)$, the off diagonal estimates in Proposition

4.1.9, and the Cauchy–Shwarz inequality, we have

$$
\begin{aligned}
\|\mathbb{1}_{4R}(\tilde{Q}_t^{B,Q} - \tilde{\gamma}_t^Q A_t)g\|_2^2 &= \sum_{S\in\Delta_t(4R)} \|\mathbb{1}_S \tilde{Q}_t^{B,Q}(g-g_S)\|_2^2 \\
&\lesssim \sum_{S\in\Delta_t(4R)} \left(\sum_{k=0}^{\tilde{K}_Q} \|\mathbb{1}_S \tilde{Q}_t^{B,Q}\mathbb{1}_{\tilde{C}_k^R(S)}\| \|\mathbb{1}_{\tilde{C}_k^R(S)}(g-g_S)\|_2\right)^2 \\
&\lesssim \sum_{S\in\Delta_t(4R)} \sum_{k=0}^{\tilde{K}_Q} 2^{-kM} \|\mathbb{1}_{\tilde{C}_k^R(S)}(g-g_S)\|_2^2.
\end{aligned}
$$

We now give a telescoping argument, by using the Poincaré inequality in [30, Equation (7.45)] noting for the constants that $\frac{1}{2}\operatorname{diam}(2^k S) \le \operatorname{diam}(2^k S\cap 4R) \le \operatorname{diam}(2^k S)$, to get

$$
\begin{aligned}
\|\mathbb{1}_{\tilde{C}_k(S)}(g-g_S)\|_2^2 &\lesssim \int_{2^kS\cap 4R} |g - g_{2^kS\cap 4R}|^2 + \int_{2^kS\cap 4R} |g_{2^kS\cap 4R} - g_R|^2 \\
&\lesssim 2^{2k}t^2 \int_{2^kS\cap 4R} |\nabla g|^2 + |2^kS\cap 4R| \left|\sum_{j=1}^{k}(g_{2^jS\cap 4R} - g_{2^{j-1}S\cap 4R})\right|^2
\end{aligned}
$$

Fix $j\in\{1,\dots,k\}$. Then, again using the Poincaré inequality in [30, Equation (7.45)] noting for the constants that $\operatorname{diam}(2^kS\cap 4R) \approx \operatorname{diam}(2^kS)$, we have

$$
\begin{aligned}
\left|\sum_{j=1}^{k}(g_{2^jS\cap 4R} - g_{2^{j-1}S\cap 4R})\right|^2 &\le \left[\sum_{j=1}^{k} \fint_{2^{j-1}S\cap 4R} |g_{2^jS\cap 4R} - g|\right]^2 \\
&\lesssim \left[\sum_{j=1}^{k} \frac{|2^jS\cap 4R|}{|2^{j-1}S\cap 4R|}\fint_{2^jS\cap 4R} |g - g_{2^jS\cap 4R}|\right]^2 \\
&\lesssim \left[\sum_{j=1}^{k} \frac{|2^jS|}{|S\cap 4R|}\left(\fint_{2^jS\cap 4R} |g - g_{2^jS\cap 4R}|^2\right)^{\frac{1}{2}}\right]^2 \\
&\lesssim \left[\sum_{j=1}^{k} \frac{2^{jn}|S|2^jt}{|S|}\left(\fint_{2^jS\cap 4R} |\nabla g|^2\right)^{\frac{1}{2}}\right]^2 \\
&\lesssim \left[t^2\int_{2^kS\cap 4R} |\nabla g|^2\right]\left[\sum_{j=1}^{k} \frac{2^{j(n+1)}}{|2^jS\cap 4R|^{\frac{1}{2}}}\right]^2
\end{aligned}
$$

Then using $j \le k$ and the identity $\sum_{j=0}^{k} 2^j = 2^{k+1} - 1$, we have

$$\sum_{j=1}^{k} \frac{2^{j(n+1)}}{|2^j S \cap 4R|^{\frac{1}{2}}} \le 2^{kn} \sum_{j=1}^{k} \frac{2^j}{|S \cap 4R|^{\frac{1}{2}}} \le \frac{2^{kn} 2^{k+1}}{|S|^{\frac{1}{2}}} = \frac{2^{k(n+2)+1}}{|S|^{\frac{1}{2}}}$$

Thus, combining the above three calculations we have

$$\begin{aligned}
\|\mathbb{1}_{\tilde{C}_k^R(S)}(g - g_S)\|_2^2 &\lesssim t^2 \left[2^{2k} + |2^k S \cap 4R| \left(\frac{2^{k(n+2)+1}}{|S|^{\frac{1}{2}}} \right)^2 \right] \left[\int_{2^k S \cap 4R} |\nabla g|^2 \right] \\
&\lesssim t^2 2^{2kn} (1 + 2^{kn} 2^{2k+1})^2 \left[\int_{2^k S \cap 4R} |\nabla g|^2 \right] \\
&\lesssim 2^{k(4n+2)} t^2 \int_{2^k S \cap 4R} |\nabla g|^2.
\end{aligned}$$

Therefore, choosing $M > 5n + 2$ and using the covering inequality $\sum_{S \in \Delta_t(4Q)} \mathbb{1}_{2^k S}(x) \lesssim 2^{kn}$, we have

$$\begin{aligned}
\sum_{S \in \Delta_t(4R)} \sum_{k=0}^{\infty} 2^{-kM} \|\mathbb{1}_{2^k S \cap 4R}(g - g_S)\|_2^2 &\lesssim t^2 \sum_{S \in \Delta_t(4R)} \sum_{k=0}^{\infty} 2^{-k(M-(4n+2))} \|\mathbb{1}_{2^k S \cap 4R} \nabla g\|_2^2 \\
&\lesssim t^2 \sum_{k=0}^{\infty} 2^{-k(M-(5n+2))} \|\nabla g\|_{L^2(4R)}^2 \\
&\lesssim t^2 \|\nabla g\|_{L^2(4R)}^2
\end{aligned}$$

Recall $g = e^{i\varphi_Q} \eta_R f$. Then, using the product rule, chain rule, the identity $\nabla \varphi = b - h_Q$ on $4Q$, and the definition of L, we have

$$\begin{aligned}
\|\nabla g\|_{L^2(4R)}^2 &= \|(\nabla e^{i\varphi_Q})(\eta_R f) + e^{i\varphi_Q} \nabla(\eta_R f)\|_{L^2(4R)}^2 \\
&= \|(i\nabla \varphi_Q)(e^{i\varphi_Q} \eta_R f) + e^{i\varphi_Q} \nabla(\eta_R f)\|_{L^2(4R)}^2 \\
&= \|i(b - h_Q)(e^{i\varphi_Q} \eta_R f) + e^{i\varphi_Q} \nabla(\eta_R f)\|_{L^2(4R)}^2 \\
&= \|e^{i\varphi_Q} (\nabla + ib)(\eta_R f) - i e^{i\varphi_Q} h_Q (\eta_R f)\|_{L^2(4R)}^2 \\
&\lesssim \|e^{i\varphi_Q} L(\eta_R f)\|_{L^2(4R)}^2 + \|h_Q g\|_{L^2(4R)}^2
\end{aligned}$$

Now, to estimate the last term, we use Hölder's inequality, the Poincaré-Sobolev inequality (see (7.45) in [30]), and $l(R) \le l(4Q)$, to get

$$\begin{aligned}
\left(\fint_{4R} |h_Q g|^2\right)^{\frac{1}{2}} &\le \left(\fint_{4Q} |h_Q|^n\right)^{\frac{1}{n}} \left(\fint_{4R} |g|^{2^*}\right)^{\frac{1}{2^*}} \\
&\lesssim l(4Q) \left(\fint_{4Q} |\mathbf{B}|^{\frac{n}{2}}\right)^{\frac{2}{n}} \left[\left(\fint_{4R} \left||g| - \left(\fint_{4R} |g|\right)\right|^{2^*}\right)^{\frac{1}{2^*}} + \fint_{4R} |g|\right] \\
&\lesssim l(4Q) \left(\fint_{4Q} |\mathbf{B}|\right) \left[l(R) \left(\fint_{4R} |\nabla |g||^2\right)^{\frac{1}{2}} + \fint_{4R} |g|\right] \\
&\le l(4Q)^2 \left(\fint_{4Q} |\mathbf{B}|\right) \left(\fint_{4R} |\nabla |g||^2\right)^{\frac{1}{2}} + l(4Q) \left(\fint_{4Q} |\mathbf{B}|\right) \fint_{4R} |g|.
\end{aligned}$$

To estimate the first term we use the smallness of $4Q$ and the diamagnetic inequality with $\tilde{L}_Q$, to obtain

$$l(4Q)^2 \left(\fint_{4Q} |\mathbf{B}|\right) \left(\fint_{4R} |\nabla |g||^2\right)^{\frac{1}{2}} \le \left(\fint_{4R} |\nabla |g||^2\right)^{\frac{1}{2}} \lesssim \left(\fint_{4R} |\tilde{L}_Q g|^2\right)^{\frac{1}{2}}$$

Now, as $4Q$ is small, we have

$$l(4Q) \left(\fint_{4Q} |\mathbf{B}|\right) = \left(l(4Q)^2 \fint_{4Q} |\mathbf{B}|\right)^{\frac{1}{2}} \left(\fint_{4Q} |\mathbf{B}|\right)^{\frac{1}{2}} \le \left(\fint_{4Q} |\mathbf{B}|\right)^{\frac{1}{2}}$$

Then, by Jensen's inequality, the Fefferman–Phong inequality for $\tilde{L}_Q$, and then the support conditions on η_R, we have

$$\begin{aligned}
l(4Q) \left(\fint_{4Q} |\mathbf{B}|\right) \fint_{4R} |g| &\lesssim l(4Q) \left(\fint_{4Q} |\mathbf{B}|\right) \left(\fint_{4R} |g|^2\right)^{\frac{1}{2}} \\
&\lesssim \left[\left(\fint_{4Q} |\mathbf{B}|\right) \left(\frac{1}{|R|} \int_{4Q} |g|^2\right)\right]^{\frac{1}{2}} \\
&\lesssim \frac{1}{|R|^{\frac{1}{2}}} \left(\int_{4Q} |\tilde{L}_Q g|^2 + \int_{4Q} ||\mathbf{B}|^{\frac{1}{2}} g|^2\right)^{\frac{1}{2}} \\
&= \left(\fint_{4R} |\tilde{L}_Q g|^2 + \fint_{4R} ||\mathbf{B}|^{\frac{1}{2}} g|^2\right)^{\frac{1}{2}}
\end{aligned}$$

Thus, by the above calculations, squaring, multiplying by $|4r|$, and the gauge invariance, we get

$$\begin{aligned}\int_{4R} |h_Q g|^2 &\lesssim \int_{4R} |\tilde{L}_Q e^{i\varphi_Q}\eta_R f|^2 + |\mathbf{B}||e^{i\varphi_Q}\eta_R f|^2 \\ &= \int_{4R} |e^{i\varphi_Q} L\eta_R f|^2 + |\mathbf{B}||\eta_R f|^2 \\ &= \int_{4R} |L\eta_R f|^2 + |\mathbf{B}||\eta_R f|^2.\end{aligned}$$

Therefore, by collecting the above estimates together, we have

$$\begin{aligned}\|\mathbb{1}_{4R}(\tilde{Q}_t^{B,Q} - \tilde{\gamma}_t^Q A_t)g\|_2^2 &\lesssim t^2\|\nabla g\|_{L^2(4R)}^2 \\ &\lesssim t^2\|e^{i\varphi_Q}L(\eta_R f)\|_{L^2(4R)}^2 + t^2\|h_Q g\|_{L^2(4R)}^2 \\ &\lesssim t^2\|\mathbb{1}_{4R}L\eta_R f\|^2 + t^2\|\mathbb{1}_{4R}|\mathbf{B}|^{\frac{1}{2}}\eta_R f\|^2.\end{aligned}$$

This completes the proof. □

Now we give the estimate for the second term in (4.3.2) using the above lemma.

Proposition 4.3.6. We have the estimate

$$\sum_{Q\in\tilde{\Delta}_T^{\mathbf{B}}} \int_0^{l(Q)} \|\mathbb{1}_{4Q}(\tilde{Q}_t^{B,Q} - \tilde{\gamma}_t^Q A_t)e^{i\varphi_Q}\eta_Q P_t u\|_2^2 \frac{\mathrm{d}t}{t} \lesssim \|u\|_2^2$$

for all $u \in \overline{\mathsf{R}(D)}$, independently of T.

Proof. We now use Lemma 4.3.5 with $R = Q$ and $f = P_t u$, and the product rule, we have

$$\begin{aligned}\|\mathbb{1}_{4Q}(\tilde{Q}_t^{B} - \tilde{\gamma}_t^Q A_t)\eta_Q e^{i\varphi} P_t u\|_2^2 &\lesssim t^2\|\mathbb{1}_{4Q}e^{i\varphi}L\eta_Q P_t u\|_2^2 + t^2\|\mathbb{1}_{4Q}e^{i\varphi}|\mathbf{B}|^{\frac{1}{2}}\eta_Q P_t u\|_2^2 \\ &\lesssim t^2\|\mathbb{1}_{4Q}LP_t u\|_2^2 + t^2\|(\nabla\eta_Q)P_t u\|_2^2 + t^2\|\mathbb{1}_{4Q}|\mathbf{B}|^{\frac{1}{2}}P_t u\|_2^2 \\ &\lesssim t^2\left[\|\mathbb{1}_{4Q}LP_t u\|_2^2 + \|\mathbb{1}_{4Q}|\mathbf{B}|^{\frac{1}{2}}P_t u\|_2^2 + \frac{1}{l(Q)^2}\|\mathbb{1}_{4Q}P_t u\|_2^2\right].\end{aligned}$$

The last term above is bounded using off-diagonal estimates in Proposition 4.1.9, the Cauchy–Schwarz inequality, Proposition 4.1.7, and choosing $M > l$ where $l_{\mathbf{B}}$ is as in

(4.1.3), we have

$$
\begin{aligned}
\sum_{Q\in\tilde{\Delta}_T^{\mathbf{B}}}\int_0^{l(Q)}\frac{t^2}{l(Q)^2}\|\mathbb{1}_{4Q}P_tu\|_2^2\frac{\mathrm{d}t}{t} &\lesssim \sum_{Q\in\tilde{\Delta}_T^{\mathbf{B}}}\frac{1}{l(Q)^2}\int_0^{l(Q)} t\left[\sum_{k=0}^{\infty}\|\mathbb{1}_{4Q}P_t\mathbb{1}_{C_k(4Q)}u\|_2\right]^2\mathrm{d}t\\
&\lesssim \sum_{Q\in\tilde{\Delta}_T^{\mathbf{B}}}\frac{1}{l(Q)^2}\left[\sum_{k=0}^{\infty}\|\mathbb{1}_{2^{k+2}Q}u\|_2^2\int_0^{l(Q)}\left(\frac{t}{2^kl(Q)}\right)^M t\,\mathrm{d}t\right]\\
&= \sum_{k=0}^{\infty}\sum_{Q\in\tilde{\Delta}_T^{\mathbf{B}}}2^{-kM}\frac{\|\mathbb{1}_{2^{k+2}Q}u\|_2^2}{l(Q)^{M+2}}\int_0^{l(Q)}t^{M+1}\,\mathrm{d}t\\
&= \sum_{k=0}^{\infty}\sum_{Q\in\tilde{\Delta}_T^{\mathbf{B}}}2^{-kM}\frac{\|\mathbb{1}_{2^{k+2}Q}u\|_2^2}{l(Q)^{M+2}}\frac{l(Q)^{M+2}}{M+2}\\
&\lesssim \|u\|_2^2\sum_{k=0}^{\infty}2^{-k(M-l)}\\
&\lesssim \|u\|_2^2,
\end{aligned}
\tag{4.3.4}
$$

Then, using Proposition 4.1.7, Corollary 4.1.6, and using the quadratic estimate for the self-adjoint operator D, we have

$$
\begin{aligned}
&\sum_{Q\in\tilde{Q}_T^B}\int_0^{l(Q)} t^2\left[\|\mathbb{1}_{4Q}LP_tu\|_2^2+\|\mathbb{1}_{4Q}|\mathbf{B}|^{\frac{1}{2}}P_tu\|_2^2\right]\frac{\mathrm{d}t}{t}\\
&\quad= \sum_{Q\in\tilde{Q}_T^B}\int_0^{\infty}\mathbb{1}_{(0,l(Q))}(t)t^2\left[\|\mathbb{1}_{4Q}LP_tu\|_2^2+\|\mathbb{1}_{4Q}|\mathbf{B}|^{\frac{1}{2}}P_tu\|_2^2\right]\frac{\mathrm{d}t}{t}\\
&\quad\le \int_0^{\infty}\sum_{Q\in\tilde{Q}_T^B}t^2\left[\|\mathbb{1}_{4Q}LP_tu\|_2^2+\|\mathbb{1}_{4Q}|\mathbf{B}|^{\frac{1}{2}}P_tu\|_2^2\right]\frac{\mathrm{d}t}{t}\\
&\quad\lesssim \int_0^{\infty}t^2\left[\|LP_tu\|_2^2+\||\mathbf{B}|^{\frac{1}{2}}P_tu\|_2^2\right]\frac{\mathrm{d}t}{t}\\
&\quad\lesssim \int_0^{\infty}t^2\|DP_tu\|_2^2\frac{\mathrm{d}t}{t}\\
&\quad= \int_0^{\infty}\|Q_tu\|_2^2\frac{\mathrm{d}t}{t}\\
&\quad\lesssim \|u\|_2^2.
\end{aligned}
$$

By the above calculation and (4.3.4), we have

$$\sum_{Q\in\tilde{\Delta}_T^{\mathbf{B}}}\int_0^{l(Q)}\|\mathbb{1}_{4Q}(\tilde{Q}_t^{B,Q}-\tilde{\gamma}_t^Q A_t)\eta_Q e^{i\varphi}P_t u\|_2^2\lesssim\|u\|_2^2$$

As required. □

Now we estimate the last term in (4.3.2). Here it is important that we are able to use Lemma 4.1.10 and this is why we needed to work on the maximal dyadic mesh $\tilde{\Delta}_T^{\mathbf{B}}$ instead of arbitrary dyadic cubes.

Proposition 4.3.7. We have the estimate

$$\sum_{Q\in\tilde{\Delta}_T^{\mathbf{B}}}\int_0^{l(Q)}\|\tilde{\gamma}_t^Q A_t^Q e^{i\varphi_Q}\eta_Q(P_t-I)u\|_2\frac{\mathrm{d}t}{t}\lesssim\|u\|_2^2$$

for all $u\in\overline{\mathsf{R}(D)}$, independently of T.

Proof. We first establish

$$\sum_{Q\in\tilde{\Delta}_T^{\mathbf{B}}}\|\mathbb{1}_{4Q}A_t^Q e^{i\varphi_Q}\eta_Q(P_t-I)Q_s u\|_2^2\lesssim\min\left\{\frac{t}{s},\frac{s}{t}\right\}\|u\|_2^2,\tag{4.3.5}$$

for all $u\in L^2(\mathbb{R}^n;\mathbb{C}^{n+1})$. For $t\le s$ we use the uniform boundedness of A_t^Q and η_Q, the fact that $(P_t-I)Q_s=\frac{t}{s}Q_t(P_s-I)$ and the the uniform boundedness of P_s and Q_t to get

$$\begin{aligned}\sum_{Q\in\tilde{\Delta}_T^{\mathbf{B}}}\|\mathbb{1}_{4Q}A_t^Q e^{i\varphi_Q}\eta_Q(P_t-I)Q_s u\|_2^2&\lesssim\sum_{Q\in\tilde{\Delta}_T^{\mathbf{B}}}\left(\frac{t}{s}\right)^2\|\mathbb{1}_{4Q}Q_t(P_s-I)u\|_2^2\\&\lesssim\frac{t}{s}\|Q_t(P_s-I)u\|_2^2\\&\le\frac{t}{s}\|u\|_2^2.\end{aligned}$$

Fix $Q\in\tilde{\Delta}_T^{\mathbf{B}}$. Now, let $s\le t$. Note that when $t>4l(Q)$ we have $\Delta_t(4Q)=\emptyset$ and so we may assume $t\le 4l(Q)$. Using the uniform boundedness of A_t, the fact that $P_tQ_s=\frac{s}{t}Q_tP_s$, Propostion 4.1.7, and the uniform boundedness of P_s and Q_t, we get

$$\begin{aligned}
&\sum_{Q\in\tilde{\Delta}_T^{\mathbf{B}}} \|\mathbb{1}_{4Q} A_t^Q e^{i\varphi_Q}\eta_Q (P_t - I) Q_s u\|_2^2 \\
&\qquad \lesssim \sum_{Q\in\tilde{\Delta}_T^{\mathbf{B}}} \|\eta_Q P_t Q_s u\|_2^2 + \sum_{Q\in\tilde{\Delta}_T^{\mathbf{B}}} \sum_{R\in\Delta_t(4Q)} \|\mathbb{1}_R A_t e^{i\varphi_Q}\eta_Q Q_s u\|_2^2 \\
&\qquad \lesssim \left(\frac{s}{t}\right)^2 \|Q_t P_s u\|_2^2 + \sum_{Q\in\tilde{\Delta}_T^{\mathbf{B}}} \sum_{R\in\Delta_t(4Q)} \int_R \left| \fint_R e^{i\varphi_Q}\eta_Q Q_s u\right|^2 \\
&\qquad \lesssim \frac{s}{t}\|u\|_2^2 + \sum_{Q\in\tilde{\Delta}_T^{\mathbf{B}}} \sum_{R\in\Delta_t(4Q)} s^2|R| \left| \fint_R e^{i\varphi_Q}(\eta_Q D - D\eta_Q + D\eta_Q) P_s u\right|^2
\end{aligned}$$

For the second term above we use the gauge invariance in (4.1.5), the commutator bounds in (4.1.1), and $t \leq 4l(Q)$, to get

$$\begin{aligned}
&\sum_{R\in\Delta_t(4Q)} s^2|R| \left| \fint_R e^{i\varphi_Q}(\eta_Q D - D\eta_Q + D\eta_Q) P_s u\right|^2 \\
&\qquad \lesssim \sum_{R\in\Delta_t(4Q)} \left[s^2|R| \left| \fint_R \tilde{D}^Q e^{i\varphi_Q}\eta_Q P_s u\right|^2 + s^2|R| \left(\fint_R |[\eta_Q, D] P_s u|^2 \right) \right] \\
&\qquad \lesssim \sum_{R\in\Delta_t(4Q)} \left[s^2|R| \left| \fint_R \tilde{D}^Q e^{i\varphi_Q}\eta_Q P_s u\right|^2 + \frac{s^2}{t^2}\|\mathbb{1}_{4Q} P_s u\|^2 \right]
\end{aligned}$$

Now by Lemma 4.1.10, the gauge invariance, adding and subtracting $e^{i\varphi_Q}\eta_Q D P_s u$, the commutator bounds, $t \leq 4l(Q)$, and the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
&\sum_{R \in \Delta_t(4Q)} s^2 |R| \left| \fint_R \tilde{D}^Q e^{i\varphi_Q} \eta_Q P_s u \right|^2 \\
&\quad \lesssim \sum_{R \in \Delta_t(4Q)} \frac{s^2 |R|}{l(R)} \left(\fint_R |\tilde{D}^Q e^{i\varphi_Q} \eta_Q P_s u|^2 \right)^{\frac{1}{2}} \left(\fint_R |\eta_Q P_s u|^2 \right)^{\frac{1}{2}} \\
&\quad \lesssim \frac{s^2}{t} \sum_{R \in \Delta_t(4Q)} \left[\left(\int_R |\eta_Q D P_s u|^2 \right)^{\frac{1}{2}} + \left(\int_R |[\eta_Q, D] P_s u|^2 \right)^{\frac{1}{2}} \right] \left(\int_R |\eta_Q P_s u|^2 \right)^{\frac{1}{2}} \\
&\quad \lesssim \frac{s}{t} \sum_{R \in \Delta_t(4Q)} \left[\left(\int_R |\eta_Q Q_s u|^2 \right)^{\frac{1}{2}} + \frac{s}{l(Q)} \left(\int_R |\mathbb{1}_{4Q} P_s u|^2 \right)^{\frac{1}{2}} \right] \left(\int_R |\eta_Q P_s u|^2 \right)^{\frac{1}{2}} \\
&\quad \lesssim \frac{s}{t} \sum_{R \in \Delta_t(4Q)} \left[\left(\int_R |\eta_Q Q_s u|^2 \right)^{\frac{1}{2}} \left(\int_R |\eta_Q P_s u|^2 \right)^{\frac{1}{2}} + \frac{s}{l(Q)} \left(\int_R |\mathbb{1}_{4Q} P_s u|^2 \right) \right] \\
&\quad \lesssim \frac{s}{t} \left(\sum_{R \in \Delta_t(4Q)} \int_R |\eta_Q Q_s u|^2 \right)^{\frac{1}{2}} \left(\sum_{R \in \Delta_t(4Q)} \int_R |\eta_Q P_s u|^2 \right)^{\frac{1}{2}} + \frac{s^2}{t^2} \sum_{R \in \Delta_t(4Q)} \int_R |\mathbb{1}_{4Q} P_s u|^2 \\
&\quad \lesssim \frac{s}{t} \left(\int_{4Q} |\eta_Q Q_s u|^2 \right)^{\frac{1}{2}} \left(\int_{4Q} |\eta_Q P_s u|^2 \right)^{\frac{1}{2}} + \frac{s^2}{t^2} \int_{4Q} |P_s u|^2
\end{aligned}$$

We collect the previous estimates and sum over all $Q \in \tilde{\Delta}_T^{\mathbf{B}}$. We use Cauchy–Schwarz, Proposition 4.1.7, $s \leq t$, and the uniform boundedness of Q_t and P_t we have

$$\begin{aligned}
&\sum_{Q \in \tilde{\Delta}_T^{\mathbf{B}}} \|\mathbb{1}_{4Q} A_t^Q e^{i\varphi_Q} \eta_Q (P_t - I) Q_s u\|_2^2 \\
&\qquad \lesssim \frac{s}{t} \|u\|_2^2 + \sum_{Q \in \tilde{\Delta}_T^{\mathbf{B}}} \left[\frac{s}{t} \left(\int_{4Q} |\eta_Q Q_s u|^2 \right)^{\frac{1}{2}} \left(\int_{4Q} |\eta_Q P_s u|^2 \right)^{\frac{1}{2}} + \left(\frac{s}{t} \right)^2 \int_{4Q} |P_s u|^2 \right] \\
&\qquad \lesssim \frac{s}{t} \|u\|_2^2 + \frac{s}{t} \left(\sum_{Q \in \tilde{\Delta}_T^{\mathbf{B}}} \int_{4Q} |\eta_Q Q_s u|^2 \right)^{\frac{1}{2}} \left(\sum_{Q \in \tilde{\Delta}_T^{\mathbf{B}}} \int_{4Q} |\eta_Q P_s u|^2 \right)^{\frac{1}{2}} + \frac{s}{t} \|P_s u\|_2^2 \\
&\qquad \lesssim \frac{s}{t} \|u\|_2^2 + \frac{s}{t} \|Q_s u\|_2 \|P_s u\|_2 + \frac{s}{t} \|P_s u\|_2^2 \\
&\qquad \lesssim \frac{s}{t} \|u\|_2^2.
\end{aligned}$$

Thus, we have established (4.3.5). Now we begin the Schur-type estimate by letting $m(s,t) \coloneqq \min\left\{\frac{t}{s}, \frac{s}{t}\right\}^{\frac{1}{2}}$. Then, we use the uniform boundedness of $\tilde{\gamma}_t^Q A_t^Q$ in Lemma 4.3.1, the Calderón reproducing formula, Minkowski's inequality, (4.3.5), the Cauchy–Schwarz inequality, and Tonneli's theorem

$$
\begin{aligned}
&\sum_{Q\in\tilde{\Delta}_t}\int_0^{l(Q)} \|\mathbb{1}_{4Q}\tilde{\gamma}_t^Q A_t^Q e^{i\varphi_Q}\eta_Q(P_t-I)u\|_2^2\frac{\mathrm{d}t}{t}\\
&\quad=\sum_{Q\in\tilde{\Delta}_T^{\mathbf{B}}}\int_0^{l(Q)} \|\mathbb{1}_{4Q}\tilde{\gamma}_t^Q A_t^Q\mathbb{1}_{4Q}A_t^Q e^{i\varphi_Q}\eta_Q(P_t-I)u\|_2\frac{\mathrm{d}t}{t}\\
&\quad\lesssim\sum_{Q\in\tilde{\Delta}_T^{\mathbf{B}}}\int_0^{l(Q)} \|\tilde{\gamma}_t^Q A_t^Q\|\left\|\mathbb{1}_{4Q}A_t^Q e^{i\varphi_Q}\eta_Q(P_t-I)\left(\int_0^\infty Q_s^2u\frac{\mathrm{d}s}{s}\right)\right\|_2^2\frac{\mathrm{d}t}{t}\\
&\quad\lesssim\int_0^T\sum_{Q\in\tilde{\Delta}_T^{\mathbf{B}}}\left[\int_0^\infty\|\mathbb{1}_{4Q}A_t^Q e^{i\varphi_Q}\eta_Q(P_t-I)Q_s(Q_su)\|_2\frac{\mathrm{d}s}{s}\right]^2\frac{\mathrm{d}t}{t}\\
&\quad\leq\int_0^\infty\left[\int_0^\infty\left[\sum_{Q\in\tilde{\Delta}_T^{\mathbf{B}}}\|\mathbb{1}_{4Q}A_t^Q e^{i\varphi_Q}\eta_Q(P_t-I)Q_s(Q_su)\|_2^2\right]^{\frac{1}{2}}\frac{\mathrm{d}s}{s}\right]^2\frac{\mathrm{d}t}{t}\\
&\quad\lesssim\int_0^\infty\int_0^\infty\left[m(s,t)\|Q_su\|_2\frac{\mathrm{d}s}{s}\right]^2\frac{\mathrm{d}t}{t}\\
&\quad\leq\int_0^\infty\left(\int_0^\infty m(s,t)\frac{\mathrm{d}s}{s}\right)\left(\int_0^\infty m(s,t)\|Q_su\|_2^2\frac{\mathrm{d}s}{s}\right)\frac{\mathrm{d}t}{t}\\
&\quad\lesssim\int_0^\infty\|Q_su\|_2^2\frac{\mathrm{d}s}{s}\\
&\quad\lesssim\|u\|_2^2
\end{aligned}
$$

This completes the proof. □

Combining Lemma 4.3.4, and Propositions 4.3.6 and 4.3.7 we have

$$
\sum_{Q\in\tilde{\Delta}_T^{\mathbf{B}}}\int_0^{l(Q)}\|\mathbb{1}_{4Q}(\tilde{Q}_t^{B,Q}-\tilde{\gamma}_t^Q A_t)e^{i\varphi_Q}\eta_Q u\|_2^2\lesssim\|u\|_2^2,
$$

for all $u\in\overline{\mathsf{R}(D)}$ and we are left to estimate the second term of (4.3.1).

4.4 Carleson Measure Estimate

We now begin to estimate the second term in (4.3.1), that is we need to prove the following proposition.

Proposition 4.4.1. If $\mathbf{B}$ satisfies (2.4.3), then we have the following estimate

$$\sum_{Q\in\tilde{\Delta}_T^{\mathbf{B}}}\int_0^{l(Q)}\|\mathbb{1}_{4Q}\tilde{\gamma}_t^Q A_t u\|_2^2\frac{\mathrm{d}t}{t}\lesssim\|u\|_2^2,\quad\forall u\in\overline{\mathsf{R}(D)},$$

where the implicit constants depend on $\mathbf{B}$, n, κ, and $\|B\|_\infty$.

Firstly, we fix $Q\in\tilde{\Delta}_T^{\mathbf{B}}$. We will do this by proving a localised Carleson measure estimate for $|\tilde{\gamma}_t^Q(x)|^2\frac{\mathrm{d}x\,\mathrm{d}t}{t}$.

Definition 4.4.2. Let μ be a measure on $4Q\times(0,l(Q)]$. Then we say μ is Carleson on $4Q\times(0,l(Q)]$ if

$$\|\mu\|_{\mathcal{C}}\coloneqq\sup_{R\in\Delta(4Q)}\frac{1}{|R|}\mu(\mathcal{C}(R))<\infty,$$

and $\|\mu\|_{\mathcal{C}}$ is independent of Q, here $\mathcal{C}(R)\coloneqq R\times(0,l(R)]$, the Carleson box of the cube R.

The following propositions proves that if $|\tilde{\gamma}_t^Q(x)|^2\frac{\mathrm{d}x\,\mathrm{d}t}{t}$ is a Carleson measure estimate then the second term in (4.3.1) is bounded and thus the quadratic estimate is established.

Lemma 4.4.3. If μ is Carleson on $4Q\times(0,l(Q)]$, then

$$\int_0^{l(Q)}\int_{4Q}|A_t e^{i\varphi_Q}\eta_Q u|\,\mathrm{d}\mu(x,t)\lesssim\|\mu\|_{\mathcal{C}}\|\eta_Q u\|_2^2$$

for all $u\in L^2(\mathbb{R}^n;\mathbb{C}^{n+1})$ where the implicit constant is independent of Q.

Proof. First, using the fact that $\Delta_t(4Q)=\Delta_{2^k}(4Q)$ for $k\in\mathbb{Z}\cap(-\infty,K_Q]$ where $2^{K_Q-1}<l(Q)\le 2^{K_Q}$, and $2^{k-1}<t\le 2^k$ and Tonelli's Theorem, we have

$$\begin{aligned}\int_0^{l(Q)}\int_{4Q}|A_t e^{i\varphi_Q}\eta_Q u(x)|^2\,\mathrm{d}\mu(x,t)&=\sum_{k=-\infty}^{K_Q}\int_{2^{k-1}}^{2^k}\sum_{R\in\Delta_{2^k}(4Q)}\int_R\left|\fint_R e^{i\varphi_Q}\eta_Q u(y)\,\mathrm{d}y\right|^2\mathrm{d}\mu(x,t)\\&\le\sum_{k=-\infty}^{K_Q}\sum_{R\in\Delta_{2^k}(4Q)}\left(\fint_R|\eta_Q u(y)|\,\mathrm{d}y\right)^2\int_{2^{k-1}}^{2^k}\int_R\mathrm{d}\mu(x,t).\end{aligned}$$

Now let $I_k\subseteq\mathbb{N}$ be an indexing set such that $\Delta_{2^k}(4Q)=\{R_\alpha^k:\alpha\in I_k\}$. We also introduce the notation

$$u_{\alpha,k} := \fint_{R^k_\alpha} |\eta_Q u(y)| \,\mathrm{d}y, \quad \text{and} \quad \mu_{\alpha,k} := \mu(R^k_\alpha \times (2^{k-1}, 2^k]).$$

Therefore, rearranging and using Tonelli's Theorem, we have that

$$\begin{aligned}
\int_0^{l(Q)} \int_{4Q} |A_t e^{i\varphi_Q} \eta_Q u(x)|^2 \,\mathrm{d}\mu(x,t) &\leq \sum_{k=-\infty}^{K_Q} \sum_{\alpha \in I_k} u^2_{\alpha,k} \mu_{\alpha,k} \\
&= \sum_{k=-\infty}^{K_Q} \sum_{\alpha \in I_k} \mu_{\alpha,k} \int_0^{u_{\alpha,k}} 2r \,\mathrm{d}r \\
&= \int_0^\infty \sum_{k=-\infty}^{K_Q} \sum_{\alpha \in I_k} \mu_{\alpha,k} \mathbb{1}_{\{u_{\alpha,k} > r\}}(r) 2r \,\mathrm{d}r,
\end{aligned}$$

where dr denotes the Lebesgue measure on $(0, \infty)$. For each $r > 0$ let $\{R_j(r)\}_{j \in \mathbb{N}}$ be an enumeration of the collection of maximal dyadic cubes $R^k_\alpha \in \Delta(4Q)$ such that $u_{\alpha,k} > r$. Define

$$\mathcal{M}^Q_\Delta u(x) := \sup \left\{ \fint_R u : R \in \Delta(4Q), \text{ with } x \in R \right\}.$$

We claim that

$$\bigcup_{j=1}^\infty R_j(r) = \{x \in \mathbb{R}^n : (\mathcal{M}^Q_\Delta |\eta_Q u|)(x) > r\}.$$

Let $x \in \bigcup_{j=1}^\infty R_j(r)$. Therefore $x \in R$ such that $R = R_j(r)$ for some $j \in \mathbb{N}$, then

$$r < \fint_R |\eta_Q u| \leq (\mathcal{M}^Q_\Delta |\eta_Q u|)(x).$$

Now if $x \in \mathbb{R}^n$ such that $(\mathcal{M}^Q_\Delta |\eta_Q u|)(x) > r$, then there exists $R' \in \Delta(4Q)$ with $x \in R'$ such that

$$r < \fint_{R'} |\eta_Q u|.$$

Then either $R' = R_j(r)$ for some $j \in \mathbb{N}$ or, as the cubes in $\{R_j(r)\}_{j \in \mathbb{N}}$ are maximal, there exists a cube $R = R_j(r)$ for some $j \in \mathbb{N}$ with $R' \subseteq R$. Therefore, $x \in \bigcup_{j=1}^\infty R_j(r)$. This proves the claim.

Now suppose that $R^k_\alpha \in \Delta(4Q)$ is such that $u_{\alpha,k} > r$. Then, as the cubes in $\{R_j(r)\}_{j \in \mathbb{N}}$

are maximal, either $R_\alpha^k = R_j(r)$ or $R_\alpha^k \subseteq R_j(r)$ for some $j \in \mathbb{N}$. Therefore, using the definition of a Carleson measure on $4Q$, the above claim, and standard results for maximal functions (see [51, p. 7]), we have

$$\begin{aligned}
\int_0^\infty \sum_{k=-\infty}^{K_Q} \sum_{\alpha \in I_k} \mu_{\alpha,k} \mathbb{1}_{\{u_{\alpha,k} > r\}}(r) 2r \, \mathrm{d}r &\leq \int_0^\infty \sum_{j=1}^\infty \sum_{\substack{S \in \Delta(4Q) \\ S \subseteq R_j(r)}} \mu(S \times (\tfrac{l(S)}{2}, l(S)]) 2r \, \mathrm{d}r \\
&\leq \int_0^\infty 2r \sum_{j=1}^\infty \mu(\mathcal{C}(R_j(r))) \, \mathrm{d}r \\
&\leq \|\mu\|_{\mathcal{C}} \int_0^\infty 2r \sum_{j=1}^\infty |R_j(r)| \, \mathrm{d}r \\
&= \|\mu\|_{\mathcal{C}} \int_0^\infty 2r \left| \bigcup_{j=1}^\infty R_j(r) \right| \mathrm{d}r \\
&= \|\mu\|_{\mathcal{C}} \int_0^\infty 2r \left| \{x \in \mathbb{R}^n : (\mathcal{M}_\Delta^Q |\eta_Q u|)(x) > r\} \right| \mathrm{d}r \\
&= \|\mu\|_{\mathcal{C}} \|\mathcal{M}_\Delta^Q |\eta_Q u|\|_2^2 \\
&\lesssim \|\mu\|_{\mathcal{C}} \|\eta_Q u\|_2^2,
\end{aligned}$$

where $\mathcal{C}(R_j(r))$ is the Carleson box of $R_j(r)$. This completes the proof. □

To see that Lemma 4.4.3 implies Proposition 4.4.1 we use Lemma 4.4.3, the Carleson norms independence of Q, and then Proposition 4.1.7, to get

$$\begin{aligned}
\sum_{Q \in \tilde{\Delta}_T^{\mathbf{B}}} \int_0^{l(Q)} \|\mathbb{1}_{4Q} \tilde{\gamma}_t^Q A_t u\|_2^2 \frac{\mathrm{d}t}{t} &= \sum_{Q \in \tilde{\Delta}_T^{\mathbf{B}}} \int_0^{l(Q)} \int_{4Q} |A_t e^{i\varphi_Q} \eta_Q u| |\tilde{\gamma}_t^Q(x)|^2 \frac{\mathrm{d}x \, \mathrm{d}t}{t} \\
&\lesssim \sum_{Q \in \tilde{\Delta}_T^{\mathbf{B}}} \|\mu\|_{\mathcal{C}} \|\eta_Q u\|_2^2 \\
&\lesssim \sum_{Q \in \tilde{\Delta}_T^{\mathbf{B}}} \|\eta_Q u\|_2^2 \\
&\lesssim \|u\|_2^2,
\end{aligned} \tag{4.4.1}$$

where $\|\mu\|_{\mathcal{C}}$ is the Carleson norm of $|\tilde{\gamma}_t^Q(x)|^2 \frac{\mathrm{d}x \, \mathrm{d}t}{t}$ for all Q, noting that by definition if $|\tilde{\gamma}_t^Q(x)|^2 \frac{\mathrm{d}x \, \mathrm{d}t}{t}$ is a Careson measure on $4Q \times (0, l(Q)]$ then the Carleson norm is independent of Q.

The rest of this section will be dedicated to proofing that $|\tilde{\gamma}_t^Q(x)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t}$ is a Carleson measure on $4Q \times (0, l(Q)]$. To begin we will fix a subcube R, of $4Q$, and construct a test function which is support on $4R$. To do this we will need consider the case when $4R \cap (\mathbb{R}^n \setminus 4Q) \neq \emptyset$ separately and follow the ideas in [14] to prove a Carleson measure estimate on such cubes. To this end, consider the measure $\chi_Q(x,t)\frac{\mathrm{d}x\,\mathrm{d}t}{t}$ where

$$\chi_Q(x,t) = \begin{cases} 0 \text{ if the cube } R \in \Delta_t(4Q) \text{ such that } x \in R \text{ satisfies } 4R \subseteq 4Q, \\ 1 \text{ otherwise,} \end{cases}$$

for all $(x,t) \in Q \times (0, l(Q)]$. We will then prove that $|\tilde{\gamma}_t^Q(x)|^2 \chi_Q(x,t)\frac{\mathrm{d}x\,\mathrm{d}t}{t}$ and $|\tilde{\gamma}_t^Q(x)|^2(1-\chi_Q(x,t))\frac{\mathrm{d}x\,\mathrm{d}t}{t}$ are Carleson measures on $4Q \times (0, l(Q)]$ separately. First we consider $|\tilde{\gamma}_t^Q(x)|^2 \chi_Q(x,t)\frac{\mathrm{d}x\,\mathrm{d}t}{t}$.

Lemma 4.4.4. $\chi_Q(x,t)\frac{\mathrm{d}x\,\mathrm{d}t}{t}$ and $|\tilde{\gamma}_t^Q(x)|^2 \chi_Q(x,t)\frac{\mathrm{d}x\,\mathrm{d}t}{t}$ are Carleson measures on $4Q \times (0, l(Q)]$.

Proof. Let $R \in \Delta(4Q)$ and fix $K_R \coloneqq l(R)$. Then

$$\int_0^{l(R)} \int_R \chi_Q(x,t)\frac{\mathrm{d}x\,\mathrm{d}t}{t} = \sum_{k=-\infty}^{K_R} \int_{2^{k-1}}^{2^k} \sum_{S \in \Delta_{2^k}(R)} \int_S \chi_Q(x,t)\frac{\mathrm{d}x\,\mathrm{d}t}{t}.$$

Now $\chi_Q(x,t) = 1$ on S if and only if $4S \cap (\mathbb{R}^n \setminus 4Q) \neq \emptyset$. That is $\chi_Q(x,t) = 1$ if and only if $\operatorname{dist}(S, \mathbb{R}^n \setminus 4Q) \leq \frac{3}{2} l(S)$, that is at most a depth of 2 cubes of $l(S)$ away from the boundary of Q. Therefore, there are at most $2 \times 2^{-(n-1)(k-K_R)}$ such cubes. Thus

$$\sum_{S \in \Delta_{2^k}(R)} \int_S \chi_Q(x,t)\,\mathrm{d}x = \sum_{\substack{S \in \Delta_{2^k}(R) \\ 4S \cap (\mathbb{R}^n \setminus 4Q) \neq \emptyset}} |S| \lesssim 2^{kn} 2^{-(n-1)(k-K_R)} \lesssim 2^k l(R)^{n-1}.$$

As, $\frac{2^k}{t} \lesssim 1$ on each integral, then

$$\begin{aligned}
\int_0^{l(R)} \int_R \chi_Q(x,t) \frac{\mathrm{d}x\,\mathrm{d}t}{t} &\lesssim \sum_{k=-\infty}^{K_R} \int_{2^{k-1}}^{2^k} 2^k l(R)^{n-1} \frac{\mathrm{d}t}{t} \\
&\lesssim l(R)^{n-1} \sum_{k=-\infty}^{K_R} \int_{2^{k-1}}^{2^k} \mathrm{d}t \\
&= l(R)^{n-1} \int_0^{l(R)} \mathrm{d}t \\
&= |R|.
\end{aligned}$$

Thus $\chi_Q(x,t)\frac{\mathrm{d}x\,\mathrm{d}t}{t}$ is a Carleson measure on $Q \times (0, l(Q)]$. Then by a similar argument and using Lemma 4.3.1, we have

$$\begin{aligned}
\int_0^{l(R)} \int_R |\tilde{\gamma}_t^Q(x)|^2 \chi_Q(x,t) \frac{\mathrm{d}x\,\mathrm{d}t}{t} &= \sum_{k=-\infty}^{K_R} \int_{2^{k-1}}^{2^k} \sum_{\substack{S\in\Delta_{2^k}(R) \\ 4S\cap(\mathbb{R}^n\setminus 4Q)\neq\emptyset}} \int_S |\tilde{\gamma}_t^Q(x)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t} \\
&\lesssim \sum_{k=-\infty}^{K_R} \int_{2^{k-1}}^{2^k} \sum_{\substack{S\in\Delta_{2^k}(R) \\ 4S\cap(\mathbb{R}^n\setminus 4Q)\neq\emptyset}} |S| \frac{\mathrm{d}t}{t} \\
&\lesssim \sum_{k=-\infty}^{K_R} \int_{2^{k-1}}^{2^k} 2^k l(R)^{n-1} \frac{\mathrm{d}t}{t} \\
&\lesssim l(R)^{n-1} \sum_{k=-\infty}^{K_R} \int_{2^{k-1}}^{2^k} \mathrm{d}t \\
&= l(R)^{n-1} \int_0^{l(R)} \mathrm{d}t \\
&\lesssim |R|.
\end{aligned}$$

This completes the proof. $\square$

Next, to prove $|\tilde{\gamma}_t^Q(x)|^2(1-\chi_Q(x,t))\frac{\mathrm{d}x\,\mathrm{d}t}{t}$ is a Carleson measure on $4Q$, it suffices to prove that

$$\int_0^{l(R)} \int_R |\tilde{\gamma}_t^Q(x)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t} \lesssim |R|$$

for all $R \in \Delta(4Q)$ with $4R \subseteq 4Q$ independently of Q. Now fix $R \in \Delta(4Q)$ with $4R \subseteq 4Q$ and set $\sigma > 0$ to be chosen later. Choose a finite set $\mathcal{V}_\sigma$ of non-zero matrices,

$\nu \in \mathcal{L}(\mathbb{C}^{n+1}) \setminus \{0\}$, with $|\nu| = 1$ such that $\bigcup_{\nu \in \mathcal{V}_\sigma} K_{\nu,\sigma} = \mathcal{L}(\mathbb{C}^{n+1}) \setminus \{0\}$ where

$$K_{\nu,\sigma} := \left\{ \mu \in \mathcal{L}(\mathbb{C}^{n+1}) \colon \left| \frac{\mu}{|\mu|} - \nu \right| \leq \sigma \right\}.$$

Then, we may prove the Carleson measure estimate for each cone $K_{\nu,\sigma}$ separately. That is,

$$\iint_{(x,t)\in\mathcal{C}(R)} |\tilde{\gamma}_t^Q(x)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t} \lesssim \sum_{\nu \in \mathcal{V}_\sigma} \iint_{\substack{(x,t)\in\mathcal{C}(R) \\ \tilde{\gamma}_t^Q(x) \in K_{\nu,\sigma}}} |\tilde{\gamma}_t^Q(x)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t}.$$

Therefore we are left to prove

$$\iint_{\substack{(x,t)\in\mathcal{C}(R) \\ \tilde{\gamma}_t^Q(x) \in K_{\nu,\sigma}}} |\tilde{\gamma}_t^Q(x)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t} \lesssim |R|, \tag{4.4.2}$$

for every $\nu \in \mathcal{V}_\sigma$ independently of Q. Let $\zeta, \xi \in \mathbb{C}^{n+1}$ with $|\zeta| = |\xi| = 1$ and $\nu^*(\zeta) = \xi$. Let $\eta_R \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ supported on $4R$, with $\eta_R \equiv 1$ on $2R$ and $\eta_R(x) \in [0,1]$ for all $x \in 4R$ with $\|\nabla \eta_R\|_\infty \lesssim \frac{1}{l}$, where $l := l(R)$. Define $\xi_R := \eta_R \xi$. We define the test functions as follows, for $\varepsilon > 0$ to be chosen, define

$$f_{R,\varepsilon}^\xi := (I + (\varepsilon l)^2 (\tilde{D}^Q B)^2)^{-1} \xi_R = \tilde{P}_{\varepsilon l}^B \xi_R$$

Some of the important properties of $f_{R,\varepsilon}^\xi$ are stated in the following lemma.

Lemma 4.4.5. We have the following estimates

1. $\|f_{R,\varepsilon}^\xi\|_2 \lesssim |R|^{\frac{1}{2}}$,

2. $\|\varepsilon l \tilde{D}^Q B f_{R,\varepsilon}^\xi\|_2 \lesssim |R|^{\frac{1}{2}}$,

3. $\left| \fint_R f_{R,\varepsilon}^\xi - \xi \right| \lesssim \varepsilon^{\frac{1}{2}}$,

for all $R \in \Delta(4Q)$ with $4R \subseteq 4Q$.

Proof. By the uniform boundedness of $\tilde{P}_{\varepsilon l}^B$ we have

$$\|f_{R,\varepsilon}^\xi\|_2^2 = \|\tilde{P}_{\varepsilon l}^B \xi_R\|_2^2 \lesssim \|\xi_R\|_2^2 \lesssim |R|.$$

Similarly, using the the uniform boundedness of $\tilde{Q}^B_{\varepsilon l}$, we have

$$\|\varepsilon l \tilde{D}^Q B f^{\xi}_{R,\varepsilon}\|_2^2 = \|\tilde{Q}^B_{\varepsilon l}\xi_R\|_2^2 \lesssim \|\xi_R\|_2^2 \lesssim |R|.$$

For (3) we use the fact that $t^2(\tilde{D}^Q B)^2 \tilde{P}^B_t = I - \tilde{P}^B_t$, Lemma 4.1.10 with $Q \in \tilde{\Delta}^{\mathbf{B}}_T$, the uniform boundedness of $I - \tilde{P}^B_t$ and $\tilde{Q}^B_t$

$$\begin{aligned}
\left| \fint_R f^{\xi}_{R,\varepsilon} - \xi \right|^2 &= \left| \fint_R (\tilde{P}^B_{\varepsilon l} - I)\xi_R \right|^2 \\
&= (\varepsilon l)^4 \left| \fint_R (\tilde{D}^Q B)^2 \tilde{P}^B_{\varepsilon l} \xi_R \right|^2 \\
&\lesssim \frac{(\varepsilon l)^4}{l} \left(\fint_R |(\tilde{D}^Q B)^2 \tilde{P}^B_{\varepsilon l} \xi_R|^2 \right)^{\frac{1}{2}} \left(\fint_R |B \tilde{D}^Q B \tilde{P}^B_{\varepsilon l} \xi_R|^2 \right)^{\frac{1}{2}} \\
&= \varepsilon \|B\|_\infty \left(\fint_R |(\varepsilon l \tilde{D}^Q B)^2 \tilde{P}^B_{\varepsilon l} \xi_R|^2 \right)^{\frac{1}{2}} \left(\fint_R |\varepsilon l \tilde{D}^Q B \tilde{P}^B_{\varepsilon l} \xi_R|^2 \right)^{\frac{1}{2}} \\
&\lesssim \varepsilon \left(\fint_R |(\tilde{P}^{B,Q}_{\varepsilon l} \xi_R|^2 \right)^{\frac{1}{2}} \left(\fint_R |\tilde{Q}^{B,Q}_{\varepsilon l} \xi_R|^2 \right)^{\frac{1}{2}} \\
&\lesssim \frac{\varepsilon}{|R|} \|\xi_R\|_2^2 \\
&\lesssim \varepsilon
\end{aligned}$$

As required. □

We prove (4.4.2) below by introducing a sub-collection of disjoint subcubes of each $R \subset Q$ with $4R \subseteq 4Q$ as below.

Proposition 4.4.6. There exists $\tau \in (0,1)$ such that for all cubes $R \in \Delta(4Q)$ and $\nu \in \mathcal{L}(\mathbb{C}^{n+1}) \setminus \{0\}$ with $|\nu| = 1$ there is a collection $\{R_k\}_{k \in I_R} \subset \Delta(R)$ of disjoint subcubes of R, where $I_R \subseteq \mathbb{N}$ is an indexing set for the collection, such that $|E_{R,\nu}| > \tau |R|$, where $E_{R,\nu} := R \setminus \bigcup_{k \in I_R} R_k$ and with

$$\iint_{\substack{(x,t) \in E^*_{R,\nu} \\ \tilde{\gamma}^Q_t(x) \in K_{\nu,\sigma}}} |\tilde{\gamma}^Q_t(x)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t} \lesssim |R| \tag{4.4.3}$$

where $E^*_{R,\nu} := \mathcal{C}(R) \setminus \bigcup_{k \in I_R} \mathcal{C}(R_k)$.

We may fix $\varepsilon_0 > 0$, defining $f^{\xi}_R := f^{\xi}_{R,\varepsilon_0}$, such that when we apply Lemma 4.4.5 we

have

$$\left| \fint_R f_R^\xi - \xi \right| \leq \frac{1}{2},$$

for all $R \in \Delta(4Q)$, with $4R \subseteq 4Q$. Therefore,

$$\mathrm{Re} \left\langle \xi, \fint_R f_R^\xi \right\rangle \geq \frac{1}{2}. \tag{4.4.4}$$

We will now give the stopping time argument to construct the collection of bad cubes described in Proposition 4.4.6.

Lemma 4.4.7. Let $R \in \Delta(4Q)$. Then, there exists constants $c_1, c_2 > 0$, and $\tau \in (0,1)$, and a disjoint collection $\{R_k\}_{k \in I_R} \subset \Delta(R)$ such that

- $R_k \subseteq R$,
- $|E_{R,\nu}| \geq \tau |R|$,

satisfying

$$\fint_S |f_R^\xi| \leq c_1 \quad \text{and} \quad \mathrm{Re} \left\langle \xi, \fint_S f_R^\xi \right\rangle \geq c_2 \tag{4.4.5}$$

for all $S \in \Delta(R)$ for which $\mathcal{C}(S) \cap E_{R,\nu}^* \neq \emptyset$, where $E_{R,\nu}^* := \mathcal{C}(R) \setminus \bigcup_{k \in I_R} \mathcal{C}(C_k)$.

Proof. Let $\alpha \in (0,1)$ to be chosen. Let $\mathcal{B}_1$ be the set maximal cubes $S \in \Delta(R)$, for which

$$\fint_S |f_R^\xi| > \frac{1}{\alpha}.$$

Then using the Cauchy–Schwarz inequality and (1) from Lemma 4.4.5, we have

$$\left| \bigcup \mathcal{B}_1 \right| = \sum_{S \in \mathcal{B}_1} |S| < \alpha \sum_{S \in \mathcal{B}_1} \int_S |f_R^\xi| \leq \alpha \int_R |f_R^\xi| \leq \alpha |R|^{\frac{1}{2}} \left(\int_R |f_R^\xi|^2 \right)^{\frac{1}{2}} \lesssim C\alpha |R|,$$

where $C > 0$ is the implicit constant in (1) from Lemma 4.4.5. Now let $\mathcal{B}_2$ be the set of maximal cubes $S \in \Delta(R)$, such that

$$\mathrm{Re} \left\langle \xi, \fint_S f_R^\xi \right\rangle < \alpha.$$

Now by (4.4.4), the Cauchy–Schwarz inequality, and (1) from Lemma 4.4.5, we have

$$\begin{aligned}\frac{1}{2} &\leq \operatorname{Re}\left\langle \xi, \fint_R f_R^\xi \right\rangle \\ &= \sum_{S\in\mathcal{B}_2} \frac{|S|}{|R|} \operatorname{Re}\left\langle \xi, \fint_S f_R^\xi \right\rangle + \frac{1}{|R|} \operatorname{Re}\left\langle \xi, \int_{R\setminus\bigcup\mathcal{B}_2} f_R^\xi \right\rangle \\ &< \alpha + \frac{1}{|R|}\left(\int |f_R^\xi|^2\right)^{\frac{1}{2}} |R\setminus \bigcup \mathcal{B}_2|^{\frac{1}{2}} \\ &\leq \alpha + C\left(\frac{|R\setminus\bigcup\mathcal{B}_2|}{|R|}\right)^{\frac{1}{2}}.\end{aligned}$$

Making the restriction $\alpha \in (0, \frac{1}{2})$, we have

$$|R\setminus \bigcup \mathcal{B}_2| \geq \left(\frac{\frac{1}{2}-\alpha}{C}\right)^2 |R|$$

Therefore, letting $\{R_k\}_{k\in I_R} := \mathcal{B}_1 \cup \mathcal{B}_2$, where I_R is an enumeration of the cubes in $\mathcal{B}_1\cup\mathcal{B}_2$, we have

$$|E_{R,\nu}| \geq |R\setminus\bigcup\mathcal{B}_2| - |\bigcup\mathcal{B}_1| \geq \left[\left(\frac{\frac{1}{2}-\alpha}{C}\right)^2 - C\alpha\right]|R|.$$

Now, choosing $\alpha \in (0, \frac{1}{2})$ sufficiently small, gives $\tau := \left[\left(\frac{\frac{1}{2}-\alpha}{C}\right)^2 - C\alpha\right] > 0$. Now let $S \in \Delta(R)$ with $\mathcal{C}(S)\cap E^*_{R,\nu} \neq \emptyset$. By the maximality of the cubes in $\mathcal{B}_1$ and $\mathcal{B}_2$ then a similar argument to Lemma 3.3.6, we have $S \notin \mathcal{B}_1\cup\mathcal{B}_2$. Therefore, S satisfies (4.4.5). This completes the proof. □

Now we choose $\sigma := \frac{c_2}{2c_1}$ and will use the following lemma to introduce the test function into our argument.

Lemma 4.4.8. Let $\nu \in \mathcal{L}(\mathbb{C}^{n+1})\setminus\{0\}$. There exists $\sigma > 0$ such that if $(x,t) \in E^*_{R,\nu}$ and $\tilde{\gamma}_t^Q(x) \in K_{\nu,\sigma}$, then

$$|\tilde{\gamma}_t^Q(x)(A_t f_R^\xi)| \geq \frac{1}{2}c_2|\tilde{\gamma}_t^Q(x)|,$$

where $c_2 > 0$ is the constant from (4.4.5).

Proof. As $(x,t) \in E^*_{R,\nu}$, there exists $S \in \Delta_t(R)$, such that $x \in S$ and $\mathcal{C}(S)\cap E^*_{R,\nu} \neq \emptyset$.

Then, recalling $\nu^*(\zeta) = \xi$ and Lemma 4.4.7, we have

$$|\nu(A_t f_R^\xi(x))| \geq \mathrm{Re}\langle \zeta, \nu(A_t f_R^\xi) \rangle = \mathrm{Re} \left\langle \xi, \fint_S f_R^\xi \right\rangle \geq c_2.$$

Now, by above and Lemma 4.4.7, we have

$$\left| \frac{\tilde{\gamma}_t^Q(x)}{|\tilde{\gamma}_t^Q(x)|} (A_t^Q f_R^\xi(x)) \right| \geq |\nu(A_t^Q f_R^\xi(x))| - \left| \left(\frac{\tilde{\gamma}_t^Q(x)}{|\tilde{\gamma}_t^Q(x)|} - \nu \right) \left(\fint_S f_R^\xi \right) \right| \geq c_2 - c_1 \sigma \geq \frac{1}{2} c_2.$$

As required. $\square$

Proof of Proposition 4.4.6. Let $E_{R,\nu}$ and τ be as constructed in Lemma 4.4.7. By Lemma 4.4.8 and as $(A_t^Q f_R^\xi)(x) = (A_t^Q \eta_R f_R^\xi)(x)$ for all $(x,t) in \mathcal{C}(R)$, we have

$$\begin{aligned} \iint_{\substack{(x,t) \in E_{R,\nu}^* \\ \tilde{\gamma}_t^Q(x) \in K_{\nu,\sigma}}} |\tilde{\gamma}_t^Q(x)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t} &\lesssim \iint_{\substack{(x,t) \in E_{R,\nu}^* \\ \tilde{\gamma}_t^Q(x) \in K_{\nu,\sigma}}} |\tilde{\gamma}_t^Q(x)(A_t f_R^\xi)(x)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t} \\ &\leq \iint_{\mathcal{C}(R)} |\tilde{\gamma}_t^Q(x)(A_t \eta_R f_R^\xi)(x)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t}. \end{aligned}$$

Now, we introduce $\tilde{Q}_t^{B,Q} \eta_R f_R^\xi$ to perform the reverse principal part approximation in reverse, to get

$$\begin{aligned} \iint_{\mathcal{C}(R)} |\tilde{\gamma}_t^Q(x)(A_t \eta_R f_R^\xi)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t} &\lesssim \iint_{\mathcal{C}(R)} |(\tilde{Q}_t^{B,Q} - \tilde{\gamma}_t^Q A_t) \eta_R f_R^\xi|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t} \\ &\quad + \iint_{\mathcal{C}(R)} |(\tilde{Q}_t^{B,Q} \eta_R f_R^\xi)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t}. \end{aligned} \tag{4.4.6}$$

Now we estimate the error term in (4.4.6), by using the uniform boundedness of $\tilde{P}_t^{B,Q}$, the gauge invariance in (4.1.5), the commutator bounds in (4.1.7), and Lemma 4.4.5, to get

$$\begin{aligned}\iint_{\mathcal{C}(R)} |(\tilde{Q}_t^{B,Q}\eta_R f_R^\xi)(x)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t} &= \int_0^l \|\mathbb{1}_R \tilde{P}_t^{B,Q} t\tilde{D}^Q B\eta_R f_R^\xi\|_2^2 \frac{\mathrm{d}t}{t}\\ &\lesssim \int_0^l t\|\eta_R \tilde{D}^Q B f_R^\xi\|_2^2 + t\|[\eta_R,\tilde{D}^Q]Bf_R^\xi\|_2^2\,\mathrm{d}t\\ &\lesssim \int_0^l \frac{t}{(\varepsilon_0 l)^2}\|\varepsilon_0 l\tilde{D}^Q B f_R^\xi\|_2^2 + \frac{t}{l^2}\|f_R^\xi\|_2^2\,\mathrm{d}t\\ &\lesssim |R|\left(\frac{1}{\varepsilon_0^2}+1\right)\int_0^l \frac{t}{l^2}\,\mathrm{d}t\\ &\lesssim |R|.\end{aligned}$$

Our aim now is to perform the principal part approximation in reverse on the remaining term in (4.4.6). In order to do this we need to be on $\overline{\mathsf{R}(D)}$ so we introduce ξ_R and note that, using the gauge invariance, we have $e^{-i\varphi_Q}(f_R^\xi - \xi_R) = e^{-i\varphi_Q}(\tilde{P}_{\varepsilon l}^{B,Q} - I)\xi_R = e^{-i\varphi_Q}(\varepsilon_0\tilde{D}^Q B)^2\tilde{P}_{\varepsilon_0 l}^{B,Q}\xi_R = D[e^{-\varphi_Q}\varepsilon_0^2 B\tilde{D}^Q B\tilde{P}_{\varepsilon_0 l}^{B,Q}\xi_R]$. That is $e^{-i\varphi_Q}(f_R^\xi - \xi_R) \in \overline{\mathsf{R}(D)}$. Therefore

$$\begin{aligned}&\iint_{\mathcal{C}(R)} |(\tilde{Q}_t^{B,Q} - \tilde{\gamma}_t^Q A_t)\eta_R f_R^\xi|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t}\\ &\lesssim \iint_{\mathcal{C}(R)} |(\tilde{Q}_t^{B,Q} - \tilde{\gamma}_t^Q A_t)\eta_R (f_R^\xi - \xi_R)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t} + \iint_{\mathcal{C}(R)} |(\tilde{Q}_t^{B,Q} - \tilde{\gamma}_t^Q A_t)\eta_R \xi_R|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t}\\ &\lesssim \iint_{\mathcal{C}(R)} |(\tilde{Q}_t^{B,Q} - \tilde{\gamma}_t^Q A_t)e^{i\varphi_Q}\eta_R P_t e^{-i\varphi_Q}(f_R^\xi - \xi_R)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t}\\ &\quad + \iint_{\mathcal{C}(R)} |\tilde{Q}_t^{B,Q}(\eta_R - e^{i\varphi_Q}\eta_R P_t e^{-i\varphi_Q})(f_R^\xi - \xi_R)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t}\\ &\quad + \iint_{\mathcal{C}(R)} |\tilde{\gamma}_t^Q A_t(\eta_R - e^{i\varphi_Q}\eta_R P_t e^{-i\varphi_Q})(f_R^\xi - \xi_R)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t}\\ &\quad + \iint_{\mathcal{C}(R)} |(\tilde{Q}_t^{B,Q} - \tilde{\gamma}_t^Q A_t)\eta_R \xi_R|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t}\\ &= I + II + III + IV\end{aligned}$$

Let $g := e^{-i\varphi_Q}(f_R^\xi - \xi_R)$. Now, by Lemma 4.3.5 since P_t preserves $\overline{\mathsf{R}(D)}$, we have

$$\|\mathbb{1}_R(\tilde{Q}_t^{B,Q} - \tilde{\gamma}_t^Q A_t)e^{i\varphi_Q}\eta_R P_t g\|_2^2 \lesssim t^2\|\mathbb{1}_{4R} L\eta_R P_t g\|_2^2 + t^2\|\mathbb{1}_{4R}|\mathbf{B}|^{\frac{1}{2}}\eta_R P_t g\|_2^2.$$

Then, the product rule, Corollary 4.1.6, and the uniform boundedness of P_t, gives

$$\begin{aligned}\|\mathbb{1}_{4R}L\eta_R P_t g\|_2^2 + \|\mathbb{1}_{4R}|\mathbf{B}|^{\frac{1}{2}}\eta_R P_t g\|_2^2 &\lesssim \|LP_t g\|_2^2 + \||\mathbf{B}|^{\frac{1}{2}}P_t g\|_2^2 + \|\mathbb{1}_{4R}(\nabla\eta_R)P_t g\|_2^2 \\ &\lesssim \|DP_t g\|_2^2 + \frac{1}{l(R)^2}\|g\|_2^2.\end{aligned}$$

Therefore, using above, the quadratic estimates for the self-adjoint operator D, and Lemma 4.4.5, we have

$$I \lesssim \int_0^{l(R)} \|tDP_t g\|_2^2 \frac{\mathrm{d}t}{t} + \int_0^{l(R)} \frac{t^2\|g\|_2^2}{l(R)^2}\frac{\mathrm{d}t}{t} \lesssim \int_0^{\infty} \|Q_t g\|_2^2 \frac{\mathrm{d}t}{t} + \frac{\|g\|_2^2}{l(R)^2}\int_0^{l(R)} t\,\mathrm{d}t \lesssim \|g\|_2^2 \lesssim |R|.$$

Now, by Lemma 4.3.3, the quadratic estimates for the self-adjoint operator D, and Lemma 4.4.5, we have

$$\begin{aligned} II &= \iint_{\mathcal{C}(R)} |\tilde{Q}_t^{B,Q} e^{i\varphi_Q}\eta_R(I-P_t)g|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t} \\ &\lesssim \int_0^l \left(\frac{t^2}{l(R)^2}+1\right)\|Q_t g\|_2^2 \frac{\mathrm{d}t}{t} \\ &\lesssim \int_0^\infty \|Q_t g\|_2^2 \frac{\mathrm{d}t}{t} \\ &\lesssim \|g\|_2^2 \\ &\lesssim |R|. \end{aligned}$$

For the third term we follow the proof of Proposition 4.3.7 an so the estimate will follow once we have established

$$\|\mathbb{1}_R A_t e^{i\varphi_Q}\eta_R(I-P_t)Q_s\| \lesssim \min\left\{\frac{s}{t},\frac{t}{s}\right\}$$

for $s \in (0,\infty)$ and $t \in (0,l)$. We note that we need to reprove this as the localisation here is different to that of Proposition 4.3.7. Firstly, suppose $t < s$ then by the uniform boundedness of A_t and η_R, the fact that $(I-P_t)Q_s = \frac{t}{s}Q_t(I-P_s)$ and the the uniform boundedness of P_s and Q_t to get

$$\|\mathbb{1}_R A_t e^{i\varphi_Q}\eta_R(I-P_t)Q_s u\|^2 \lesssim \left(\frac{t}{s}\right)^2 \|Q_t(I-P_s)u\|_2^2 \lesssim \frac{t}{s}\|Q_t(I-P_s)u\|_2^2 \leq \frac{t}{s}\|u\|_2^2.$$

For $s \leq t \leq l$, we use the uniform boundedness of A_t, the fact that $P_tQ_s = \frac{s}{t}Q_tP_s$, and the uniform boundedness of P_s and Q_t to get

$$\begin{aligned}\|\mathbb{1}_R A_t e^{i\varphi_Q}\eta_R(I-P_t)Q_s u\|_2^2 &\lesssim \left(\frac{s}{t}\right)^2 \|Q_tP_s u\|_2^2 + \|\mathbb{1}_R A_t e^{i\varphi_Q}\eta_R Q_s u\|_2^2 \\ &\lesssim \frac{s}{t}\|u\|_2^2 + \sum_{S\in\Delta_t(R)} \|\mathbb{1}_S A_t e^{i\varphi_Q}\eta_R Q_s u\|_2^2.\end{aligned}$$

Now we interchange the η_R and D and use the gauge invarince and Jensen's inequality, to obtain

$$\begin{aligned}\sum_{S\in\Delta_t(R)} \|\mathbb{1}_S A_t e^{i\varphi_Q}\eta_R Q_s u\|_2^2 &= \sum_{S\in\Delta_t(R)} \int_S \left|\fint_S e^{i\varphi_Q}\eta_R Q_s u\right|^2 \\ &= \sum_{S\in\Delta_t(R)} s^2|S| \left|\fint_S e^{i\varphi_Q}(\eta_R D - D\eta_R + D\eta_R)P_s u\right|^2 \\ &\lesssim \sum_{S\in\Delta_t(R)} s^2|S| \left[\left|\fint_S \tilde{D}^Q e^{i\varphi_Q}\eta_R P_s u\right|^2 + \left(\fint_S |[\eta_R, D]P_s u|^2\right)\right]\end{aligned}$$

Now using the commutator bounds and $t < l$, we get

$$\sum_{S\in\Delta_t(R)} s^2|S|\fint_S |[\eta_R, D]P_s u|^2 \lesssim \frac{s^2}{t^2}\|P_s u\|_2^2$$

Now by Lemma 4.1.10, the gauge invariance, commutator bounds, and the Cauchy–

Schwarz inequality, we have

$$
\begin{aligned}
&\sum_{S\in\Delta_t(R)} s^2|S|\left|\fint_S \tilde{D}^Q e^{i\varphi_Q}\eta_R P_s u\right|^2 \\
&\qquad\lesssim \sum_{S\in\Delta_t(R)} \frac{s^2|S|}{l(S)}\left(\fint_S |\tilde{D}^Q e^{i\varphi_Q}\eta_R P_s u|^2\right)^{\frac{1}{2}}\left(\fint_S |\eta_R P_s u|^2\right)^{\frac{1}{2}} \\
&\qquad\lesssim \frac{s^2}{t}\sum_{S\in\Delta_t(R)}\left[\left(\int_S |\eta_R D P_s u|^2\right)^{\frac{1}{2}} + \left(\int_S |[\eta_R, D]P_s u|^2\right)^{\frac{1}{2}}\right]\left(\int_S |\eta_R P_s u|^2\right)^{\frac{1}{2}} \\
&\qquad\lesssim \frac{s}{t}\sum_{S\in\Delta_t(R)}\left[\left(\int_S |\eta_R Q_s u|^2\right)^{\frac{1}{2}} + \frac{s}{l}\left(\int_S |P_s u|^2\right)^{\frac{1}{2}}\right]\left(\int_S |\eta_R P_s u|^2\right)^{\frac{1}{2}} \\
&\qquad\lesssim \frac{s}{t}\sum_{S\in\Delta_t(R)}\left[\left(\int_S |Q_s u|^2\right)^{\frac{1}{2}}\left(\int_S |P_t u|^2\right)^{\frac{1}{2}} + \frac{s}{l}\left(\int_S |P_s u|^2\right)\right] \\
&\qquad\lesssim \frac{s}{t}\left(\int_R |Q_s u|^2\right)^{\frac{1}{2}}\left(\int_R |P_s u|^2\right)^{\frac{1}{2}} + \frac{s^2}{t^2}\int_R |P_s u|^2
\end{aligned}
$$

Combining the above and using and the uniform boundedness of Q_t and P_t, we have

$$\|\mathbb{1}_R A_t e^{i\varphi_Q}\eta_Q(I-P_t)Q_s u\|_2^2 \lesssim \frac{s}{t}\left[\|u\|_2^2\|Q_s u\|_2\|P_s u\|_2 + \|P_s u\|_2^2\right] \lesssim \frac{s}{t}\|u\|_2^2$$

Thus, we will replicate the Schur estimate in Proposition 4.3.7. First, let $m(s,t) \coloneqq \min\left\{\frac{t}{s}, \frac{s}{t}\right\}$. Then, we use the uniform boundedness of $\tilde{\gamma}_t^Q A_t^Q$ in Lemma 4.3.1, the Calderón reproducing formula, Minkowski's inequality, Tonneli's theorem, and Lemma 4.4.5, we

have

$$
\begin{aligned}
III &\lesssim \int_0^l \|\tilde{\gamma}_t^Q A_t \mathbb{1}_R A_t e^{i\varphi_Q} \eta_R (I - P_t) g\|_2^2 \frac{\mathrm{d}t}{t} \\
&\lesssim \int_0^l \left(\int_0^\infty \|\mathbb{1}_R A_t e^{i\varphi_Q} \eta_R (I - P_t) Q_s^2 g\|_2 \frac{\mathrm{d}s}{s} \right)^2 \frac{\mathrm{d}t}{t} \\
&\lesssim \int_0^l \left(\int_0^\infty \|\mathbb{1}_R A_t e^{i\varphi_Q} \eta_R (I - P_t) Q_s\| \|Q_s g\|_2 \frac{\mathrm{d}s}{s} \right)^2 \frac{\mathrm{d}t}{t} \\
&\lesssim \int_0^\infty \left(\int_0^\infty m(s,t) \frac{\mathrm{d}s}{s} \right) \left(\int_0^\infty m(s,t) \|Q_s g\|_2^2 \frac{\mathrm{d}s}{s} \right) \frac{\mathrm{d}t}{t} \\
&\lesssim \int_0^\infty \|Q_s g\|_2^2 \frac{\mathrm{d}s}{s} \\
&\lesssim \|g\|_2^2 \\
&\lesssim |R|.
\end{aligned}
$$

Now we begin the last estimate. Then, as $\tilde{\gamma}_t^Q A_t \xi = (\tilde{Q}_t^{B,Q} \xi)$ and as $\eta_R \equiv 1$ on $2R$ then $A_t^Q(\eta_R^2 \xi)(x) = \xi$ for all $(x,t) \in \mathcal{C}(R)$. Thus, we have

$$
(\tilde{Q}_t^{B,Q} - \tilde{\gamma}_t^Q A_t) \eta_R \xi_R = \tilde{Q}_t^{B,Q} \left(\eta_R^2 \xi - A_t(\eta_R^2 \xi) \right) = \tilde{Q}_t^{B,Q} (\eta_R^2 \xi - \xi)
$$

on $\mathcal{C}(R)$. Recall the definition of $\tilde{C}_k^{4Q}(R)$ from the proof of Lemma 4.3.1. Then, by off-diagonal estimates in Proposition 4.1.9 noting that $\operatorname{supp}(\eta_R^2 - 1) \cap 2R = \emptyset$, the Cauchy–Schwarz inequality, and choosing $M > n$, we obtain

$$
\begin{aligned}
\|\mathbb{1}_R \tilde{Q}_t^{B,Q} (\eta_R^2 - 1) \xi\|_2^2 &\leq \left(\sum_{k=1}^{K_R^{4Q}} \|\mathbb{1}_R \tilde{Q}_t^{B,Q} \mathbb{1}_{\tilde{C}_k^{4Q}(R)}\| \|\mathbb{1}_{\tilde{C}_k^{4Q}(R)} (\eta_R^2 - 1) \xi\|_2 \right)^2 \\
&\lesssim \left(\frac{t}{l} \right)^M \sum_{k=1}^\infty 2^{-kM} \|\mathbb{1}_{2^k R} (\eta_R^2 - 1) \xi\|_2^2 \\
&\lesssim \left(\frac{t}{l} \right)^M |R| \sum_{k=1}^\infty 2^{-k(M-n)} \\
&\lesssim \left(\frac{t}{l} \right)^M |R|
\end{aligned}
$$

Thus, integrating in t gives

$$IV = \iint_{\mathcal{C}(R)} |(\tilde{Q}_t^{B,Q} - \tilde{\gamma}_t^Q A_t)\eta_R \xi_R|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t} \lesssim \frac{|R|}{l^M}\int_0^l t^{M-1}\,\mathrm{d}t \lesssim \frac{|R|}{l^M}\frac{l^M}{M} \lesssim |R|$$

This completes the proof. $\square$

Thus we are able to prove the last term we need to estimate.

Proof of Proposition 4.4.1. As Proposition 4.4.6 implies (4.4.2) which then implies the measure $|\tilde{\gamma}_t^Q(x)|^2 \frac{\mathrm{d}t\,\mathrm{d}x}{t}$ is a Carleson measure on $4Q \times (0, l(Q)]$. Then by Lemma 4.4.3 and (4.4.1), we have

$$\sum_{Q \in \tilde{\Delta}_T^{\mathbf{B}}} \int_0^{l(Q)} \|\mathbb{1}_{4Q}\tilde{\gamma}_t^Q A_t u\|_2^2 \frac{\mathrm{d}t}{t} \lesssim \|u\|_2^2,$$

for all $u \in \overline{\mathsf{R}(D)}$, as required. $\square$

We are finally able to prove our main theorem of the chapter.

Proof of Theorem 4.0.2. We start by showing that Proposition 4.4.6 implies (4.4.2). Consider an arbitrary $R \in \Delta(4Q)$ and fix $\nu \in \mathcal{V}_\sigma$. Then, for all $\delta \in (0,1)$, using Lemma 4.3.1 gives

$$\begin{aligned}
\iint_{\mathcal{C}(R)} \mathbb{1}_{\tilde{\gamma}_t^Q(x)\in K_{\nu,\sigma}}(x,t)\mathbb{1}_{(\delta,\delta^{-1})}(t)|\tilde{\gamma}_t^Q(x)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t} &\le \frac{1}{\delta}\int_0^{\delta^{-1}}\int_R |\tilde{\gamma}_t^Q(x)|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\le \frac{1}{\delta}\int_0^{\delta^{-1}} \sum_{S\in\Delta_t(R)} \left(\int_S |\tilde{\gamma}_t^Q(x)|^2\,\mathrm{d}x\right)\mathrm{d}t \\
&\lesssim \frac{1}{\delta}\int_0^{\delta^{-1}} \left(\sum_{S\in\Delta_t(R)} |S|\right)\mathrm{d}t \\
&\lesssim \frac{|R|}{\delta^2},
\end{aligned}$$

where the implicit constant does not depend on Q. That is

$$\mu_\delta := \mathbb{1}_{\tilde{\gamma}_t^Q(x)\in K_{\nu,\sigma}}(x,t)\mathbb{1}_{(\delta,\delta^{-1})}(t)|\tilde{\gamma}_t^Q(x)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t}$$

is a Carleson measure on $4Q \times (0, l(Q)]$. We now show that $\|\mu_\delta\|_{\mathcal{C}}$ does not depend on

$\delta \in (0,1)$. Therefore, as $\mathcal{C}(R) = E^*_{R,\nu} \cup \big(\bigcup_{k\in I_R} \mathcal{C}(R)\big)$, and using (4.4.3), μ_δ being Carleson on $4Q \times (0, l(Q)]$, and the definition of $\{R_k\}_{k\in I_R}$ we have

$$\begin{aligned}
\iint_{\mathcal{C}(R)} \mathbb{1}_{\tilde{\gamma}_t^Q(x)\in K_{\nu,\sigma}}(x,t)\mathbb{1}_{(\delta,\delta^{-1})}(t)|\tilde{\gamma}_t^Q(x)|^2\frac{\mathrm{d}x\,\mathrm{d}t}{t} &= \iint_{\substack{(x,t)\in E^*_{R,\nu}\\ \tilde{\gamma}_t^Q(x)\in K_{\nu,\sigma}}} |\tilde{\gamma}_t^Q(x)|^2\frac{\mathrm{d}x\,\mathrm{d}t}{t} \\
&\quad + \sum_{k\in I_R}\mu_\delta(\mathcal{C}(R_k)) \\
&\leq C_0|R| + \|\mu_\delta\|_{\mathcal{C}}\sum_{k\in I_R}|R_k| \\
&\leq C_0|R| + \|\mu_\delta\|_{\mathcal{C}}|R\setminus E_{R,\nu}|,
\end{aligned}$$

where $C_0 > 0$ depends only on the constants in (4.4.3) and n. Note that the above is true for all $\nu \in \mathcal{V}_\sigma$. Also, as $|E_{R,\nu}| > \tau|R|$ then $|R \setminus E_{R,\nu}| < (1-\tau)|R|$. Therefore, dividing by $|R|$ and taking supremum over $R \in \Delta(4Q)$ with $4R \subset 4Q$, we have

$$\sup_{\substack{R\in\Delta(4Q)\\ 4R\subset 4Q}}\frac{1}{|R|}\mu_\delta(\mathcal{C}(R)) \leq \sup_{\substack{R\in\Delta(4Q)\\ 4R\subset 4Q}}\frac{1}{|R|}\iint_{\substack{(x,t)\in\mathcal{C}(R)\\ \tilde{\gamma}_t^Q(x)\in K_{\nu,\sigma}}}\mathbb{1}_{(\delta,\delta^{-1})}(t)|\tilde{\gamma}_t^Q(x)|^2\frac{\mathrm{d}x\,\mathrm{d}t}{t} \leq C_0+(1-\tau)\|\mu_\delta\|_{\mathcal{C}}$$

Then using Lemma 4.4.4 and above we have

$$\begin{aligned}
\|\mu_\delta\|_{\mathcal{C}} &= \sup_{R\in\Delta(4Q)}\frac{1}{|R|}\mu_\delta(\mathcal{C}(R)) \\
&\leq \sup_{\substack{R\in\Delta(4Q)\\ 4R\cap(\mathbb{R}^n\setminus 4Q)\neq\emptyset}}\frac{1}{|R|}\mu_\delta(\mathcal{C}(R)) + \sup_{\substack{R\in\Delta(4Q)\\ 4R\subset 4Q}}\frac{1}{|R|}\mu_\delta(\mathcal{C}(R)) \\
&\leq \sup_{\substack{R\in\Delta(4Q)\\ 4R\cap(\mathbb{R}^n\setminus 4Q)\neq\emptyset}}\frac{1}{|R|}\mu_0(\mathcal{C}(R)) + C_0 + (1-\tau)\|\mu_\delta\|_{\mathcal{C}} \\
&\leq C_1 + C_0 + (1-\tau)\|\mu_\delta\|_{\mathcal{C}},
\end{aligned}$$

where $C_1 > 0$ is the constant coming from Lemma 4.4.4. Therefore, rearranging gives

$$\|\mu_\delta\|_{\mathcal{C}} < \frac{C}{\tau}$$

Thus μ_δ is a Carleson measure on $4Q \times (0, l(Q)]$ independent of δ. Also, $\mathbb{1}_{(\delta,\delta^{-1})}(t)$ is a pointwise increasing function, by the monotone convergence theorem we have

$$\begin{aligned}
\iint_{\substack{(x,t)\in\mathcal{C}(R)\\ \tilde{\gamma}_t^Q(x)\in K_{\nu,\sigma}}} |\tilde{\gamma}_t^Q(x)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t} &= \iint_{\substack{(x,t)\in\mathcal{C}(R)\\ \tilde{\gamma}_t^Q(x)\in K_{\nu,\sigma}}} \lim_{\delta\to 0} \mathbb{1}_{(\delta,\delta^{-1})}(t)|\tilde{\gamma}_t^Q(x)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t} \\
&= \lim_{\delta\to 0} \iint_{\substack{(x,t)\in\mathcal{C}(R)\\ \tilde{\gamma}_t^Q(x)\in K_{\nu,\sigma}}} \mathbb{1}_{(\delta,\delta^{-1})}(t)|\tilde{\gamma}_t^Q(x)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t} \\
&\lesssim |R|,
\end{aligned}$$

where the implicit constant does not depend on Q. Therefore, as $\mathcal{V}_\sigma$ is finite, we have

$$\iint_{(x,t)\in\mathcal{C}(R)} |\tilde{\gamma}_t^Q(x)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t} \lesssim \sum_{\nu\in\mathcal{V}_\sigma} \iint_{\substack{(x,t)\in\mathcal{C}(R)\\ \tilde{\gamma}_t^Q(x)\in K_{\nu,\sigma}}} |\tilde{\gamma}_t^Q(x)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t} \lesssim |R|,$$

where the implicit constant does not depend on Q. That is, $|\tilde{\gamma}_t^Q(x)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t}$ is Carleson on $4Q \times (0, l(Q)]$. Thus by Lemma 4.4.3 we have

$$\sum_{Q\in\tilde{\Delta}_T^{\mathbf{B}}} \int_0^{l(Q)} \|\mathbb{1}_{4Q}\tilde{\gamma}_t^Q A_t u\|_2^2 \frac{\mathrm{d}t}{t} \lesssim \sum_{Q\in\tilde{\Delta}_T^{\mathbf{B}}} \|\eta_Q u\|_2^2 \lesssim \|u\|_2^2,$$

for all $u \in \overline{\mathsf{R}(D)}$. This proves Proposition 4.4.1. Thus combining this with Proposition 4.3.2 proves (4.2.4) and thus using Proposition 4.2.3 completes the proof. $\square$

CHAPTER 5

APPLICATIONS OF QUADRATIC ESTIMATES

In this chapter we will discuss applications of an operator of the form DB, where D is self-adjoint operator on a Hilbert space $\mathcal{H}$ and B is bound on $\mathcal{H}$ and elliptic on $\overline{\mathsf{R}(D)}$ as in Section 2.5, satisfying the quadratic estimate

$$\int_0^\infty \|tDB(I+(tDB)^2)^{-1}u\|_2^2 \frac{\mathrm{d}t}{t} \approx \|u\|_2^2, \tag{5.0.1}$$

for all $u \in \overline{\mathsf{R}(D)}$. These will include the famed Kato square root type estimate, perturbation results, and applications to initial value problems for a Cauchy–Riemann type equation. We first prove the operators of the form DB have a bounded holomorphic functional calculus. We note that in order to invoke Theorem 2.1.4 we need to restrict to an injective operator; however, by Proposition 2.5.1 the operators we will be considering will be injective on $\overline{\mathsf{R}(D)}$.

Theorem 5.0.1. Let $\mu \in (\omega, \frac{\pi}{2})$, where $\omega \in [0, \frac{\pi}{2})$ is the angle of ellipticity of B. Let DB satisfy the quadratic estimate (5.0.1). Then, there exists $c_\mu > 0$ which depends only on n, κ, and $\|B\|_\infty$, for all $f \in H^\infty(S_\mu^o)$

$$\|f(T)u\|_2 \leq c_\mu \|f\|_\infty \|u\|_2,$$

for all $u \in \overline{\mathsf{R}(D)}$, where T is the restriction of DB to $\overline{\mathsf{R}(D)}$. That is, $T\colon \overline{\mathsf{R}(D)} \to \overline{\mathsf{R}(D)}$ defined by $Tu := DBu$.

Proof. By Proposition 2.5.1 we have that T is a densely defined, closed, and injective operator. Also, by the hypothesis, we have

$$\int_0^\infty \|tT(I+t^2T^2)^{-1}u\|_2^2 \frac{\mathrm{d}t}{t} \approx \|u\|_2^2, \quad \text{for all } u \in \overline{\mathsf{R}(D)}.$$

Thus by Theorem 2.1.4 this is equivalent to T having bounded holomorphic functional calculus, as required. □

5.1 Kato Square Root Type Estimates

In this section we discuss the application of quadratic estimates to Kato square root type estimates. In particular, we give as a corollary the Kato square root type estimate for the Schrödiger operator and the purely magnetic Schrödinger operator. We note that the Kato result was proven for V with small $L^{\frac{n}{2}}$-norm in [29] and the results have recently been expanded in [16] to include $B^2(\mathbb{R}^n)$ and $L^{\frac{n}{2}}(\mathbb{R}^n)$ for $n \geq 4$. We reproduce these results

In order to give the Kato square root estimate we need to define the square root of an operator, to this we define the following sesquilinear form

$$J_{A,a,V}(u,v) := \int_{\mathbb{R}^n} A\nabla u \cdot \overline{\nabla v} + \int_{\mathbb{R}^n} aVu\overline{v} = \int_{\mathbb{R}^n} \mathcal{A}^{\mathcal{V}} \nabla_\mu u \cdot \overline{\nabla_\mu v},$$

for all $u, v \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^n)$. If $\operatorname{Re} J_{A,a,V}(u,u) \gtrsim \|u\|^2_{\dot{\mathcal{V}}^{1,2}(\mathbb{R}^n)}$ for all $u \in \mathcal{V}^{1,2}_c(\mathbb{R}^n)$, then $J_{A,a,V}$ is an accretive sesquilinear form, and we define $H_{\mathcal{A},V}$ to be the associated maximal accretive operator whereby $J_{A,a,V}(u,v) = \langle H_{\mathcal{A},V}u, v\rangle$ for all u in a dense domain $\mathscr{D}(H_{\mathcal{A},V})$ in L^2 (see, for instance, Chapter 6 in [37] for details on how this is done). We are now able to define the square root operator, $\sqrt{H_{\mathcal{A},V}}$, of $H_{\mathcal{A},V}$ as the unique maximal accretive operator such that $\sqrt{H_{\mathcal{A},V}}\sqrt{H_{\mathcal{A},V}} = H_{\mathcal{A},V}$, see [37, Theorem V.3.35] for more detail. Now the form satisfies

$$|J_{A,a,V}(u,v)| \leq \|\mathcal{A}^{\mathcal{V}}\|_\infty \int_{\mathbb{R}^n} |\nabla_\mu u||\nabla_\mu v| \leq \max\{\|A\|_\infty, \|a\|_\infty\}\|\nabla_\mu u\|_2 \|\nabla_\mu v\|_2.$$

Also, by (2.6.2) we have

$$|J_{A,a,V}(u,u)| \geq \mathrm{Re}\langle A\nabla u, \nabla u\rangle + \langle aV^{1/2}u, V^{1/2}u\rangle \geq \kappa\|\nabla_\mu u\|_2^2.$$

We now give a Kato square root type estimate for reverse Hölder potentials as a corollary of Theorem 5.0.1.

Corollary 5.1.1. Let $V \in B^{\frac{n}{2}}(\mathbb{R}^n)$. Let $A \in L^\infty(\mathbb{R}^n; \mathcal{L}(\mathbb{C}^n))$ and $a \in L^\infty(\mathbb{R}^n; \mathcal{L}(\mathbb{C}))$ be such that there exists $\kappa > 0$ satisfying

$$\mathrm{Re}\langle A\nabla u, \nabla u\rangle + \mathrm{Re}\langle a|V|^{\frac{1}{2}}u, |V|^{\frac{1}{2}}u\rangle \geq \kappa(\|\nabla u\|_2^2 + \||V|^{\frac{1}{2}}u\|_2^2),$$

for all $u \in \mathcal{V}^{1,2}(\mathbb{R}^n)$. Then we have

$$\|\sqrt{-\operatorname{div} A\nabla + aV}u\|_2 \eqsim \|\nabla u\|_2 + \||V|^{\frac{1}{2}}u\|_2,$$

for all $u \in \mathcal{V}^{1,2}(\mathbb{R}^n)$.

Proof. Let $\mu \in (\omega, \frac{\pi}{2})$, where ω is the angle of ellipticity of $\mathcal{A}$ Note that $f \colon S_\mu \to \mathbb{C}$ defined by $f(z) = \frac{\sqrt{z^2}}{z}$ is bounded and holomorphic. Then, define the operators

$$D := \begin{bmatrix} 0 & \operatorname{div} & -|V|^{\frac{1}{2}} \\ -\nabla & 0 & 0 \\ -|V|^{\frac{1}{2}} & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} I & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & a \end{bmatrix}.$$

Now, let $u \in \overline{\mathsf{R}(D)}$ and so using Lemma 2.3.6 we have

$$u = \begin{bmatrix} u_\perp \\ \nabla_\mu v \end{bmatrix},$$

for some $v \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^n)$. Then by the ellipticity of A and a we have

$$\begin{aligned}
\operatorname{Re}\langle Bu, u\rangle &= \langle u_\perp, u_\perp\rangle + \operatorname{Re}\langle A\nabla v, \nabla v\rangle + \operatorname{Re}\langle a|V|^{\frac{1}{2}}v, |V|^{\frac{1}{2}}v\rangle \\
&\geq \min\{1,\kappa\}(\|u_\perp\|_2^2 + \|\nabla v\|_2^2 + \||V|^{\frac{1}{2}}v\|_2^2) \\
&= \min\{1,\kappa\}\|u\|_2^2.
\end{aligned}$$

Therefore, by Theorem 3.0.1 we have that DB satisfies the quadratic estimate (5.0.1). Thus, by Theorem 5.0.1 we have

$$\|f(DB)u\|_2 \lesssim \|f\|_\infty \|u\|_2 = \|u\|_2.$$

We also have

$$(DB)^2 = \begin{bmatrix} -\operatorname{div} A\nabla + aV & 0 & 0 \\ 0 & -\nabla \operatorname{div} A & \nabla |V|^{\frac{1}{2}} a \\ 0 & -|V|^{\frac{1}{2}} \operatorname{div} A & |V|a \end{bmatrix}.$$

Therefore, square rooting the above and restricting u to the first component, that is letting $u = (u_\perp, 0, 0)^T$, we have

$$\|\sqrt{-\operatorname{div} A\nabla + aV}u_\perp\| = \|\sqrt{(DB)^2}u\|_2 \lesssim \|DBu\|_2 = \|Du\|_2 = \|\nabla u_\perp\|_2 + \||V|^{\frac{1}{2}}u_\perp\|_2,$$

where in the penultimate equality we use $B_{\perp\perp} = I$. The reverse estimate comes from considering $g(z) = \frac{z}{\sqrt{z^2}}$. This completes the proof. $\square$

We also get a Kato square root type estimate when $V \in L^{\frac{n}{2}}(\mathbb{R}^n)$ without the restriction of small norm. We remove this restriction by hiding the size of the norm in the perturbation a and proceeding as if V has small norm.

Corollary 5.1.2. Let $V \in L^{\frac{n}{2}}(\mathbb{R}^n)$. Let $A \in L^\infty(\mathbb{R}^n; \mathcal{L}(\mathbb{C}^n))$ and $a \in L^\infty(\mathbb{R}^n; \mathcal{L}(\mathbb{C}))$ be such that there exists $\kappa > 0$ satisfying

$$\operatorname{Re}\langle A\nabla u, \nabla u\rangle + \operatorname{Re}\langle a|V|^{\frac{1}{2}}u, |V|^{\frac{1}{2}}u\rangle \geq \kappa(\|\nabla u\|_2^2 + \||V|^{\frac{1}{2}}u\|_2^2),$$

for all $u \in W^{1,2}(\mathbb{R}^n)$. Then,

$$\|\sqrt{-\operatorname{div} A\nabla + aV}u\|_2 \eqsim \|\nabla u\|_2,$$

for all $u \in W^{1,2}(\mathbb{R}^n)$.

Proof. Define $\tilde{V}(x) := \frac{\varepsilon|V(x)|}{\|V\|_{\frac{n}{2}}}$ and $\tilde{a} := e^{i\arg(V(x))}\frac{a(x)\|V\|_{\frac{n}{2}}}{\varepsilon}$ where $\varepsilon > 0$ is such that $\|\tilde{V}\|_{\frac{n}{2}} = \varepsilon$ is sufficiently small. Therefore, $\tilde{a}\tilde{V} = aV$. Then, define the operators

$$D = \begin{bmatrix} 0 & \operatorname{div} & -\tilde{V}^{\frac{1}{2}} \\ -\nabla & 0 & 0 \\ -\tilde{V}^{\frac{1}{2}} & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} I & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & \tilde{a} \end{bmatrix}.$$

Now, let $u \in \overline{\mathsf{R}(D)}$, then by the ellipticity of A and a we have that

$$\begin{aligned}
\operatorname{Re}\langle Bu, u\rangle &= \langle u_\perp, u_\perp\rangle + \operatorname{Re}\langle A\nabla v, \nabla v\rangle + \operatorname{Re}\langle \tilde{a}\tilde{V}^{\frac{1}{2}}v, \tilde{V}^{\frac{1}{2}}v\rangle \\
&= \langle u_\perp, u_\perp\rangle + \operatorname{Re}\langle A\nabla v, \nabla v\rangle + \operatorname{Re}\langle a|V|^{\frac{1}{2}}v, |V|^{\frac{1}{2}}v\rangle \\
&\geq \min\{1, \kappa\}(\|u_\perp\|_2^2 + \|\nabla v\|_2^2 + \||V|^{\frac{1}{2}}v\|_2^2) \\
&= \min\{1, \kappa\}\|u\|_2^2.
\end{aligned}$$

Then the same argument as in Corollary 5.1.1 and then Hölder's inequality, gives

$$\|\sqrt{-\operatorname{div} A\nabla + aV}u\|_2 = \|\sqrt{-\operatorname{div} A\nabla + \tilde{a}\tilde{V}}u\|_2 \eqsim \|\nabla u\|_2 + \|\tilde{V}^{\frac{1}{2}}u\|_2 \eqsim \|\nabla u\|_2.$$

for all $u \in W^{1,2}(\mathbb{R}^n)$. As required. □

We also have a Kato square root type estimates associated with the purely magnetic Schrödinger operator.

Corollary 5.1.3. Let $b \in L^2_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^n)$ with $\operatorname{curl} b = \mathbf{B}$ satisfying the conditions (2.4.3). Let $A \in L^\infty(\mathbb{R}^n; \mathcal{L}(\mathbb{C}^n))$ be such that there exists $\kappa > 0$ satisfying

$$\operatorname{Re}\langle A(\nabla + ib)u, (\nabla + ib)u\rangle + \geq \kappa\|(\nabla + ib)u\|_2^2,$$

for all $u \in W_b^{1,2}(\mathbb{R}^n)$. Then we have that

$$\|\sqrt{(\nabla + ib)^* A(\nabla + ib)}u\|_2 \approx \|(\nabla + ib)u\|_2,$$

for all $u \in W_b^{1,2}(\mathbb{R}^n)$.

Proof. Again we apply Theorem 5.0.1 to the operators

$$D = \begin{bmatrix} 0 & (\nabla + ib)^* \\ (\nabla + ib) & 0 \end{bmatrix}, \quad B = \begin{bmatrix} I & 0 \\ 0 & A \end{bmatrix}.$$

Then the result follows in a similar fashion to Corollaries 5.1.1 and 5.1.2. □

5.2 Analytic Dependence and Lipschitz Estimates

Here we show that the functional calculus depends analytically on the perturbation B equipped with the L^∞-norm. We follow the same method as in [13, Section 6] by first showing that the resolvents depend analytically on B and then building up to functions in the class $\Phi(S_\mu^o)$, and finally to all functions in $H^\infty(S_\mu^o)$, where $\mu \in (\omega, \frac{\pi}{2})$ and $\omega \in [0, \frac{\pi}{2})$ is the angle of ellipticity of B. Let $\mathcal{H} := L^2(\mathbb{R}^n; \mathbb{C}^N)$ for some $n, N\mathbb{N}$, for the rest of this section.

Theorem 5.2.1. Let $D \colon \mathcal{H} \to \mathcal{H}$, be a self-adjoint operator. Let $U \subseteq \mathbb{C}$ be open. Let $B : U \to L^\infty(\mathbb{R}^n; \mathcal{L}(\mathbb{C}^N))$ be holomorphic, such that B is uniformly bounded in U and there exists $\kappa > 0$ such that

$$\mathrm{Re}\langle B_z u, u\rangle \geq \kappa \|u\|_2^2, \quad \forall u \in \mathcal{H}, \forall z \in U.$$

Let $\mu \in (\omega, \frac{\pi}{2})$ where $\omega \in [0, \frac{\pi}{2})$ is the angle of ellipticity of B. Then

1. $z \mapsto (I + tDB_z)^{-1}$ is holomorphic in U for all $t \in \mathbb{C} \setminus S_\mu^o$;

2. $z \mapsto \mathbb{P}_{\mathsf{N}(DB_z)}$ is holomorphic in U, where $\mathbb{P}_{\mathsf{N}(DB_z)}$ is the projection onto the subspace $\mathsf{N}(DB_z)$;

3. $z \mapsto \psi(DB_z)$, for all $\psi \in \Psi(S^o_\mu)$ is holomorphic in U.

Moreover, if there exists $C_\mu > 0$ such that $\|f(DB_z)u\|_2 \leq C_\mu \|f\|_\infty \|u\|_2$ for all $f \in H^\infty(S^o_\mu)$, uniformly in $z \in U$, then $z \mapsto f(DB_z)$ is holomorphic in U for all $f \in H^\infty(S^o_\mu)$.

Proof. Let $z \in U$ and $t \in \mathbb{C} \setminus S^o_\mu$. Then we have

$$\frac{\mathrm{d}}{\mathrm{d}z}(I + tDB_z)^{-1}u = -(I + tDB_z)^{-1}tD\left(\frac{\mathrm{d}}{\mathrm{d}z}B_z\right)(I + tDB_z)^{-1}u.$$

Then, using the fact that $Q_t = P_t tDB$ is uniformly bounded and $\mathcal{H} = \overline{\mathsf{R}(D)} \oplus N(D)$, we have

$$\begin{aligned}
\|\frac{\mathrm{d}}{\mathrm{d}z}(I + tDB_z)^{-1}u\|_2 &= \|(I + tDB_z)^{-1}tD\left(\frac{\mathrm{d}}{\mathrm{d}z}B_z\right)(I + tDB_z)^{-1}u\|_2, \\
&= \|(I + tDB_z)^{-1}tD(\mathbb{P}_{\overline{\mathsf{R}(D)}} + \mathbb{P}_{\mathsf{N}(D)})\left(\frac{\mathrm{d}}{\mathrm{d}z}B_z\right)(I + tDB_z)^{-1}u\|_2, \\
&= \|\left((I + tDB_z)^{-1}tDB_z\right)\left(B_z^{-1}\mathbb{P}_{\overline{\mathsf{R}(D)}}\right)\left(\frac{\mathrm{d}}{\mathrm{d}z}B_z\right)(I + tDB_z)^{-1}u\|_2, \\
&\lesssim \|u\|_2,
\end{aligned}$$

here the bound is independent of $z \in U$. Therefore, $z \mapsto (I + tDB_z)^{-1}$ is holomorphic on U. In particular, we have that $z \mapsto (I + inDB_z)^{-1}$ is holomorphic for all $n \in \mathbb{N}$.

We claim that $\mathbb{P}_{\mathsf{N}(DB_z)}u = \lim_{n\to\infty}(I + tDB_z)^{-1}u$ in $\mathcal{H}$ for all $u \in \mathcal{H}$. Let $u \in \mathsf{N}(DB_z)$. Then

$$(I + inDB_z)^{-1}u = (I + inDB_z)^{-1}(u + inDB_z u) = u,$$

for all $n \in \mathbb{N}$. Now, let $u \in \mathsf{R}(DB_z)$. Then, there exists $v \in \mathscr{D}(DB_z)$ such that $u = DB_z v$. Therefore,

$$\begin{aligned}
\|(I + inDB_z)^{-1}u\|_2 &= \frac{1}{n}\|(I + inDB_z)^{-1}inDB_z v\|_2 \\
&\leq \frac{1}{n}\left(\|(I + inDB_z)^{-1}(v + inDB_z v)\|_2 + \|v\|_2\right) \\
&\lesssim \frac{1}{n}\|v\|_2,
\end{aligned}$$

where the implicit constant is independent of n and z. That is, $\lim_{n\to\infty}(I + inDB_z)^{-1}u = 0$

for all $u \in \mathsf{R}(DB_z)$. Now, let $u \in \overline{\mathsf{R}(DB_z)}$. Then, there exists $\{u_m\}_{m\in\mathbb{N}} \subseteq \mathsf{R}(DB_z)$ such that $u_m \to u$ as $m \to \infty$ in L^2. Let $\varepsilon > 0$. Then

$$\begin{aligned}\|(I + inDB_z)^{-1}u\|_2 &\leq \|(I + inDB_z)^{-1}(u - u_m)\|_2 + \|(I + inDB_z)^{-1}u_m\|_2 \\ &\lesssim \|u - u_m\|_2 + \|(I + inDB_z)^{-1}u_m\|_2,\end{aligned}$$

where the implicit constant is independent of n, m, and z. Now choose $m \in \mathbb{N}$ be such that $\|u - u_m\|_2 \leq \frac{\varepsilon}{2}$. Now, let $N \in \mathbb{N}$ be such that $\|(I + inDB_z)^{-1}u_m\|_2 \leq \frac{\varepsilon}{2}$ for all $n > N$. Thus $\|(I + inDB_z)^{-1}u\|_2 \lesssim \varepsilon$ for all $n > N$. This proves the claim. As $\mathbb{P}_{\mathsf{N}(DB_z)}u$ is the limit of holomorphic operators and so is holomorphic itself (see for example [37]).

Let $\psi \in \Psi(S^o_\mu)$. Then

$$\psi(DB_z) = \int_\gamma \psi(\lambda)(\lambda - DB_z)^{-1}\, \mathrm{d}\lambda.$$

Using the approximation of the contour integral by Riemann sums and the fact that the Riemann sums are holomorphic gives the desired result.

Now assume further that $\|f(DB_z)u\|_2 \leq C_\mu\|f\|_\infty\|u\|_2$ for all $f \in H^\infty(S^o_\mu)$. Let $f \in H^\infty(S^o_\mu)$. Then, choose a uniformly bounded sequence $\{\psi_n\} \subseteq \Psi(S^o_\mu)$ which converges uniformly on compact sets to f (to see that we may choose such a sequence, see [1, Lecture 3]). Then, by the convergence lemma we have that $f(DB_z)u = \lim_{n\to\infty} \psi_n(DB_z)u$ in L^2. Thus, $f(DB_z)$ is holomorphic on U. This completes the proof □

Theorem 5.2.2. Let $D \colon \mathcal{H} \to \mathcal{H}$, be a self-adjoint operator and let $B \in L^\infty(\mathbb{R}^n; \mathcal{L}(\mathbb{C}^N))$ be elliptic as in (2.5.1). Suppose further that DB has a bounded holomorphic functional calculus. Let $\mu \in (\omega, \frac{\pi}{2})$ where $\omega \in [0, \frac{\pi}{2})$ is the angle of ellipticity of B. Let $0 < \delta < \kappa$, and $\tilde{B} \in L^\infty(\mathbb{R}^n; \mathcal{L}(\mathbb{C}^N))$ such that $\|B - \tilde{B}\|_\infty < \delta$. Then

$$\|f(DB)u - f(D\tilde{B})u\|_2 \lesssim \|B - \tilde{B}\|_\infty\|f\|_\infty\|u\|_2, \quad \forall f \in H^\infty(S^o_\mu),$$

where the implicit constant depends only on n, κ, $\|B\|_\infty$, and δ.

Proof. Let $f \in H^\infty(S^o_\mu)$. Define $B\colon \{z \in \mathbb{C}\colon |z| < \delta\} \to L^\infty(\mathbb{R}^n; \mathcal{L}(\mathbb{C}^n))$, given by

$$B(z) := B_z := B + \frac{z(\tilde{B} - B)}{\|B - \tilde{B}\|_\infty}$$

Then B_z is holomorphic and we have that

$$\begin{aligned}
\operatorname{Re}\langle B_z u, u\rangle &= \operatorname{Re}\langle Bu, u\rangle - \operatorname{Re}\left\langle \frac{z}{\|B - \tilde{B}\|_\infty}(\tilde{B} - B)u, u\right\rangle \\
&\geq \kappa\|u\|_2^2 - \delta \operatorname{Re}\langle u, u\rangle \\
&\geq (\kappa - \delta)\|u\|_2^2.
\end{aligned}$$

We also have that

$$\|B_z\| \leq \|B\|_\infty + |z| < \|B\|_\infty + \delta.$$

Thus, B_z is uniformly bounded and uniformly elliptic. Therefore, by Theorem 5.0.1 we have $\|f(DB_z)u\|_2 \lesssim \|f\|_\infty \|u\|_2$ uniformly in z. Thus, by Theorem 5.2.1 we have $z \mapsto f(DB_z)$ is holomorphic. Now fix $u \in \overline{\mathsf{R}(D)}$ and define $G_u : \{z \in \mathbb{C}\colon |z| < \delta\} \to \mathcal{L}(L^2(\mathbb{R}^n))$, given by

$$G_u(z) := \frac{f(DB)u - f(DB_z)u}{2c\|f\|_\infty \|u\|_2},$$

where c is the uniform constant such that $\|f(DB_z)u\|_2 \leq c\|f\|_\infty\|u\|_2$. By Theorem 5.2.1 and the bounded holomorphic functional calculus of DB_z, we have G_u is is holomorphic and

$$\|G_u(z)\| \leq \frac{1}{2c\|f\|_\infty\|u\|_2}\|f(DB)u - f(DB_z)u\|_2 \leq 1.$$

As G_u is holomorphic then the pairing $(G_u(z), f)$ is holomorphic as a function, for all $f \in L^2(\mathbb{R}^n; \mathbb{C}^{n+2})'$. In particular, we have for all $f \in L^2(\mathbb{R}^n; \mathbb{C}^{n+2})'$ with $\|f\| \leq 1$ then

$$|(G_u(z), f)| \leq \|f\|\|G_u(z)\|_2 \leq 1$$

Therefore, by Schwarz's lemma we have that

$$\|G_u(z)\|_2 = \sup_{\|f\|=1} |(G_u(z), f)| \leq |z|.$$

Thus, choosing $z = \|B - \tilde{B}\|_\infty < \delta$ gives

$$\|f(DB)u - f(D\tilde{B})u\|_2 \leq 2c\|f\|_\infty \|B - \tilde{B}\|_\infty \|u\|_2.$$

Here we note that $u \in \overline{\mathsf{R}(D)}$ was arbitrary and the constants are all independent of u. This completes the proof. □

5.3 The Global Well-Posedness for First-Order Initial Value Problems

In this section we discuss the applications quadratic estimates and bounded holomorphic functional calculus to solving initial value problems to first-order Cauchy–Riemann type equations of the following form

$$\begin{cases} \partial_t F + DBF = 0, & \text{in } \mathbb{R}_+, \\ F(t) \in \overline{\mathsf{R}(D)}, & \forall t > 0, \end{cases}$$

where $D\colon \mathscr{D}(D) \to \mathcal{H}$ is a self-adjoint operator, B is a bounded operator on $\mathcal{H}$ and elliptic on $\overline{\mathsf{R}(D)}$ as in Section 2.5, we also make the additional assumption that $\mathcal{H} = L^2(\mathbb{R}^n; \mathbb{C}^N)$ for some $n, N \in \mathbb{N}$. To begin, we make precise the definition of a weak solution of

$$\partial_t F + DBF = 0 \quad \text{in } \mathbb{R}^{n+1}_+. \tag{5.3.1}$$

We will adopt the convention for functions $\phi\colon \mathbb{R}^{n+1}_+ \to \mathbb{C}^N$ and $t \in \mathbb{R}_+$, whereby $\phi(t)\colon \mathbb{R}^n \to \mathbb{C}^N$ is defined by $(\phi(t))(x) := \phi(t, x)$ for all $x \in \mathbb{R}^n$.

Definition 5.3.1. We shall write that F *is a weak solution of* $\partial_t F + DBF = 0$ *in* $\mathbb{R}_+$, or simply $\partial_t F + DBF = 0$ *in* $\mathbb{R}_+$, if $F \in L^2_{\text{loc}}(\mathbb{R}^{n+1}_+; \mathbb{C}^N)$ and

$$\int_0^\infty \langle F(t), \partial_t \varphi(t) \rangle \, \mathrm{d}t = \int_0^\infty \langle BF(t), D\varphi(t) \rangle \, \mathrm{d}t,$$

for all $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^{n+1}_+; \mathbb{C}^N)$.

To solve these equations we will use the theory of analytic semigroups to generate solutions. However, it is well known that analytic semigroups are generated by sectorial operators (see [25] for details) but, the operator DB is ω-bisectorial for some $\omega \in [0, \frac{\pi}{2})$. To resolve this we restrict to a subspaces on which DB is sectorial and then the theory of analytic semigroups states that on this domain DB will generate an analytic semigroup. This is where the bounded holomorphic functional calculus of DB is seen to be critical as this provides a splitting of $\overline{\mathsf{R}(D)}$. For $\mu \in (\omega, \frac{\pi}{2})$ we start by defining the following holomorphic functions on S^o_μ in a similar manner as in [4]:

$$\chi^\pm(z) := \begin{cases} 1 & \text{if } \pm \operatorname{Re}(z) > 0 \\ 0 & \text{if } \pm \operatorname{Re}(z) \le 0. \end{cases}, \quad \operatorname{sgn}(z) := \chi^+(z) - \chi^-(z), \text{ and } [z] := z \operatorname{sgn}(z),$$

for all $z \in S^o_\mu$. Then let $E^\pm_{DB} := \chi^\pm(DB)$ be the generalised Hardy-type projections of DB. Let $E_{DB} := \operatorname{sgn}(DB) = E^+_{DB} - E^-_{DB}$. Let $\mathcal{H}^0_{DB} := \overline{\mathsf{R}(D)}$, and define $\mathcal{H}^{0,\pm}_{DB} := E^\pm_{DB}\overline{\mathsf{R}(D)} = \{E^\pm_{DB} f : f \in \overline{\mathsf{R}(D)}\}$. Note $\chi^+(z) + \chi^-(z) = 1$ for all $z \in S^o_\mu$. Then for $f \in \mathcal{H}^0_{DB}$ we have $f = E^+_{DB} f + E^-_{DB} f$ so

$$\|f\|_2 \le \|E^+_{DB} f\|_2 + \|E^-_{DB} f\|_2.$$

Now as DB has bounded H^∞ functional calculus, then

$$\|E^+_{DB} f\|_2 + \|E^-_{DB} f\|_2 \lesssim (\|\chi^+\|_\infty + \|\chi^-\|_\infty)\|f\|_2 = 2\|f\|_2.$$

Therefore we have the topological splitting $\mathcal{H}^0_{DB} = \mathcal{H}^{0,+}_{DB} \oplus \mathcal{H}^{0,-}_{DB}$. We use the $\mathcal{F}$-functional calculus (as defined in (2.1.1)) to define the operator $[DB] = DB \operatorname{sgn}(DB)$. We see that for $f \in \mathcal{H}^{0,\pm}_{DB}$

$$[DB]f = DB(\chi^+(DB) - \chi^-(DB))\chi^\pm(DB)f = \pm DB\chi^\pm(DB)f = \pm DBf.$$

We give a notion of the Cauchy problem for the first-order equation so that we can solve the first-order initial value problems on $\mathcal{H}^{0,\pm}_{DB}$.

Definition 5.3.2. We shall write that (5.3.2) is globally well-posed in $\mathcal{H}^{0,+}_{DB}$ if for each $f \in \mathcal{H}^{0,+}_{DB}$, there exists a unique $F \in \mathcal{C}^1(\mathbb{R}_+; \mathcal{H}^{0,+}_{DB})$ such that

$$\begin{cases} \partial_t F + DBF = 0 \text{ in } \mathbb{R}_+ \\ \sup_{t>0} \fint_t^{2t} \|F(s)\|^2_{L^2(\mathbb{R}^n,\mathbb{C}^N)} \,\mathrm{d}s < \infty, \\ \lim_{t\to 0} F(t) = f, \end{cases} \tag{5.3.2}$$

where the limit converges pointwise on Whitney averages as in (2.6.3) (and in L^2).

We will do this by constructing an analytic semigroup which solves (5.3.2). First note that as $f_t(z) := e^{-t[z]}$ is a bounded holomorphic function for all $t \in \mathbb{R}^n$. Therefore, we can define the family of bounded operators $\left(e^{-t[DB]}\right)_{t>0}$ by using the bounded holomorphic functional calculus of DB.

Lemma 5.3.3. Let $\mu \in (0,\omega)$. Then the family of operators $\left(e^{-z[DB]}\right)$, where $z \in S^o_{(\frac{\pi}{2}-\mu)+}$ forms an analytic semigroup on $L^2(\mathbb{R}^n;\mathbb{C}^N)$ with generator $[DB]$. Moreover, $[DB]$ is a sectorial operator on $L^2(\mathbb{R}^n;\mathbb{C}^N)$ of type $S_{\mu+}$.

Proof. First notice that $z\,\mathrm{sgn}(z) = (z^2)^{\frac{1}{2}}$. Then by [8, Proposition 8.2] we have $[DB] = DB\,\mathrm{sgn}(DB)((DB)^2)^{\frac{1}{2}}$ is sectorial of type $S_{\mu+}$. Then as $[DB]$ is sectorial we have $\left(e^{-z[DB]}\right)$, where $z \in S^o_{(\frac{\pi}{2}-\mu)+}$ is an analytic semigroup from classical semigroup theory (see [25, Theorem 4.6] for details). □

We now give a proposition which shows that the semigroup is a solution of (5.3.1) and satisfies some important estimates.

Proposition 5.3.4. Let $f \in \mathcal{H}^{0,+}_{DB}$ and define $F(t,x) := e^{-t[DB]}f(x)$. Then we have $F \in \mathcal{C}^\infty(\mathbb{R}_+;\mathcal{H}^{0,+}_{DB})$ and $\partial_t F + DBF = 0$ on $\mathbb{R}^{n+1}_+$ in the strong sense with bounds

$$\sup_{t>0}\|F(t)\|_2^2 \eqsim \|f\|_2^2 \eqsim \sup_{t>0} \fint_t^{2t} \|F(s)\|_2^2 \,\mathrm{d}s,$$

and limits, where the convergence is in the L^2 sense,

$$\lim_{t\to 0} F(t) = f, \quad \lim_{t\to\infty} F(t) = 0.$$

Proof. That fact that $\partial_t F + DBF = 0$ on $\mathbb{R}^{n+1}_+$ in the strong sense and the limits at 0 and infinity come from the theory of analytic semigroups. Now using the bounded holomorphic functional calculus we have

$$\sup_{t>0}\|F(t)\|_2 \lesssim \sup_{t>0}\|e^{-tz}\|_\infty\|f\|_2 \le \|f\|_\infty.$$

Similarly, we have

$$\sup_{t>0}\fint_t^{2t}\|F(s)\|_2^2\,\mathrm{d}s \lesssim \|f\|_2^2.$$

Let $\varepsilon > 0$, and $T > 0$ be such that $\|f - F(s)\|_2 < \varepsilon$ for all $s < 2T$. Then

$$\|f\|_2^2 = \fint_T^{2T}\|f\|_2^2\,\mathrm{d}s \le \fint_T^{2T}\|f - F(s)\|_2^2\,\mathrm{d}s + \fint_T^{2T}\|F(s)\|_2^2\,\mathrm{d}s < \varepsilon + \sup_{t>0}\fint_t^{2t}\|F(s)\|_2^2\,\mathrm{d}s.$$

Then as $\varepsilon > 0$ was arbitrary we have

$$\|f\|_2^2 \le \sup_{t>0}\fint_t^{2t}\|F(s)\|_2^2\,\mathrm{d}s.$$

A similar argument gives

$$\|f\|_2 \le \sup_{t>0}\|F(t)\|_2.$$

This completes the proof. $\square$

From this point on we will specialise to the case in Chapter 3 and Section 2.3. That is

$$D = \begin{bmatrix} 0 & -(\nabla_\mu)^* \\ -\nabla_\mu & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} B_{\perp\perp} & B_{\perp\|} & 0 \\ B_{\|\perp} & B_{\|\|} & 0 \\ 0 & 0 & b \end{bmatrix}$$

as defined in Section 2.3. We will work to prove the converse of Proposition 5.3.4 following the methods in [2]. We require a lemma that is analogous to [2, Proposition 4.4]. But, first we have the following lemma about the test functions we will use in Lemma 5.3.7.

Lemma 5.3.5. Let $\psi \in \overline{\mathsf{R}(D)}$. Let $t > 0$ and $\eta_+ \in \mathcal{C}_c^\infty(0,t)$. For $s \in (0,t)$, define $\varphi(s) \coloneqq \eta_+(s)\left(e^{-(t-s)[DB]}E_{DB}^+\right)^* \psi$. If $F \in L^2_{\text{loc}}(\mathbb{R}_+; \mathsf{R}(D))$ is a weak solution to $\partial_t F + DBF = 0$ in $\mathbb{R}_+$, then we have the following

$$\int_0^t \langle F(s), \partial_s \varphi(s)\rangle \, \mathrm{d}s = \int_0^t \langle BF(s), D\varphi(s)\rangle \, \mathrm{d}s.$$

Proof. Let $\psi \in \overline{\mathsf{R}(D)}$. Let $t > 0$ and consider a test function $\eta_+ \in \mathcal{C}_c^\infty(0,t)$. For $s \in (0,t)$, define $\varphi(s) \coloneqq \eta_+(s)\left(e^{-(t-s)[DB]}E_{DB}^+\right)^* \psi$. Then by the properties of semigroups we have $\varphi \in \mathcal{C}_c^1(\mathbb{R}_+; \mathscr{D}(D))$. We aim to use Definition 5.3.2, and so we construct smooth functions $\varphi^{k,r,\delta} \in \mathcal{C}_c^\infty(\mathbb{R}_+^{n+1})$ for which $\partial_s \varphi^{k,r,\delta} \to \partial_s \varphi$ and $D\varphi^{k,r,\delta} \to D\varphi$ in $L^2((0,t)\times\mathbb{R}^n; \mathbb{C}^{n+2})$ as $k \to \infty$, $r \to \infty$, and $\delta \to 0$. To this end, let $\eta \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ with $\eta(x) = 1$ on a neighbourhood of 0 and $\int_{\mathbb{R}^n} \eta = 1$, then define

$$\varphi^{k,r,\delta}(s) \coloneqq \left[\eta_\delta * \left(\eta\left(\frac{\cdot}{r}\right) g_k(\eta_+(s)(e^{-(t-s)[DB]}E_{DB}^+)^*\psi)\right)\right](x),$$

where $\eta_\delta(x) \coloneqq \delta^{-n}\eta(\frac{x}{\delta})$ and

$$g_k(x) \coloneqq k \tanh\left(\frac{x}{k}\right),$$

where tanh is the hyperbolic tangent. We note that the convolution and g_k are applied componentwise. Note that, $\tanh\colon \mathbb{R} \to (-1,1)$ is smooth and $\tanh'(x) = 1 - \tanh^2(x) \in (0,1]$. Therefore by the mean value theorem, for all $x, y \in \mathbb{R}$, we have

$$|\tanh(x) - \tanh(y)| \leq |x - y|.$$

In particular, as $\tanh(0) = 0$, for all $x \in |\mathbb{R}$, we have

$$|\tanh(x)| \leq |x|.$$

Thus we have the following two bounds for all $x \in \mathbb{R}$

$$|g_k(x)| \leq k|\tanh\left(\frac{x}{k}\right)| \leq k \quad \text{ and } \quad |g_k(x)| \leq k\left|\frac{x}{k}\right| \leq |x|.$$

We aim to prove that $\lim_{k\to\infty} g_k(x) = x$ for all $x \in \mathbb{R}$. Therefore, fix $x \in \mathbb{R}$. Recall the definition for the hyperbolic tangent as

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

Then, we have the following

$$|g_k(x) - x| = \left|\frac{k(e^{\frac{x}{k}} - e^{-\frac{x}{k}})}{e^{\frac{x}{k}} + e^{-\frac{x}{k}}} - x\right| = \frac{1}{e^{\frac{x}{k}} + e^{-\frac{x}{k}}}|k(e^{\frac{x}{k}} - e^{-\frac{x}{k}}) - x(e^{\frac{x}{k}} + e^{-\frac{x}{k}})|$$

Now using the Taylor expansion we have

$$k(e^{\frac{x}{k}} - e^{-\frac{x}{k}}) = \sum_{l=0}^{\infty} \frac{2x^{2l+1}}{k^{2l}(2l+1)!} \quad \text{and} \quad x(e^{\frac{x}{k}} + e^{-\frac{x}{k}}) = \sum_{l=0}^{\infty} \frac{2x^{2l+1}}{k^{2l}(2l)!}.$$

Now, as $e^{\frac{x}{k}} + e^{-\frac{x}{k}} \geq 1$ for all $x \in \mathbb{R}^n$ and $k \in \mathbb{N}$, and note that the zeroth order terms in the Taylor series cancel. Thus, we have

$$|g_k(x) - x| \leq \left|\sum_{l=1}^{\infty} \frac{2x^{2l+1}}{k^{2l}} \left(\frac{1}{(2l+1)!} - \frac{1}{(2l)!}\right)\right| \to 0, \text{ as } k \to \infty.$$

We now begin to prove that $D\varphi^{k,r,\delta} \to D\varphi$ in $L^2((0,t)\times\mathbb{R}^n;\mathbb{C}^{n+2}))$ as $k \to \infty$, $r \to \infty$, and $\delta \to 0$. Let $f := \varphi_\perp \in \mathcal{C}_c^\infty(\mathbb{R}_+;\dot{\mathcal{V}}^{1,2}(\mathbb{R}^n))$ and $h := \varphi_r \in \mathcal{C}_c^\infty(\mathbb{R}_+;\mathscr{D}((\nabla_\mu^\|)^*))$. We will use a triangularisation argument. First consider

$$\begin{aligned}
&\left\|\nabla_\mu^\| \left(f - \left[\eta_\delta * \left(\eta\left(\frac{\cdot}{r}\right) g_k(f)\right)\right]\right)\right\|_{L^2((0,t)\times\mathbb{R}^n)} \\
&\quad \leq \|\nabla_\mu^\| \left(f - g_k(f)\right)\|_{L^2((0,t)\times\mathbb{R}^n)} + \|\nabla_\mu^\| \left(g_k(f) - \eta\left(\frac{\cdot}{r}\right) g_k(f)\right)\|_{L^2((0,t)\times\mathbb{R}^n)} \\
&\qquad + \left\|\nabla_\mu^\| \left(\eta\left(\frac{\cdot}{r}\right) g_k(f) - \left[\eta_\delta * \left(\eta\left(\frac{\cdot}{r}\right) g_k(f)\right)\right]\right)\right\|_{L^2((0,t)\times\mathbb{R}^n)},
\end{aligned}$$

Note, using Theorem 7.8 in [30] and the bounds on g_k', we have

$$|\nabla_\| g_k(f)| = |g_k'(f) \cdot \nabla_\| f| \leq |\nabla_\| f|,$$

again, where g_k' is applied componentwise. Thus, we get

$$|\nabla_\|(f - g_k(f))|^2 \leq 2^2(|\nabla_\| f|^2 + |\nabla_\| g_k(f)|^2) \leq 2^3|\nabla_\| f|^2 \in L^1((0,t)\times\mathbb{R}^n),$$

for all $k \in \mathbb{N}$, and, similarly

$$||V|^{\frac{1}{2}}(f - g_k(f))|^2 \leq 2^3|V||f|^2 \in L^1((0,t)\times\mathbb{R}^n),$$

for all $k \in \mathbb{N}$. Here the integrability is because $f \in \mathcal{C}_c^\infty(\mathbb{R}_+; \dot{\mathcal{V}}^{1,2}(\mathbb{R}^n))$. Hence, by the dominated convergence theorem we have

$$\|\nabla_\mu^\|(f - g_k(f))\|_{L^2((0,t)\times\mathbb{R}^n)} \to 0,$$

as $k \to \infty$. The second term follows similarly using Theorem 6.13 [39], $\eta \in L^\infty(\mathbb{R}^n)$, and then the dominated convergence theorem. To prove the last term converges to 0 we will use a standard mollifying argument. To this end let $\tilde{f} := \eta\left(\frac{\cdot}{r}\right) g_k(f)$. Notice that from the definition of η we have $\operatorname{supp}(\tilde{f}(s)) \subseteq B_r(0)$ and the definition of g_k and η gives $\|\tilde{f}\|_\infty \leq \|\eta\|_\infty k$. Then

$$|(\eta_\delta * \tilde{f}(s))(x)| = \left|\int \delta^{-n}\eta\left(\frac{x-y}{\delta}\right)\tilde{f}(s,y)\,\mathrm{d}y\right| \leq \int_{B_r(0)\cap B_\delta(x)} \left|\eta\left(\frac{x-y}{\delta}\right)\tilde{f}(s,y)\right|\mathrm{d}y.$$

Now making the additional restriction $\delta \in (0,1)$ we see

$$|(\eta_\delta * \tilde{f}(s))(x)| \leq \begin{cases} 0, & \text{if } x \in \mathbb{R}^n \setminus B_{r+1}(0) \\ \fint_{B_\delta(x)} \|\eta\|_\infty \|\tilde{f}\|_\infty, & \text{if } x \in B_{r+1}(0) \end{cases} \leq \|\eta\|_\infty^2 k \mathbb{1}_{B_{r+1}(0)}(x).$$

Therefore,

$$|V||(\eta_\delta * \tilde{f})|^2 \leq |V|\|\eta\|_\infty^4 k^2 \mathbb{1}_{B_{1+1}(0)}(x)\mathbb{1}_{(0,t)}(s) \in L^1((0,t)\times\mathbb{R}^n),$$

Thus, by the dominated convergence theorem and Theorem 7.6 in [39], we have

$$\left\|\nabla_\mu \left(\eta\left(\frac{\cdot}{r}\right) g_k(f) - \left[\eta_\delta * \left(\eta\left(\frac{\cdot}{r}\right) g_k(f)\right)\right]\right)\right\|_{L^2((0,t)\times\mathbb{R}^n)} \to 0$$

as $\delta \to 0$.

Next we will consider

$$\begin{aligned} &\left\|(\nabla_\mu^\parallel)^* \left(h - \left[\eta_\delta * \left(\eta\left(\frac{\cdot}{r}\right) g_k(h)\right)\right]\right)\right\|_{L^2((0,t)\times\mathbb{R}^n)} \\ &\quad \leq \|(\nabla_\mu^\parallel)^* \left(h - g_k(h)\right)\|_{L^2((0,t)\times\mathbb{R}^n)} + \|(\nabla_\mu^\parallel)^* \left(g_k(h) - \eta\left(\frac{\cdot}{r}\right) g_k(h)\right)\|_{L^2((0,t)\times\mathbb{R}^n)} \\ &\qquad + \left\|(\nabla_\mu^\parallel)^* \left(\eta\left(\frac{\cdot}{r}\right) g_k(h) - \left[\eta_\delta * \left(\eta\left(\frac{\cdot}{r}\right) g_k(h)\right)\right]\right)\right\|_{L^2((0,t)\times\mathbb{R}^n)}, \end{aligned}$$

Now

$$|(\nabla_\mu^\parallel)^* \left(h - g_k(h)\right)|^2 \leq 2^2|(\nabla_\mu^\parallel)^* h|^2 + |(\nabla_\mu^\parallel)^* g_k(h)|^2 \leq 2|(\nabla_\mu^\parallel)^* h|^2 \in L^1((0,t)\times\mathbb{R}^n),$$

as $h \in \mathcal{C}_c^\infty(\mathbb{R}_+; \mathscr{D}((\nabla_\mu^\parallel)^*))$. Then the dominated convergence theorem gives

$$\|(\nabla_\mu^\parallel)^* \left(h - g_k(h)\right)\|_{L^2((0,t)\times\mathbb{R}^n)} \to 0$$

as $k \to \infty$. Next, using the product rule for the divergence and the chain rule gives

$$\begin{aligned}
&\|(\nabla_\mu^\|)^* \left[\eta\left(\frac{\cdot}{r}\right) g_k(h)\right] - (\nabla_\mu^\|)^* g_k(h)\|_{L^2((0,t)\times\mathbb{R}^n)} \\
&\quad \leq \| \operatorname{div}_\| \left[\eta\left(\frac{\cdot}{r}\right) g_k(h)\right] + |V|^{\frac{1}{2}}\left[\eta\left(\frac{\cdot}{r}\right) g_k(h)\right] - (\nabla_\mu^\|)^* g_k(h)\|_{L^2((0,t)\times\mathbb{R}^n)} \\
&\quad = \|\nabla\left[\eta\left(\frac{\cdot}{r}\right)\right] + \eta\left(\frac{\cdot}{r}\right)(\nabla_\mu^\|)^* g_k(h) - (\nabla_\mu^\|)^* g_k(h)\|_{L^2((0,t)\times\mathbb{R}^n)} \\
&\quad \leq \|\frac{1}{r}(\nabla\eta)\left(\frac{\cdot}{r}\right)\cdot g_k(h_\|)\|_{L^2((0,t)\times\mathbb{R}^n)} + \|\eta\left(\frac{\cdot}{r}\right)(\nabla_\mu^\|)^* g_k(h) - (\nabla_\mu^\|)^* g_k(h)\|_{L^2((0,t)\times\mathbb{R}^n)}.
\end{aligned}$$

Since $\eta = 1$ on a neighbourhood of 0 then $\nabla\eta = 0$ on a neighbourhood of 0. Thus $\frac{1}{r}(\nabla\eta)\left(\frac{x}{r}\right) \to 0$ as $r \to \infty$ for all $x \in \mathbb{R}^n$. Also, for all $r \in \mathbb{N}$

$$\left|\frac{1}{r}(\nabla\eta)\left(\frac{x}{r}\right)\cdot g_k(h_\|)(x)\right| \leq \|\nabla\eta\|_\infty |h_\|| \in L^2((0,t)\times\mathbb{R}^n).$$

Moreover,

$$\begin{aligned}
\left|\eta\left(\frac{x}{r}\right)(\nabla_\mu^\|)^* g_k(h)(x) - (\nabla_\mu^\|)^* g_k(h)(x)\right| &\leq \|\eta\|_\infty |(\nabla_\mu^\|)^* g_k(h)(x)| \\
&\leq \|\eta\|_\infty |(\nabla_\mu^\|)^* h(x)| \in L^2((0,t)\times\mathbb{R}^n).
\end{aligned}$$

Then, the dominated convergence theorem gives

$$\|(\nabla_\mu^\|)^* \left[\eta\left(\frac{\cdot}{R}\right) g_k(h)\right] - (\nabla_\mu^\|)^* g_k(h)\|_{L^2((0,t)\times\mathbb{R}^n)} \to 0,$$

as $r \to \infty$. Now for the final term we let $\tilde{h} := \eta\left(\frac{\cdot}{r}\right) g_k(h)$. Now we differentiate under the integral sign and then use the chain rule to get

$$\begin{aligned}
(\nabla_\mu^\|)^*(\tilde{h}(s) * \eta_\delta)(x) &= \sum_{j=1}^n \partial_{x_j} \int_{\mathbb{R}^n} \tilde{h}_j(s,y)\eta_\delta(x-y)\,\mathrm{d}y + |V|^{\frac{1}{2}}(x)(\tilde{h}_{n+1}(s) * \eta_\delta)(x) \\
&= \int_{\mathbb{R}^n} \tilde{h}_j(s,y) \sum_{j=1}^n \partial_{x_j}\eta_\delta(x-y)\,\mathrm{d}y + |V|^{\frac{1}{2}}(x)(\tilde{h}_{n+1}(s) * \eta_\delta)(x) \\
&= -\sum_{j=1}^n \int_{\mathbb{R}^n} \tilde{h}_j(s,y)\partial_{y_j}\eta_\delta(x-y)\,\mathrm{d}y + |V|^{\frac{1}{2}}(x)(\tilde{h}_{n+1}(s) * \eta_\delta)(x)
\end{aligned}$$

Then as $\tilde{h}(s) \in \mathscr{D}((\nabla_\mu^\|)^*)$, we use the definition of weak derivative to get

$$
\begin{aligned}
-\sum_{j=1}^n \int_{\mathbb{R}^n} & \tilde{h}_j(s,y) \partial_{y_j} \eta_\delta(x-y)\,\mathrm{d}y \\
&= \int_{\mathbb{R}^n} \sum_{j=1}^n \partial_{y_j} \tilde{h}_j(s,y) \eta_\delta(x-y)\,\mathrm{d}y + (|V|^{\frac{1}{2}} \tilde{h}_{n+1}(s) * \eta_\delta)(x) \\
&\quad - (|V|^{\frac{1}{2}} \tilde{h}_{n+1}(s) * \eta_{\delta_m})(x) \\
&= \int_{\mathbb{R}^n} (\nabla_\mu^\|)^* \tilde{h}(s,y) \eta_\delta(x-y)\,\mathrm{d}y - (|V|^{\frac{1}{2}} \tilde{h}_{n+1}(s) * \eta_\delta)(x) \\
&= ((\nabla_\mu^\|)^* \tilde{h}(s) * \eta_\delta)(x) - (|V|^{\frac{1}{2}} \tilde{h}_{n+1}(s) * \eta_\delta)(x).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|(\nabla_\mu^\|)^*(\tilde{h} * \eta_\delta) & - (\nabla_\mu^\|)^* \tilde{h}\|_{L^2((0,t)\times\mathbb{R}^n)} \\
&\leq \|((\nabla_\mu^\|)^* \tilde{h} * \eta_\delta) - (\nabla_\mu^\|)^* \tilde{h}\|_{L^2((0,t)\times\mathbb{R}^n)} \\
&\quad + \||V|^{\frac{1}{2}}(\tilde{h}_{n+1} * \eta_\delta) - (|V|^{\frac{1}{2}} \tilde{h}_{n+1} * \eta_\delta)\|_{L^2((0,t)\times\mathbb{R}^n)}.
\end{aligned}
$$

As $\tilde{h}(s) \in L_c^\infty(\mathbb{R}^n)$ then $|V|^{\frac{1}{2}} \tilde{h}_{n+1} \in L^2(\mathbb{R}^n)$. Therefore, by [30, Lemma 7.2], we have

$$
\||V|^{\frac{1}{2}} \tilde{h}_{n+1} - (|V|^{\frac{1}{2}} \tilde{h}_{n+1} * \eta_\delta)\|_{L^2((0,t)\times\mathbb{R}^n)} \to 0,
$$

as $\delta \to 0$. As before we have $|\tilde{h}_{n+1}(s) * \eta_\delta| \leq \|\eta\|_\infty^2 k \mathbb{1}_{r+1(0)}$ and

$$
|V||\tilde{h}_{n+1} * \eta_\delta|^2 \leq |V| \|\eta\|_\infty^4 k^2 \mathbb{1}_{r+1(0)} \in L^1((0,t) \times \mathbb{R}^n).
$$

Thus, by the dominated convergence theorem we have

$$
\||V|^{\frac{1}{2}}(\tilde{h}_{n+1} * \eta_\delta) - |V|^{\frac{1}{2}} \tilde{h}_{n+1}\|_{L^2((0,t)\times\mathbb{R}^n)} \to 0,
$$

as $\delta \to 0$. Hence

$$
\|(\nabla_\mu^\|)^*(\tilde{h} * \eta_\delta) - (\nabla_\mu^\|)^* \tilde{h}\|_{L^2((0,t)\times\mathbb{R}^n)} \to 0,
$$

as $\delta \to 0$.

Now fix $m \in \mathbb{N}$. Then, there exists $k_m \in \mathbb{N}$ such that

$$\|\nabla_\mu(f - g_{k_m}(f))\|_{L^2((0,t)\times\mathbb{R}^n)} < \frac{1}{6m}, \quad \text{and} \quad \|(\nabla_\mu^{\|})^* \left(h - g_{k_m}(h)\right)\|_{L^2((0,t)\times\mathbb{R}^n)} < \frac{1}{6m}.$$

Then, there exists $r_m \in \mathbb{N}$ (depending on k_m) such that

$$\begin{aligned} \|\nabla_\mu(g_{k_m}(f) - \eta\left(\frac{\cdot}{r_m}\right) g_{k_m}(f))\|_{L^2((0,t)\times\mathbb{R}^n)} &< \frac{1}{6m}, \text{ and} \\ \|(\nabla_\mu^{\|})^* \left[\eta\left(\frac{\cdot}{r_m}\right) g_{k_m}(h)\right] - (\nabla_\mu^{\|})^* g_{k_m}(h)\|_{L^2((0,t)\times\mathbb{R}^n)} &< \frac{1}{6m}. \end{aligned}$$

Finally, there exists $\delta_m \in (0,1)$ (depending on K_m and R_m) such that

$$\left\|\nabla_\mu\left(\eta\left(\frac{\cdot}{r_m}\right) g_{k_m}(f) - \left[\eta_{\delta_m} * \left(\eta\left(\frac{\cdot}{r_m}\right) g_{k_m}(f)\right)\right]\right)\right\|_{L^2((0,t)\times\mathbb{R}^n)} < \frac{1}{6m},$$

and

$$\left\|(\nabla_\mu^{\|})^*\left(\eta\left(\frac{\cdot}{r_m}\right) g_{k_m}(h) - \left[\eta_{\delta_m} * \left(\eta\left(\frac{\cdot}{r_m}\right) g_{k_m}(h)\right)\right]\right)\right\|_{L^2((0,t)\times\mathbb{R}^n)} < \frac{1}{6m}.$$

Now, define $\varphi_m \coloneqq \eta_{\delta_m} * \left(\eta\left(\frac{\cdot}{r_m}\right) g_{k_m}(f)\right)$. Now define $f_m \coloneqq (\varphi_m)_\perp$ and $h_m \coloneqq (\varphi_m)_r$. Therefore $\varphi_m \in \mathcal{C}_c^\infty((0,t)\times\mathbb{R}^n;\mathbb{C}^{n+2})$ for all $m \in \mathbb{N}$. Thus, for all $\varepsilon > 0$, choose $M \in \mathbb{N}$ to be $M > \frac{1}{\varepsilon}$. Therefore,

$$\|D(\varphi_m - \varphi)\|_{L^2((0,t)\times\mathbb{R}^n)} \le \|\nabla_\mu^{\|}(f_m - f)\|_2 + \|(\nabla_\mu^{\|})^*(h_m - h)\|_{L^2((0,t)\times\mathbb{R}^n)} < \frac{1}{m} < \varepsilon$$

for all $m > M$. That is $\{\varphi_m\}_{m\in\mathbb{N}} \subset \mathcal{C}_c^\infty((0,t)\times\mathbb{R}^n;\mathbb{C}^{n+2})$ such that $D\varphi_m$ converges to $D\varphi$ as $m \to \infty$. A similar mollifier argument yields $\partial_s\varphi_m$ converges to $\partial_s\varphi$ as $m \to \infty$.

Now we have a sequence of smooth functions which converges to φ we proceed to prove the identity. Fix $\varepsilon > 0$ and choose $M \in \mathbb{N}$ such that

$$\|\partial_s(\varphi_m - \varphi)\|_{L^2((0,t)\times\mathbb{R}^n)} < \frac{\varepsilon^2}{2\|F(s)\|_{L^2((0,t)\times\mathbb{R}^n)}}$$

and

$$\|D(\varphi_m - \varphi)\|_{L^2((0,t)\times\mathbb{R}^n)} < \frac{\varepsilon^2}{\|BF(s)\|_{L^2((0,t)\times\mathbb{R}^n)}},$$

for all $m > M$. Since $\varphi_m \in \mathcal{C}_c^\infty(\mathbb{R}_+^{n+1}; \mathbb{C}^{n+2})$ and F is a weak solution to $-\operatorname{div} A\nabla F + DBF = 0$ in $\mathbb{R}_+$ we use Definition 5.3.2 and the Cauchy–Schwarz inequality, to get

$$\begin{aligned}
&\left|\int_0^t \langle F(s), \partial_s\varphi(s)\rangle - \langle BF(s), D\varphi(s)\rangle \,\mathrm{d}s\right| \\
&\quad = \left|\int_0^t \langle F(s), \partial_s(\varphi(s) - \varphi_m)\rangle + \langle F(s), \partial_s\varphi_m)\rangle - \langle BF(s), D\varphi(s)\rangle \,\mathrm{d}s\right| \\
&\quad = \left|\int_0^t \langle F(s), \partial_s(\varphi(s) - \varphi_m)\rangle + \langle BF(s), D\varphi_m)\rangle - \langle BF(s), D\varphi(s)\rangle \,\mathrm{d}s\right| \\
&\quad = \left|\int_0^t \langle F(s), \partial_s(\varphi(s) - \varphi_m)\rangle + \langle BF(s), D(\varphi_m(s) - \varphi(s))\rangle \,\mathrm{d}s\right| \\
&\quad = \int_0^t \|F(s)\|_2\|\partial_s(\varphi(s) - \varphi_m)\|_2 + \|BF(s)\|_2\|D(\varphi_m(s) - \varphi(s))\|_2 \,\mathrm{d}s \\
&\quad < \varepsilon.
\end{aligned}$$

As $\varepsilon > 0$ was arbitrary then we must have

$$\int_0^t \langle F(s), \partial_s\varphi(s)\rangle - \langle BF(s), D\varphi(s)\rangle \,\mathrm{d}s = 0.$$

This completes the proof. □

The following result is similar to above.

Lemma 5.3.6. Let $\psi \in \overline{\mathsf{R}(D)}$. Let $t > 0$ and $\eta_- \in \mathcal{C}_c^\infty(t,\infty)$. For $s \in (t,\infty)$, define $\varphi(s) := \eta_-(s)\left(e^{-(s-t)[DB]}E_{DB}^+\right)^* \psi$. If $F \in L^2_{\mathrm{loc}}(\mathbb{R}_+; \mathsf{R}(D))$ is a weak solution to $\partial_t F + DBF = 0$ in $\mathbb{R}_+$, then we have the following

$$\int_t^\infty \langle F(s), \partial_s\varphi(s)\rangle \,\mathrm{d}s = \int_t^\infty \langle BF(s), D\varphi(s)\rangle \,\mathrm{d}s.$$

Proof. The proof is similar to Lemma 5.3.5. □

Lemma 5.3.7. Let $t > 0$ and consider non-negative functions $\eta_+ \in \mathcal{C}_c^\infty((0,t); \mathbb{R})$ and $\eta_- \in \mathcal{C}_c^\infty((t,\infty); \mathbb{R})$. If $F \in L^2_{\mathrm{loc}}(\mathbb{R}_+; \overline{\mathsf{R}(D)})$ is a weak solution of $\partial_t F + DBF = 0$ in $\mathbb{R}_+$, then

$$\int_0^t \eta_+'(s)e^{-(t-s)[DB]}E_{DB}^+F(s)\,\mathrm{d}s = \int_t^\infty \eta_-'(s)e^{-(s-t)[DB]}E_{DB}^-F(s)\,\mathrm{d}s = 0.$$

Proof. Let $\psi \in \overline{\mathsf{R}(D)}$ and define $\varphi(s) \coloneqq \eta_+(s)\left(e^{-(t-s)[DB]}E_{DB}^+\right)^*\psi$. Now, as F is a solution of the first-order equation (5.3.1) we have from Lemma 5.3.5 that

$$\int_0^t \langle \partial_s\varphi(s), F(s)\rangle\,\mathrm{d}s = \int_0^t \langle D\varphi(s), BF(s)\rangle\,\mathrm{d}s \tag{5.3.3}$$

Now, by the definition of φ, the self-adjointness of D, and the algebra homomorphism property of the functional calculus, we have

$$\begin{aligned}\int_0^t \langle D\varphi(s), BF(s)\rangle\,\mathrm{d}s &= \int_0^t \eta_+(s)\langle\psi, e^{-(t-s)DB}E_{DB}^+DBF(s)\rangle\,\mathrm{d}s\\ &= \int_0^t \eta_+(s)\langle\psi, DBe^{-(t-s)DB}E_{DB}^+F(s)\rangle\,\mathrm{d}s.\end{aligned}$$

As DB has bounded holomorphic functional calculus, then $e^{-tDB} \in \mathcal{L}(\mathcal{H})$ and therefore $(e^{-tDB})^* = e^{-tB^*D}$ (see [25, Section I.3.15]). Therefore,

$$\begin{aligned}&\int_0^t \langle \partial_s\varphi(s), F(s)\rangle\,\mathrm{d}s\\ &\quad= \int_0^t \eta_+'(s)\left\langle\left(e^{-(t-s)DB}E_{DB}^+\right)^*\psi, F(s)\right\rangle\mathrm{d}s + \int_0^t \eta_+(s)\left\langle\partial_s\left(e^{-(t-s)DB}E_{DB}^+\right)^*\psi, F(s)\right\rangle\mathrm{d}s\\ &\quad= \int_0^t \eta_+'(s)\left\langle\left(e^{-(t-s)DB}E_{DB}^+\right)^*\psi, F(s)\right\rangle\mathrm{d}s + \int_0^t \eta_+(s)\left\langle\partial_s e^{-(t-s)B^*D}\psi, E_{DB}^+F(s)\right\rangle\mathrm{d}s\\ &\quad= \int_0^t \eta_+'(s)\left\langle\psi, e^{-(t-s)DB}E_{DB}^+F(s)\right\rangle\mathrm{d}s + \int_0^t \eta_+(s)\langle\psi, DBe^{-(t-s)DB}E_{DB}^+F(s)\rangle\,\mathrm{d}s.\end{aligned}$$

Using (5.3.3) and the above two calculations, we have

$$\begin{aligned} 0 &= \int_0^t \langle \partial_s \varphi(s), F(s)\rangle \,\mathrm{d}s - \int_0^t \langle D\varphi(s), BF(s)\rangle \,\mathrm{d}s \\ &= \int_0^t \eta_+'(s) \left\langle \psi, e^{-(t-s)DB} E_{DB}^+ F(s)\right\rangle \mathrm{d}s + \int_0^t \eta_+(s)\langle \psi, DBe^{-(t-s)DB} E_{DB}^+ F(s)\rangle \,\mathrm{d}s \\ &\quad - \int_0^t \eta_+(s)\langle \psi, DBe^{-(t-s)DB} E_{DB}^+ F(s)\rangle \,\mathrm{d}s \\ &= \int_0^t \eta_+'(s) \left\langle \psi, e^{-(t-s)DB} E_{DB}^+ F(s)\right\rangle \mathrm{d}s \end{aligned}$$

Therefore, using Fubini's Theorem, we have that

$$0 = \int_0^t \eta_+'(s) \left\langle \psi, e^{-(t-s)DB} E_{DB}^+ F(s)\right\rangle \mathrm{d}s = \int_{\mathbb{R}^n} \psi(x) \left(\int_0^t \eta_+'(s) e^{-(t-s)[DB]} E_{DB}^+ F(s) \,\mathrm{d}s \right) \mathrm{d}x.$$

Then as ψ was arbitrary, we have

$$\int_0^t \eta_+'(s) e^{-(t-s)[DB]} E_{DB}^+ F(s) \,\mathrm{d}s = 0.$$

A similar argument using $\varphi(s) := \eta_-(s) \left(e^{-(s-t)DB} E_{DB}^-\right)^* \psi$ as the test function, gives that

$$\int_t^\infty \eta_-'(s) e^{-(s-t)[DB]} E_{DB}^- F(s) \,\mathrm{d}s = 0.$$

This completes the proof. □

Now let $\varepsilon > 0$ and construct the functions $\eta_\varepsilon^\pm$, in the same way as in [2], as follows: First define $\eta^0\colon [0,\infty) \to [0,1]$ to be a smooth function supported in $[1,\infty)$, where $\eta^0(t) = 1$ for all $t \in (2,\infty)$; then define $\eta_\varepsilon(t) := \eta^0(\frac{t}{\varepsilon})(1 - \eta^0(2\varepsilon t))$; finally we define

$$\eta_\varepsilon^\pm(t,s) := \eta^0\left(\frac{\pm(t-s)}{\varepsilon}\right) \eta_\varepsilon(t)\eta_\varepsilon(s).$$

Then η_ε^+ is uniformly bounded and compactly supported in the set $\{(s,t) \in \mathbb{R}^2 : 0 < s < t\}$ and approximates the characteristic function of this set. Similarly η_ε^- approximates the characteristic function of the set $\{(s,t) \in \mathbb{R}^2 : 0 < t < s\}$.

Theorem 5.3.8. If $F \in L^2_{\text{loc}}(\mathbb{R}_+; \overline{\mathsf{R}(D)})$ is a weak solution of $\partial_t F + DBF = 0$ in $\mathbb{R}_+$ such that

$$\sup_{t>0} \fint_t^{2t} \|F(s)\|_2^2 \,\mathrm{d}s < \infty,$$

then there exists $f \in \mathcal{H}^{0,+}_{DB}$ such that $\lim_{t\to 0} F(t) = f$ in L^2 and $F(t,x) = e^{-tDB} f(x)$.

Proof. As $F \in L^2_{\text{loc}}(\mathbb{R}_+; \overline{\mathsf{R}(D)})$ is a weak solution of $\partial_t F + DBF = 0$ in $\mathbb{R}^{n+1}_+$, using Lemma 5.3.7 with $\eta^\pm_\varepsilon$ used instead of $\eta_\pm$, we obtain

$$\int_0^t (\partial_s \eta^+_\varepsilon)(t,s) e^{-(t-s)[DB]} E^+_{DB} F(s) \,\mathrm{d}s + \int_t^\infty (\partial_s \eta^-_\varepsilon)(t,s) e^{-(s-t)[DB]} E^-_{DB} F(s) \,\mathrm{d}s = 0.$$

We can then follow the abstract approach in [2, Theorem 8.2 (i)], to complete the proof verbatim. □

We are now ready to return to discuss the global well-posedness of (5.3.2). The following corollary is a consequence of Theorem 5.3.8 and Proposition 5.3.4.

Corollary 5.3.9. We have (5.3.2) is globally well-posed in $\mathcal{H}^{0,+}_{DB}$ with convergence in L^2. Moreover, solutions to (5.3.2) are of the form $e^{-tDB} f$ for $t > 0$ for initial data $f \in \mathcal{H}^{0,+}_{DB}$.

Proof. We have existence of solutions from Proposition 5.3.4. We also have a clasification of all solutions from Theorem 5.3.8. □

We remark that Theorem 5.3.8 and Proposition 5.3.4 give a classification of the solutions for (5.3.2) as those that arise from the semigroup applied to the initial data. We now give a Fatou type result for the first-order equation.

Proposition 5.3.10. If F is a solution of the first-order equation (5.3.1) such that

$$\sup_{t>0} \fint_t^{2t} \|F(s)\|_2^2 \,\mathrm{d}s < \infty,$$

then there exists $f \in \mathcal{H}^{0,+}_{DB}$ such that

$$\lim_{t\to 0} \fint_t^{2t} \|F(s) - f\|_2^2 \,\mathrm{d}s = 0 = \lim_{t\to\infty} \fint_t^{2t} \|F(s)\| \,\mathrm{d}s.$$

Proof. Fix $\varepsilon > 0$. Then by Theorem 5.3.8, there exists $f \in \mathcal{H}^{0,+}_{DB}$ such that $\lim_{t\to 0} F(t) = f$ in L^2, in particular, we have $F(t) = e^{-tDB} f$. Now let $\delta > 0$ be such that $\|F(s)-f\|_2 < \varepsilon$ whenever $0 \le s < \delta$. If $t < \frac{\delta}{2}$, then

$$\fint_t^{2t} \|F(s) - f\|_2^2 \,\mathrm{d}s < \fint_t^{2t} \varepsilon \,\mathrm{d}s = \varepsilon.$$

Thus

$$\lim_{t\to 0} \fint_t^{2t} \|F(s) - f\|_2^2 \,\mathrm{d}s = 0.$$

The other limit is proved similarly. $\square$

CHAPTER 6

BOUNDARY VALUE PROBLEMS FOR THE ELECTRIC SCHÖDINGER EQUATION

From this Chapter onward we will focus on solving boundary value problems for the Schrödinger equation (2.6.1) where the potential is in the reverse Hölder class i.e. $V \in B^{\frac{n}{2}}(\mathbb{R}^n)$. This Chapter is dedicated to discovering the connection between second-order equation $H_{A,a,V}\, u = 0$, and the first-order equation

$$\partial_t F + DBF = 0,$$

as discussed in Section 5.3. Once this connection has been established we will introduce boundary mappings which will map the initial data for the first-order system of equations to boundary data for the second order equation. Thus, solvabilty will be be reduced to inverting these mappings. The majority of the rest of this thesis will focused on proving the following two theorems. The first theorem is about the well-posedness of the the second-order equation, see (2.6.4) and (2.6.5) for the definitions of well-posedness of the Neumann and Dirichlet regularity problems respectively, when the matrix $\mathcal{A}$ is self-adjoint or block-type, where $\mathcal{A}$ is considered block-type if it is of the form

$$\mathcal{A}^{\mathcal{V}} = \begin{bmatrix} A_{\perp\perp} & 0 & 0 \\ 0 & A_{\parallel\parallel} & 0 \\ 0 & 0 & ae^{i \arg V(x)} \end{bmatrix}.$$

Theorem 6.0.1. Let $V \in B^{\frac{n}{2}}(\mathbb{R}^n)$ and let $\mathcal{A} \in L^\infty(\mathbb{R}^{n+1}_+; \mathcal{L}(\mathbb{C}^{n+2}))$ be elliptic s in (2.5.1). Then the following are true:

1. The boundary value problems $(\mathcal{R})^{\mathcal{A}}_{L^2}$ and $(\mathcal{N})^{\mathcal{A}}_{L^2}$ are well-posed if $\mathcal{A}$ is self-adjoint or block-type;

2. The sets $WP(\mathcal{R})$ and $WP(\mathcal{N})$ are open;

3. If $\mathcal{A} \in WP(\mathcal{R})$, then for each $\varphi \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^n)$ we have the estimates

$$\int_0^\infty \|t\partial_t(\nabla_\mu u)\|_2^2 \frac{\mathrm{d}t}{t} \eqsim \|\widetilde{N}_*(\nabla_\mu u)\|_2^2 \eqsim \|\nabla_\mu \varphi\|_2^2$$

   where u is the solution for the initial data $\varphi \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^n)$;

4. If $\mathcal{A} \in WP(\mathcal{N})$, then for each $\varphi \in L^2(\mathbb{R}^n)$ we have the estimates

$$\int_0^\infty \|t\partial_t(\nabla_\mu u_t)\|_2^2 \frac{\mathrm{d}t}{t} \eqsim \|\widetilde{N}_*(\nabla_\mu u)\|_2^2 \eqsim \|\varphi\|_2^2,$$

   where u is the solution for the initial data $\varphi \in L^2(\mathbb{R}^n)$.

The following theorem is a Fatou-type theorem. That is, if a solution to the second-order equation has non-tangential control of its gradient then there exists some boundary data which solves the Neumann or Dirichlet boundary problem.

Theorem 6.0.2. Let $V \in B^{\frac{n}{2}}(\mathbb{R}^n)$ and let $\mathcal{A} \in L^\infty(\mathbb{R}^{n+1}_+; \mathcal{L}(\mathbb{C}^{n+2}))$ be elliptic. Let $u \in \mathcal{V}^{1,2}_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$ be such that $H_{A,a,V}\, u = 0$ with $\widetilde{N}_*(\nabla_\mu u) \in L^2(\mathbb{R}^n)$. Then we have:

1. There exists $\varphi \in L^2(\mathbb{R}^n)$ such that $\lim_{t\to 0} \partial_{\nu_A} u(\cdot, t) = \varphi$ in L^2 and pointwise on Whitney averages;

2. There exists $\varphi \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^n)$ such that $\lim_{t\to 0} \nabla_\mu^{\|} u(t, \cdot) = \nabla_\mu \varphi$ in L^2 and pointwise on Whitney averages.

6.1 Reduction to a First-Order System

We now work towards showing that there is some equivalence between the first-order system and second-order equation. The fist step is to choose the correct perturbation matrix B depending on $\mathcal{A}^{\mathcal{V}}$. We do this in a similar way as in [4] where it was shown that if A is bounded and elliptic then a transformed matrix, $\widehat{A}$, is also bounded and elliptic. We replicate these results using the bounded operators $\mathcal{A}^{\mathcal{V}}$, $\overline{\mathcal{A}}$ and $\underline{\mathcal{A}^{\mathcal{V}}}$ defined on $L^2(\mathbb{R}^n, \mathbb{C}^{n+2})$ by

$$\mathcal{A}^{\mathcal{V}} := \begin{bmatrix} A_{\perp\perp} & A_{\perp\|} & 0 \\ A_{\|\perp} & A_{\|\|} & 0 \\ 0 & 0 & ae^{i\arg V(x)} \end{bmatrix}, \quad \overline{\mathcal{A}} := \begin{bmatrix} A_{\perp\perp} & A_{\perp\|} & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix},$$

and

$$\underline{\mathcal{A}^{\mathcal{V}}} := \begin{bmatrix} I & 0 & 0 \\ A_{\|\perp} & A_{\|\|} & 0 \\ 0 & 0 & ae^{i\arg V(x)} \end{bmatrix}.$$

Note that since A and a are bounded then so are $\mathcal{A}^{\mathcal{V}}$, $\overline{\mathcal{A}}$ and $\underline{\mathcal{A}^{\mathcal{V}}}$. Also, as $A_{\perp\perp}$ is pointwise strictly elliptic then $A_{\perp\perp}$ is invertible, and so $\overline{\mathcal{A}}$ is invertible with inverse,

$$\overline{\mathcal{A}}^{-1} = \begin{bmatrix} A_{\perp\perp}^{-1} & -A_{\perp\perp}^{-1}A_{\perp\|} & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}.$$

Now define

$$\widehat{\mathcal{A}^{\mathcal{V}}} = \underline{\mathcal{A}^{\mathcal{V}}}\,\overline{\mathcal{A}}^{-1} = \begin{bmatrix} A_{\perp\perp}^{-1} & -A_{\perp\perp}^{-1}A_{\perp\|} & 0 \\ A_{\|\perp}A_{\perp\perp}^{-1} & A_{\|\|} - A_{\|\perp}A_{\perp\perp}^{-1}A_{\perp\|} & 0 \\ 0 & 0 & ae^{i\arg V(x)} \end{bmatrix}.$$

Let $u \in \overline{\mathsf{R}(D)}$. Then, as $A_{\perp\perp}^{-1}(u_\perp - A_{\perp\|}u_\|) \in L^2(\mathbb{R}^n)$, we have

$$v = \begin{bmatrix} A_{\perp\perp}^{-1}(u_\perp - A_{\perp\|}u_\|) \\ u_\| \\ u_\mu \end{bmatrix} \in \overline{\mathsf{R}(D)}.$$

Then $\overline{\mathcal{A}}v = u$. That is $\overline{\mathcal{A}} \colon \overline{\mathsf{R}(D)} \to \overline{\mathsf{R}(D)}$ is surjective. Also, $\overline{A}$ is invertible and therefore is injective. Thus $\overline{\mathcal{A}} \colon \overline{\mathsf{R}(D)} \to \overline{\mathsf{R}(D)}$ is an isomorphism. We now prove the result that $\widehat{\mathcal{A}}$ preserves the important properties of boundedness and ellipticity from $\mathcal{A}^{\mathcal{V}}$.

Proposition 6.1.1. We have that $\mathcal{A}^{\mathcal{V}}$ is bounded and elliptic on $\overline{\mathsf{R}(D)}$ if and only if the matrix $\widehat{\mathcal{A}}^{\mathcal{V}}$ is bounded and elliptic on $\overline{\mathsf{R}(D)}$.

Proof. We first prove that if $\mathcal{A}^{\mathcal{V}}$ is bounded and elliptic on $\overline{\mathsf{R}(D)}$ then so is $\widehat{\mathcal{A}}^{\mathcal{V}}$. Now for any $f \in \overline{\mathsf{R}(D)}$, let $g \in \overline{\mathsf{R}(D)}$ such that $\overline{\mathcal{A}}g = f$ recalling that $\overline{\mathcal{A}} \colon \overline{\mathsf{R}(D)} \to \overline{\mathsf{R}(D)}$ is an isomorphism. Then

$$\begin{aligned}
\mathrm{Re}\langle \widehat{\mathcal{A}}^{\mathcal{V}} f, f\rangle &= \mathrm{Re}\langle \widehat{\mathcal{A}}^{\mathcal{V}}\overline{\mathcal{A}}g, \overline{\mathcal{A}}g\rangle \\
&= \mathrm{Re}\langle \underline{\mathcal{A}}^{\mathcal{V}} g, \overline{\mathcal{A}}g\rangle \\
&= \mathrm{Re}\left\langle \begin{bmatrix} g_\perp \\ A_{\|\perp}g_\perp + A_{\|\|}g_\| \\ ae^{i\arg V(x)}g_\mu \end{bmatrix}, \begin{bmatrix} A_{\perp\perp}g_\perp + A_{\perp\|}g_\| \\ g_\| \\ g_\mu \end{bmatrix} \right\rangle \\
&= \mathrm{Re}(\langle g_\perp, A_{\perp\perp}g_\perp + A_{\perp\|}g_\|\rangle + \langle A_{\|\perp}g_\perp + A_{\|\|}g_\|, g_\|\rangle + \langle ae^{i\arg V(x)}g_\mu, g_\mu\rangle \\
&= \mathrm{Re}(\langle A_{\perp\perp}g_\perp + A_{\perp\|}g_\|, g_\perp\rangle + \langle A_{\|\perp}g_\perp + A_{\|\|}g_\|, g_\|\rangle + \langle ae^{i\arg V(x)}g_\mu, g_\mu\rangle.
\end{aligned}$$

We also have

$$\begin{aligned}
\operatorname{Re}\langle \mathcal{A}^{\mathcal{V}} g, g\rangle &= \operatorname{Re}\left\langle \begin{bmatrix} A_{\perp\perp} g_{\perp} + A_{\perp\|} g_{\|} \\ A_{\|\perp} g_{\perp} + A_{\|\|} g_{\|} \\ a e^{i \arg V(x)} g_{\mu} \end{bmatrix}, \begin{bmatrix} g_{\perp} \\ g_{\|} \\ g_{\mu} \end{bmatrix} \right\rangle \\
&= \operatorname{Re}(\langle A_{\perp\perp} g_{\perp} + A_{\perp\|} g_{\|}, g_{\perp}\rangle + \langle A_{\|\perp} g_{\perp} + A_{\|\|} g_{\|}, g_{\|}\rangle + \langle a e^{i \arg V(x)} g_{\mu}, g_{\mu}\rangle \\
&= \operatorname{Re}\langle \widehat{\mathcal{A}}^{\mathcal{V}} f, f\rangle.
\end{aligned}$$

Now as $\mathcal{A}^{\mathcal{V}}$ is elliptic we have

$$\kappa \|g\|_2^2 \leq \operatorname{Re}\left\langle \mathcal{A}^{\mathcal{V}} g, g\right\rangle = \operatorname{Re}\langle \widehat{\mathcal{A}}^{\mathcal{V}} f, f\rangle.$$

Then

$$\|f\|_2^2 = \|\overline{\mathcal{A}}^{\mathcal{V}} (\overline{\mathcal{A}}^{\mathcal{V}})^{-1} f\|_2^2 \leq \|\overline{\mathcal{A}}^{\mathcal{V}}\|^2 \|g\|_2^2 \lesssim \operatorname{Re}\langle \widehat{\mathcal{A}}^{\mathcal{V}} f, f\rangle.$$

That is $\widehat{\mathcal{A}}^{\mathcal{V}}$ is elliptic. Also as $\mathcal{A}^{\mathcal{V}}$ is bounded then $\underline{\mathcal{A}}^{\mathcal{V}}$ and $\overline{\mathcal{A}}^{\mathcal{V}}$ are bounded and as $\overline{\mathcal{A}}^{\mathcal{V}}$ is invertible then $(\overline{\mathcal{A}}^{\mathcal{V}})^{-1}$ is bounded. Then $\widehat{\mathcal{A}}^{\mathcal{V}} = \underline{\mathcal{A}}^{\mathcal{V}} (\overline{\mathcal{A}}^{\mathcal{V}})^{-1}$ is bounded.

Note that $\widehat{(\widehat{\mathcal{A}}^{\mathcal{V}})} = \mathcal{A}^{\mathcal{V}}$. Therefore, if $\widehat{\mathcal{A}}^{\mathcal{V}}$ is bounded and elliptic then so is $\mathcal{A}^{\mathcal{V}}$, by the above argument. This completes the proof. $\square$

From now on we let $B = \widehat{\mathcal{A}}^{\mathcal{V}}$, and by Proposition 6.1.1 we have B is elliptic and bounded. We now show that this is indeed the correct B to obtain the correspondence between the first-order system and the second-order equation. We recall the notation

$$\nabla_{\mu}^{\|} u := \begin{bmatrix} \nabla_{\|} u \\ |V|^{\frac{1}{2}} u \end{bmatrix} \text{ and } \nabla_{\mu} u := \begin{bmatrix} \nabla u \\ |V|^{\frac{1}{2}} u \end{bmatrix} = \begin{bmatrix} \partial_t u \\ \nabla_{\|} u \\ |V|^{\frac{1}{2}} u \end{bmatrix}, \text{ for } u \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^{n+1}_{+})$$

so to make it clear whether we are referring to the tangential adapted gradient or the full adapted gradient.

Proposition 6.1.2. Let u be such that $H_{A,a,V}\, u = 0$ in $\mathbb{R}^{n+1}_+$. Assume further that

$$\nabla_{\mathcal{A},\mu} u := \begin{bmatrix} \partial_{\nu_A} u \\ \nabla_{\|} u \\ |V|^{\frac{1}{2}} u \end{bmatrix} \in L^2_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^n; \mathbb{C}^{n+2})),$$

then $F := \nabla_{\mathcal{A},\mu} u$ is a weak solution of $\partial_t F + DBF = 0$ in $\mathbb{R}_+$, with $F \in L^2_{\mathrm{loc}}(\mathbb{R}_+; \overline{\mathsf{R}(D)})$. Conversely, if $F \in L^2_{\mathrm{loc}}(\mathbb{R}_+; \overline{\mathsf{R}(D)})$ is a weak solution of $\partial_t F + DBF = 0$ in $\mathbb{R}_+$, then there exists u such that $H_{A,a,V}\, u = 0$ in $\mathbb{R}^{n+1}_+$ with $F = \nabla_{\mathcal{A},\mu} u$.

Proof. Let u be such that $H_{A,a,V}\, u = 0$ where $\nabla_{\mathcal{A},\mu} u \in L^2_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^n; \mathbb{C}^{n+2}))$. Define $F := \nabla_{\mathcal{A},\mu} u$. Then for all $\varphi \in \mathcal{C}^\infty_c(\mathbb{R}^{n+1}_+)$ we have

$$\iint_{\mathbb{R}^{n+1}_+} A\nabla u \cdot \overline{\nabla\varphi} + aVu\overline{\varphi}\, \mathrm{d}x\, \mathrm{d}t = 0.$$

Note that for each fixed $t > 0$, by Lemma 2.3.6, we have

$$F(t) = \begin{bmatrix} (\partial_{\nu_A} u)(t) \\ (\nabla_{\|} u)(t) \\ (|V|^{\frac{1}{2}} u)(t) \end{bmatrix} \in \overline{\mathsf{R}(D)}.$$

Therefore, $F \in L^2_{\mathrm{loc}}(\mathbb{R}_+; \overline{\mathsf{R}(D)})$. Also, the definition of F and ∂_{ν_A}, we have

$$\begin{aligned}
(BF)_{\|} &= A_{\|\perp} A_{\perp\perp}^{-1} F_{\perp} + (A_{\|\|} - A_{\|\perp} A_{\perp\perp}^{-1} A_{\perp\|}) F_{\|} \\
&= A_{\|\perp} A_{\perp\perp}^{-1} \partial_{\nu_A} u + (A_{\|\|} - A_{\|\perp} A_{\perp\perp}^{-1} A_{\perp\|}) \nabla_{\|} u \\
&= A_{\|\perp} A_{\perp\perp}^{-1} (A_{\perp\perp} \partial_t u + A_{\perp\|} \nabla_{\|} u) + (A_{\|\|} - A_{\|\perp} A_{\perp\perp}^{-1} A_{\perp\|}) \nabla_{\|} u \\
&= A_{\|\perp} \partial_t u + A_{\|\|} \nabla_{\|} u \\
&= (A\nabla u)_{\|}.
\end{aligned}$$

We also have $ae^{i \arg V(x)} |V|^{\frac{1}{2}} u = (BF)_{\mu}$. Now, for any $\varphi \in \mathcal{C}^\infty_c(\mathbb{R}^{n+1}_+; \mathbb{R}^{n+2})$, as $H_{A,a,V}\, u =$

0 and the above, then

$$\begin{aligned}\int_0^\infty \langle F_\perp, \partial_t \varphi_\perp\rangle \,\mathrm{d}t &= \int_0^\infty \langle (A\nabla u)_\perp, \partial_t\varphi_\perp\rangle \,\mathrm{d}t \\ &= -\int_0^\infty \langle (A\nabla u)_\parallel, \nabla_\parallel \varphi_\perp\rangle - \langle a e^{i\arg V(x)}|V|^{\frac{1}{2}}u, |V|^{\frac{1}{2}}\varphi_\perp\rangle \,\mathrm{d}t \\ &= \int_0^\infty \langle (BF)_\parallel, (D\varphi)_\parallel\rangle + \langle (BF)_\mu, (D\varphi)_\mu\rangle \,\mathrm{d}t.\end{aligned}$$

A direct calculation gives us that $\partial_t u = (BF)_\perp$. Therefore, integrating by parts and using that $(D\varphi)_\perp = \operatorname{div}_\parallel \varphi_\parallel - |V|^{\frac{1}{2}}\varphi_\mu$, we have

$$\begin{aligned}\int_0^\infty \langle F_\parallel, \partial_t\varphi_\parallel\rangle \,\mathrm{d}t &= \int_0^\infty \langle \nabla_\parallel u, \partial_t \varphi_\parallel\rangle \,\mathrm{d}t, \\ &= \int_0^\infty \langle \partial_t u, \operatorname{div}_\parallel \varphi_\parallel\rangle \,\mathrm{d}t, \\ &= \int_0^\infty \langle \partial_t u, \operatorname{div}_\parallel \varphi_\parallel - |V|^{\frac{1}{2}}\varphi_\mu\rangle + \langle \partial_t u, |V|^{\frac{1}{2}}\varphi_\mu\rangle \,\mathrm{d}t, \\ &= \int_0^\infty \langle (BF)_\perp, (D\varphi)_\perp\rangle \,\mathrm{d}t + \int_0^\infty \langle \partial_t u, |V|^{\frac{1}{2}}\varphi_\mu\rangle \,\mathrm{d}t.\end{aligned}$$

Also,

$$\int_0^\infty \langle F_\mu, \partial_t\varphi_\mu\rangle \,\mathrm{d}t = \int_0^\infty \langle |V|^{\frac{1}{2}}u, \partial_t\varphi_\mu\rangle \,\mathrm{d}t = -\int_0^\infty \langle \partial_t u, |V|^{\frac{1}{2}}\varphi_\mu\rangle \,\mathrm{d}t.$$

Combining these gives

$$\int_0^\infty \langle F, \partial_t\varphi\rangle \,\mathrm{d}t = \int_0^\infty \langle BF, D\varphi\rangle \,\mathrm{d}t.$$

Thus F is a weak solution of $\partial_t F + DBF = 0$ in $\mathbb{R}_+$.

Now let $F \in L^2_{\mathrm{loc}}(\mathbb{R}_+; \overline{\mathsf{R}(D)})$ be a weak solution of $\partial_t F + DBF = 0$ in $\mathbb{R}_+$. Then, for each $t > 0$ there exists $g_t \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^n)$ such that

$$\begin{bmatrix} F_\parallel(t,x) \\ F_\mu(t,x)\end{bmatrix} = \begin{bmatrix} \nabla_\parallel g_t(x) \\ |V|^{\frac{1}{2}} g_t(x)\end{bmatrix}.$$

Define $g(t,x) \coloneqq g_t(x)$. Fix $0 < c_0 < \infty$. Now define

$$u(t,x) \coloneqq \int_{c_0}^t (BF)_\perp(s,x)\,\mathrm{d}s + g(c_1, x), \quad \forall t > 0, \text{ and a.e. } x \in \mathbb{R}^n.$$

Then, for fixed $x_0 \in \mathbb{R}^n$ we have $u(t, x_0)$ is absolutely continuous in t and, by the Fundamental Theorem of Calculus, $(\partial_t u)(t, x_0) = (BF)_\perp(t, x_0)$. Let $\psi \in \mathcal{C}_c^\infty(\mathbb{R}_+)$, $\eta \in \mathcal{C}_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$ and

$$\varphi(t, x) \coloneqq \begin{bmatrix} 0 \\ \psi(t)\eta(x) \\ 0 \end{bmatrix} \in \mathcal{C}_c^\infty(\mathbb{R}_+^{n+1}; \mathbb{R}^{n+2}).$$

As F is a weak solution of $\partial_t F + DBF = 0$ in $\mathbb{R}_+$ we have

$$\int_0^\infty \langle F(t), (\partial_t \varphi)(t)\rangle \,\mathrm{d}t = \int_0^\infty \langle (BF)(t), (D\varphi)(t)\rangle \,\mathrm{d}t \tag{6.1.1}$$

Therefore, the structure of φ gives

$$\int_0^\infty \langle F(t), (\partial_t \varphi)(t)\rangle \,\mathrm{d}t = \int_0^\infty \int_{\mathbb{R}^n} F_\parallel(t, x) \cdot \overline{\eta(x)(\partial_t \psi)(t)} \,\mathrm{d}x \,\mathrm{d}t.$$

Also, using the structure of φ, Fubini's Theorem and integrating by parts, we have

$$\begin{aligned} \int_0^\infty \langle (BF)(t), (D\varphi)(t)\rangle \,\mathrm{d}t &= \int_0^\infty \int_{\mathbb{R}^n} (BF)_\perp(t, x)\overline{\psi(t)(\operatorname{div}_\parallel \eta)(x)} \,\mathrm{d}x \,\mathrm{d}t \\ &= \int_{\mathbb{R}^n} \left(\int_0^\infty (\partial_t u)(t, x)\overline{\psi(t)} \,\mathrm{d}t \right) \overline{(\operatorname{div}_\parallel \eta)(x)} \,\mathrm{d}x \\ &= -\int_{\mathbb{R}^n} \left(\int_0^\infty u(t, x)\overline{(\partial_t \psi)(t)} \,\mathrm{d}t \right) \overline{(\operatorname{div}_\parallel \eta)(x)} \,\mathrm{d}x. \end{aligned}$$

Then, by (6.1.1), we get

$$\int_0^\infty \int_{\mathbb{R}^n} F_\parallel(t, x) \cdot \overline{\eta(x)(\partial_t \psi)(t)} \,\mathrm{d}x \,\mathrm{d}t = -\int_{\mathbb{R}^n} \left(\int_0^\infty u(t, x)\overline{(\partial_t \psi)(t)} \,\mathrm{d}t \right) \overline{(\operatorname{div}_\parallel \eta)(x)} \,\mathrm{d}x.$$

Then using Fubini's Theorem and rearranging gives

$$\int_0^\infty \left(\int_{\mathbb{R}^n} F_\parallel(t, x) \cdot \overline{\eta(x)} + u(t, x)\overline{(\operatorname{div}_\parallel \eta)(x)} \,\mathrm{d}x \right) \overline{(\partial_t \psi)(t)} \,\mathrm{d}t = 0.$$

Then as ψ was arbitrary we use integration by parts to deduce that there exists a constant

c such that

$$c = \int_{\mathbb{R}^n} F_{\|}(t,x) \cdot \overline{\eta(x)} + u(t,x)\overline{(\operatorname{div}_{\|} \eta)(x)} \, \mathrm{d}x.$$

Also, using (6.1.1) gives

$$\int_0^\infty \left(\int_{\mathbb{R}^n} F_{\|}(t,x) \cdot \overline{\eta(x)} \right) \overline{(\partial_t \psi)(t)} \, \mathrm{d}t = \int_0^\infty \left(\int_{\mathbb{R}^n} (BF)_\perp(t,x) \overline{(\operatorname{div}_{\|} \eta)(x)} \, \mathrm{d}x \right) \overline{\psi(t)} \, \mathrm{d}t$$

Then, by the definition of a weak derivative, we get

$$\partial_t \left(\int_{\mathbb{R}^n} F_{\|}(t,x) \cdot \overline{\eta(x)} \, \mathrm{d}x \right) = - \int_{\mathbb{R}^n} (BF)_\perp(t,x) \overline{\operatorname{div}_{\|} \eta(x)} \, \mathrm{d}x,$$

in the weak sense. Recall that a function is weakly differentiable if and only if it is locally absolutely continuous (see [30]). Therefore, c_1 is a Lebesgue point for

$$\int_{\mathbb{R}^n} F_{\|}(t,x) \cdot \overline{\eta(x)} \, \mathrm{d}x \quad \text{for a.e. } x \in \mathbb{R}^n \text{ and } \forall \eta \in \mathcal{C}_c^\infty(\mathbb{R}^n; \mathbb{R}^n).$$

Let $\varepsilon > 0$. Let $\delta \in (0, c_0)$. Thus, using the Lebesgue differentiation Theorem we may choose $\delta > 0$ such that

$$\left| \fint_{c_0 - \delta}^{c_0 + \delta} \left(\int_{\mathbb{R}^n} F_{\|}(t,x) \cdot \overline{\eta(x)} \, \mathrm{d}x \right) \mathrm{d}t - \int_{\mathbb{R}^n} F_{\|}(c_1, x) \cdot \overline{\eta(x)} \, \mathrm{d}x \right| < \frac{\varepsilon}{2}, \quad \text{and}$$
$$\left| \fint_{c_0 - \delta}^{c_0 + \delta} \int_{c_0}^{t} (BF)_\perp(s,x) \, \mathrm{d}s \, \mathrm{d}t \right| < \frac{\varepsilon}{2 \| \operatorname{div}_{\|} \eta \|_1}.$$

Recalling the definition of u, we have

$$c = \fint_{c_0 - \delta}^{c_0 + \delta} \int_{\mathbb{R}^n} F_{\|}(t,x) \cdot \overline{\eta(x)} - \nabla_{\|} g(c_1, x) \cdot \overline{\eta(x)} + \left(\int_{c_0}^{t} (BF)_\perp(s,x) \, \mathrm{d}s \right) \overline{(\operatorname{div}_{\|} \eta)(x)} \, \mathrm{d}x \, \mathrm{d}t.$$

Using integration by parts, the definition of g, and Fubini's Theorem we get

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n}\left(\fint_{c_0-\delta}^{c_0+\delta} F_{\|}(t,x)\cdot\overline{\eta(x)}\,\mathrm{d}t\right)-\nabla_{\|}g(c_0,x)\cdot\overline{\eta(x)}\,\mathrm{d}x\right|\\
&\quad=\left|\int_{\mathbb{R}^n}\left(\fint_{c_0-\delta}^{c_0+\delta} F_{\|}(t,x)\,\mathrm{d}t-\nabla_{\|}g(c_1,x)\right)\cdot\overline{\eta(x)}\,\mathrm{d}x\right|\\
&\quad=\left|\fint_{c_0-\delta}^{c_0+\delta}\left(\int_{\mathbb{R}^n} F_{\|}(t,x)\cdot\overline{\eta(x)}\,\mathrm{d}x\right)\mathrm{d}t-\int_{\mathbb{R}^n} F_{\|}(c_0,x)\cdot\overline{\eta(x)}\,\mathrm{d}x\right|\\
&\quad<\frac{\varepsilon}{2}.
\end{aligned}$$

Similarly, we have that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n}\left(\fint_{c_0-\delta}^{c_0+\delta}\int_{c_0}^{t}(BF)_{\perp}(s,x)\,\mathrm{d}s\,\mathrm{d}t\right)\overline{(\operatorname{div}_{\|}\eta)(x)}\,\mathrm{d}x\right|\\
&\quad\leq\int_{\mathbb{R}^n}\left|\fint_{c_0-\delta}^{c_0+\delta}\int_{c_0}^{t}(BF)_{\perp}(s,x)\,\mathrm{d}s\,\mathrm{d}t\right|\left|(\operatorname{div}_{\|}\eta)(x)\right|\mathrm{d}x\\
&\quad<\frac{\varepsilon}{2}.
\end{aligned}$$

Thus $|c|<\varepsilon$. As $\varepsilon>0$ was arbitrary we have that $c=0$. We now proceed in a similar manner for the third component. Redefine $\eta\in\mathcal{C}_c^\infty(\mathbb{R}^n)$ and

$$\varphi(t,x):=\begin{bmatrix}0\\0\\\psi(t)\eta(x)\end{bmatrix}\in\mathcal{C}_c^\infty(\mathbb{R}^{n+1}_+;\mathbb{R}^{n+2}).$$

Then, using that F is a solution of $\partial_t F+DBF=0$, $\partial_t u=(BF)_{\perp}$, and integration by parts, we have

$$\int_0^\infty\left(\int_{\mathbb{R}^n}F_\mu(t,x)\overline{\eta(x)}\,\mathrm{d}x\right)\overline{(\partial_t\psi)(t)}\,\mathrm{d}t=\int_0^\infty\left(\int_{\mathbb{R}^n}u(t,x)\overline{(|V|^{\frac{1}{2}}\eta)(x)}\,\mathrm{d}x\right)\overline{(\partial_t\psi)(t)}\,\mathrm{d}t.$$

Therefore, we have

$$c=\int_{\mathbb{R}^n}\left(F_\mu(t,x)-|V|^{\frac{1}{2}}u(t,x)\right)\overline{\eta(x)}\,\mathrm{d}x$$

As before, let $\varepsilon>0$. Let $\delta\in(0,c_0)$. Then, using the Lebesgue differentiation Theorem we may choose $\delta > 0$ such that

$$\left| \fint_{c_0-\delta}^{c_0+\delta} \left(\int_{\mathbb{R}^n} F_\mu(t,x)\overline{\eta(x)}\,\mathrm{d}x \right) \mathrm{d}t - \int_{\mathbb{R}^n} F_\mu(c_1,x)\overline{\eta(x)}\,\mathrm{d}x \right| < \frac{\varepsilon}{2}, \quad \text{and}$$
$$\left| \fint_{c_0-\delta}^{c_0+\delta} \int_{c_0}^{t} (BF)_\perp(s,x)\,\mathrm{d}s\,\mathrm{d}t \right| < \frac{\varepsilon}{2\||V|^{\frac{1}{2}}\eta\|_1}.$$

Replicating the argument for $F_\|$, we obtain $|c| < \varepsilon$ for all $\varepsilon > 0$. Hence $c = 0$. That is $(BF)_\perp(t,x) = \partial_t u(t,x)$, $F_\|(t,x) = \nabla_\| u(t,x)$, and $F_\mu(t,x) = |V|^{\frac{1}{2}} u(t,x)$. Then, rearranging gives $F_\perp = \partial_{\nu_A} u$. Therefore,

$$F(t,x) = \nabla_{\mathcal{A},\mu} u(t,x).$$

We proceed by proving $H_{A,a,V}\, u = 0$. To this end, let $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^{n+1}_+)$, and define

$$\varphi = \begin{bmatrix} \psi \\ 0 \\ 0 \end{bmatrix} \in \mathcal{C}_c^\infty(\mathbb{R}^{n+1}_+, \mathbb{C}^{n+2}).$$

A direct calculation leads to $(BF)_\| = (A\nabla u)_\|$ and $(BF)_\mu = a e^{i \arg V(x)} |V|^{\frac{1}{2}} u$. Therefore, using (6.1.1), we obtain

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}^n} \partial_{\nu_A} u\, \overline{\partial_t \psi}\,\mathrm{d}x\,\mathrm{d}t &= -\int_0^\infty \int_{\mathbb{R}^n} (BF)_\| \cdot \overline{\nabla_\| \psi} - (BF)_\mu |V|^{\frac{1}{2}} \overline{\psi}\,\mathrm{d}x\,\mathrm{d}t \\ &= -\int_0^\infty \int_{\mathbb{R}^n} (A\nabla u)_\| \cdot \overline{\nabla_\| \psi} - aVu\overline{\psi}\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Thus

$$\int_0^\infty \int_{\mathbb{R}^n} (A\nabla u) \cdot \overline{\nabla \psi} + aVu\overline{\psi}\,\mathrm{d}x\,\mathrm{d}t = 0.$$

Hence, $H_{A,a,V}\, u = 0$. This completes the proof. □

Note that to go from a second-order solution of a first-order solution we need to assume some kind of control on the adapted gradient. A natural estimate to have is L^2 control on the non-tangential maximal function of the adapted gradient. We give the proof of that this is sufficient in Proposition 8.1.1.

6.2 Boundary Isomorphisms for Block Type Matrices

To recap, the set $\mathcal{H}^{0,+}_{DB}$ is the set of all initial data for the solutions satisfying the first-order Cauchy problem (5.3.2) on $\mathbb{R}^{n+1}_+$. We also have solutions satisfying (5.3.2) arise from an analytic semigroup generated by DB applied to the initial data in $\mathcal{H}^{0,+}_{DB}$, and that these solutions are equal to the adapted gradients of a solution of second order equation (2.6.1). That is $F = e^{-tDB}f = \nabla_{\mathcal{A},\mu} u$ for some u such that $H_{A,a,V}\, u = 0$. We define the mappings

$$\begin{aligned} \Phi_N \colon \mathcal{H}^{0,+}_{DB} &\to L^2(\mathbb{R}^n), \text{ given by } \Phi_N(f) \coloneqq f_\perp, \\ \Phi_R \colon \mathcal{H}^{0,+}_{DB} &\to \{\nabla_\mu u : u \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^n)\}, \text{ given by } \Phi_R(f) \coloneqq \begin{bmatrix} f_\parallel \\ f_\mu \end{bmatrix} = f_r. \end{aligned} \tag{6.2.1}$$

These mappings are seen to be sending the initial values for the Cauchy problem $\partial_t F + DBF = 0$ to the boundary conditions for equation $H_{A,a,V}\, u = 0$. Therefore, if the mappings are isomorphisms we will be able to invert them and uniquely assign any given boundary data for the boundary value problem with a solution of the first-order equation. Then using Proposition 6.1.2 will give a solution u such that $H_{A,a,V}\, u = 0$. In other words, if Φ_N and Φ_R are isomorphisms then the second-order equating is well-posed. We will formalise this in Section 8.1

We now proceed by proving that the mappings Φ_R and Φ_N from (6.2.1) are isomorphisms in the case when $\mathcal{A}^{\mathcal{V}}$ is block-type, that is

$$\mathcal{A}^{\mathcal{V}} = \begin{bmatrix} A_{\perp\perp} & 0 & 0 \\ 0 & A_{\parallel\parallel} & 0 \\ 0 & 0 & ae^{i\arg V(x)} \end{bmatrix}, \text{ and so } \widehat{\mathcal{A}}^{\mathcal{V}} = B = \begin{bmatrix} A_{\perp\perp}^{-1} & 0 & 0 \\ 0 & A_{\parallel\parallel} & 0 \\ 0 & 0 & ae^{i\arg V(x)} \end{bmatrix}.$$

We do this in a similar way to the methods used in [4]. Define the bounded linear operator

$N\colon \overline{\mathsf{R}(D)} \to \overline{\mathsf{R}(D)}$ given by

$$N := \begin{bmatrix} -1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Note that $N^{-1} = N$. We start with a lemma from the theory of functional calculus so that we may take advantage of the symmetry of the operator E_{DB}.

Lemma 6.2.1. Let $\omega \in (0, \frac{\pi}{2})$. If T is a closed densely defined ω-bisectorial operator with bounded holomorphic functional calculus, then $S := NTN$, is a closed densely defined ω-bisectorial operator with bounded holomorphic functional calculus and $f(S) = Nf(T)N$ for all $f \in H^\infty(S^o_\mu)$ and all $\mu \in (\omega, \frac{\pi}{2})$.

Proof. First we prove $\sigma(T) = \sigma(S)$. Let $\lambda \in \rho(T)$. Then we have that $\lambda I - S = \lambda NN - NSN = N(\lambda I - T)N$. As N and $\lambda I - T$ are invertible then $\lambda I - S$ is invertible. That is $\lambda \in \rho(S)$. Now suppose $\lambda \in \rho(S)$. Then $\lambda I - T = NN(\lambda I - T)NN = N(\lambda I - S)N$. Then as N and $\lambda I - S$ are invertible so is $\lambda I - T$. Thus $\lambda \in \rho(T)$. Thus $\rho(T) = \rho(S)$. Equivalently, we have $\sigma(T) = \sigma(S)$. Now let $\mu \in (\omega, \pi/2)$.

Now we prove resolvent bounds for S. Note that T satisfies the resolvent bounds and consider

$$\begin{aligned} \|(\lambda I - S)^{-1}u\|_2 &= \|(\lambda I - NTN)^{-1}u\|_2 \\ &= \|N(\lambda I - T)^{-1}Nu\|_2 \\ &\lesssim \|(\lambda I - T)^{-1}Nu\|_2 \\ &\leq \frac{C_\mu}{|\lambda|}\|Nu\|_2 \\ &\lesssim \frac{C_\mu}{|\lambda|}\|u\|_2, \end{aligned}$$

for all $\lambda \notin S^o_\mu$. Thus S is closed densely defined ω-bisectorial operator.

To prove the functional calculus of S let $\psi \in \Psi(S^o_\mu)$. Then by the Dunford–Riesz functional calculus we have that

$$\begin{aligned}\psi(S) &= \frac{1}{2\pi i}\int_\gamma \psi(z)(zI-S)^{-1}\,\mathrm{d}z \\ &= \frac{1}{2\pi i}\int_\gamma \psi(z)N(zI-T)^{-1}N\,\mathrm{d}z \\ &= N\left(\frac{1}{2\pi i}\int_\gamma \psi(z)(zI-T)^{-1}\,\mathrm{d}z\right)N \\ &= N\psi(T)N.\end{aligned}$$

Now as T has bounded holomorphic functional calculus we have

$$\|\psi(S)u\|_2 = \|N\psi(T)Nu\|_2 \lesssim \|\psi(T)Nu\|_2 \lesssim \|\psi\|_\infty\|Nu\|_2 \lesssim \|\psi\|_\infty\|u\|_2.$$

Thus we have that there exists $\mu \in (\omega, \pi/2)$ such that $\|\psi(S)u\|_2 \lesssim \|\psi\|_\infty\|u\|_2$ for all $\psi \in \Psi(S^0_\mu)$ and $u \in \mathcal{H}$. Hence, S has bound holomorphic functional calculus by Theorem 2.1.4. Then there exists a sequence $\{\psi_n\}_{n\in\mathbb{N}} \subseteq \Psi(S^o_\mu)$ such that $\lim_{n\to\infty}((f\psi_n)(S)u) = f(S)u$ for all $f \in H^\infty(S^o_\mu)$ and $u \in \mathcal{H}$. Then

$$f(S)u = \lim_{n\to\infty}((f\psi_n)(S)u) = \lim_{n\to\infty}(N(f\psi_n)(T)Nu) = N\lim_{n\to\infty}((f\psi_n)(T)Nu) = Nf(T)Nu.$$

This completes the proof. $\square$

Now as DB has bounded holomorphic functional calculus and $\operatorname{sgn} \in H^\infty(S^o_\mu)$, then the previous lemma gives

$$NE_{DB} = N\operatorname{sgn}(DB)NN = \operatorname{sgn}(NDBN)N = \operatorname{sgn}(-DB)N = -\operatorname{sgn}(DB)N = -E_{DB}N. \tag{6.2.2}$$

Now define the bounded linear operators $N^\pm \colon \overline{\mathsf{R}(D)} \to \overline{\mathsf{R}(D)}$ given by

$$N^- := \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} = \frac{1}{2}(I-N), \quad N^+ := \begin{bmatrix}0 & 0 & 0\\ 0 & I & 0\\ 0 & 0 & 1\end{bmatrix} = \frac{1}{2}(I+N).$$

We can see that if $f \in \mathcal{H}^{0,+}_{DB}$, then $N^{\pm} f$ corresponds to the Neumann and Regularity boundary conditions respectively. The next lemma formalises this idea.

Lemma 6.2.2. If $N^+ \colon \mathcal{H}^{0,+}_{DB} \to N^+\overline{\mathsf{R}(D)}$ is an isomorphism, then $\Phi_R \colon \mathcal{H}^{0,+}_{DB} \to \{\nabla_\mu u : u \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^n)\}$ is an isomorphism. Also, if $N^- \colon \mathcal{H}^{0,+}_{DB} \to N^-\overline{\mathsf{R}(D)}$ is an isomorphism, then $\Phi_N \colon \mathcal{H}^{0,+}_{DB} \to L^2(\mathbb{R}^n)$ is an isomorphism.

Proof. Suppose N^+ is an isomorphism. Note by Lemma 2.3.6 we have

$$N^+\overline{\mathsf{R}(D)} = \left\{ \begin{bmatrix} 0 \\ \nabla_\mu u \end{bmatrix} : \text{ for some } u \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^n) \right\}.$$

Then let $F \in N^+\overline{\mathsf{R}(D)}$. Therefore $F = (0, \nabla_\mu u)^T$ for some $u \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^n)$. Then as N^+ is an isomorphism we have there exists a unique $f \in \mathcal{H}^{0,+}_{DB}$ such that $N^+ f = F$. Therefore, $\Phi_R(f) = \nabla_\mu u$. That is Φ_R is surjective. Now assume that there exists $f, g \in \mathcal{H}^{0,+}_{DB}$ such that $\Phi_R(f) = \Phi_R(g) = \nabla_\mu u$. Then $N^+ f = N^+ g$. Thus as N^+ is an isomorphism (and therefore injective) we have $f = g$. Then Φ_R is injective. Thus Φ_R is an isomorphism.

The second statement is proved similarly. $\square$

We now prove that these mappings are indeed isomorphisms, in the block case.

Proposition 6.2.3. If $\mathcal{A}$ is block type then the mappings $\Phi_R \colon \mathcal{H}^{0,+}_{DB} \to \{\nabla_\mu u : u \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^n)\}$ and $\Phi_N \colon \mathcal{H}^{0,+}_{DB} \to L^2(\mathbb{R}^n)$ are isomorphisms.

Proof. By Lemma 6.2.2 it suffices to show $N^+ \colon \mathcal{H}^{0,+}_{DB} \to N^+\overline{\mathsf{R}(D)}$ and $N^- \colon \mathcal{H}^{0,+}_{DB} \to N^-\overline{\mathsf{R}(D)}$ are isomorphisms.

To prove surjectivity, let $g \in N^+\overline{\mathsf{R}(D)}$ so $g = (0, g_\parallel, g_\mu)^T$ and $Ng = g$. We have

$$2E^+_{DB} g = (\chi^+(DB) - \chi^-(DB))g + (\chi^-(DB) + \chi^+(DB))g = E_{DB} g + g.$$

Therefore, using the above calculation and (6.2.2), we have

$$
\begin{aligned}
N^+(2E_{DB}^+g) &= N^+(E_{DB}+I)g \\
&= \frac{1}{2}(I+N)(I+E_{DB})g \\
&= \frac{1}{2}(g+Ng)+\frac{1}{2}(E_{DB}g-E_{DB}Ng) \\
&= g.
\end{aligned}
$$

Then, for any $g \in N^+\overline{\mathsf{R}(D)}$ we have $N^+(2E_{DB}^+g) = g$. That is $N^+ \colon \mathcal{H}_{DB}^{0,+} \to N^+\overline{\mathsf{R}(D)}$ is surjective.

To prove injectivity let $f \in \mathcal{H}_{DB}^{0,+}$ with $N^+f = 0$, so $Nf = -f$ and $f_\parallel = f_\mu = 0$. Then, as $E_{DB}f = f$ and (6.2.2), we have

$$
0 = E_{DB}Nf + NE_{DB}f = -E_{DB}f + Nf = -f - f = -2f.
$$

Thus $f = 0$. That is N^+ is injective and so bijective. Thus, by Lemma 6.2.2 we have $\Phi_R \colon E_{DB}^+ \to \{\nabla_\mu u : u \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^n)\}$ is an isomorphism.

The case for $\Phi_N \colon \mathcal{H}_{DB}^{0,+} \to L^2(\mathbb{R}^n)$ is similar. □

6.3 Boundary Isomorphisms for Self-Adjoint Matrices

We now move to the self adjoint case, that is $\mathcal{A}^* = \mathcal{A}$. Then we have

$$
\mathcal{A}^* = \begin{bmatrix} A & 0 \\ 0 & a \end{bmatrix}^* = \begin{bmatrix} A_{\perp\perp}^* & A_{\parallel\perp}^* & 0 \\ A_{\perp\parallel}^* & A_{\parallel\parallel}^* & 0 \\ 0 & 0 & a^* \end{bmatrix} = \begin{bmatrix} A_{\perp\perp} & A_{\perp\parallel} & 0 \\ A_{\parallel\perp} & A_{\parallel\parallel} & 0 \\ 0 & 0 & a \end{bmatrix} = \mathcal{A}
$$

Then, by a direct computation we have

$$
(\widehat{\mathcal{A}})^* = B^* = NBN. \tag{6.3.1}
$$

We now aim to establish a Rellich type estimate. This will be used to prove that the mappings Φ_N and Φ_R from (6.2.1) are injective. We will then use the method of continuity to prove the surjectivity of the mappings.

Proposition 6.3.1. If $\mathcal{A}$ is self-adjoint then the mappings $\Phi_R \colon \mathcal{H}^{0,+}_{DB} \to \{\nabla_\mu u : u \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^n)\}$ and $\Phi_N \colon \mathcal{H}^{0,+}_{DB} \to L^2(\mathbb{R}^n)$ are injective. Moreover, they satisfy the estimate:

$$\|\Phi_N(f)\|_2 \eqsim \|\Phi_R(f)\|_2$$

for all $f \in \mathcal{H}^{0,+}_{DB}$.

Proof. Let $f \in \mathcal{H}^{0,+}_{DB}$. Then by Corollary 5.3.9 we have there exists a unique solution, $F \in \mathcal{C}^1(\mathbb{R}_+; \overline{\mathsf{R}(D)})$ satisfying (5.3.2), and in particular, $F(t,x) = e^{-tDB}f(x)$ Then using the Fundamental Theorem of Calculus to get

$$\int_0^\infty \partial_t \langle NBF(t), F(t)\rangle \, \mathrm{d}t = \lim_{t\to\infty} \langle NBF(t), F(t)\rangle - \lim_{t\to 0} \langle NBF(t), F(t)\rangle = -\langle NBf, f\rangle.$$

Now

$$\begin{aligned} DN + ND &= \begin{bmatrix} 0 & (-\nabla_\mu^{\|})^* \\ -\nabla_\mu^{\|} & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & (-\nabla_\mu^{\|})^* \\ -\nabla_\mu^{\|} & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & (-\nabla_\mu^{\|})^* \\ \nabla_\mu^{\|} & 0 \end{bmatrix} + \begin{bmatrix} 0 & -(-\nabla_\mu^{\|})^* \\ -\nabla_\mu^{\|} & 0 \end{bmatrix} \\ &= 0 \end{aligned} \tag{6.3.2}$$

Now, using $\partial_t F = -DBF$, $N^2 = I$, (6.3.1), and (6.3.2), we have

$$\begin{aligned}
-\int_0^\infty \partial_t \langle NBF(t), F(t)\rangle \,\mathrm{d}t &= -\int_0^\infty (\langle NB\partial_t F(t), F(t)\rangle + \langle NBF(t), \partial_t F(t)\rangle)\,\mathrm{d}t \\
&= \int_0^\infty (\langle NBDBF(t), F(t)\rangle + \langle NBF(t), DBF(t)\rangle)\,\mathrm{d}t \\
&= \int_0^\infty (\langle (NBN)NDBF(t), F(t)\rangle + \langle DNBF(t), BF(t)\rangle)\,\mathrm{d}t \\
&= \int_0^\infty (\langle NDBF(t), BF(t)\rangle + \langle DNBF(t), BF(t)\rangle)\,\mathrm{d}t \\
&= \int_0^\infty \langle (ND + DN)BF(t), BF(t)\rangle \,\mathrm{d}t \\
&= 0.
\end{aligned}$$

That is $\langle NBf, f\rangle = 0$. Or, equivalently $\langle (Bf)_\perp, f_\perp\rangle = \langle (Bf)_\parallel, f_\parallel\rangle + \langle (Bf)_\mu, f_\mu\rangle$. Then as B is elliptic on $\overline{\mathsf{R}(D)}$, as in (2.5.1), we have

$$\|f\|_2^2 \lesssim \mathrm{Re}\langle Bf, f\rangle = 2\,\mathrm{Re}\langle (Bf)_\perp, f_\perp\rangle \lesssim \|(Bf)_\perp\|_2 \|f_\perp\|_2 \le \|f\|_2 \|\Phi_N(f)\|_2.$$

Thus $\|f\|_2 \lesssim \|\Phi_N(f)\|_2$ and so Φ_N is injective. We also have

$$\|f\|_2^2 \lesssim \mathrm{Re}\langle Bf, f\rangle = 2\,\mathrm{Re}\left(\langle (Bf)_\parallel, f_\parallel\rangle + \langle (Bf)_\mu, f_\mu\rangle\right) \lesssim \|f\|_2 \|\Phi_R(f)\|_2.$$

Hence $\|f\|_2 \lesssim \|\Phi_R(f)\|_2$ and so Φ_R is injective. Together, this gives the Rellich estimate

$$\|\Phi_N(f)\|_2 = \|f_\perp\|_2 \eqsim \left\| \begin{bmatrix} f_\parallel \\ f_\mu \end{bmatrix} \right\|_2 = \|\Phi_R(f)\|_2,$$

as required. $\square$

We now turn to surjectivity where we will use the method of continuity. It is important to note that this depends on the analytic dependence of the functional calculus of the operator DB as in Theorem 5.2.1. Define the self-adjoint matrix $\mathcal{A}_\tau := \tau\mathcal{A} + (1-\tau)I$ for

$\tau \in [0,1]$. Also, define $B_\tau = \widehat{\mathcal{A}}_\tau$ For all $u \in \overline{\mathsf{R}(D)}$, we have

$$\operatorname{Re}\langle B_\tau u, u\rangle = \tau \operatorname{Re}\langle Bu, u\rangle + (1-\tau)\langle u, u\rangle \geq \kappa\tau\|u\|_2^2 + (1-\tau)\|u\|^2 \geq \min\{\kappa, 1\}\|u\|_2^2,$$

and,

$$\|B_\tau\|_\infty \leq \tau\|B\|_\infty + (1-\tau)\|I\|_\infty \leq \|B\|_\infty + 1,$$

So B_τ is uniformly bounded and uniformly elliptic on $\overline{\mathsf{R}(D)}$ for all $\tau \in [0,1]$. If $0 < \omega < \mu$, then by Theorems 3.0.1 and 2.1.4, we have there exists $c_\mu > 0$, depending only on n, κ, and $\|B\|_\infty$, such that

$$\|f(DB_\tau)\| \leq c_\mu \|f\|_\infty, \tag{6.3.3}$$

for all $f \in H^\infty(S_\mu^o)$ and for all $\tau \in [0,1]$. Here we use the fact that the constant in Theorem 3.0.1 depends only on n, κ, and $\|B\|_\infty$. Now, for every $\tau \in [0,1]$ define the spectral projection associated with DB_τ by $E_\tau^+ \coloneqq \chi^+(DB_\tau)$. Also, define the bounded linear operator

$$\Phi_N^\tau \colon E_\tau^+\overline{\mathsf{R}(D)} \to L^2(\mathbb{R}^n) \quad \text{given by } \Phi_N^\tau(f) \coloneqq f_\perp,$$
$$\Phi_R^\tau \colon E_\tau^+\overline{\mathsf{R}(D)} \to \{\nabla_\mu u : u \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^n)\} \quad \text{given by } \Phi_R^\tau(f) \coloneqq \begin{bmatrix} f_\parallel \\ f_\mu \end{bmatrix},$$

for all $\tau \in [0,1]$, so $\Phi_N = \Phi_N^1$.

Lemma 6.3.2. There exists $\varepsilon > 0$ such that, if $|\tau - \sigma| < \varepsilon$, then $E_\tau^+ \colon E_\sigma^+\overline{\mathsf{R}(D)} \to E_\tau^+\overline{\mathsf{R}(D)}$ is bijective.

Proof. We claim $(I - E_\sigma^+(E_\sigma^+ - E_\tau^+))^{-1}E_\sigma^+$ and $E_\sigma^+(I - E_\tau^+(E_\tau^+ - E_\sigma^+))^{-1}$ are the left and the right inverse respectively. First, by the quadratic estimate in Theorem 3.0.1 we have the bounded holomorphic functional calculus in Theorem 5.0.1, and so

$$\begin{aligned}\|E_\tau^+(E_\tau^+ - E_\sigma^+)f\|_2 &\leq c_\mu \|\chi^+\|_\infty \|(E_\tau^+ - E_\sigma^+)f\|_2 \\ &\leq c\|B_\tau - B_\sigma\|_\infty \|f\|_2 \\ &= c|\tau - \sigma| \|B - I\|_\infty \|f\|_2.\end{aligned}$$

where $c > 0$ depends on the constant from the analytic dependence in Theorem 5.2.1 and on $c_\mu > 0$ from (6.3.3). Thus, if $|\tau - \sigma| < \frac{1}{c\|B-I\|_\infty}$, then the Neumann series gives us $I - E_\tau^+(E_\tau^+ - E_\sigma^+)$ is invertible. Using a direct computation we see

$$E_\tau^+ E_\sigma^+ = E_\tau^+(I - E_\tau^+(E_\tau^+ - E_\sigma^+)).$$

And so, for all $f \in E_\tau^+\overline{\mathsf{R}(D)}$, we have

$$E_\tau^+ E_\sigma^+ (I - E_\tau^+(E_\tau^+ - E_\sigma^+))^{-1} f = E_\tau^+ f = f.$$

Thus, $E_\sigma^+(I - E_\tau^+(E_\tau^+ - E_\sigma^+))^{-1}$ is the right inverse of $E_\tau^+ \colon E_\sigma^+\overline{\mathsf{R}(D)} \to E_\tau^+\overline{\mathsf{R}(D)}$, that is $E_\tau^+ \colon E_\sigma^+\overline{\mathsf{R}(D)} \to E_\tau^+\overline{\mathsf{R}(D)}$ is surjective. Similarly, if $|\tau - \sigma| < \frac{1}{c\|B-I\|_\infty}$, then $I - E_\sigma^+(E_\sigma^+ - E_\tau^+)$ is also invertible, again using the Neumann series. Now, another computation gives

$$E_\sigma^+ E_\tau^+ E_\sigma^+ = (I - E_\sigma^+(E_\sigma^+ - E_\tau^+))E_\sigma^+.$$

Then, for all $f \in E_\sigma^+\overline{\mathsf{R}(D)}$, we have

$$(I - E_\sigma^+(E_\sigma^+ - E_\tau^+))^{-1} E_\sigma^+ E_\tau^+ f = (I - E_\sigma^+(E_\sigma^+ - E_\tau^+))^{-1} E_\sigma^+ E_\tau^+ E_\sigma^+ = E_\sigma^+ f = f.$$

Thus $(I - E_\sigma^+(E_\sigma^+ - E_\tau^+))^{-1} E_\sigma^+$ is the left inverse of $E_\tau^+ \colon E_\sigma^+\overline{\mathsf{R}(D)} \to E_\tau^+\overline{\mathsf{R}(D)}$. That is $E_\tau^+ \colon E_\sigma^+\overline{\mathsf{R}(D)} \to E_\tau^+\overline{\mathsf{R}(D)}$ is injective. Thus, $E_\tau^+ \colon E_\sigma^+\overline{\mathsf{R}(D)} \to E_\tau^+\overline{\mathsf{R}(D)}$ is a bijection. □

Lemma 6.3.3. If $E_\tau^+ \colon E_\sigma^+\overline{\mathsf{R}(D)} \to E_\tau^+\overline{\mathsf{R}(D)}$ is a bijection, then

- $\Phi_N^\tau \colon E_\tau^+\overline{\mathsf{R}(D)} \to L^2(\mathbb{R}^n)$ is bijective if and only if $\Phi_N E_\tau^+ \colon E_\sigma^+\overline{\mathsf{R}(D)} \to L^2(\mathbb{R}^n)$ is bijective.

- $\Phi_R^\tau\colon E_\tau^+\overline{\mathsf{R}(D)} \to \{\nabla_\mu u : u \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^n)\}$ is bijective if and only if $\Phi_R E_\tau^+\colon E_\sigma^+\overline{\mathsf{R}(D)} \to \{\nabla_\mu u : u \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^n)\}$ is bijective.

Proof. Suppose that $\Phi_N^\tau\colon E_\tau^+\overline{\mathsf{R}(D)} \to L^2(\mathbb{R}^n)$ is bijective. Therefore, $\Phi_N E_\tau^+\colon E_\sigma^+\overline{\mathsf{R}(D)} \to L^2(\mathbb{R}^n)$ is the composition of two bijective operators and so is a bijection.

Suppose that $\Phi_N E_\tau^+\colon E_\sigma^+\overline{\mathsf{R}(D)} \to L^2(\mathbb{R}^n)$ is bijective. Let $f \in L^2(\mathbb{R}^n)$. Then as the mapping $\Phi_N E_\tau^+\colon E_\tau^+\overline{\mathsf{R}(D)} \to L^2(\mathbb{R}^n)$ is bijective we have there exists $g \in E_\sigma^+\overline{\mathsf{R}(D)}$ such that $\Phi_N E_\tau^+ g = f$. Then there exists $h \in E_\tau^+\overline{\mathsf{R}(D)}$ (namely $h = E_\tau^+$ g) such that $\Phi_N^\tau h = f$. Thus, $\Phi_N^\tau\colon E_\tau^+\overline{\mathsf{R}(D)} \to L^2(\mathbb{R}^n)$ is surjective. Let $f \in E_\sigma^+\overline{\mathsf{R}(D)}$ be such that $\Phi_N^\tau f = 0$. As $E_\tau^+\colon E_\sigma^+\overline{\mathsf{R}(D)} \to E_\tau^+\overline{\mathsf{R}(D)}$ is a bijection, so invertible, we have $0 = \Phi_N^\tau f = \Phi_N E_\tau^+ E_\tau^{+^{-1}} f$. then as $\Phi_N E_\tau^+\colon E_\sigma^+\overline{\mathsf{R}(D)} \to L^2(\mathbb{R}^n)$ is bijective we have $E_\tau^{+^{-1}} f = 0$. Thus $f = 0$ and $\Phi_N^\tau\colon E_\tau^+\overline{\mathsf{R}(D)} \to L^2(\mathbb{R}^n)$ is injective, and thus bijective.

The case for $\Phi_R^\tau\colon E_\tau^+\overline{\mathsf{R}(D)} \to \{\nabla_\mu u : u \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^n)\}$ is similar. □

Lemma 6.3.4. Let $\sigma \in [0, 1]$. The following hold:

- If $\Phi_N^\sigma\colon E_\sigma^+\overline{\mathsf{R}(D)} \to L^2(\mathbb{R}^n)$ is bijective, then there exists $\varepsilon > 0$ such that for all $|\tau - \sigma| < \varepsilon$ we have $\Phi_N^\tau\colon E_\tau^+\overline{\mathsf{R}(D)} \to L^2(\mathbb{R}^n)$ is bijective.

- If $\Phi_R^\sigma\colon E_\sigma^+\overline{\mathsf{R}(D)} \to \{\nabla_\mu u : u \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^n)\}$ is bijective, then there exists $\varepsilon > 0$ such that for all $|\tau - \sigma| < \varepsilon$ we have $\Phi_R^\tau\colon E_\tau^+\overline{\mathsf{R}(D)} \to \{\nabla_\mu u : u \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^n)\}$ is bijective.

Proof. Consider $\Phi_N^\tau E_\tau^+\colon E_\sigma^+\overline{\mathsf{R}(D)} \to L^2(\mathbb{R}^n)$. Then fix $\varepsilon > 0$ such that $E_\tau^+\colon E_\sigma^+\overline{\mathsf{R}(D)} \to E_\tau^+\overline{\mathsf{R}(D)}$ is a bijection for all $|\tau - \sigma| < \varepsilon$. Let $f \in E_\sigma^+\overline{\mathsf{R}(D)}$ such that $\Phi_N^\tau E_\tau^+ f = 0$. Then, using the Rellich estimates in Proposition 6.3.1 (as $\mathcal{A}_\tau$ is self-adjoint), we have

$$0 = \|\Phi_N^\tau E_\tau^+ f\|_2 \gtrsim \|E_\tau^+ f\|_2.$$

Thus, $E_\tau^+ f = 0$ and as $E_\tau^+\colon E_\sigma^+\overline{\mathsf{R}(D)} \to E_\tau^+\overline{\mathsf{R}(D)}$ is a bijection then $f = 0$. Hence, $\Phi_N^\tau E_\tau^+\colon E_\sigma^+\overline{\mathsf{R}(D)} \to L^2(\mathbb{R}^n)$ is injective.

Let $g \in L^2(\mathbb{R}^n)$. Then as $\Phi_N^\sigma \colon E_\sigma^+ \overline{\mathsf{R}(D)} \to L^2(\mathbb{R}^n)$ is bijective we have there exists $h \in E_\sigma^+ \overline{\mathsf{R}(D)}$ such that $\Phi_N^\sigma h = g$. Therefore

$$g = \Phi_N^\sigma h = (E_\tau^+ h + (I - E_\tau^+)h)_\perp = \Phi_N^\tau E_\tau^+ h + ((I - E_\tau^+)h)_\perp.$$

Now using the fact that E_τ^+ and E_σ^+ are projections and $h \in E_\sigma^+ \overline{\mathsf{R}(D)}$ we have

$$(I - E_\tau^+)h = {E_\tau^+}^{-1} E_\tau^+ (E_\sigma^+ - E_\tau^+)h = {E_\tau^+}^{-1} (E_\tau^+ E_\sigma^+ - E_\tau^+ E_\sigma^+)h = 0.$$

Thus $\Phi_N^\tau E_\tau^+ h = g$. That is $\Phi_N E_\tau^+ \colon E_\sigma^+ \overline{\mathsf{R}(D)} \to L^2(\mathbb{R}^n)$ is surjective and so bijective. Then by Lemma 6.3.3 we have that $\Phi_N^\tau \colon E_\tau^+ \overline{\mathsf{R}(D)} \to L^2(\mathbb{R}^n)$ is a bijection.

A similar argument proves that $\Phi_R^\tau \colon E_\tau^+ \overline{\mathsf{R}(D)} \to \{\nabla_\mu u : u \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^n)\}$ is a bijection.

□

Proposition 6.3.5. If $\mathcal{A}$ is self-adjoint, then the mappings $\Phi_R \colon \mathcal{H}_{DB}^{0,+} \to \{\nabla_\mu u : u \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^n)\}$ and $\Phi_N \colon \mathcal{H}_{DB}^{0,+} \to L^2(\mathbb{R}^n)$ are isomorphisms.

Proof. By Proposition 6.3.1 we have $\Phi_R \colon \mathcal{H}_{DB}^{0,+} \to \{\nabla_\mu u : u \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^n)\}$ and $\Phi_N \colon \mathcal{H}_{DB}^{0,+} \to L^2(\mathbb{R}^n)$ are injective.

Now as Φ_N^0 corresponds to $\mathcal{A} = I$ and I is a block type matrix then by Proposition 6.2.3 we have $\Phi_N^0 \colon E_0^+ \overline{\mathsf{R}(D)} \to L^2(\mathbb{R}^n)$ is an isomorphism. Now by Lemma 6.3.4 we have there exists $\varepsilon > 0$ such that for all $|\tau| < \varepsilon$ then $\Phi_N^\tau \colon E_\tau^+ \overline{\mathsf{R}(D)} \to L^2(\mathbb{R}^n)$ is an isomorphism. We then iterate this argument a finite number of times to give us $\Phi_N^1 = \Phi_N \colon \mathcal{H}_{DB}^{0,+} \to L^2(\mathbb{R}^n)$ is a bijection.

A similar argument gives that $\Phi_R^\tau \colon E_\tau^+ \overline{\mathsf{R}(D)} \to \{\nabla_\mu u : u \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^n)\}$ is also surjective and so an isomorphism. This completes the proof. □

CHAPTER 7

NON-TANGENTIAL MAXIMAL FUNCTION BOUNDS

We are now ready to prove the non-tangential maximal function bounds in Theorem 6.0.1 parts 3 and 4. These are needed to show that the second-order equation $H_{A,a,V}\, u = 0$ is well-posed, as in (2.6.4) and (2.6.5), by proving $N^*(\nabla_\mu u) \in L^2(\mathbb{R}^n)$ and the convergence to the boundary data is pointwise on Whitney averages. We do this by estimating the non-tangential operator by a quadratic estimate term and a term involving the Hardy–Littlewood maximal function. However, in order to do this we need to reduce the exponent from 2 to some $p \in (1, 2)$, since otherwise we will end up trying to bound the Hardy–Littlewood maximal function on L^1. But the Hardy–Littlewood maximal function is not a bounded operator on L^1. Therefore we need to prove a weak reverse Hölder estimate on $\nabla_\mu u$. We note when $V = 0$ then this is easy as $H_{A,a,V}(u - c) = 0$ for all constants $c \in \mathbb{C}$. Then for a fixed cube we choose c to be the average of u on the cube. Thus, using the Caccioppoli inequality and then the Sobolev–Poincaré lemma gives the desired result. We aim to replicate this approach with the zeroth-order term V included.

Hence, we start by giving a Caccioppoli inequality adapted to the potential V in the sense that we bound ∇_μ rather than ∇, and the inhomogeneous term on right-hand side depends on $(\nabla_\mu)^*$ not $-\operatorname{div}$. The result is proved similarly to the standard inhomogeneous Caccioppoli inequality. We will adapt the method from the parabolic equation in [6] to our case with a potential. From this point on we specialise to the case when $V \in B^{\frac{n}{2}}(\mathbb{R}^n)$ as in Section 2.3. Therefore we have $\mathcal{A} = \mathcal{A}^{\mathcal{V}}$.

7.1 Reverse Hölder Estimates for Solutions

Throughout this section, suppose that $d > 2$ is an integer and that Ω is an open subset of $\mathbb{R}^d$. We begin with the following version of Caccioppoli's inequality to account for the presence of an inhomogeneity $f \in L^2(\Omega, \mathbb{C}^{d+1})$ from the domain $\mathscr{D}(\nabla_\mu^*)$ in (2.3.5). In particular, we shall say that u is a weak solution of $-\operatorname{div} A\nabla u + Vu = \nabla_\mu^* f$ in Ω, or simply that $H_{A,a,V}\, u = \nabla_\mu^* f$ in Ω, if $u \in \mathcal{V}^{1,2}_{\mathrm{loc}}(\Omega)$ and $\int_\Omega A\nabla u \cdot \overline{\nabla v} + aVu\overline{v} = \int_\Omega f \cdot \overline{\nabla_\mu v}$ for all $v \in \mathcal{C}_c^\infty(\Omega)$.

Proposition 7.1.1. Suppose that $V \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. If $f \in \mathscr{D}(\nabla_\mu^*)$ and $H_{A,a,V}\, u = \nabla_\mu^* f$ in Ω, then

$$\int_Q |\nabla u|^2 + \int_Q |V||u|^2 \lesssim \frac{1}{l(Q)^2} \int_{2Q} |u|^2 + \int_{2Q} |f|^2,$$

for all cubes $Q \subset 2Q \subset\subset \Omega$, where the implicit constant depends only on κ, $\|\mathcal{A}\|_\infty$ and d.

Proof. Let $\eta \in \mathcal{C}_c^\infty(\Omega)$ be supported in $2Q \subset\subset \Omega$ such that $0 \leq \eta(x) \leq 1$ for all $x \in \Omega$ and $\eta(x) = 1$ for all $x \in Q$ whilst $\|\nabla\eta\|_\infty \lesssim \frac{1}{l(Q)}$. If $H_{A,a,V}\, u = \nabla_\mu^* f$ in Ω, then $u\eta^2 \in \dot{\mathcal{V}}^{1,2}_c(\Omega)$, so by the definition of a weak solution we have

$$\int_\Omega \mathcal{A}\nabla_\mu u \cdot \overline{\nabla_\mu(u\eta^2)} = \int_\Omega f \cdot \overline{\nabla_\mu(u\eta^2)}, \tag{7.1.1}$$

since $\mathcal{C}_c^\infty(\Omega)$ is dense in $\dot{\mathcal{V}}^{1,2}_c(\Omega)$. Now let $\varepsilon \in (0,1)$ to be chosen. The product rule gives

$$\int_Q |\nabla_\mu u|^2 \leq \int_\Omega |\nabla_\mu u|^2\eta^2 \lesssim \int_\Omega |\nabla_\mu(\eta u)|^2 + \int_\Omega |u|^2|\nabla\eta|^2.$$

From Lemma 2.3.6 we have $\nabla_\mu(\eta u) \in \overline{\mathsf{R}(D)}$. Then, using the ellipticity, the product rule repeatedly, and (7.1.1), gives

$$
\begin{aligned}
\int_\Omega |\nabla_\mu(\eta u)|^2 &\lesssim \left| \int_\Omega \mathcal{A}\nabla_\mu(\eta u)\cdot\overline{\nabla_\mu(\eta u)} \right| \\
&= \left| \int_\Omega uA\nabla\eta\cdot\overline{\nabla(u\eta)} + \eta A\nabla u\overline{\nabla(u\eta)} + a|V|^{\frac{1}{2}}u\overline{|V|^{\frac{1}{2}}(u\eta^2)} \right| \\
&= \left| \int_\Omega u\overline{u}A\nabla\eta\cdot\overline{\nabla\eta} + u\eta A\nabla\eta\cdot\overline{\nabla u} + A\nabla_\mu u\overline{\nabla_\mu(u\eta^2)} + u\eta A\nabla u\cdot\overline{\nabla\eta} \right| \\
&= \left| \int_\Omega u\overline{u}A\nabla\eta\cdot\overline{\nabla\eta} + u\eta A\nabla\eta\cdot\overline{\nabla u} + f\cdot\overline{\nabla_\mu(u\eta^2)} + u\eta A\nabla u\cdot\overline{\nabla\eta} \right|
\end{aligned}
$$

Now, the above two calculations, the boundedness of $\mathcal{A}$, the product rule and the ε-version of Young's inequality, we have

$$
\begin{aligned}
\int_\Omega |\nabla_\mu u|^2\eta^2 &\lesssim \int_\Omega |u|^2|\nabla\eta|^2 + |u||\eta||\nabla\eta||\nabla u| + |f||\nabla_\mu(u\eta^2)| \\
&\lesssim \int_\Omega |\nabla u||\eta\nabla\eta||u| + \int_\Omega |f|\left(|u||\eta\nabla\eta| + |\nabla u|\eta^2 + ||V|^{\frac{1}{2}}u|\eta^2\right) \\
&\lesssim \varepsilon\int_\Omega |\nabla u|^2\eta^2 + \left(\frac{1}{\varepsilon}+\varepsilon\right)\int_\Omega |u|^2|\nabla\eta|^2 + \varepsilon\int_\Omega ||V|^{\frac{1}{2}}u|^2\eta^2 + \frac{1}{\varepsilon}\int_\Omega |f|^2\eta^2.
\end{aligned}
$$

Combining the above estimates gives

$$
\int_\Omega |\nabla_\mu u|^2\eta^2 \lesssim \varepsilon\int_\Omega |\nabla_\mu u|^2\eta^2 + \frac{1}{\varepsilon}\int_\Omega |u|^2|\nabla\eta|^2 + \frac{1}{\varepsilon}\int_\Omega |f|^2\eta^2,
$$

where the implicit constant depends only on κ, $\|\mathcal{A}\|_\infty$ and d.

We now choose $\varepsilon \in (0,1)$ sufficiently small, and recall the properties of η, to obtain

$$
\int_Q |\nabla_\mu u|^2 \le \int_\Omega |\nabla_\mu u|^2\eta^2 \lesssim \frac{1}{l(Q)^2}\int_{2Q} |u|^2 + \int_{2Q} |f|^2,
$$

as required. □

We can use the Caccioppoli inequality to lower the exponent of a weak solution. We recall that $2^* := 2d/(d-2)$ denotes the Sobolev exponent for $\mathbb{R}^d$.

Proposition 7.1.2. Suppose that $V \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. If $\delta > 0$ and $H_{A,a,V}\, u = 0$ in Ω, then

$$\left(\fint_Q |u|^{2^*}\right)^{1/2^*} \lesssim_\delta \left(\fint_{2Q} |u|^\delta\right)^{1/\delta},$$

for all cubes $Q \subset 2Q \subset\subset \Omega$, where the implicit constant depends only on κ, $\|\mathcal{A}\|_\infty$, d and δ.

Proof. As $u \in W^{1,2}(Q)$, then using the Sobolev–Poincaré inequality (see (7.45) in [30]) we have

$$\left(\fint_Q |u|^{2^*}\right)^{1/2^*} \lesssim \left(\fint_Q |u - (\textstyle\fint_Q u)|^{2^*}\right)^{1/2^*} + \fint_Q |u| \lesssim l(Q) \left(\fint_Q |\nabla u|^2\right)^{1/2} + \left(\fint_Q |u|^2\right)^{1/2}.$$

Therefore, by the version of Caccioppoli's inequality in Proposition 7.1.1 in the case $f = 0$, we have the weak reverse Hölder estimate

$$\left(\fint_Q |u|^{2^*}\right)^{1/2^*} \lesssim \left(\fint_{2Q} |u|^2\right)^{1/2},$$

for all cubes $Q \subset 2Q \subset\subset \Omega$ whenever $H_{A,a,V}\, u = 0$. The self-improvement of the exponent in the right-hand side of such estimates (see [33, Theorem 2]) completes the proof. $\square$

To prove the non-tangential maximal bounds we need to be able to lower the exponent, on the adapted gradient ∇_μ, from 2 to some $p < 2$. In the homogeneous case (when $V = 0$) this is relatively straight forward as if u is a solution then $\operatorname{div} A\nabla(u - u_W) = 0$. Therefore, we can use Caccioppoli's inequality on $u - u_W$ followed by the Poincare Inequality. However, in the inhomogeneous case we need to control the potential term. To do this we will use the Fefferman–Phong inequality (Proposition 2.2.5) with exponent 1. Moreover, we will make crucial use of the right-hand side self-improvement property, proved by Iwaniec and Nolder in [33, Theorem 2], for reverse Hölder inequalities. Specifically, if $\delta \in (0, \infty)$ and $V \in B^q(\mathbb{R}^d)$ for some $q \in (1, \infty)$, then $(\fint_Q V^q)^{1/q} \lesssim_\delta (\fint_Q V^\delta)^{1/\delta}$ for all cubes Q in $\mathbb{R}^d$. Hence, if $V \in A_\infty(\mathbb{R}^d) = \cup_{q>1} B^q(\mathbb{R}^d)$, then $V^s \in B^{\frac{1}{s}}(\mathbb{R}^d)$ for each $s \in (0, 1)$ and

$$\fint_Q V \eqsim \left(\fint_Q V^{\frac{1}{2}}\right)^2, \tag{7.1.2}$$

for all cubes Q in $\mathbb{R}^d$. To see this we use Jensen's Inequality and the reverse Hölder inequality, to get

$$l(Q)\fint_Q V^{\frac{1}{2}} \leq l(Q)\left(\fint_Q V\right)^{\frac{1}{2}} = \left(l(Q)^2\fint_Q V\right)^{\frac{1}{2}} = l(Q)\left(\fint_Q (V^{\frac{1}{2}})^2\right)^{\frac{1}{2}} \lesssim l(Q)\fint_Q V^{\frac{1}{2}}.$$

We are now able to give a weak reverse Hölder type inequality for the ∇_μ of a solution of the second-order equation.

An inspection of the proofs above provides for the following routine extension. This will be used henceforth without further reference.

Remark 7.1.3. The results in Propositions 7.1.1 and 7.1.2 also hold when $2Q$ is replaced by αQ for any $\alpha > 1$, except then the implicit constants in the estimates will also depend on α.

Proposition 7.1.4. Suppose that $V \in A_\infty(\mathbb{R}^d)$. If $\delta > 0$ and $H_{A,a,V}\, u = 0$ in Ω, then

$$\left(\fint_Q |\nabla_\mu u|^2\right)^{\frac{1}{2}} \lesssim_\delta \left(\fint_{2Q} |\nabla_\mu u|^\delta\right)^{1/\delta},$$

for all cubes $Q \subset 2Q \subset\subset \Omega$, where the implicit constant depend only on κ, $\|\mathcal{A}\|_\infty$, d and δ.

Proof. Suppose that $H_{A,a,V}\, u = 0$ in Ω and let Q denote an arbitrary cube such that $2Q \subset\subset \Omega$. If $l(2Q)\fint_{2Q} V^{\frac{1}{2}} \geq 1$, then by Caccioppoli's inequality in Lemma 7.1.1 with $f = 0$, followed by the reverse Hölder estimate in Proposition 7.1.2 with $\delta = 1$, we have

$$\begin{aligned}
\left(\fint_Q |\nabla_\mu u|^2\right)^{1/2} &\lesssim \frac{1}{l(Q)}\left(\fint_{(3/2)Q} |u|^2\right)^{1/2} \\
&\lesssim \frac{1}{l(Q)}\fint_{2Q} |u| \\
&\lesssim \frac{1}{l(2Q)}\left(l(2Q)\fint_{2Q} V^{\frac{1}{2}}\right)^{\beta}\fint_{2Q} |u| \\
&\lesssim \fint_{2Q} |\nabla_\mu u|,
\end{aligned}$$

where $\beta \in (0,1)$ denotes the constant from the Fefferman–Phong inequality in Proposition 2.2.5 applied here with $p = 1$ and $\omega = V^{\frac{1}{2}} \in A_\infty(\mathbb{R}^d)$.

If $l(2Q) \fint_{2Q} V^{\frac{1}{2}} \leq 1$, then we set $u_Q := \fint_{2Q} u$ and write

$$\fint_Q |\nabla_\mu u|^2 \lesssim \fint_Q |\nabla_\mu (u - u_Q)|^2 + \fint_Q V |u_Q|^2.$$

For all $\varphi \in \mathcal{C}_c^\infty(\Omega)$, since $H_{A,a,V}\, u = 0$ in Ω, we have

$$\int_\Omega \mathcal{A} \nabla_\mu (u - u_Q) \cdot \nabla_\mu \varphi = \int_\Omega \mathcal{A} \nabla_\mu u \cdot \nabla_\mu \varphi - \int V u_Q \varphi.$$

Thus, $H_{A,a,V}(u - u_Q) = -V u_Q$ in Ω. We now define $f = (f_1, \ldots, f_{d+1}) \in L^2(\Omega; \mathbb{C}^d)$ by setting $f_1 = \ldots = f_d \equiv 0$ and $f_{d+1} := -V^{\frac{1}{2}} u_Q$, so $-V u_Q = (-\operatorname{div}_\parallel, V^{\frac{1}{2}})(0, \ldots, 0, -V^{\frac{1}{2}} u_Q) = (\nabla_\mu)^* f$ and $H_{A,a,V}(u - u_Q) = (\nabla_\mu)^* f$ in Ω. The inhomogeneous version of Caccioppoli's inequality in Proposition 7.1.1 can then be applied to show that

$$\fint_Q |\nabla_\mu u|^2 \lesssim \frac{1}{l(Q)^2} \fint_{2Q} |u - u_Q|^2 + \fint_{2Q} V |u_Q|^2 \lesssim \left(\fint_{2Q} |\nabla u|^{2_*} \right)^{2/2_*} + \fint_{2Q} V |u_Q|^2,$$

where we used the Sobolev–Poincaré inequality (see (7.45) in [30]) in the second estimate with $2_* := 2d/(d+2)$. Using (7.1.2) followed by the Fefferman–Phong inequality in Proposition 2.2.5, applied again with $p = 1$ and $\omega = V^{\frac{1}{2}}$ but now in the case when $l(2Q) \fint_{2Q} V^{\frac{1}{2}} \leq 1$, we have

$$\left(\fint_{2Q} V |u_Q|^2 \right)^{1/2} \leq \left(\fint_{2Q} V \right)^{1/2} \fint_{2Q} |u| \lesssim \left(\fint_{2Q} V^{\frac{1}{2}} \right) \fint_{2Q} |u| \lesssim \fint_{2Q} |\nabla_\mu u|.$$

Combining these estimates with Jensen's inequality we get

$$\left(\fint_Q |\nabla_\mu u|^2 \right)^{1/2} \lesssim \left(\fint_{2Q} |\nabla_\mu u|^{2_*} \right)^{1/2_*},$$

since $1 < 2_* < 2$.

We can now conclude that the preceding weak reverse Hölder estimate holds for all cubes $Q \subset 2Q \subset\subset \Omega$. The self-improvement of the exponent in the right-hand side of such estimates (see [33, Theorem 2]) completes the proof. $\square$

We note that Remark 7.1.3 also applies to Proposition 7.1.4.

7.2 Off-Diagonal Estimates

The next step to proving the non-tangential maximal bounds is to show that DB has $L^q \to L^q$ off-diagonal estimates for some $q < 2$. First we need to know that DB is bisectorial in L^q and so we need to prove the L^q-resolvent bounds for DB. To do this we follow the methods of [2] and [10], adapting them to the potential V.

Lemma 7.2.1. There exists $1 < p_1 < 2 < p_2$ such that for $q \in (p_1, p_2)$ we have the $L^q \to L^q$ resolvent bounds

$$\|(I + itDB)^{-1} f\|_q \lesssim \|f\|_q,$$

for all $f \in L^2(\mathbb{R}^n; \mathbb{C}^{n+2}) \cap L^q(\mathbb{R}^n; \mathbb{C}^{n+2})$.

Proof. Let $f \in L^2(\mathbb{R}^n; \mathbb{C}^{n+2}) \cap L^q(\mathbb{R}^n; \mathbb{C}^{n+2})$ where q is to be chosen later. Then define $\tilde{f}$ such that $(I + itDB)\tilde{f} = f$. Define

$$g = \begin{bmatrix} (Bf)_\perp \\ f_\| \\ f_\mu \end{bmatrix}, \quad \tilde{g} = \begin{bmatrix} (B\tilde{f})_\perp \\ \tilde{f}_\| \\ \tilde{f}_\mu \end{bmatrix} \quad \text{then} \quad f = \begin{bmatrix} (\mathcal{A}g)_\perp \\ g_\| \\ g_\mu \end{bmatrix}, \quad \tilde{f} = \begin{bmatrix} (\mathcal{A}\tilde{g})_\perp \\ \tilde{g}_\| \\ \tilde{g}_\mu \end{bmatrix}.$$

Now let $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n; \mathbb{C}^{n+2})$. Then

$$\int f \cdot \psi = \int (I + itDB)\tilde{f} \cdot \psi = \int \tilde{f} \cdot \psi + \int B\tilde{f} \cdot (itD\psi). \tag{7.2.1}$$

Now let $\psi = (\varphi, 0, 0)^T$, where $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$. Noting that $(B\tilde{f})_\| = (\mathcal{A}\tilde{g})_\|$ and $(B\tilde{f})_\mu =$

$(\mathcal{A}\tilde{g})_\mu = a\tilde{g}_\mu$, then, by (7.2.1), we obtain

$$\begin{aligned}
\int (\mathcal{A}g)_\perp \varphi &= \int \tilde{f}_\perp \varphi + \int (B\tilde{f})_\parallel \cdot (it\nabla_\parallel \varphi) + \int (B\tilde{f})_\mu (it|V|^{\frac{1}{2}}\varphi) \\
&= \int_{\mathbb{R}^n} (\mathcal{A}\tilde{g})_\perp \varphi + \int (\mathcal{A}\tilde{g})_\parallel \cdot (it\nabla_\parallel \varphi) + \int (\mathcal{A}\tilde{g})_\mu (it|V|^{\frac{1}{2}}\varphi) \\
&= \int \begin{bmatrix} (\mathcal{A}\tilde{g})_\perp \\ (\mathcal{A}\tilde{g})_\parallel \\ (\mathcal{A}\tilde{g})_\mu \end{bmatrix} \cdot \begin{bmatrix} \varphi \\ it\nabla_\parallel \varphi \\ it|V|^{\frac{1}{2}}\varphi \end{bmatrix}
\end{aligned} \tag{7.2.2}$$

Letting $\psi = (0, \varphi, 0)^T$, where $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$, in (7.2.1). Then we have that

$$\int g_\parallel \cdot \varphi = \int \tilde{g}_\parallel \cdot \varphi + \int (B\tilde{f})_\perp (it \operatorname{div}_\parallel \varphi) = \int \tilde{g}_\parallel \cdot \varphi - \int it\nabla_\parallel \tilde{g}_\perp \cdot \varphi. \tag{7.2.3}$$

Similarly, letting $\psi = (0, 0, \varphi)^T$, where $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, in (7.2.1), we obtain

$$\int g_\mu \varphi = \int \tilde{g}_\mu \varphi - \int (B\tilde{f})_\perp (it|V|^{\frac{1}{2}}\varphi) = \int \tilde{g}_\mu \varphi - \int (it|V|^{\frac{1}{2}}\tilde{g}_\perp)\varphi. \tag{7.2.4}$$

Therefore, we have $g_\parallel = \tilde{g}_\parallel - it\nabla_\parallel \tilde{g}_\perp$ and $g_\mu = \tilde{g}_\mu - it|V|^{\frac{1}{2}}\tilde{g}_\perp$. For $t > 0$ we define the space $\mathcal{V}_t^{1,q}(\mathbb{R}^n)$ to be $\mathcal{V}^{1,q}(\mathbb{R}^n)$ equipped with the norm $\|u\|_q + t\|\nabla_\mu u\|_q$. Also, define $(\mathcal{V}_t^{1,q}(\mathbb{R}^n))^*$ to be the dual space equipped with the dual norm. Define the operator $L_{t,V} \colon \mathcal{V}_t^{1,q}(\mathbb{R}^n) \to (\mathcal{V}_t^{1,q'}(\mathbb{R}^n))^*$ such that for $u \in \mathcal{V}_t^{1,q}(\mathbb{R}^n)$ we define the linear functional $L_{t,V}u \in (\mathcal{V}_t^{1,q'}(\mathbb{R}^n))^*$ defined, for all $\varphi \in \mathcal{V}_t^{1,q'}(\mathbb{R}^n)$, by

$$(L_{t,V}u)(\varphi) \coloneqq \int \mathcal{A} \begin{bmatrix} u \\ it\nabla_\parallel u \\ it|V|^{\frac{1}{2}}u \end{bmatrix} \cdot \begin{bmatrix} \varphi \\ it\nabla_\parallel \varphi \\ it|V|^{\frac{1}{2}}\varphi \end{bmatrix}.$$

Now, using Hölder's inequality, we have

$$
\begin{aligned}
|(L_{t,V}u)(\varphi)| &\leq \|\mathcal{A}\|_\infty \int |u||\varphi| + t^2|\nabla_\mu u||\nabla_\mu \varphi|, \\
&\leq \|\mathcal{A}\|_\infty \left(\|u\|_q\|\varphi\|_{q'} + t^2\|\nabla_\mu u\|_q\|\nabla_\mu \varphi\|_{q'}\right), \\
&\leq \|\mathcal{A}\|_\infty \left(\|u\|_q + t\|\nabla_\mu u\|_q\right)\left(\|\varphi\|_{q'} + t\|\nabla_\mu \varphi\|_{q'}\right), \\
&= \|\mathcal{A}\|_\infty \|u\|_{\mathcal{V}_t^{1,q}}\|\varphi\|_{\mathcal{V}_t^{1,q'}}.
\end{aligned}
$$

Therefore, $\|L_{t,V}u\|_{\left(\mathcal{V}_t^{1,q'}\right)^*} \leq \|\mathcal{A}\|_\infty\|u\|_{\mathcal{V}_t^{1,q}}$. That is, $L_{t,V}$ is bounded for $q \in (1,\infty)$ independently of q. Now recall

$$
\overline{\mathsf{R}(D)} = \left\{ \begin{bmatrix} h \\ \nabla_\| g \\ |V|^{\frac{1}{2}} g \end{bmatrix} : h \in L^2(\mathbb{R}^n),\ g \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^n) \right\}, \quad \text{then} \quad \begin{bmatrix} u \\ \nabla_\|(itu) \\ |V|^{\frac{1}{2}}(itu) \end{bmatrix} \in \overline{\mathsf{R}(D)}.
$$

Therefore, by the ellipticity of $\mathcal{A}$, since $\mathcal{V}_t^{1,2} \subseteq \dot{\mathcal{V}}^{1,2}$, and so for any $u \in \mathcal{V}_t^{1,2}(\mathbb{R}^n)$, we have

$$
|(L_{t,V}u)(u)| \geq \operatorname{Re}\left(\int_{\mathbb{R}^n} \mathcal{A} \begin{bmatrix} u \\ \nabla_\|(itu) \\ |V|^{\frac{1}{2}}(itu) \end{bmatrix} \cdot \begin{bmatrix} u \\ \nabla_\|(itu) \\ |V|^{\frac{1}{2}}(itu) \end{bmatrix} \right) \gtrsim \kappa(\|u\|_2^2 + t^2\|\nabla_\mu u\|_2^2) \eqsim \kappa\|u\|^2_{\mathcal{V}_t^{1,2}}.
$$

That is $\|L_{t,V}u\|_{(\mathcal{V}_t^{1,2})^*} \gtrsim \|u\|_{\mathcal{V}_t^{1,2}}$. There exists $\varepsilon > 0$ such that $|V|^{\frac{1}{2}} \in B_{2+\varepsilon}$, therefore using [15] we have $\mathcal{V}_t^{1,q}(\mathbb{R}^n)$ is an interpolation space for $q \in (1, 2+\varepsilon)$. Then by Šneĭberg's Lemma in [49] we have there exists p_1, p_2 with $1 < p_1 < 2 < p_2 < 2+\varepsilon$ such that $\|L_{t,V}u\|_{(\mathcal{V}_t^{1,q})^*} \gtrsim_q \|u\|_{\mathcal{V}_t^{1,q}}$ for all $q \in (p_1, p_2)$. That is $L_{t,V}$ is invertible for $q \in (p_1, p_2)$. For

$\varphi \in \mathcal{V}_t^{1,q'}(\mathbb{R}^n)$ use (7.2.3) and (7.2.4), and then (7.2.2) to obtain

$$
\begin{aligned}
(L_{t,V}\tilde{g}_\perp)(\varphi) &= \int_{\mathbb{R}^n} \mathcal{A}\begin{bmatrix} \tilde{g}_\perp \\ it\nabla_\parallel \tilde{g}_\perp \\ it|V|^{\frac{1}{2}}\tilde{g}_\perp \end{bmatrix} \cdot \begin{bmatrix} \varphi \\ it\nabla_\parallel \varphi \\ it|V|^{\frac{1}{2}}\varphi \end{bmatrix} \\
&= \int_{\mathbb{R}^n} \begin{bmatrix} A_{\perp\perp}\tilde{g}_\perp + itA_{\perp\parallel}\nabla_\parallel \tilde{g}_\perp \\ A_{\parallel\perp}\tilde{g}_\perp + itA_{\parallel\parallel}|V|^{\frac{1}{2}}\tilde{g}_\perp \\ a(\tilde{g}_\mu - g_\mu) \end{bmatrix} \cdot \begin{bmatrix} \varphi \\ it\nabla_\parallel \varphi \\ it|V|^{\frac{1}{2}}\varphi \end{bmatrix} \\
&= \int_{\mathbb{R}^n} \begin{bmatrix} A_{\perp\perp}\tilde{g}_\perp + A_{\perp\parallel}(\tilde{g}_\parallel - g_\parallel) \\ A_{\parallel\perp}\tilde{g}_\perp + A_{\parallel\parallel}(\tilde{g}_\parallel - g_\parallel) \\ a(\tilde{g}_\mu - g_\mu) \end{bmatrix} \cdot \begin{bmatrix} \varphi \\ it\nabla_\parallel \varphi \\ it|V|^{\frac{1}{2}}\varphi \end{bmatrix} \\
&= \int_{\mathbb{R}^n} \begin{bmatrix} (\mathcal{A}\tilde{g})_\perp - A_{\perp\parallel}g_\parallel \\ (\mathcal{A}\tilde{g})_\parallel - A_{\parallel\parallel}g_\parallel \\ (\mathcal{A}\tilde{g})_\mu - ag_\mu \end{bmatrix} \cdot \begin{bmatrix} \varphi \\ it\nabla_\parallel \varphi \\ it|V|^{\frac{1}{2}}\varphi \end{bmatrix} \\
&= \int_{\mathbb{R}^n} \begin{bmatrix} (\mathcal{A}\tilde{g})_\perp \\ (\mathcal{A}\tilde{g})_\parallel \\ (\mathcal{A}\tilde{g})_\mu \end{bmatrix} \cdot \begin{bmatrix} \varphi \\ it\nabla_\parallel \varphi \\ it|V|^{\frac{1}{2}}\varphi \end{bmatrix} - \begin{bmatrix} A_{\perp\parallel}g_\parallel \\ A_{\parallel\parallel}g_\parallel \\ ag_\mu \end{bmatrix} \cdot \begin{bmatrix} \varphi \\ it\nabla_\parallel \varphi \\ it|V|^{\frac{1}{2}}\varphi \end{bmatrix} \\
&= \int (\mathcal{A}g)_\perp \varphi - \begin{bmatrix} A_{\perp\parallel}g_\parallel \\ A_{\parallel\parallel}g_\parallel \\ ag_\mu \end{bmatrix} \cdot \begin{bmatrix} \varphi \\ it\nabla_\parallel \varphi \\ it|V|^{\frac{1}{2}}\varphi \end{bmatrix} \\
&= \int_{\mathbb{R}^n} \begin{bmatrix} A_{\perp\perp}g_\perp \\ -A_{\parallel\parallel}g_\parallel \\ -ag_\mu \end{bmatrix} \cdot \begin{bmatrix} \varphi \\ it\nabla_\parallel \varphi \\ it|V|^{\frac{1}{2}}\varphi \end{bmatrix}.
\end{aligned}
$$

Define $F\colon L^q(\mathbb{R}^n;\mathbb{C}^{n+2}) \to (\mathcal{V}_t^{1,q'}(\mathbb{R}^n))^*$ such that for $u \in L^q(\mathbb{R}^n;\mathbb{C}^{n+2})$ and $\varphi \in \mathcal{V}_t^{1,q'}(\mathbb{R}^n)$ then

$$
(Fu)(\varphi) := \int_{\mathbb{R}^n} \begin{bmatrix} A_{\perp\perp}u_\perp \\ -A_{\parallel\parallel}u_\parallel \\ -au_\mu \end{bmatrix} \cdot \begin{bmatrix} \varphi \\ it\nabla_\parallel \varphi \\ it|V|^{\frac{1}{2}}\varphi \end{bmatrix}.
$$

Therefore, $L_{t,V}\tilde{g}_\perp = Fg$. Now for any $\varphi \in \mathcal{V}_t^{1,q'}(\mathbb{R}^n)$, we have

$$\begin{aligned}|(Fu)(\varphi)| &\leq \|\mathcal{A}\|_\infty \int_{\mathbb{R}^n} \left(|u_\perp||\varphi| + t|u_\parallel||\nabla_\parallel \varphi| + t|u_\mu|||V|^{\frac{1}{2}}\varphi|\right) \\ &\leq \|\mathcal{A}\|_\infty \left(\|u\|_q\|\varphi\|_{q'} + t\|u\|_q\|\nabla_\mu\varphi\|_{q'}\right) \\ &= \|\mathcal{A}\|_\infty\|u\|_q\|\varphi\|_{\mathcal{V}_t^{1,q'}}.\end{aligned}$$

Thus

$$\|Fu\|_{\left(\mathcal{V}_t^{1,q'}\right)^*} \leq \|\mathcal{A}\|_\infty\|u\|_q. \tag{7.2.5}$$

Therefore, using (7.2.2), (7.2.3), (7.2.4), the ellipticity of $L_{t,V}$, (7.2.5), and the definition of g, we obtain

$$\|\tilde{f}\|_q \eqsim \|\tilde{g}\|_q \lesssim \|\tilde{g}_\perp\|_{\mathcal{V}_t^{1,q}} + \|g\|_q \lesssim \|L_{t,V}\tilde{g}_\perp\|_{\mathcal{V}_t^{1,q}} + \|g\|_q = \|Fg\|_{(\mathcal{V}_t^{1,q})^*} + \|g\|_q \lesssim \|g\|_q \eqsim \|f\|_q.$$

Recalling the definition of $\tilde{f}$, gives

$$\|(I + itDB)^{-1}f\|_q \lesssim \|f\|_q.$$

Now as $L^2(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ is dense in $L^q(\mathbb{R}^n)$ a density argument completes the proof. □

Therefore, we have the off-diagonal estimates

Proposition 7.2.2. Let $E, F \subset \mathbb{R}^n$ and $f \in \overline{\mathsf{R}(D)}$ with $\operatorname{supp}(f) \subset F$. Then, there exists $1 < p_1 < 2 < p_2$ such that for $q \in (p_1, p_2)$, we have the following estimate

$$\|(I + itDB)^{-1}f\|_{L^q(E)} \leq C_M \left(1 + \frac{\operatorname{dist}(E, F)}{t}\right)^{-M} \|f\|_{L^q(F)},$$

where C_M does not depend on E, F, f, and t.

Proof. By Lemma 7.2.1 we have there exists $1 < p_1 < 2 < p_2$ such that, for $p \in (p_1, p_2)$, we have

$$\|(I + itDB)^{-1}f\|_{L^p(E)} \leq C_p\|(I + itDB)^{-1}f\|_p \leq C_p\|f\|_p = C_p\|f\|_{L^p(F)},$$

where C_p is independent of E, F, f, and t. We also have, from Proposition 3.1.3, for any $N \in \mathbb{N}$ then

$$\|(I + itDB)^{-1} f\|_{L^2(E)} \leq C_N \left(1 + \frac{\operatorname{dist}(E, F)}{t}\right)^{-N} \|f\|_{L^2(F)},$$

where C_N is independent of E, F, f, and t. Then by Riesz–Thorin interpolation we have for any $\theta \in (0, 1)$ and $\frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{2}$ then

$$\|(I + itDB)^{-1} f\|_{L^q(E)} \leq C_p^{1-\theta} C_N^{\theta} \left(1 + \frac{\operatorname{dist}(E, F)}{t}\right)^{-N\theta} \|f\|_{L^q(F)}.$$

Now choosing $N \in \mathbb{N}$ such that $N\theta \geq M$ gives the required result. $\square$

7.3 Non-Tangential Estimates

Now that we have weak reverse Hölder estimates for the gradient of solutions and $L^q \to L^q$ off diagonal estimates we are ready to prove the non-tangential maximal function estimates. We first give the following lemma.

Lemma 7.3.1. If $F \in L^2_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^n; \mathbb{C}^{n+2}))$, then

$$\sup_{t>0} \fint_t^{2t} \|F(s)\|_2^2 \, \mathrm{d}s \lesssim \|\widetilde{N}_* F\|_2^2 \lesssim \int_0^\infty \|F(s)\|_2^2 \frac{\mathrm{d}s}{s}.$$

Proof. Firstly, by the definition of the non-tangential maximal function we have

$$|\widetilde{N}_* F(x)|^2 \eqsim \sup_{t>0} \int_t^{2t} \fint_{Q_t(x)} |F(s, y)|^2 \frac{\mathrm{d}y \, \mathrm{d}s}{s} \lesssim \int_0^\infty \frac{1}{s^n} \int_{\mathbb{R}^n} \mathbb{1}_{Q_s(y)}(x) |F(s, y)|^2 \frac{\mathrm{d}y \, \mathrm{d}s}{s},$$

where $Q_t(x)$ is the cube with side-length $l(Q_t(x)) = t$, centred at x. By integrating in x

and then using Tonelli's Theorem, we obtain

$$\begin{aligned}
\|\widetilde{N}_*F\|_2^2 &\lesssim \int_{\mathbb{R}^n}\int_0^\infty \frac{1}{s^n}\int_{\mathbb{R}^n} \mathbb{1}_{Q_s(y)}(x)|F(s,y)|^2 \frac{\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x}{s} \\
&= \int_0^\infty \frac{1}{s^n}\int_{\mathbb{R}^n} |Q_s(y)||F(s,y)|^2 \frac{\mathrm{d}y\,\mathrm{d}s}{s} \\
&\eqsim \int_0^\infty \|F(s)\|_2^2 \frac{\mathrm{d}s}{s}.
\end{aligned}$$

For the lower inequality let $t_0 > 0$ be fixed and arbitrary. Therefore, by definition of supremum, we have

$$\begin{aligned}
\sup_{t>0}\int_t^{2t} \fint_{Q_t(x)} |F(s,y)|^2 \frac{\mathrm{d}y\,\mathrm{d}s}{s} &\geq \int_{t_0}^{2t_0} \fint_{Q_{t_0}(x)} |F(s,y)|^2 \frac{\mathrm{d}y\,\mathrm{d}s}{s} \\
&\eqsim \int_{t_0}^{2t_0} \frac{1}{s^n}\int_{\mathbb{R}^n} \mathbb{1}_{Q_s(y)}(x)|F(s,y)|^2 \frac{\mathrm{d}y\,\mathrm{d}s}{s}.
\end{aligned}$$

Again, integrating in x and then using Tonelli's Theorem gives

$$\begin{aligned}
\|\widetilde{N}_*F\|_2^2 &\gtrsim \int_{t_0}^{2t_0} \frac{1}{s^n}\int_{\mathbb{R}^n} |Q_s(y)||F(s,y)|^2 \frac{\mathrm{d}y\,\mathrm{d}s}{s} \\
&\eqsim \fint_{t_0}^{2t_0} \int_{\mathbb{R}^n} |F(s,y)|^2 \,\mathrm{d}y\,\mathrm{d}s \\
&= \fint_{t_0}^{2t_0} \|F(s)\|_2^2 \,\mathrm{d}s.
\end{aligned}$$

Finally, noting that t_0 was arbitrary so the above is true for all $t_0 > 0$. Thus, taking supremum over $t_0 > 0$ completes the proof. $\square$

We are finally ready to prove the non-tangential maximal function estimates and square function estimates for first-order solutions when $V \in B^{\frac{n}{2}}(\mathbb{R}^n)$.

Theorem 7.3.2. If $F \in L^2_{\mathrm{loc}}(\mathbb{R}_+; \overline{R(D)})$ is a weak solution of $\partial_t F + DBF = 0$ in $\mathbb{R}_+$ such that

$$\sup_{t>0}\fint_t^{2t} \|F(s)\|_2^2 \,\mathrm{d}s < \infty,$$

then

$$\int_0^\infty \|t\partial_t F\|_2^2 \frac{\mathrm{d}t}{t} \eqsim \|f\|_2^2 \eqsim \|\widetilde{N}_*F\|_2^2,$$

where $f \in E^+_{DB}\mathcal{H}$ and $F(t) = e^{-tDB}f$ as in Theorem 5.3.8.

Proof. The ω-bisectorial operator DB has a bounded $H^\infty(S^o_\mu)$-functional calculus on $\overline{\mathsf{R}(D)}$ for all $\mu > \omega$ by Theorem 5.0.1. Therefore, applying the equivalence in property 4 of Theorem 2.1.4 with $\psi \in \Psi(S^o_\mu)$ defined by $\psi(z) := [z]e^{-[z]}$, where $[z] := z \operatorname{sgn} z$, for all $z \in S^o_\mu$, we have

$$\begin{aligned}
\int_0^\infty \|t\partial_t F\|_2^2 \frac{\mathrm{d}t}{t} &= \int_0^\infty \|t\partial_t(e^{-t[DB]}f)\|_2^2 \frac{\mathrm{d}t}{t} \\
&= \int_0^\infty \|t[DB]e^{-t[DB]}f\|_2^2 \frac{\mathrm{d}t}{t} \\
&= \int_0^\infty \|\psi(tDB)f\|_2^2 \frac{\mathrm{d}t}{t} \\
&\eqsim \|f\|_2^2,
\end{aligned}$$

where the differentiation in the second equality is justified because $(e^{-t[DB]})_{t>0}$ is an analytic semi-group on $\overline{R(D)}$ by Lemma 5.3.3.

It remains to prove that $\|\widetilde{N}_* F\|_2^2 \eqsim \|f\|_2^2$. To begin, by Lemma 7.3.1 and Proposition 5.3.10, we have

$$\|\widetilde{N}_* F\|_2^2 \gtrsim \sup_{t>0} \fint_t^{2t} \|F(s)\|_2^2 \,\mathrm{d}s \geq \lim_{t\to 0} \fint_t^{2t} \|F(s)\|_2^2 \,\mathrm{d}s = \|f\|_2^2.$$

To prove the reverse estimate, consider a Whitney box $W(t,x) := [t,2t] \times Q_t(x) \subset \mathbb{R}^{n+1}_+$ for some $x \in \mathbb{R}^n$ and $t > 0$. Using Proposition 6.1.2, since $F \in L^2_{\mathrm{loc}}(\mathbb{R}_+; \overline{\mathsf{R}(D)})$ is a weak solution of $\partial_t F + DBF = 0$ in $\mathbb{R}_+$, there exists a weak solution u such that $H_{A,a,V}\, u = 0$ in $\mathbb{R}^{n+1}_+$ and $F = \nabla_{\mathcal{A},\mu} u$. We now choose $p \in (p_1, 2)$, where p_1 is the exponent from Lemma 7.2.1. Applying Proposition 7.1.4 on W, since $2W = [t/2, 5t/2] \times Q_{2t}(x) \subset\subset \mathbb{R}^{n+1}_+$, and the fact that $\overline{\mathcal{A}}$ is bounded and invertible on $\overline{\mathsf{R}(D)}$, we have

$$\begin{aligned}\left(\fint\!\!\!\fint_W |F|^2\right)^{\frac{1}{2}} &= \left(\fint\!\!\!\fint_W |\nabla_{\mathcal{A},\mu} u|^2\right)^{\frac{1}{2}} \\ &\lesssim \left(\fint\!\!\!\fint_W |\nabla_\mu u|^2\right)^{\frac{1}{2}} \\ &\lesssim_p \left(\fint\!\!\!\fint_{2W} |\nabla_\mu u|^p\right)^{\frac{1}{p}} \\ &\lesssim \left(\fint\!\!\!\fint_{2W} |F|^p\right)^{\frac{1}{p}}.\end{aligned}$$

Now, using $F(t) = e^{-t|DB|}f$ for some $f \in \overline{\mathsf{R}(D)}$, recalling that $R_s = (I + isDB)^{-1}$, then we have

$$\begin{aligned}\left(\fint\!\!\!\fint_{2W} |F(s,y)|^p \,\mathrm{d}y\,\mathrm{d}s\right)^{\frac{1}{p}} &= \left(\fint\!\!\!\fint_{2W} |e^{-s|DB|}f(y)|^p \,\mathrm{d}y\,\mathrm{d}s\right)^{\frac{1}{p}} \\ &\lesssim \left(\fint\!\!\!\fint_{2W} |(e^{-s|DB|} - R_s)f(y)|^2 \,\mathrm{d}y\,\mathrm{d}s\right)^{\frac{1}{2}} \\ &\qquad + \left(\fint\!\!\!\fint_{2W} |R_s f(y)|^p \,\mathrm{d}y\,\mathrm{d}s\right)^{\frac{1}{p}}.\end{aligned}$$

Therefore,

$$\|\widetilde{N}_* F\|_2 \lesssim \left\|\widetilde{N}_*\left((e^{-t|DB|} - R_s)f\right)\right\|_2 + \left\|\sup_{t>0}\left(\fint\!\!\!\fint_{2W(t,x)} |R_s f(y)|^p \,\mathrm{d}y\,\mathrm{d}s\right)^{\frac{1}{p}}\right\|_2.$$

Then, by Lemma 7.3.1, letting $\psi(z) \coloneqq e^{-[z]} - (1+iz)^{-1}$ so $\psi \in \Psi(S^o_\mu)$, and the quadratic estimates for DB in Theorem 3.0.1, we have

$$\left\|\widetilde{N}_*\left((e^{-t|DB|} - R_s)f\right)\right\|_2^2 \lesssim \int_0^\infty \|(e^{-t|DB|} - R_s)f\|_2^2 \frac{\mathrm{d}t}{t} = \int_0^\infty \|\psi(tDB)f\|_2^2 \frac{\mathrm{d}t}{t} \lesssim \|f\|_2^2.$$

Now

$$\left(\fint\!\!\!\fint_{2W(x,t)} |R_s f(y)|^p \,\mathrm{d}y\,\mathrm{d}s\right)^{\frac{1}{p}} \eqsim \left(\int_{\frac{t}{2}}^{\frac{5t}{2}} \int_{\mathbb{R}^n} \mathbb{1}_{2Q_t(x)}(y) |R_s f(y)|^p \frac{\mathrm{d}y\,\mathrm{d}s}{s^{n+1}}\right)^{\frac{1}{p}}$$

If $s \in (\frac{t}{2}, \frac{5t}{2})$, then using the off-diagonal estimates in Proposition 7.2.2, we have

$$\begin{aligned}
\|\mathbb{1}_{2Q_t(x)} R_s f\|_p &\le \sum_{j=0}^{\infty} \|\mathbb{1}_{Q_{2t}(x)} R_s \mathbb{1}_{C_j(Q_{2t}(x))} f\|_p \\
&\lesssim \sum_{j=0}^{\infty} \left(1 + \frac{\operatorname{dist}(Q_{2t}(x), C_j(Q_{2t}(x)))}{s}\right)^{-M} \|\mathbb{1}_{C_j(Q_{2t}(x))} f\|_p \\
&\lesssim \sum_{j=0}^{\infty} 2^{-jM} (2^j t)^{\frac{n}{p}} \left(\fint_{2^{j+1}Q_t(x)} |f|^p\right)^{\frac{1}{p}} \\
&\lesssim t^{\frac{n}{p}} \left(\mathcal{M}(|f|^p)(x)\right)^{\frac{1}{p}} \sum_{j=0}^{\infty} 2^{-j(M-\frac{n}{p})}
\end{aligned}$$

where $C_0(Q_{2t}(x)) \coloneqq Q_{2t}(x)$ and $C_j(Q_t(x)) \coloneqq Q_{2^{j+1}t}(x) \setminus Q_{2^j t}(x)$ for all $j \in \mathbb{N}$. Then taking $M > \frac{n}{p}$ gives

$$\|\mathbb{1}_{2Q_t(x)} R_s f\|_p^p \lesssim t^n \mathcal{M}(|f|^p)(x), \quad \forall s \in \left(\frac{t}{2}, \frac{5t}{2}\right).$$

Thus, using the above calculations, we have

$$\begin{aligned}
\left(\int_{\frac{t}{2}}^{\frac{5t}{2}} \int_{\mathbb{R}^n} \mathbb{1}_{2Q_t(x)}(y) |R_s f(y)|^p \frac{\mathrm{d}y\,\mathrm{d}s}{s^{n+1}}\right)^{\frac{1}{p}} &= \left(\int_{\frac{t}{2}}^{\frac{5t}{2}} \|\mathbb{1}_{2Q_t(x)}(y) R_s f\|_p^p \frac{\mathrm{d}s}{t^{n+1}}\right)^{\frac{1}{p}} \\
&\lesssim \left(\int_{\frac{t}{2}}^{\frac{5t}{2}} \left(\mathcal{M}(|f|^p)(x)\right) \frac{\mathrm{d}s}{t}\right)^{\frac{1}{p}} \\
&\eqsim \left(\mathcal{M}(|f|^p)(x)\right)^{\frac{1}{p}}.
\end{aligned}$$

Therefore by the boundedness of the Hardy–Littlewood maximal function on $L^{\frac{2}{p}}$, where $\frac{2}{p} > 1$, we have

$$\begin{aligned}
\left\| \sup_{t>0} \left(\fint\!\!\fint_{2W(x,t)} |R_s f(y)|^p \,\mathrm{d}y\,\mathrm{d}s \right)^{\frac{1}{p}} \right\|_2 &\lesssim \| \left(\mathcal{M}(|f|^p)\right)^{\frac{1}{p}} \|_2 \\
&= \| \left(\mathcal{M}(|f|^p)\right) \|_{\frac{2}{p}}^{\frac{1}{p}} \\
&\lesssim \| |f|^p \|_{\frac{2}{p}}^{\frac{1}{p}} \\
&= \|f\|_2.
\end{aligned}$$

Thus

$$\|\widetilde{N}_*(F)\|_2 \lesssim \|f\|_2.$$

This completes the proof. $\square$

We are now left to prove that first-order solutions, F, converge pointwise on Whitney averages to the initial data f.

Proposition 7.3.3. If $F(t,x) = e^{-t|DB|}f(x)$ for some $f \in \overline{\mathsf{R}(D)}$, then we have almost everywhere convergence of Whitney averages to f as $t \to 0$. That is

$$\lim_{t\to 0} \fint\!\!\fint_{W(t,x)} |F(s,y) - f(x)|^2 \, \mathrm{d}y \, \mathrm{d}s = 0,$$

for almost every $x \in \mathbb{R}^n$.

Proof. We proceed as in [6] by proving the estimate on a dense subspace of $\overline{\mathsf{R}(D)}$, namely $\{h \in \mathsf{R}(DB) \cap \mathscr{D}(DB) : DBh \in L^p(\mathbb{R}^n;\mathbb{C}^{n+2})\}$, for some $p \in (2, p_2)$ where p_2 is as in Proposition 7.2.2. To prove that this set is dense let $m \in \mathbb{N}$. Define $T_m \in \mathcal{L}(L^2(\mathbb{R}^n;\mathbb{C}^{n+2}))$ by

$$T_m h := R_{\frac{1}{m}} imDBR_m h,$$

for each $h \in \overline{R(D)}$ where $R_k := (I + ikDB)^{-1}$. As DB is densely defined and bisectorial then the T_m are uniformly bounded with respect to $m \in \mathbb{N}$. Now as $h \in \mathscr{D}(DB)$ we have

$$\|(I - R_{\frac{1}{m}})h\|_2 = \|\frac{i}{m} DBR_{\frac{1}{m}} h\|_2 \lesssim \frac{1}{m}\|DBh\|_2 \to 0,$$

as $m \to \infty$. Also, as $h \in R(DB)$ then there exists $u \in \mathscr{D}(DB)$ such that $h = DBu$. Then

$$\|(I - imDBR_m)h\|_2 = \|R_m h\|_2 = \|\frac{im}{im} DBR_m u\|_2 = \frac{1}{m}\|(I - R_m)u\|_2 \lesssim \frac{1}{m}\|u\|_2 \to 0,$$

as $m \to \infty$. Therefore

$$\begin{aligned}\|(I - T_m)h\|_2 &\leq \|(I - R_{\frac{1}{m}})h\|_2 + \|R_{\frac{1}{m}}h - R_{\frac{1}{m}}imDBR_m h\| \\ &\lesssim \|(I - R_{\frac{1}{m}})h\|_2 + \|(I - imDBR_m)h\|_2.\end{aligned}$$

Thus $\|(I - T_m)u\|_2 \to 0$, as $m \to \infty$. Hence we have proved T_m converges strongly to the identity as $m \to \infty$. Now let $h \in \overline{\mathsf{R}(D)}$. Let $h_m \in L^2(\mathbb{R}^n;\mathbb{C}^{n+2}) \cap L^p(\mathbb{R}^n;\mathbb{C}^{n+2})$ such that $h_m \to h$ in L^2 and $p \in (2, p_2)$. Now, as T_m is uniformly bounded in m and converges to the identity we have

$$\|T_m h_m - h\|_2 \leq \|T_m h_m - T_m h\|_2 + \|T_m h - h\|_2 \lesssim \|h_m - h\|_2 + \|T_m h - h\|_2 \to 0,$$

as $m \to \infty$. Now by Lemma 7.2.1 we have that there exists $p > 2$ such that

$$\|DBT_m h_m\|_p = \|DBR_{\frac{1}{m}}(I - R_m)h_m\|_p = m\|(R_{\frac{1}{m}} - I)(I - R_m)h_m\|_p \lesssim m\|h_m\|_p < \infty.$$

Thus $DBT_m h_m \in L^p(\mathbb{R}^{n+1};\mathbb{C}^{n+2})$. Hence $\{h \in \mathsf{R}(DB) \cap \mathscr{D}(DB) : DBh \in L^p(\mathbb{R}^n;\mathbb{C}^{n+2})\}$ is a dense subspace of $\overline{\mathsf{R}(D)}$.

Now let $f \in \{h \in \mathsf{R}(DB) \cap \mathscr{D}(DB) : DBh \in L^p(\mathbb{R}^n;\mathbb{C}^{n+2}\}$ and let $x \in \mathbb{R}^n$ be a Lebesgue point. Then

$$\begin{aligned}\fint\!\!\!\fint_{W(t,x)} |F(s,y) - f(x)|^2 \,\mathrm{d}y\,\mathrm{d}s \lesssim &\fint\!\!\!\fint_{W(t,x)} |e^{-tDB}f(y) - R_t f(y)|^2 \,\mathrm{d}y\,\mathrm{d}s \\ &+ \fint\!\!\!\fint_{W(t,x)} |R_t f(y) - f(y)|^2 \,\mathrm{d}y\,\mathrm{d}s \\ &+ \fint\!\!\!\fint_{W(t,x)} |f(y) - f(x)|^2 \,\mathrm{d}y\,\mathrm{d}s.\end{aligned}$$

Now the third term above converges to 0 as $t \to 0$ by the Lebesgue differentiation theorem. For the first term let $\psi \in \Psi(S^o_\mu)$ given by $\psi(z) := e^{-z} - (1 + iz)^{-1}$. Now define

$$h(t,x) := \iint_{W(t,x)} |\psi(sDB)f(y)|^2 \frac{\mathrm{d}y\,\mathrm{d}s}{s^{n+1}}.$$

Note for almost all $x \in \mathbb{R}^n$ we have that $0 \leq h(t_0, x) \leq h(t_1, x)$ for $0 \leq t_0 \leq t_1$. Also by Proposition 7.3.1 the quadratic estimates for DB we have that

$$\int_{\mathbb{R}^n} h(t,x)\,\mathrm{d}x \lesssim \int_0^t \|\psi(sDB)f\|_2^2 \frac{\mathrm{d}s}{s} \lesssim \|f\|_2^2.$$

Therefore, by the monotone convergence theorem and h being continuous in t, we have

$$0 \leq \int_{\mathbb{R}^n} h(0,x)\,\mathrm{d}x = \lim_{t\to 0} \int_{\mathbb{R}^n} h(t,x)\,\mathrm{d}x \lesssim \lim_{t\to 0} \int_0^t \|\psi(sDB)f\|_2^2 \frac{\mathrm{d}s}{s} = 0.$$

Thus, $h(0,x) = 0$ for almost every $x \in \mathbb{R}^n$. Therefore

$$\lim_{t\to 0} \fint\!\!\fint_{W(t,x)} |e^{-tDB} f(y) - R_t f(y)|^2 \,\mathrm{d}y\,\mathrm{d}s \eqsim \lim_{t\to 0} h(t,x) = 0,$$

for almost every $x \in \mathbb{R}^n$. Now for the second term we use the off diagonal argument used in the proof of Theorem 7.3.2 to obtain

$$\begin{aligned}
\fint\!\!\fint_{W(t,x)} |R_t f(y) - f(y)|^2 \,\mathrm{d}y\,\mathrm{d}s &= \fint\!\!\fint_{W(t,x)} |sR_t DBf(y)|^2 \,\mathrm{d}y\,\mathrm{d}s \\
&\eqsim t^2 \fint\!\!\fint_{W(t,x)} |R_t DBf(y)|^2 \,\mathrm{d}y\,\mathrm{d}s \\
&\lesssim t^2 \left(\mathcal{M}(|DBf|^2)(x)\right)^2.
\end{aligned}$$

As there exists $p > 2$ such that $DBf \in L^p(\mathbb{R}^{n+1}; \mathbb{C}^{n+2})$ we have that $\mathcal{M}(|DBf|^2) \in L^p(\mathbb{R}^n)$. Thus $\mathcal{M}(|DBf|^2)(x) < \infty$ almost everywhere. Then, as $\mathcal{M}(|DBf|^2)$ is independent of t, we have that

$$\lim_{t\to 0} \fint\!\!\fint_{W(t,x)} |R_t f(y) - f(y)|^2 \,\mathrm{d}y\,\mathrm{d}s \lesssim \left(\mathcal{M}(|DBf|^2)(x)\right)^2 \lim_{t\to 0} t^2 = 0,$$

for almost every $x \in \mathbb{R}^n$.

Now let $f \in \overline{\mathsf{R}(D)}$ and $\varepsilon > 0$. Let $\{f_k\}_{k\in\mathbb{N}} \subset \{h \in \mathsf{R}(DB) \cap \mathscr{D}(DB) : DBh \in L^p(\mathbb{R}^n; \mathbb{C}^{n+2}\}$ such that $f_k \to f$ as $n \to 0$. Now choose $K \in \mathbb{N}$ and $t > 0$ (depending on

$K \in \mathbb{N}$) such that

$$\|f - f_K\|_2^2 < \frac{t^n \varepsilon}{3} \quad \text{and} \quad \fint\!\!\fint_{W(t,x)} |F_K(s,y) - f_K(y)|^2 \, \mathrm{d}y \, \mathrm{d}s < \frac{\varepsilon}{3}$$

Now defining $F_K(t) := e^{-t[DB]} f_K$. Then using the bounded holomorphic functional calculus of DB we have

$$\begin{aligned}
\fint\!\!\fint_{W(t,x)} |F(s,y) - f(x)|^2 \, \mathrm{d}y \, \mathrm{d}s &\lesssim \fint\!\!\fint_{W(t,x)} |e^{-t[DB]}[f(y) - f_K(y)]|^2 \, \mathrm{d}y \, \mathrm{d}s \\
&\qquad + \fint\!\!\fint_{W(t,x)} |F_K(s,y) - f_K(y)|^2 \, \mathrm{d}y \, \mathrm{d}s \\
&\qquad + \fint\!\!\fint_{W(t,x)} |f_K(y) - f(y)| \, \mathrm{d}y \, \mathrm{d}s \\
&\lesssim \fint_t^{2t} t^{-n} \|e^{-t[DB]}[f(y) - f_K(y)]\|_2^2 \, \mathrm{d}t + \frac{\varepsilon}{3} + t^{-n}\|f_K - f\|_2^2 \\
&\lesssim t^{-n}\|f_K - f\|_2^2 + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&< \varepsilon.
\end{aligned}$$

As required $\square$

Combining Proposition 7.3.3 with Theorem 5.3.8 and Proposition 5.3.4 we gain the following corollary, which is the same as Corollary 5.3.9 with convergence on Whitney averages as in (2.6.3) instead of in L^2.

Corollary 7.3.4. We have (5.3.2) is globally well-posed in $E_{DB}^{\pm}\mathcal{H}$ with pointwise convergence on Whitney averages. Moreover, solutions to (5.3.2) are of the form $e^{-tDB}f$ for $\pm t > 0$ for initial data $f \in E_{DB}^{\pm}\mathcal{H}$

CHAPTER 8

SOLVABILITY RESULTS FOR THE ELECTRIC SCHÖDINGER EQUATION

In this chapter we discuss results concerning the solvbility of the second-order equation $H_{A,a,V}\,u = 0$. We are finial in a position to prove Theorems 6.0.1 and 6.0.2 which will resolve the problems of well-posedness of the boundary value problems in the sense of (2.6.4) and (2.6.5). We will also consider the notion of compatible well-posedness, where we will prove that the solutions from Theorem 6.0.1 are equivalent the energy solutions, which are constructed from the Lax–Milgram theorem. To do this we will need to know the trace space of $\dot{\mathcal{V}}^{1,2}(\mathbb{R}^{n+1}_+)$ which will rely on the construction of the fractional counterpart $\dot{\mathcal{V}}^{s,2}(\mathbb{R}^n)$.

8.1 Well-posedness of the Second-Order Equation

We now are ready to transfer results about well-posedness from the first-order setting in Chapter 6 to the second order setting as in Theorem 6.0.1. We first show the equivalence between the invertability of the mappings (6.2.1) and the well-posedness of the first-order equation as in definition 5.3.2.

8.1.1 Equivalences of well-posedness

We first show that non-tangential control is sufficient to give a correspondence between the first-order and the second-order solutions.

Proposition 8.1.1. If $H_{A,a,V}\, u = 0$ with $\widetilde{N}_*(\nabla_\mu u) \in L^2(\mathbb{R}^n)$, then $F \coloneqq \nabla_{\mathcal{A},\mu} u$ is a weak solution of $\partial_t F + DBF = 0$ in $\mathbb{R}_+$.

Proof. By Proposition 6.1.2, it suffices to prove that $\nabla_{\mathcal{A},\mu} u \in L^2_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^{n+1}; \mathbb{C}^{n+2}))$. Now let $K \subset \mathbb{R}_+$ be compact. Then, there exists an interval (t_0, t_1) such that $K \subseteq (t_0, t_1)$. Then, letting $l = \log_2(t_1) - \log_2(t_0)$ and using Lemma 7.3.1, we have

$$\begin{aligned}
\int_K \int_{\mathbb{R}^n} |\nabla_{\mathcal{A},\mu} u|^2 \,\mathrm{d}x\,\mathrm{d}s &\lesssim \int_{t_0}^{t_1} \int_{\mathbb{R}^n} |\nabla_\mu u|^2 \,\mathrm{d}x\,\mathrm{d}s, \\
&\leq \sum_{k=0}^{l} \int_{2^k t_0}^{2^{k+1} t_0} \|\nabla_\mu u\|_2^2 \,\mathrm{d}s, \\
&= \sum_{k=0}^{l} 2^k t_0 \fint_{2^k t_0}^{2^{k+1} t_0} \|\nabla_\mu u\|_2^2 \,\mathrm{d}s, \\
&\leq \sum_{k=0}^{l} 2^k t_0 \sup_{t>0} \fint_t^{2t} \|\nabla_\mu u\|_2^2 \,\mathrm{d}s, \\
&\lesssim \sum_{k=0}^{l} 2^k t_0 \|\widetilde{N}_*(\nabla_\mu u)\|_2^2, \\
&< \infty.
\end{aligned}$$

Thus $\nabla_{\mathcal{A},\mu} u \in L^2_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^{n+1}; \mathbb{C}^{n+2}))$, as required. □

Remark 8.1.2. If $V \in L^{\frac{n}{2}}(\mathbb{R}^n)$ then we have $\dot{\mathcal{V}}^{1,2}(\mathbb{R}^{n+1}_+) = \dot{W}^{1,2}(\mathbb{R}^{n+1}_+)$. That is, $\|\nabla_\mu u\|_2 \eqsim \|\nabla u\|_2$. Therefore, in the case when $V \in L^{\frac{n}{2}}(\mathbb{R}^n)$ we may replace the condition $\widetilde{N}_*(\nabla_\mu u) \in L^2(\mathbb{R}^n)$ with $\widetilde{N}_*(\nabla u) \in L^2(\mathbb{R}^n)$.

Now we show that the notions of well-posedness transfer across from the first-order system to the second-order equations.

Proposition 8.1.3. $(\mathcal{R})^{\mathcal{A}}_{L^2}$ is well-posed if and only if $\Phi_R \colon \mathcal{H}^{0,+}_{DB} \to \{\nabla_\mu u : u \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^n)\}$ is an isomorphism.

Proof. First suppose $(\mathcal{R})^{\mathcal{A}}_{L^2}$ is well-posed. Let $\varphi \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^n)$. Let u be the unique solution of the Regularity problem with boundary data φ. As $H_{A,a,V}\, u = 0$ and $\widetilde{N}_*(\nabla_\mu u) \in L^2(\mathbb{R}^n)$, then by Proposition 8.1.1, we have $F \coloneqq \nabla_{\mathcal{A},\mu} u$ is a weak solution of $\partial_t F + DBF = 0$ in

$\mathbb{R}_+$. Thus, by Theorem 5.3.8 we have there exists $f \in \mathcal{H}^{0,+}_{DB}$ such that $\lim_{t\to 0^+} F(t) = f$ in L^2. Now

$$\left\| \begin{pmatrix} F_\|(t,\cdot) \\ F_\mu(t,\cdot) \end{pmatrix} - \nabla_\mu \varphi \right\|_2 = \|\nabla_\mu^\| u(t,\cdot) - \nabla_\mu\varphi\|_2 \to 0,$$

as $t \to 0$. That is $(f_\|, f_\mu)^T = \nabla_\mu \varphi$. That is $\Phi_R \colon \mathcal{H}^{0,+}_{DB} \to \{\nabla_\mu u : u \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^n)\}$ is surjective as for every $\varphi \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^n)$ there exists $f \in \mathcal{H}^{0,+}_{DB}$ such that $\Phi_R(f) = \nabla_\mu\varphi$.

Suppose there exists $f \in \mathcal{H}^{0,+}_{DB}$ such that $\Phi_R(f) = 0$. By Corollary 5.3.9 we have (5.3.2) is globally well-posed and so there exists a unique F which satisfies (5.3.2) with initial data f. Also, let u be the unique solution of the regularity problem with initial data 0. Since $H_{A,a,V}\, 0 = 0$ and the solution 0 satisfies the boundary data 0, therefore, by uniqueness, we have $u = 0$. Then, $G = \nabla_{\mathcal{A},\mu} u = 0$ satisfies (5.3.2) with initial data 0. Hence, by uniqueness, $F = G = 0$. Thus, $f = 0$. That is $\Phi_R \colon \mathcal{H}^{0,+}_{DB} \to \{\nabla_\mu u : u \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^n)\}$ is injective.

Conversely, suppose $\Phi_R \colon \mathcal{H}^{0,+}_{DB} \to \{\nabla_\mu u : u \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^n)\}$ is an isomorphism. Let $\varphi \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^n)$. Then we have a unique $f \in \mathcal{H}^{0,+}_{DB}$ such that $\Phi_R(f) = \nabla_\mu\varphi$. By corollary 5.3.9 there exists a unique $F \in L^2_{\text{loc}}(\mathbb{R}_+; \overline{\mathsf{R}(D)})$ satisfying (5.3.2) with initial data f. Then, by Proposition 6.1.2 there exists u such that $H_{A,a,V}\, u = 0$ and $F = \nabla_{\mathcal{A},\mu} u$. Therefore

$$\|u(t,\cdot) - \varphi\|_{\dot{\mathcal{V}}^{1,2}} = \left\| \begin{pmatrix} F_\|(t,\cdot) \\ F_\mu(t,\cdot) \end{pmatrix} - \nabla_\mu \varphi \right\|_2 \to 0,$$

as $t \to 0$. We also have convergence pointwise on Whitney averages

$$\lim_{t\to 0} \fint\!\!\!\fint_{W(t,x)} |\nabla_\mu^\| u - \nabla_\mu \varphi|^2 \, \mathrm{d}y \, \mathrm{d}s = \lim_{t\to 0} \fint\!\!\!\fint_{W(t,x)} \left| \begin{pmatrix} F_\| \\ F_\mu \end{pmatrix} - \nabla_\mu\varphi \right|^2 \mathrm{d}y \, \mathrm{d}s = 0.$$

That is for each $\varphi \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^n)$ there exists u a solution to $H_{A,a,V}\, u = 0$. Now by Theorem 5.3.8 we have $F(t) = e^{-tDB} f$. Thus, by Theorem 7.3.2 we have $\tilde{N}_*(F) \in L^2(\mathbb{R}^n)$. Therefore, as $|\nabla_\mu u| = |\overline{\mathcal{A}}^{-1}||\nabla_{\mathcal{A},\mu} u| \lesssim |F|$. And so we have $\tilde{N}_*(\nabla_\mu u) \in L^2(\mathbb{R}^n)$. Thus, there exists u solving the problem $(\mathcal{R})^{\mathcal{A}}_{L^2}$.

Now we prove uniqueness. Let $\varphi \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^n)$ be such that there exists u, v satisfying $H_{A,a,V}\, u = H_{A,a,V}\, v = 0$, where $\tilde{N}_*(\nabla_\mu u), \tilde{N}_*(\nabla_\mu v) \in L^2(\mathbb{R}^n)$, and u and v converge to the boundary data $\nabla_\mu \varphi$. Now, by Proposition 8.1.1 we have $G = \nabla_{\mathcal{A},\mu} u \in L^2_{\text{loc}}(\mathbb{R}_+; \overline{\mathsf{R}(D)})$ and $H = \nabla_{\mathcal{A},\mu} v \in L^2_{\text{loc}}(\mathbb{R}_+; \overline{\mathsf{R}(D)})$ are weak solutions to $\partial_t F + DBF = 0$. Thus, by Theorem 5.3.8, there exists $g, h \in \mathcal{H}^{0,+}_{DB}$ such that $G(t) = e^{-tDB} g$ and $H(t) = e^{-tDB} h$. Now

$$\left\| \begin{pmatrix} G_\parallel(t,\cdot) \\ G_\mu(t,\cdot) \end{pmatrix} - \nabla_\mu \varphi \right\|_2 = \|\nabla_\mu^\parallel u(t,\cdot) - \nabla_\mu \varphi\|_2 \to 0,$$

and

$$\left\| \begin{pmatrix} H_\parallel(t,\cdot) \\ H_\mu(t,\cdot) \end{pmatrix} - \nabla_\mu \varphi \right\|_2 = \|\nabla_\mu^\parallel v(t,\cdot) - \nabla_\mu \varphi\|_2 \to 0,$$

as $t \to 0$. Hence $\Phi_R(g) = \Phi_R(h) = \nabla_\mu \varphi$. Then, as $\Phi_R \colon \mathcal{H}^{0,+}_{DB} \to \{\nabla_\mu u : u \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^n)\}$ is an isomorphism, we have $g = h$. Thus, $G = H$. Therefore, $\nabla_{\mathcal{A},\mu} u = \nabla_{\mathcal{A},\mu} v$. That is $u = v$. Hence, $(\mathcal{R})^{\mathcal{A}}_{L^2}$ is well-posed. $\square$

Proposition 8.1.4. $(\mathcal{N})^{\mathcal{A}}_{L^2}$ is well-posed if and only if $\Phi_N \colon \mathcal{H}^{0,+}_{DB} \to L^2(\mathbb{R}^n)$ is an isomorphism.

Proof. Proved similarly to the regularity case. $\square$

8.1.2 Proofs of Main Theorems

We now give the proofs of the main theorems. We start with the well-posedness theorem

Proof of Theorem 6.0.1. Let $\mathcal{A}$ be a block type matrix. Then by Lemma 6.2.3 we have the mappings $\Phi_R \colon \mathcal{H}^{0,+}_{DB} \to \{\nabla_\mu u : u \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^n)\}$ and $\Phi_N \colon \mathcal{H}^{0,+}_{DB} \to L^2(\mathbb{R}^n)$ are isomorphisms. Thus, by Propositions 8.1.3 and 8.1.4 we have $(\mathcal{N})^{\mathcal{A}}_{L^2}$ and $(\mathcal{R})^{\mathcal{A}}_{L^2}$ are well-posed.

Now let $\mathcal{A}$ be self-adjoint. Then by Proposition 6.3.5 we have that $\Phi_R \colon \mathcal{H}^{0,+}_{DB} \to \{\nabla_\mu u : u \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^n)\}$ and $\Phi_N \colon \mathcal{H}^{0,+}_{DB} \to L^2(\mathbb{R}^n)$ are isomorphisms. Thus, by Propositions 8.1.3 and 8.1.4 we have $(\mathcal{N})^{\mathcal{A}}_{L^2}$ and $(\mathcal{R})^{\mathcal{A}}_{L^2}$ are well-posed.

To prove openness first let $\mathcal{A}_0 \in WP(\mathcal{N})$. Let $\mathcal{A} \in L^\infty(\mathbb{R}^n; \mathbb{C}^{n+2})$ with $\|\mathcal{A} - \mathcal{A}_0\|_\infty < \varepsilon$ for some $\varepsilon \in (0, \kappa)$, to be chosen later, where κ is the ellipticity constant of $\mathcal{A}_0$. Now define

$$\mathcal{A}(z) := \mathcal{A}_0 - \frac{z(\mathcal{A}_0 - \mathcal{A})}{\|\mathcal{A}_0 - \mathcal{A}\|_\infty},$$

for $z \in \Omega := \{w \in \mathbb{C} : |w| < \varepsilon\}$. Then $z \mapsto A(z)$ is holomorphic in Ω. $A(z)$ is bounded and elliptic (with ellipticity constant $\kappa - \varepsilon > 0$) uniformly in Ω. Now define $B(z) := \widehat{\mathcal{A}(z)}$. As $\mathcal{A}(z)$ is elliptic uniformly in Ω then $A_{\perp\perp}(z)$ is invertible and $A_{\perp\perp}(z)^{-1}$ is holomorphic in Ω. Therefore $B(z)$ is holomorphic in Ω. Note

$$\begin{aligned} B_0 - B(z) &= \underline{\mathcal{A}_0}\overline{\mathcal{A}_0}^{-1} - \underline{\mathcal{A}(z)}\overline{\mathcal{A}_0}^{-1} + \underline{\mathcal{A}(z)}\overline{\mathcal{A}_0}^{-1} - \underline{\mathcal{A}(z)}\overline{\mathcal{A}(z)}^{-1} \\ &= (\underline{\mathcal{A}_0} - \underline{\mathcal{A}(z)})\overline{\mathcal{A}_0}^{-1} + \underline{\mathcal{A}(z)}\overline{\mathcal{A}_0}^{-1}(\overline{\mathcal{A}(z)} - \overline{\mathcal{A}_0})\overline{\mathcal{A}(z)}^{-1}, \end{aligned}$$

where $B_0 = \widehat{\mathcal{A}_0}$. Therefore, we have $\|B_0 - B(z)\|_\infty \leq C_0 \|\mathcal{A}_0 - \mathcal{A}(z)\|_\infty$, where $C_0 > 0$ depends only on n, κ, and the bounds of $\overline{\mathcal{A}_0}^{-1}$, $\overline{\mathcal{A}(z)}^{-1}$, and $\underline{\mathcal{A}(z)}$. Note that, as $\mathcal{A} = \widehat{\widehat{\mathcal{A}}}$ we have the lower bound as well so $\|B_0 - B(z)\|_\infty \eqsim \|\mathcal{A}_0 - \mathcal{A}(z)\|_\infty$. Now choose $z_0 = \|\mathcal{A}_0 - \mathcal{A}\|_\infty$, $\mathcal{A} = \mathcal{A}(z_0)$, and $\varepsilon < \frac{\kappa}{C_0}$. Thus $\|B_0 - B\| < C_0\varepsilon < \kappa$, where $B = \widehat{\mathcal{A}}$. Therefore, by Theorem 5.2.2, we have

$$\|f(DB_0)u - f(DB)u\|_2 \lesssim \|B_0 - B\|_\infty \|f\|_\infty \|u\|_2 \lesssim \|\mathcal{A}_0 - \mathcal{A}\|_\infty \|f\|_\infty \|u\|_2,$$

for all $f \in H^\infty(S^o_\mu)$. Choosing $f = \chi^+$ gives

$$\|E^+_{DB}u - E^+_{DB_0}u\|_\infty \lesssim \|\mathcal{A} - \mathcal{A}_0\|_\infty \|u\|_2.$$

That is the projections E^+_{DB} depend continuously on $\mathcal{A}$. Then as $\mathcal{A}_0 \in WP(\mathcal{N})$ we have $\Phi_N \colon \mathcal{H}^{0,+}_{DB_0} \to L^2(\mathbb{R}^n)$ is an isomorphism. Therefore, by [4, Lemma 4.3] we have for $\mathcal{A}$ sufficiently close to $\mathcal{A}_0$ then $\Phi_N \colon \mathcal{H}^{0,+}_{DB} \to L^2(\mathbb{R}^n)$ is an isomorphism as well. The Regularity case is identical.

We now prove the equivalence of norms for the Regularity problem. Let $A \in WP(\mathcal{R})$. Let $\varphi \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^n)$ be the boundary data. Then, as $(\mathcal{R})^{\mathcal{A}}_{L^2}$ is well-posed we have $\widetilde{N}_*(\nabla_\mu u) \in L^2(\mathbb{R}^n)$. Therefore, from Proposition 8.1.1, we have $F = \nabla_{\mathcal{A},\mu} u$ a solution of the first-order equation $\partial_t F + DBF = 0$. By Theorem 5.3.8 we have there exists $f \in \mathcal{H}^{0,+}_{DB}$ such that $F_t = e^{-tDB} f$. Then by Theorem 7.3.2 we have

$$\|\widetilde{N}_* F\|_2^2 \eqsim \int_0^\infty \|t\partial_t F\|_2^2 \frac{\mathrm{d}t}{t} \eqsim \|f\|_2^2$$

As $A \in WP(\mathcal{R})$, then by Proposition 8.1.3, we have $\Phi_R \colon \mathcal{H}^{0,+}_{DB} \to \{\nabla_\mu u : u \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^n)\}$ is an isomorphism and in particular $\Phi_R(f) = \nabla_\mu \varphi$. Also, we have $\|\Phi_R(f)\|_2 \leq \|f\|_2$ and as $\{\nabla_\mu u : u \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^n)\}$ is a closed subspace of $L^2(\mathbb{R}^n)$ and so is a Banach space. Thus, by the bounded inverse theorem we have $\Phi_R^{-1} \colon \{\nabla_\mu u : u \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^n)\} \to \overline{\mathsf{R}(D)}$ is bounded. That is $\|\Phi_R^{-1}(f)\|_2 \lesssim \|f\|_2$. Therefore, $\|f\|_2 \lesssim \|\Phi_R(f)\|_2$. Hence, we have

$$\|f\|_2 \eqsim \|\Phi_R(f)\|_2 = \left\| \begin{pmatrix} f_\| \\ f_\mu \end{pmatrix} \right\|_2 .$$

Then, recalling that $f \in \overline{\mathsf{R}(D)}$, we have

$$\|\widetilde{N}_*(\nabla_\mu u)\|_2^2 \eqsim \int_0^\infty \|t\partial_t \nabla_\mu u\|_2^2 \frac{\mathrm{d}t}{t} \eqsim \|\nabla_\mu \varphi\|_2^2.$$

The proof is similar for the Neumann problem. □

Proposition 8.1.5. Let $\mathcal{A} \in L^\infty(\mathbb{R}^{n+1}_+; \mathcal{L}(\mathbb{C}^{n+2}))$ be elliptic. Now let u be such that $H_{A,a,V}\, u = 0$, with $\|\widetilde{N}_*(\nabla_\mu u)\|_2 < \infty$. Then, there exist $\varphi \in L^2(\mathbb{R}^n; \mathbb{C}^{n+2})$, such that

$$\lim_{t\to 0} \fint_t^{2t} \|\nabla_\mu u(s) - \varphi\|_2^2 \,\mathrm{d}s = 0 = \lim_{t\to\infty} \fint_t^{2t} \|\nabla_\mu u(s)\|_2^2 \,\mathrm{d}s.$$

Proof. This follows from Proposition 8.1.1 and Proposition 5.3.10. □

Proof of Theorem 6.0.2. Let u be a weak solution of $H_{A,a,V}\, u = 0$ in $\mathbb{R}^{n+1}_+$ with $\widetilde{N}_*(\nabla_\mu u) \in L^2(\mathbb{R}^n)$. By Proposition 8.1.1, there exists F a weak solution of $\partial_t F + DBF = 0$ in $\mathbb{R}_+$ with $F = \nabla_{\mathcal{A},\mu} u$. Note, as F is a solution of the first-order equation $\partial_t F + DBF = 0$ using Theorem 5.3.8 we have there exists $f \in \mathcal{H}^{0,+}_{DB}$ such that $F(t,x) = e^{-tDB} f$. Then by

Proposition 7.3.3 we have

$$\lim_{t\to 0} \fint\!\!\fint_{W(t,x)} |F(s,y) - f(x)|^2 \,\mathrm{d}y\,\mathrm{d}s = 0,$$

for almost every $x \in \mathbb{R}^n$. Note as $f \in \overline{\mathsf{R}(D)}$ then for some $\varphi \in L^2(\mathbb{R}^n)$ and $\phi \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^n)$, we have that

$$f = \begin{bmatrix} \varphi \\ \nabla_{\|}\phi \\ |V|^{\frac{1}{2}}\phi \end{bmatrix},$$

Therefore,

$$\lim_{t\to 0} \fint\!\!\fint_{W(t,x)} |\partial_{\nu_A} u(t,x) - \varphi(x)|^2 \,\mathrm{d}x\,\mathrm{d}t = \lim_{t\to 0} \fint\!\!\fint_{W(t,x)} \left|\nabla_\mu^{\|} u(t,x) - \nabla_\mu \phi(x)\right|^2 \mathrm{d}x\,\mathrm{d}t = 0.$$

As required. □

8.2 Trace Spaces for Adapted Sobolev Spaces

We now turn our attention to compatible well-posedness. But, first we need to construct the trace space for $\dot{\mathcal{V}}^{1,2}(\mathbb{R}^{n+1}_+)$ and the definition of the trace operator.

Define the Schrödinger operator

$$H \coloneqq -\Delta_{\|} + V : \mathscr{D}(H) \subset L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n).$$

We use the Schrödinger operator to define adapted fractional Sobolev spaces in a similar fashion to the fractional Sobolev spaces defined via the Laplacian. First, note that as H is self-adjoint then H has a bounded Borel functional calculus and so we may define $H^{\frac{\alpha}{2}}$.

Definition 8.2.1. Let $\alpha > 0$. We define the fractional homogeneous Sobolev spaces, $\dot{\mathcal{V}}^{\alpha,2}(\mathbb{R}^n)$, as the completion of $\mathcal{C}_c^\infty(\mathbb{R}^n)$ under the norm $\|H^{\frac{\alpha}{2}}u\|_2$. We define the norm as follows

$$\|u\|_{\dot{\mathcal{V}}^{\alpha,2}} := \|H^{\frac{\alpha}{2}}u\|_2.$$

we also define $\dot{\mathcal{V}}^{-\alpha,2}(\mathbb{R}^n) := (\dot{\mathcal{V}}^{\alpha,2}(\mathbb{R}^n))^*$, the dual space of $\dot{\mathcal{V}}^{\alpha,2}(\mathbb{R}^n)$.

Now, as H, Δ, and $V^{\frac{1}{2}}$ are all self-adjoint, we have

$$\begin{aligned}
\|H^{\frac{1}{2}}u\|_2^2 &= \langle H^{\frac{1}{2}}u, H^{\frac{1}{2}}u\rangle \\
&= \langle Hu, u\rangle \\
&= \langle \Delta_{\|}u, u\rangle + \langle Vu, u\rangle \\
&= \langle \Delta_{\|}^{\frac{1}{2}}u, \Delta_{\|}^{\frac{1}{2}}u\rangle + \langle V^{\frac{1}{2}}u, V^{\frac{1}{2}}\rangle \\
&= \|\nabla u\|_2^2 + \||V|^{\frac{1}{2}}u\|_2^2 \\
&= \|\nabla_\mu u\|_2^2,
\end{aligned} \tag{8.2.1}$$

for all $u \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^n)$. We would like similar equivalence for $H^{\frac{1}{4}}$. It is conjectured in [5] that the following inequality holds for all $\alpha \in (0,1)$ and a range of p, but we only require the one direction in the case when $\alpha = \frac{1}{4}$ and $p = 2$. To prove this we use the Heinz–Kato inequality, see [32] and [36].

Theorem 8.2.2 (Heinz–Kato)**.** Let $\mathcal{H}$ be a Hilbet space and $T\colon \mathcal{H} \to \mathcal{H}$ be a bounded linear operator. If A and B are positive operators with $\|Tx\| \le \|Ax\|$ and $\|T^*y\| \le \|By\|$ for all $x, y \in \mathcal{H}$, then for all $x, y \in \mathcal{H}$ we have

$$|\langle Tx, y\rangle| \le \|A^\alpha x\|\|B^{1-\alpha}y\|$$

for all $\alpha \in (0,1)$.

Lemma 8.2.3. Let $V(x) \ge 0$ for all $x \in \mathbb{R}^n$. Then we have the following bound

$$\|\Delta_{\|}^{\frac{1}{4}}f\|_2^2 + \|V^{\frac{1}{4}}f\|_2^2 \lesssim \|H^{\frac{1}{4}}f\|_2^2,$$

for all $f \in \dot{\mathcal{V}}^{\frac{1}{2},2}(\mathbb{R}^n)$.

Proof. As V is a positive operator we have the following

$$\begin{aligned}
\|\Delta_{\|}^{\frac{1}{2}} f\|_2^2 &= \langle \Delta_{\|}^{\frac{1}{2}} f, \Delta_{\|}^{\frac{1}{2}} f\rangle \\
&= \langle \Delta_{\|} f, f\rangle \\
&\leq \langle \Delta_{\|} f, f\rangle + \langle V f, f\rangle \\
&= \langle (\Delta_{\|} + V) f, f\rangle \\
&= \langle H^{\frac{1}{2}} f, H^{\frac{1}{2}} f\rangle \\
&= \|H^{\frac{1}{2}} f\|_2^2,
\end{aligned}$$

for all $f \in \dot{\mathcal{V}}^{\frac{1}{2},2}(\mathbb{R}^n)$. Also, as $\Delta_{\|}^{\frac{1}{2}}$ is self-adjoint then $\|(\Delta_{\|}^{\frac{1}{2}})^* f\|_2^2 \leq \|H^{\frac{1}{2}} f\|_2^2$. Thus, by the Heinz-Kato inequality with $\alpha = \frac{1}{2}$, we have

$$\|\Delta_{\|}^{\frac{1}{4}} f\|_2^2 = |\langle \Delta_{\|}^{\frac{1}{4}} f, \Delta_{\|}^{\frac{1}{4}} f\rangle| = |\langle \Delta_{\|}^{\frac{1}{2}} f, f\rangle| \leq \|H^{\frac{1}{4}} f\| \|H^{\frac{1}{4}} f\|_2 = \|H^{\frac{1}{4}} f\|_2^2,$$

for all $f \in \dot{\mathcal{V}}^{\frac{1}{2},2}(\mathbb{R}^n)$.

A similar argument gives that $\|V^{\frac{1}{2}} f\|_2^2 \leq \|H^{\frac{1}{2}} f\|_2^2$. for all $f \in \dot{\mathcal{V}}^{\frac{1}{2},2}(\mathbb{R}^n)$. Therefore again by the Heinz–Kato inequality we have $\|V^{\frac{1}{4}} f\|_2 \leq \|H^{\frac{1}{4}} f\|_2$ for all $f \in \dot{\mathcal{V}}^{\frac{1}{2},2}(\mathbb{R}^n)$. This completes the proof. □

We now introduce the trace operator and trace space for $\dot{\mathcal{V}}^{1,2}(\mathbb{R}^{n+1}_+)$. We define the trace operator in the classical way by defining it on smooth functions and using density to extend to the whole of $\dot{\mathcal{V}}^{1,2}(\mathbb{R}^{n+1}_+)$. We define the trace operator on $\mathcal{C}_c^\infty(\overline{\mathbb{R}^{n+1}_+})$ by the linear operator $\operatorname{Tr}\colon \mathcal{C}_c^\infty(\overline{\mathbb{R}^{n+1}_+}) \to \dot{\mathcal{V}}^{\frac{1}{2},2}(\mathbb{R}^n)$, given by $(\operatorname{Tr} u)(x) := u(0, x)$. We have the following bound on the trace operator

Lemma 8.2.4. We have the following

$$\|\operatorname{Tr} u\|_{\dot{\mathcal{V}}^{\frac{1}{2},2}(\mathbb{R}^n)} \lesssim \|u\|_{\dot{\mathcal{V}}^{1,2}(\mathbb{R}^{n+1}_+)}.$$

Moreover, there exists a continuous extension $\operatorname{Tr}\colon \dot{\mathcal{V}}^{1,2}(\mathbb{R}^{n+1}_+) \to \dot{\mathcal{V}}^{\frac{1}{2},2}(\mathbb{R}^n)$.

Proof. Let $u \in \mathcal{C}_c^\infty(\overline{\mathbb{R}^{n+1}_+})$. Now using the Fundamental Theorem of calculus, the self-adjointness of $H^{\frac{1}{4}}$, the Cauchy–Schwartz inequality, and (8.2.1), we have

$$\begin{aligned}\int_{\mathbb{R}^n} |H^{\frac{1}{4}}u(0,x)|^2 \,\mathrm{d}x &= -\int_{\mathbb{R}^n} \left(\int_0^\infty \partial_t |H^{\frac{1}{4}}u(t,x)|^2 \,\mathrm{d}t\right) \mathrm{d}x \\ &= -\iint_{\mathbb{R}^{n+1}_+} \overline{H^{\frac{1}{4}}u(t,x)}\partial_t H^{\frac{1}{4}}u(t,x) + H^{\frac{1}{4}}u(t,x)\partial_t \overline{H^{\frac{1}{4}}u(t,x)} \,\mathrm{d}x\,\mathrm{d}t \\ &= -\iint_{\mathbb{R}^{n+1}_+} \overline{H^{\frac{1}{4}}u(t,x)}H^{\frac{1}{4}}\partial_t u(t,x) + H^{\frac{1}{4}}u(t,x)\overline{H^{\frac{1}{4}}\partial_t u(t,x)} \,\mathrm{d}x\,\mathrm{d}t \\ &= -\iint_{\mathbb{R}^{n+1}_+} \overline{H^{\frac{1}{2}}u(t,x)}\partial_t u(t,x) + H^{\frac{1}{2}}u(t,x)\overline{\partial_t u(t,x)} \,\mathrm{d}x\,\mathrm{d}t \\ &\lesssim \left(\iint_{\mathbb{R}^{n+1}_+} |H^{\frac{1}{2}}u(t,x)|^2 \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}} \left(\iint_{\mathbb{R}^{n+1}_+} |\partial_t u(t,x)|^2 \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}} \\ &\le \iint_{\mathbb{R}^{n+1}_+} |\nabla_\mu u(t,x)|^2 \,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

The remaining part follows from the density of $\mathcal{C}_c^\infty(\overline{\mathbb{R}^{n+1}_+})$ in $\dot{\mathcal{V}}^{1,2}(\mathbb{R}^{n+1}_+)$ in Proposition 2.3.2. □

We are now able to define $\dot{\mathcal{V}}^{1,2}_0(\mathbb{R}^{n+1}_+)$ as the set of functions in $\dot{\mathcal{V}}^{1,2}(\mathbb{R}^{n+1}_+)$ with zero trace and so $\dot{\mathcal{V}}^{1,2}_0(\mathbb{R}^{n+1}_+) \subset \dot{\mathcal{V}}^{1,2}(\mathbb{R}^{n+1}_+)$

It is important that we are able to extend functions in $\dot{\mathcal{V}}^{\frac{1}{2},2}(\mathbb{R}^n)$ to the upper half-space. To do this we prove that the trace operator Tr is a surjection and so for each function $f \in \dot{\mathcal{V}}^{\frac{1}{2},2}(\mathbb{R}^n)$ there exists an extension $F \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^{n+1}_+)$ such that $\mathrm{Tr}\, F = f$. We note such an F may not be unique.

Lemma 8.2.5. The trace map $\mathrm{Tr}\colon \dot{\mathcal{V}}^{1,2}(\mathbb{R}^{n+1}_+) \to \dot{\mathcal{V}}^{\frac{1}{2},2}(\mathbb{R}^n)$ is surjective.

Proof. Let $f \in \dot{\mathcal{V}}^{\frac{1}{2},2}(\mathbb{R}^n)$. Let $\{f_n\}_{n\in\mathbb{N}} \subseteq \mathcal{C}_c^\infty(\mathbb{R}^n)$ be a sequence converging to f in $\dot{\mathcal{V}}^{\frac{1}{2},2}$-norm. Then define $F_n(t,x) := e^{-t\sqrt{H}} f_n(x)$ for $t > 0$. As $\sqrt{H}$ is a self-adjoint operator we have $\{e^{-t\sqrt{H}}\}_{t>0}$ is an analytic semi-group and so $\lim_{t\to 0} F_n(t) = f_n$ in $\dot{\mathcal{V}}^{1,2}$ (see [25] for more detail on analytic semi-groups). As H is a self-adjoint operator then H has a bounded holomorphic functional calculus and satisfies quadratic estimates. Now, splitting the components of ∇_μ, the semi-group properties, (8.2.1), and the quadratic estimates for H with $\psi(z) := z^{\frac{1}{2}}e^{-z}$, we have

$$\begin{aligned}
\|F_n\|^2_{\dot{\mathcal{V}}^{1,2}(\mathbb{R}^{n+1}_+)} &= \iint_{\mathbb{R}^{n+1}_+} |\nabla_\mu e^{-t\sqrt{H}} f_n(x)|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^\infty \|\partial_t e^{-t\sqrt{H}} f_n\|_2^2 \,\mathrm{d}t + \int_0^\infty \|\nabla_\mu^{\|} e^{-t\sqrt{H}} f_n\|_2^2 \,\mathrm{d}t \\
&\lesssim \int_0^\infty \|\sqrt{H} e^{-t\sqrt{H}} f_n\|_2^2 \,\mathrm{d}t \\
&= \int_0^\infty \|(t\sqrt{H})^{\frac{1}{2}} e^{-t\sqrt{H}} H^{\frac{1}{4}} f_n\|_2^2 \frac{\mathrm{d}t}{t} \\
&\lesssim \|H^{\frac{1}{4}} f_n\|_2^2.
\end{aligned}$$

Thus $\|F_n\|_{\dot{\mathcal{V}}^{1,2}(\mathbb{R}^{n+1}_+)} \lesssim \|f_n\|_{\dot{\mathcal{V}}^{\frac{1}{2},2}(\mathbb{R}^n)}$ with $\operatorname{Tr} F_n = f_n$. A standard density argument shows that $\{F_n\}_{n\in\mathbb{N}} \subseteq \dot{\mathcal{V}}^{1,2}(\mathbb{R}^{n+1}_+)$ converges to some $F \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^{n+1}_+)$ with $\operatorname{Tr} F = f$ and $\|F\|_{\dot{\mathcal{V}}^{1,2}(\mathbb{R}^{n+1}_+)} \lesssim \|f\|_{\dot{\mathcal{V}}^{\frac{1}{2},2}(\mathbb{R}^n)}$. This completes the proof. $\square$

We now investigate how the adapted gradient ∇_μ behaves on the fractional adapted Sobolev space $\dot{\mathcal{V}}^{\frac{1}{2},2}(\mathbb{R}^n)$. To do this define the linear operator

$$\nabla_\mu \colon \dot{\mathcal{V}}^{\frac{1}{2},2}(\mathbb{R}^n) \to \dot{\mathcal{V}}^{-\frac{1}{2},2}(\mathbb{R}^n;\mathbb{C}^{n+1}) \quad \text{given by} \quad \nabla_\mu f = \begin{bmatrix} \partial_1 f \\ \vdots \\ \partial_n f \\ V^{\frac{1}{2}} f \end{bmatrix}.$$

where for fixed $k \in \{1,\dots,n\}$ we define the functionals

$$\begin{aligned}
\partial_k f \colon \dot{\mathcal{V}}^{\frac{1}{2},2}(\mathbb{R}^n) \to \mathbb{C} \quad &\text{by } (\partial_k f)(\varphi) := -\int_{\mathbb{R}^n} f\overline{\partial_k \varphi}, \\
\text{and } V^{\frac{1}{2}} f \colon \dot{\mathcal{V}}^{\frac{1}{2},2}(\mathbb{R}^n) \to \mathbb{C} \quad &\text{by } (V^{\frac{1}{2}} f)(\varphi) := -\int_{\mathbb{R}^n} f\overline{V^{\frac{1}{2}}\varphi}.
\end{aligned}$$

We describe the boundedness properties of ∇_μ is the following lemma.

Lemma 8.2.6. Let $u \in \dot{\mathcal{V}}^{\frac{1}{2},2}(\mathbb{R}^n)$. Then $\nabla_\mu u \in \dot{\mathcal{V}}^{-\frac{1}{2},2}(\mathbb{R}^n;\mathbb{C}^{n+1})$ and $\nabla_\mu \colon \dot{\mathcal{V}}^{\frac{1}{2},2}(\mathbb{R}^n) \to \dot{\mathcal{V}}^{-\frac{1}{2},2}(\mathbb{R}^n;\mathbb{C}^{n+1})$ is an injective and bounded operator.

Proof. Let $f, \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$. Then by Parseval's Identity we have

$$\begin{aligned} |(\partial_k f)(\varphi)| &= \left| \int_{\mathbb{R}^n} \hat{f}(\xi) \overline{\widehat{\partial_k \varphi}(\xi)} \, \mathrm{d}\xi \right| \\ &\leq \int_{\mathbb{R}^n} |\hat{f}(\xi)||\xi||\hat{\varphi}(\xi)| \, \mathrm{d}\xi \\ &\leq \left(\int_{\mathbb{R}^n} |\xi||\hat{f}(\xi)|^2 \, \mathrm{d}\xi \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} |\xi||\hat{\varphi}(\xi)|^2 \, \mathrm{d}\xi \right)^{\frac{1}{2}}. \end{aligned} \tag{8.2.2}$$

That is $|(\partial_k f)(\varphi)| \leq \|f\|_{\dot{H}^{\frac{1}{2}}} \|\varphi\|_{\dot{H}^{\frac{1}{2}}}$, where $\dot{H}^{\frac{1}{2}}(\mathbb{R}^n)$ is the standard fractional Sobolev space of order $\frac{1}{2}$. Then using the fact that $\|f\|_{\dot{H}^{\frac{1}{2}}} \eqsim \|\Delta_{\|}^{\frac{1}{4}} f\|_2$ and Lemma 8.2.3 we have $|(\partial_k f)(\varphi)| \leq \|f\|_{\dot{H}^{\frac{1}{2}}} \|\varphi\|_{\dot{H}^{\frac{1}{2}}} \leq \|f\|_{\dot{\mathcal{V}}^{\frac{1}{2},2}} \|\varphi\|_{\dot{\mathcal{V}}^{\frac{1}{2},2}}$. Thus, by the density of $\mathcal{C}_c^\infty(\mathbb{R}^n)$ in $\dot{\mathcal{V}}^{\frac{1}{2},2}(\mathbb{R}^n)$ we have $(\partial_k f)$ is a bounded linear functional on $\dot{\mathcal{V}}^{\frac{1}{2},2}(\mathbb{R}^n)$ and $\|\partial_k f\|_{\dot{\mathcal{V}}^{-\frac{1}{2},2}} \leq \|f\|_{\dot{\mathcal{V}}^{\frac{1}{2},2}}$ for all $f \in \mathcal{C}_c^\infty(\mathbb{R}^n)$. Another density argument gives that $(\partial_k f)$ is a bounded linear functional on $\dot{\mathcal{V}}^{\frac{1}{2},2}(\mathbb{R}^n)$ for all $f \in \dot{\mathcal{V}}^{\frac{1}{2},2}(\mathbb{R}^n)$. Similarly we have

$$|(V^{\frac{1}{2}} f)(\varphi)| \leq \int_{\mathbb{R}^n} |V^{\frac{1}{4}} f||V^{\frac{1}{4}} \varphi| \leq \left(\int_{\mathbb{R}^n} |V^{\frac{1}{4}} f|^2 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} |V^{\frac{1}{4}} \varphi|^2 \right)^{\frac{1}{2}} \leq \|f\|_{\dot{\mathcal{V}}^{\frac{1}{2},2}} \|\varphi\|_{\dot{\mathcal{V}}^{\frac{1}{2},2}}, \tag{8.2.3}$$

for all $f, \varphi \in \dot{\mathcal{V}}^{\frac{1}{2},2}(\mathbb{R}^n)$. Thus, $V^{\frac{1}{2}} f \in \dot{\mathcal{V}}^{-\frac{1}{2},2}(\mathbb{R}^n)$. From (8.2.2) and (8.2.3) we have that $|(\nabla_\mu f)(\varphi)| \lesssim \|f\|_{\dot{\mathcal{V}}^{\frac{1}{2},2}} \|\varphi\|_{\dot{\mathcal{V}}^{\frac{1}{2},2}}$ for all $f, \varphi \in \dot{\mathcal{V}}^{\frac{1}{2},2}(\mathbb{R}^n)$. Therefore, $\nabla_\mu \colon \dot{\mathcal{V}}^{\frac{1}{2},2}(\mathbb{R}^n) \to \dot{\mathcal{V}}^{-\frac{1}{2},2}(\mathbb{R}^n; \mathbb{C}^{n+1})$ is a bounded operator.

To prove injectivity, let $u \in \dot{\mathcal{V}}^{\frac{1}{2},2}(\mathbb{R}^n)$ with $\nabla_\mu u = 0$. That is, for all $\varphi \in \dot{\mathcal{V}}^{\frac{1}{2},2}(\mathbb{R}^n)$ the $(\nabla_\mu u)(\varphi) = 0$. In particular, we have

$$0 = (\nabla_\mu u)(\varphi) = \int_{\mathbb{R}^n} u \cdot \overline{(\nabla_\mu)^* \varphi} \quad \forall \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n).$$

Thus $u = 0$ and $\nabla_\mu \colon \dot{\mathcal{V}}^{\frac{1}{2},2}(\mathbb{R}^n) \to \dot{\mathcal{V}}^{-\frac{1}{2},2}(\mathbb{R}^n; \mathbb{C}^{n+1})$ is injective. $\square$

8.3 Sobolev Spaces Associated to an Operator

In this section we will give a description of fractional Sobolev spaces associate with an bisectorial operator. The results in this section are from [8]. For a Hilbert space $\mathcal{H}$ and a closed ω-bisectorial operator T we first introduce the Hilbert space $\mathcal{H}_T$ defined to be the completion of $\{f \in \mathcal{H} : \|f\|_T < \infty\}$ of the norm $\|\cdot\|_T$ where

$$\|f\|_T := \int_0^\infty \|\psi_t(T)f\|_2^2 \frac{\mathrm{d}t}{t},$$

for some $\psi \in \Psi(S_\mu^o)$, where $\mu \in (\omega, \frac{\pi}{2})$. Note that the choice of μ and ψ give rise to equivalent norms, see [1] for more detail. We now introduce the following definition.

Definition 8.3.1. Let $s \in \mathbb{R}$ and $\omega \in (0, \frac{\pi}{2})$. Let T be a closed ω-bisectorial operator of type S_ω. Then, define the space $\dot{\mathcal{H}}_T^s$ to be the completion of $\mathcal{H}_T$ under the quadratic norm

$$\|f\|_{T,s}^2 := \int_0^\infty t^{-2s}\|\psi_t(T)f\|_2^2 \frac{\mathrm{d}t}{t}$$

where $\psi \in \Psi(S_\mu^o)$ such that $z^s\psi \in \Psi(S_\mu^o)$ and $\mu \in (\omega, \frac{\pi}{2})$.

Note, by Theorem 4.1 in [8] we have that different choices of ψ and μ give rise to equivalent norms. We also have T is a natural isomorphism between $\dot{\mathcal{H}}_T^s$ and $\dot{\mathcal{H}}_T^{s-1}$.

Proposition 8.3.2. Let $s \in \mathbb{R}$. Let T be a closed ω-bisectorial operator. Then T extends to an isomorphism from $\dot{\mathcal{H}}_T^s$ to $\dot{\mathcal{H}}_T^{s-1}$.

Proof. Similar to Proposition 5.2 in [8] or Proposition 6.4.1 in [31]. □

The following interpolation result is from [8, Theorem 5.3].

Proposition 8.3.3. Let $s, t \in \mathbb{R}$, with $s \neq t$. Let T be a closed ω-bisectorial operator. Let $\alpha \in (0, 1)$. Then

$$\dot{\mathcal{H}}_T^{s+\alpha(t-s)} = (\dot{\mathcal{H}}_T^s, \dot{\mathcal{H}}_T^t)_\alpha$$

Another important result is that the intersection of two fractional Sobolev spaces is dense in the fractional Sobolev spaces in the intermediate regularities.

Proposition 8.3.4. Let $s, t \in \mathbb{R}$, with $s < t$. Let T be a closed ω-bisectorial operator. Then $\dot{\mathcal{H}}_T^s \cap \dot{\mathcal{H}}_T^t$ is dense in $\dot{\mathcal{H}}_T^\alpha$ for all $\alpha \in (s, t)$.

Proof. Note that by construction we have $\mathcal{H}_T$ is dense in $\mathcal{H}_T^s$ for all $s \in \mathbb{R}$, and by the interpolation in Proposition 8.3.3 we have $\dot{\mathcal{H}}_T^s \cap \dot{\mathcal{H}}_T^t \subseteq \dot{\mathcal{H}}_T^\alpha$. Thus $\dot{\mathcal{H}}_T \subseteq \dot{\mathcal{H}}_T^s \cap \dot{\mathcal{H}}_T^t \subseteq \dot{\mathcal{H}}_T^\alpha$. The result follows by properties of closures. $\square$

From [8, Theorem 4.1 (ii)] we have the following

Proposition 8.3.5. Let $s \in \mathbb{R}$. Let T be a closed ω-bisectorial operator. If $\mu > \omega$ and $s \in \mathbb{R}$ then we have

$$\|f(T)u\|_{T,s} \leq C_{\mu,s}\|f\|_\infty \|u\|_{T,s} \quad \text{for all } f \in H^\infty(S_\mu^o) \text{ and for all } u \in \dot{\mathcal{H}}_T^s.$$

Now we have collected some of the important abstract results for fractional Sobolev spaces we turn our attention to the operator DB. We start by defining our base Hilbert space as $\mathcal{H} = \overline{\mathsf{R}(D)}$. As DB has a bounded holomorphic functional calculus we have $\mathcal{H}_{DB} = \overline{\mathsf{R}(D)}$. In particular, we obtain the following.

Corollary 8.3.6. The bounded holomorphic functional calculus for DB defined on $\overline{\mathsf{R}(D)}$ extends to a bounded holomorphic functional calculus on $\dot{\mathcal{H}}_{DB}^s$. In particular, we have the topological splitting

$$\dot{\mathcal{H}}_{DB}^s = \dot{\mathcal{H}}_{DB}^{s,+} \oplus \dot{\mathcal{H}}_{DB}^{s,-},$$

where $\dot{\mathcal{H}}_{DB}^{s,+} := \chi^+(DB)\dot{\mathcal{H}}_{DB}^s$ and $\dot{\mathcal{H}}_{DB}^{s,-} := \chi^-(DB)\dot{\mathcal{H}}_{DB}^s$.

Now define the Hilbert space $\mathcal{D}_{T,s}$ to be the completion of $\mathscr{D}(T^s)$ under the norm $\|T^s u\|_2$. If $s \in (-1,0)$ then by [8, Theorem 8.3] we have $\dot{\mathcal{H}}_{DB}^s = \mathcal{D}_{[D],s}$ with $\|u\|_{DB,s} \eqsim \|[D]^s u\|_2$. We also have the following lemma

Lemma 8.3.7. If $s \in [-1,0]$, then $\dot{\mathcal{H}}_{DB}^s = \dot{\mathcal{H}}_D^s$ with equivalent norms.

Proof. Similar to [7, Proposition 4.5 (4)]. $\square$

Lemma 8.3.8. We have $(\dot{\mathcal{H}}_D^s)_\perp = \dot{\mathcal{V}}^{s,2}(\mathbb{R}^n)$ and $(\dot{\mathcal{H}}_D^s)_r = \nabla_\mu H^{-\frac{1}{2}}\dot{\mathcal{V}}^{s,2}(\mathbb{R}^n)$.

Proof. Define the operator $U\colon L^2(\mathbb{R}^n;\mathbb{C}^2) \to \overline{\mathsf{R}(D)}$ by

$$U := \begin{bmatrix} I & 0 \\ 0 & -\nabla_\mu H^{-\frac{1}{2}} \end{bmatrix}.$$

Notice that U is an isometry and has inverse $U^{-1}\colon \overline{\mathsf{R}(D)} \to L^2(\mathbb{R}^n;\mathbb{C}^2)$ given by

$$U^{-1} := \begin{bmatrix} I & 0 \\ 0 & -H^{-\frac{1}{2}}(\nabla_\mu)^* \end{bmatrix}.$$

Moreover, recalling that $(\nabla_\mu)^*\nabla_\mu = H$ gives

$$\begin{aligned} U^{-1}DU &= \begin{bmatrix} I & 0 \\ 0 & -H^{-\frac{1}{2}}(\nabla_\mu)^* \end{bmatrix} \begin{bmatrix} 0 & -(\nabla_\mu)^* \\ -\nabla_\mu & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -\nabla_\mu H^{-\frac{1}{2}} \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ 0 & -H^{-\frac{1}{2}}(\nabla_\mu)^* \end{bmatrix} \begin{bmatrix} 0 & H^{\frac{1}{2}} \\ -\nabla_\mu & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & H^{\frac{1}{2}} \\ H^{\frac{1}{2}} & 0 \end{bmatrix}. \end{aligned}$$

Then $D = UTU^{-1}$ where $T\colon L^2(\mathbb{R}^n;\mathbb{C}^2) \to L^2(\mathbb{R}^n;\mathbb{C}^2)$ is given by

$$T := \begin{bmatrix} 0 & H^{\frac{1}{2}} \\ H^{\frac{1}{2}} & 0 \end{bmatrix}.$$

Then by Theorem 4.1 in [8] we have $\dot{\mathcal{H}}^s_D = U\dot{\mathcal{H}}^s_T$ and $\|u\|_{T,s} \eqsim \|Uu\|_{D,s}$ for all $u \in \dot{\mathcal{H}}^s_T$. Now as

$$T^2 = \begin{bmatrix} H & 0 \\ 0 & H \end{bmatrix}.$$

Then as T^2 is diagonal we have

$$[T]^s = (T^2)^{\frac{s}{2}} = \begin{bmatrix} H^{\frac{s}{2}} & 0 \\ 0 & H^{\frac{s}{2}} \end{bmatrix}.$$

As U is an isomorphism we have for all $f \in \dot{\mathcal{H}}^s_D$ there exists $u \in \dot{\mathcal{H}}^s_T$ such that $f = Uu$.

Now using the comparison in norm and Theorem 8.3 in [8], we have

$$\begin{aligned}
\|f\|_{D,s} &= \|Uu\|_{D,s} \\
&\eqsim \|u\|_{T,s} \\
&\eqsim \|[T]^s Uu\|_2 \\
&= \left\| \begin{bmatrix} H^{\frac{s}{2}} & 0 \\ 0 & H^{\frac{s}{2}} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right\| \\
&= \|H^{\frac{s}{2}} u_1\|_2 + \|H^{\frac{s}{2}} u_2\|_2.
\end{aligned}$$

Therefore, $f_\perp = u_1$ and $f_r = -\nabla_\mu H^{-\frac{1}{2}} u_2$ where $u_1, u_2 \in \dot{\mathcal{V}}^{s,2}(\mathbb{R}^n)$. That is $(\dot{\mathcal{H}}^s_D)_\perp = \dot{\mathcal{V}}^{s,2}(\mathbb{R}^n)$ and $(\dot{\mathcal{H}}^s_D)_r = \nabla_\mu H^{-\frac{1}{2}} \dot{\mathcal{V}}^{s,2}(\mathbb{R}^n)$, as required. $\square$

8.4 Energy Solutions

In this section we will construct the variational or energy solutions to the Schrödinger equation. This follows [7] which in turn is based on [4].

Definition 8.4.1. We say u is an energy solution of $-\operatorname{div} A\nabla u + aVu = 0$ in $\mathbb{R}^{n+1}_+$ if $u \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^{n+1}_+)$ and

$$\iint \mathcal{A}\nabla_\mu u \cdot \overline{\nabla_\mu \varphi} = 0,$$

for all $\varphi \in \mathcal{C}^\infty_c(\mathbb{R}^{n+1}_+)$. That is u is globally in $\dot{\mathcal{V}}^{1,2}(\mathbb{R}^{n+1}_+)$ and a weak solution.

We make the following definition for the trace of the conormal derivative, ∂_{ν_A}, in the case of energy solutions

Definition 8.4.2. Let $u \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^{n+1}_+)$ such that $H_{A,a,V}\, u = 0$. Then we define $\operatorname{Tr}(\partial_{\nu_A} u) \in \dot{\mathcal{V}}^{-\frac{1}{2},2}(\mathbb{R}^{n+1}_+)$ defined by

$$[\operatorname{Tr}(\partial_{\nu_A} u)](\varphi) := \iint \mathcal{A}\nabla_\mu u \cdot \overline{\nabla_\mu \Phi}$$

for all $\varphi \in \dot{\mathcal{V}}^{\frac{1}{2},2}(\mathbb{R}^{n+1}_+)$ and where Φ is an extension of φ.

We note that this definition is well-posed as by Lemma 8.2.5 the trace operator is surjective so there exists $\Phi \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^{n+1}_+)$ such that $\mathrm{Tr}(\Phi) = \varphi$. Also, for any $\Phi, \Phi' \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^{n+1}_+)$ such that $\mathrm{Tr}(\Phi) = \mathrm{Tr}(\Phi') = \varphi$, then $\mathrm{Tr}(\Phi - \Phi') = 0$ so $\Phi - \Phi' \in \dot{\mathcal{V}}^{1,2}_0(\mathbb{R}^{n+1}_+)$. Therefore, as $H_{A,a,V}\, u = 0$, we have

$$\iint \mathcal{A}\nabla u \cdot \overline{\nabla(\Phi - \Phi')} = 0.$$

That is $\mathrm{Tr}(\partial_{\nu_A} u)(\varphi)$ is independent of our choice of extension for φ.

We have the following propositions concerning the well-posedness of the boundary value problems for solutions in the energy class. We say u is an energy solution to the Neumann problem for data $\varphi \in \dot{\mathcal{V}}^{-\frac{1}{2},2}(\mathbb{R}^n)$ if:

$$\begin{cases} H_{A,a,V}\, u = 0, \\ u \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^{n+1}_+), \\ \mathrm{Tr}(\partial_{\nu_A} u) = -\varphi. \end{cases}$$

The following proposition states the Neumann problem with data in $\dot{\mathcal{V}}^{-\frac{1}{2},2}(\mathbb{R}^n)$ is well-posed.

Proposition 8.4.3. For each $\varphi \in \dot{\mathcal{V}}^{-\frac{1}{2},2}(\mathbb{R}^n)$, there exists a unique $u \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^{n+1}_+)$ such that $H_{A,a,V}\, u = 0$ and

$$[\mathrm{Tr}(\partial_{\nu_A} u)](\phi) = -\varphi(\phi) \quad \text{for all } \phi \in \dot{\mathcal{V}}^{\frac{1}{2},2}(\mathbb{R}^n).$$

That is u is an energy solution for the boundary data φ.

Proof. Let $\varphi \in \dot{\mathcal{V}}^{-\frac{1}{2},2}(\mathbb{R}^n)$. Define the linear functional $F_\varphi \colon \dot{\mathcal{V}}^{1,2}(\mathbb{R}^{n+1}_+) \to \mathbb{C}$, given by

$$F_\varphi v := -\varphi(\mathrm{Tr}\, v).$$

Then F_φ is a bounded linear functional since using Lemma 8.2.4, we have

$$|F_\varphi v| = |\varphi(\operatorname{Tr} v)| \leq \|\varphi\|_{\dot{\mathcal{V}}^{-\frac{1}{2},2}} \|\operatorname{Tr} v\|_{\dot{\mathcal{V}}^{\frac{1}{2},2}} \lesssim \|\varphi\|_{\dot{\mathcal{V}}^{-\frac{1}{2},2}} \|v\|_{\dot{\mathcal{V}}^{1,2}},$$

for all $v \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^{n+1}_+)$. As $J_\mathcal{A}$ is a coercive and bounded sesquilinear form on $\dot{\mathcal{V}}^{1,2}(\mathbb{R}^{n+1}_+)$, using the Lax–Milgram Theorem there exists a unique $u \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^{n+1}_+)$ such that

$$J_\mathcal{A}(u,v) = -\varphi(\operatorname{Tr} v) \quad \text{for all } v \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^{n+1}_+).$$

Now, recall by definition $[\operatorname{Tr}(\partial_{\nu_A} u)](\operatorname{Tr} v) = J_\mathcal{A}(u,v)$. Then, Lemma 8.2.5 gives

$$[\operatorname{Tr}(\partial_{\nu_A} u)](\phi) = -\varphi(\phi) \quad \text{for all } \phi \in \dot{\mathcal{V}}^{\frac{1}{2},2}(\mathbb{R}^n).$$

As required. □

We say u is an energy solution to the regularity problem for data $\varphi \in \dot{\mathcal{V}}^{\frac{1}{2},2}(\mathbb{R}^n)$ if:

$$\begin{cases} H_{A,a,V}\, u = 0, \\ u \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^{n+1}_+), \\ \nabla_\mu(\operatorname{Tr} u) = \nabla_\mu \varphi. \end{cases}$$

We have the analogous proposition for the well-posedness of the regularity problem.

Proposition 8.4.4. For each $f \in \dot{\mathcal{V}}^{\frac{1}{2},2}(\mathbb{R}^n)$, there exists a unique $u \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^{n+1}_+)$ such that $H_{A,a,V}\, u = 0$ and $\operatorname{Tr} u = f$ in $\dot{\mathcal{V}}^{-\frac{1}{2},2}(\mathbb{R}^n)$.

Proof. Now, let $f \in \dot{\mathcal{V}}^{\frac{1}{2},2}(\mathbb{R}^n)$. Then by Lemma 8.2.5 we have $e^{-t\sqrt{H}} f \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^{n+1}_+)$. Define the functional $F_f \colon \dot{\mathcal{V}}^{1,2}_0(\mathbb{R}^{n+1}_+) \to \mathbb{C}$

$$F_f(v) := -\iint_{\mathbb{R}^{n+1}_+} \mathcal{A}\nabla_\mu(e^{-t\sqrt{H}} f) \cdot \overline{\nabla_\mu v}.$$

Then, F_f is a bounded linear functional and since $J_\mathcal{A}$ is a coercive and bounded sesquilinear form on $\dot{\mathcal{V}}^{1,2}_0(\mathbb{R}^{n+1}_+) \subset \dot{\mathcal{V}}^{1,2}(\mathbb{R}^{n+1}_+)$. Therefore, by the Lax–Milgram Theorem there exists a unique $w \in \dot{\mathcal{V}}_0^{1,2}(\mathbb{R}^{n+1}_+)$ such that

$$J_{\mathcal{A}}(w, v) = -\iint_{\mathbb{R}^{n+1}_+} \mathcal{A}\nabla_\mu(e^{-t\sqrt{H}}f) \cdot \overline{\nabla_\mu v} \quad \text{for all } \varphi \in \dot{\mathcal{V}}_0^{1,2}(\mathbb{R}^{n+1}_+).$$

Now define $u = w + e^{-t\sqrt{H}}f$. Note, u is an energy solution and $\operatorname{Tr} u = f$ in $\dot{\mathcal{V}}^{\frac{1}{2},2}(\mathbb{R}^n)$. Suppose there exists another solution, $u' \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^{n+1}_+)$, for the boundary data f. Then, as $u - u' \in \dot{\mathcal{V}}_0^{1,2}(\mathbb{R}^{n+1}_+)$ and u and u' are energy solutions, we have

$$J_{\mathcal{A}}(u-u', u-u') = \iint_{\mathbb{R}^{n+1}_+} \mathcal{A}\nabla_\mu u \cdot \overline{\nabla_\mu(u-u')} - \iint_{\mathbb{R}^{n+1}_+} \mathcal{A}\nabla_\mu u' \cdot \overline{\nabla_\mu(u-u')} = 0$$

and so by the coercivity of $J_{\mathcal{A}}$, we have

$$\|u - u'\|_{\dot{\mathcal{V}}^{1,2}} \lesssim J_{\mathcal{A}}(u-u', u-u') = 0.$$

Thus, u is unique. $\square$

We define the Neumann to Dirichlet operator as

$$\Gamma^+_{ND} \colon (\dot{\mathcal{H}}_D^{-\frac{1}{2}})_\perp \to (\dot{\mathcal{H}}_D^{-\frac{1}{2}})_r \quad \text{by } \Gamma^+_{ND} f := \nabla_\mu(\operatorname{Tr} u),$$

where $u \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^{n+1}_+)$ is the unique energy solution with Neumann data f. Also, define the Dirichlet to Neumann operator

$$\Gamma^+_{DN} \colon (\dot{\mathcal{H}}_D^{-\frac{1}{2}})_r \to (\dot{\mathcal{H}}_D^{-\frac{1}{2}})_\perp \quad \text{by } \Gamma^+_{DN} g := \operatorname{Tr}(\partial_{\nu_A} v),$$

where $v \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^{n+1}_+)$ is the unique energy solution with regularity data g.

Proposition 8.4.5. The operator $\Gamma^+_{ND} \colon (\dot{\mathcal{H}}_D^{-\frac{1}{2}})_\perp \to (\dot{\mathcal{H}}_D^{-\frac{1}{2}})_r$ is bounded and invertible and has inverse $\Gamma^+_{DN} \colon (\dot{\mathcal{H}}_D^{-\frac{1}{2}})_r \to (\dot{\mathcal{H}}_D^{-\frac{1}{2}})_\perp$.

Proof. Let $f, f' \in (\dot{\mathcal{H}}_D^{-\frac{1}{2}})_\perp$ be such that $\Gamma^+_{ND} f = \Gamma^+_{ND} f'$. Then by definition $\Gamma^+_{ND} f = \nabla_\mu(\operatorname{Tr} u)$ and $\Gamma^+_{ND} f' = \nabla_\mu(\operatorname{Tr} u')$ where $u, u' \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^{n+1}_+)$ are the unique energy solutions with Neumann data f and f' respectively. By assumption $\nabla_\mu(\operatorname{Tr} u) = \Gamma^+_{ND} f = \Gamma^+_{ND} f' =$

$\nabla_\mu(\operatorname{Tr} u')$. That is, u and u' are energy solutions with the same regularity data. Thus, by the uniqueness property in Proposition 8.4.4 we have $u = u'$. Hence $f = f'$. That is $\Gamma^+_{ND}\colon (\dot{\mathcal{H}}_D^{-\frac{1}{2}})_\perp \to (\dot{\mathcal{H}}_D^{-\frac{1}{2}})_r$ is injective.

Now let $g \in (\dot{\mathcal{H}}_D^{-\frac{1}{2}})_r$. By Lemma 8.3.8 we have $(\dot{\mathcal{H}}_D^{-\frac{1}{2}})_r = \nabla_\mu(\dot{\mathcal{V}}^{\frac{1}{2},2}(\mathbb{R}^n))$. Then $g = \nabla_\mu \varphi$ for some $\varphi \in \dot{\mathcal{V}}^{\frac{1}{2},2}(\mathbb{R}^n)$. Then by Proposition 8.4.4 we have there exists a unique $u \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^{n+1}_+)$ with regularity data $\nabla_\mu\varphi$. That is $\nabla_\mu(\operatorname{Tr} u) = \nabla_\mu \varphi = g$. As $H_{A,a,V}\, u = 0$ then $\operatorname{Tr}(\partial_{\nu_A} u)$ exists and is well defined and $\operatorname{Tr}(\partial_{\nu_A} u) \in \dot{\mathcal{V}}^{-\frac{1}{2},2}(\mathbb{R}^n)$. Therefore, $\Gamma^+_{ND}[\operatorname{Tr}(\partial_{\nu_A} u)] = g$ and $\Gamma^+_{ND}\colon (\dot{\mathcal{H}}_D^{-\frac{1}{2}})_\perp \to (\dot{\mathcal{H}}_D^{-\frac{1}{2}})_r$ is surjective.

Note that the inverse is $\Gamma^+_{DN}\colon (\dot{\mathcal{H}}_D^{-\frac{1}{2}})_r \to (\dot{\mathcal{H}}_D^{-\frac{1}{2}})_\perp$. Thus, we have $\Gamma^+_{ND}\colon (\dot{\mathcal{H}}_D^{-\frac{1}{2}})_\perp \to (\dot{\mathcal{H}}_D^{-\frac{1}{2}})_r$ is an isomorphism $\square$

Now we have defined energy solutions we give a presentation in terms of semi-groups. This is based on [7, Proposition 4.7].

Proposition 8.4.6. Let $u \in \dot{\mathcal{V}}^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$. Then

1. If $u \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^n)$ with $H_{A,a,V}\, u = 0$ in $\mathbb{R}^{n+1}_+$ (u is an energy solution of $-\operatorname{div} A\nabla u + aVu = 0$), then there exists $f \in \dot{\mathcal{H}}^{-\frac{1}{2},+}_{DB}$ such that $\nabla_{\mathcal{A},\mu} u = e^{-tDB} f$.

2. If $f \in \dot{\mathcal{H}}^{-\frac{1}{2},+}_{DB}$, then there exists $u \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^n)$ with $H_{A,a,V}\, u = 0$ in $\mathbb{R}^{n+1}_+$ (u is an energy solution of $-\operatorname{div} A\nabla u + aVu = 0$) such that $\nabla_{\mathcal{A},\mu} u = e^{-tDB} f$.

Moreover, f is unique and $\|f\|_{DB,-\frac{1}{2}} \eqsim \|u\|_{\dot{\mathcal{V}}^{1,2}}$.

Proof. We prove (2) first. Let $u \in \dot{\mathcal{V}}^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$. Suppose there exists $f \in \dot{\mathcal{H}}^{-\frac{1}{2},+}_{DB}$ such that $\nabla_{\mathcal{A},\mu} u = e^{-tDB} f$. Define $f_\varepsilon := e^{-\varepsilon DB} f \in \dot{\mathcal{H}}^{-\frac{1}{2},+}_{DB}$ for all $\varepsilon > 0$. Then by [8, Theorem 8.3] and using the bounded holomorphic functional calculus of DB, we have

$$\|f_\varepsilon\|_{DB,\frac{1}{2}} = \varepsilon^{-\frac{1}{2}} \|[\varepsilon DB]^{\frac{1}{2}} e^{-\varepsilon DB} f\|_2 \lesssim \varepsilon^{-\frac{1}{2}} \|f\|_2 < \infty, \quad \text{for all } \varepsilon > 0.$$

Therefore, $f_\varepsilon \in \dot{\mathcal{H}}^{-\frac{1}{2},+}_{DB} \cap \dot{\mathcal{H}}^{\frac{1}{2},+}_{DB}$. Thus, by Proposition 8.3.4, we have $f_\varepsilon \in \dot{\mathcal{H}}^{s,+}_{DB}$ for all $s \in [-\frac{1}{2}, \frac{1}{2}]$. So, in particular $f_\varepsilon \in \dot{\mathcal{H}}^{0,+}_{DB} = \chi^+(DB)\overline{R(D)}$. Hence, by the Theorem 6.0.1 there exists a weak solution $u_\varepsilon \in \dot{\mathcal{V}}^{1,2}_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$ such that $e^{-tDB} f_\varepsilon = \nabla_{\mathcal{A},\mu} u_\varepsilon$. As DB is sectorial on $\dot{\mathcal{H}}^{-\frac{1}{2},+}_{DB}$ we have $\{e^{-\varepsilon DB}\}_{\varepsilon>0}$ is an analytic semigroup and thus $f_\varepsilon \to f$ as $\varepsilon \to 0$

in $\dot{\mathcal{H}}^{-\frac{1}{2},+}_{DB}$. Now, using the ellipticity of A, the extension of $DB\colon \dot{\mathcal{H}}^{\frac{1}{2},+}_{DB} \to \dot{\mathcal{H}}^{-\frac{1}{2},+}_{DB}$ being an isomorphism in Proposition 8.3.2, the property $\psi(DB)D = D\psi(BD)$, and Proposition 8.3.2 again, we have

$$
\begin{aligned}
\iint_{\mathbb{R}^{n+1}_+} |\nabla_\mu u - \nabla_\mu u_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t &\lesssim \int_0^\infty t\|\nabla_{\mathcal{A},\mu}(u-u_\varepsilon)\|_2^2 \frac{\mathrm{d}t}{t} \\
&= \int_0^\infty t\|e^{-tDB}(f-f_\varepsilon)\|_2^2 \frac{\mathrm{d}t}{t} \\
&= \int_0^\infty t\|e^{-tDB}DB(DB)^{-1}(f-f_\varepsilon)\|_2^2 \frac{\mathrm{d}t}{t} \\
&\le \int_0^\infty t^{-1}\|\psi(tDB)(DB)^{-1}(f-f_\varepsilon)\|_2^2 \frac{\mathrm{d}t}{t} \\
&\lesssim \|(DB)^{-1}(f-f_\varepsilon)\|_{DB,\frac{1}{2}} \\
&= \|f-f_\varepsilon\|_{DB,-\frac{1}{2}} \\
&\to 0, \quad \text{as } \varepsilon \to 0,
\end{aligned} \tag{8.4.1}
$$

where $\psi(z) = ze^{-z}$. Thus, $u_\varepsilon \to u$ as $\varepsilon \to 0$ in $\dot{\mathcal{V}}^{1,2}(\mathbb{R}^{n+1}_+)$. Let $\varphi \in \mathcal{C}^\infty_c(\mathbb{R}^{n+1}_+)$ not identically zero. Fix $\delta > 0$ arbitrarily. Let $\varepsilon > 0$ be such that $\|u-u_\varepsilon\|_{\dot{\mathcal{V}}^{1,2}} < \frac{\delta}{\|\nabla_\mu\varphi\|_2}$. Therefore, as u_ε is an energy solution and Cauchy–Schwarz inequality, we have

$$
\begin{aligned}
\left|\iint_{\mathbb{R}^{n+1}_+} \mathcal{A}\nabla_\mu u \cdot \overline{\nabla_\mu\varphi}\right| &\le \iint_{\mathbb{R}^{n+1}_+} |\mathcal{A}\nabla_\mu(u-u_\varepsilon)\cdot\overline{\nabla_\mu\varphi}| \\
&\lesssim \left(\iint_{\mathbb{R}^{n+1}_+} |\nabla_\mu(u-u_\varepsilon)|^2\right)^{\frac{1}{2}} \left(\iint_{\mathbb{R}^{n+1}_+} |\nabla_\mu\varphi|^2\right)^{\frac{1}{2}} \\
&< \delta.
\end{aligned}
$$

As $\delta > 0$ was arbitrary we have that $H_{A,a,V}\,u = 0$. Thus u is an energy solution.

Now we prove (1). Let u be an energy solution. Let $f \in \dot{\mathcal{H}}^{-\frac{1}{2}}_{DB}$ be defined by

$$
f := (\nabla_{\mathcal{A},\mu}u)(0) = \begin{bmatrix} \operatorname{Tr}(\partial_{\nu_A}u) \\ \nabla_\mu(\operatorname{Tr} u) \end{bmatrix}.
$$

Now, using Lemma 8.3.7, we have $\dot{\mathcal{H}}^{-\frac{1}{2}}_{D} = \dot{\mathcal{H}}^{-\frac{1}{2}}_{DB} = \dot{\mathcal{H}}^{-\frac{1}{2},+}_{DB} \oplus \dot{\mathcal{H}}^{-\frac{1}{2},-}_{DB}$, and so splitting f as follows $f = f^+ + f^-$, where $f^\pm \in \dot{\mathcal{H}}^{\pm\frac{1}{2}}_{DB}$. Then, by [8, Theorem 8.3] and using the bounded holomorphic functional calculus of DB, we have

$$\|e^{-tDB}f^+\|_{DB,\frac{1}{2}} = t^{-\frac{1}{2}}\||tDB|^{\frac{1}{2}}e^{-tDB}f^+\|_2 \lesssim \|f^+\|_2 < \infty, \quad \text{for all } t > 0.$$

That is $e^{-tDB}f^+ \in \dot{\mathcal{H}}^{0,+}_{DB}$ for all $t > 0$, and by the Theorem 6.0.1 there exists $u^+ \in \dot{\mathcal{V}}^{1,2}_{\text{loc}}(\mathbb{R}^{n+1}_+)$ such that $e^{-tDB}f^+ = \nabla_{\mathcal{A},\mu}u^+$ with $H_{A,a,V}\, u^+ = 0$ in $\mathbb{R}^{n+1}_+$. Also as $f^+ \in \dot{\mathcal{H}}^{-\frac{1}{2}}_{DB}$ a similar argument to (8.4.1) gives

$$\iint_{\mathbb{R}^{n+1}_+} |\nabla_\mu u^+|^2 \,\mathrm{d}x\,\mathrm{d}t < \infty.$$

That is $u^+ \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^{n+1}_+)$, and so is an energy solution. Using a similar argument there exists $u^- \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^{n+1}_-)$ such that $e^{-tDB}f^- = \nabla_{\mathcal{A},\mu}u^-$ with $H_{A,a,V}\, u^- = 0$ in $\mathbb{R}^{n+1}_-$. Now, define

$$v = \begin{cases} u - u^+ & \text{in } \mathbb{R}^{n+1}_+ \\ u^- & \text{in } \mathbb{R}^{n+1}_- \end{cases}.$$

Now notice by construction that u is the weak solution with Neumann data f and $u^\pm$ are the weak solutions with Neumann data $f^\pm$. Therefore

$$\begin{aligned}\iint_{\mathbb{R}^{n+1}} \mathcal{A}\nabla_\mu v \cdot \nabla_\mu \varphi &= \iint_{\mathbb{R}^{n+1}_+} \mathcal{A}\nabla_\mu u \cdot \nabla_\mu \varphi - \iint_{\mathbb{R}^{n+1}_+} \mathcal{A}\nabla_\mu u^+ \cdot \nabla_\mu \varphi + \iint_{\mathbb{R}^{n+1}_-} \mathcal{A}\nabla_\mu u^- \cdot \nabla_\mu \varphi \\ &= -f(\operatorname{Tr}\varphi) + f^+(\operatorname{Tr}\varphi) + f^-(\operatorname{Tr}\varphi) \\ &= -f(\operatorname{Tr}\varphi) + f(\operatorname{Tr}\varphi) \\ &= 0,\end{aligned}$$

for all $\varphi \in \mathcal{C}^\infty_c(\mathbb{R}^{n+1})$. Thus,

$$\iint_{\mathbb{R}^{n+1}} \mathcal{A}\nabla_\mu v \cdot \nabla_\mu \varphi = 0$$

for all $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^{n+1})$. Hence, by density and coercivity, we have

$$0 = \iint_{\mathbb{R}^{n+1}} \mathcal{A}\nabla_\mu v \cdot \nabla_\mu v \gtrsim \|\nabla_\mu v\|_2^2.$$

Thus, $v = 0$. That is $u^- = 0$ and $u = u^+$. Hence, $e^{-tDB}f = e^{-tDB}f^+ + e^{-tDB}f^- = \nabla_{\mathcal{A},\mu}u^+ + \nabla_{\mathcal{A},\mu}u^- = \nabla_{\mathcal{A},\mu}u$. This completes the proof. $\square$

We are now able to use the Neumann to Dirichlet and Dirichlet operators to describe the class $\dot{\mathcal{H}}_{DB}^{-\frac{1}{2},+}$.

Lemma 8.4.7. We have the characterisation

$$\dot{\mathcal{H}}_{DB}^{-\frac{1}{2},+} = \{h \in \dot{\mathcal{H}}_D^{-\frac{1}{2}} : h_r = \Gamma_{ND}^+ h_\perp\} = \{h \in \dot{\mathcal{H}}_D^{-\frac{1}{2}} : h_\perp = \Gamma_{DN}^+ h_r\}.$$

Proof. Let $f \in \dot{\mathcal{H}}_{DB}^{-\frac{1}{2},+}$. Then, by Proposition 8.4.6 we have there exists an energy solution $u \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}_+^{n+1})$ such that $e^{-tDB}f = \nabla_{\mathcal{A},\mu}u$. Thus $\Gamma_{ND}f_\perp = \nabla_\mu(\operatorname{Tr} u) = f_r$. Thus $\dot{\mathcal{H}}_{DB}^{-\frac{1}{2},+} \subseteq \{h \in \dot{\mathcal{H}}_D^{-\frac{1}{2}} : h_r = \Gamma_{ND}^+ h_\perp\}$.

Now let $g \in \{h \in \dot{\mathcal{H}}_D^{-\frac{1}{2}} : h_r = \Gamma_{ND}^+ h_\perp\}$. As $(\dot{\mathcal{H}}_D^{-\frac{1}{2}})_\perp = \dot{\mathcal{V}}^{-\frac{1}{2},2}(\mathbb{R}^n)$ then $g_\perp \in \dot{\mathcal{V}}^{-\frac{1}{2},2}(\mathbb{R}^n)$ and so by the well-posedness of energy solutions in Proposition 8.4.3, there exists a weak solution $u \in \dot{\mathcal{V}}^{1,2}(\mathbb{R}^n)$ with $\operatorname{Tr}(\partial_{\nu_A}u) = g$. Therefore, by definition the Neumann-to-Dirichlet operator we have $\Gamma_{ND}^+ g = \nabla_\mu(\operatorname{Tr} u)$. Also, by Proposition 8.4.6 there exists $f \in \dot{\mathcal{H}}_{DB}^{-\frac{1}{2},+}$ such that $e^{-tDB}f = \nabla_{\mathcal{A},\mu}u$. Thus

$$\begin{bmatrix} g \\ \Gamma_{ND}^+ g \end{bmatrix} = \begin{bmatrix} \operatorname{Tr}(\partial_{\nu_A}u) \\ \nabla_\mu(\operatorname{Tr} u) \end{bmatrix} = f.$$

That is $[g, \Gamma_{ND}^+ g]^T \in \dot{\mathcal{H}}_{DB}^{-\frac{1}{2},+}$.

The second equality is proven in a similar way. $\square$

Then as for $f \in \dot{\mathcal{H}}_{DB}^{-\frac{1}{2},+}$ we have $\operatorname{sgn}(DB)h = h$ and $h_r = \Gamma_{ND}^+ h_\perp$. Therefore, combining the two characterisations gives,

$$
\begin{aligned}
h &= \operatorname{sgn}(DB)h \\
&= \begin{bmatrix} s_{\perp\perp}(DB) & s_{\perp r}(DB) \\ s_{r\perp}(DB) & s_{rr}(DB) \end{bmatrix} \begin{bmatrix} h_\perp \\ \Gamma^+_{ND} h_\perp \end{bmatrix} \\
&= \begin{bmatrix} s_{\perp\perp}(DB)h_\perp + s_{\perp r}(DB)\Gamma^+_{ND} h_\perp \\ s_{r\perp}(DB)h_\perp + s_{rr}(DB)\Gamma^+_{ND} h_\perp \end{bmatrix}.
\end{aligned}
$$

Thus, equating the first component followed by equating the second component gives

$$
\Gamma^+_{ND} = s_{\perp r}(DB)^{-1}(1 - s_{\perp\perp}(DB)) = (I - s_{rr}(DB))^{-1} s_{r\perp}(DB) \tag{8.4.2}
$$

A similar argument gives

$$
\Gamma^+_{DN} = (1 - s_{\perp\perp}(DB))^{-1} s_{\perp r}(DB) = s_{r\perp}(DB)^{-1}(I - s_{rr}(DB)) \tag{8.4.3}
$$

8.5 Compatible Well-Posedness

In this section we prove the compatible well-posedness of the purely electric Schrödinger equation.

Definition 8.5.1. We say $(\mathcal{N})^{\mathcal{A}}_{L^2}$ (or $(\mathcal{R})^{\mathcal{A}}_{L^2}$) is compatibly well-posed if $(\mathcal{N})^{\mathcal{A}}_{L^2}$ (or $(\mathcal{R})^{\mathcal{A}}_{L^2}$) is well-posed and the solution agrees with the energy solution when the data is in $L^2(\mathbb{R}^n) \cap \dot{\mathcal{V}}^{-\frac{1}{2},2}(\mathbb{R}^n)$ (or $\dot{\mathcal{V}}^{1,2}(\mathbb{R}^n) \cap \dot{\mathcal{V}}^{\frac{1}{2},2}(\mathbb{R}^n)$).

We start by introducing the following decomposition of $\operatorname{sgn}(DB)$,

$$
E_{DB} := \operatorname{sgn}(DB) = \begin{bmatrix} s_{\perp\perp}(DB) & s_{\perp r}(DB) \\ s_{r\perp}(DB) & s_{rr}(DB) \end{bmatrix}
$$

Using the identity $\chi^{\pm}(DB) = \frac{1}{2}(I \pm \operatorname{sgn}(DB))$ we obtain

$$E^{\pm}_{DB} \coloneqq \chi^{\pm}(DB) = \frac{1}{2}\begin{bmatrix} I \pm s_{\perp\perp}(DB) & \pm s_{\perp r}(DB) \\ \pm s_{r\perp}(DB) & I \pm s_{rr}(DB) \end{bmatrix}.$$

We are now ready to explore the relationship between the above representation of $E^{\pm}_{DB}$ and the well-posedness of boundary value problems.

Proposition 8.5.2. $(\mathcal{R})^{\mathcal{A}}_{L^2}$ is well-posed if and only if $s_{r\perp}(DB)\colon (\dot{\mathcal{H}}^0_D)_\perp \to (\dot{\mathcal{H}}^0_D)_r$ is an isomorphism. Also, $(\mathcal{N})^{\mathcal{A}}_{L^2}$ is well-posed if and only if $s_{\perp r}(DB)\colon (\dot{\mathcal{H}}^0_D)_r \to (\dot{\mathcal{H}}^0_D)_\perp$ is an isomorphism.

Proof. Note that $s_{r\perp}(DB) = 2\Phi_R\chi^+(DB)\mathbb{P}_\perp$, where Φ_R is as in 6.0.1. Then as Φ_R and $\mathbb{P}_\perp$ are a pair of complimentary projection, as are $\chi^{\pm}(DB)$. Therefore, by [9, Lemma 13.6] we have $2\Phi_R\chi^+(DB)\mathbb{P}_\perp$ is an isomorphism if and only if both $\Phi_R\chi^{\pm}\colon \dot{\mathcal{H}}^0_D \to (\dot{\mathcal{H}}^0_D)_r$ are isomorphisms. This is true by the well-posedness of $(\mathcal{R})^{\mathcal{A}}_{L^2}$ in Theorem 6.0.1. Hence, $s_{r\perp}(DB)\colon (\dot{\mathcal{H}}^0_D)_\perp \to (\dot{\mathcal{H}}^0_D)_r$ is an isomorphism.

The Neumann problem is similar. □

We have the same when the solutions are energy solutions.

Proposition 8.5.3. $s_{r\perp}(DB)\colon (\dot{\mathcal{H}}^{-\frac{1}{2}}_D)_\perp \to (\dot{\mathcal{H}}^{-\frac{1}{2}}_D)_r$ and $s_{\perp r}(DB)\colon (\dot{\mathcal{H}}^{-\frac{1}{2}}_D)_r \to (\dot{\mathcal{H}}^{-\frac{1}{2}}_D)_\perp$ are both invertible.

Proof. As the energy problem is always well-posed then the proof is the same as Proposition 8.5.2. □

We now give a condition for compatibly well-posedness of the Neumann problem in terms of s_{rr}.

Proposition 8.5.4. We have the following:

1. If $I - s_{rr}\colon (\dot{\mathcal{H}}^{0,+}_D)_r \to (\dot{\mathcal{H}}^{0,+}_D)_r$ is invertible and the inverse agrees with when $I - s_{rr}$ is restricted to $(\dot{\mathcal{H}}^{-\frac{1}{2},+}_D)_r$, then $(\mathcal{N})^{\mathcal{A}}_{L^2}$ is compatibly well-posed.

2. If $I - s_{\perp\perp} \colon (\dot{\mathcal{H}}_D^{0,+})_\perp \to (\dot{\mathcal{H}}_D^{0,+})_\perp$ is invertible and the inverse agrees with $I - s_{\perp\perp}$ when restrited to $(\dot{\mathcal{H}}_D^{-\frac{1}{2},+})_\perp$ then $(\mathcal{R})_{L^2}^{\mathcal{A}}$ is compatibly well-posed.

Proof. Define the operator

$$T \colon (\dot{\mathcal{H}}_D^{0,+})_\perp \cap (\dot{\mathcal{H}}_D^{-\frac{1}{2},+})_\perp \to \dot{\mathcal{H}}_D^{0,+} \cap \dot{\mathcal{H}}_D^{-\frac{1}{2},+} \quad \text{given by } Th := \begin{bmatrix} h \\ \Gamma_{DB}^+ h \end{bmatrix}.$$

Note, as $I - s_{rr} \colon (\dot{\mathcal{H}}_D^{0,+})_\perp \to (\dot{\mathcal{H}}_D^{0,+})_\perp$ is invertible and the inverse agrees with the case when on $(\dot{\mathcal{H}}_D^{-\frac{1}{2},+})_\perp$ then by (8.4.2) we have Γ_{ND}^+, and therefore T, are well defined on $(\dot{\mathcal{H}}_D^{0,+})_\perp \cap (\dot{\mathcal{H}}_D^{-\frac{1}{2},+})_\perp$. Clearly $\Phi_N Th = h$. Let $f \in \dot{\mathcal{H}}_D^{0,+} \cap \dot{\mathcal{H}}_D^{-\frac{1}{2},+}$. Then by Lemma 8.4.7 we have $f = [f_\perp, \Gamma_{ND}^+ f_\perp]^T$. Therefore, $T\Phi_N f = f$. That is, $\Phi_N \colon \dot{\mathcal{H}}_D^{0,+} \cap \dot{\mathcal{H}}_D^{-\frac{1}{2},+} \to (\dot{\mathcal{H}}_D^{0,+})_\perp \cap (\dot{\mathcal{H}}_D^{-\frac{1}{2},+})_\perp$ is an isomophism with inverse T. Then $(\mathcal{N})_{L^2}^{\mathcal{A}}$ is compatibly well-posed.

The case for the regularity problem is proved similarly to the Neumann case. □

We are now in a position to prove compatible well-posedness results. We start with the case when $\mathcal{A}$ is block-type.

Theorem 8.5.5. If $\mathcal{A}$ is block-type, then $(\mathcal{N})_{L^2}^{\mathcal{A}}$ and $(\mathcal{R})_{L^2}^{\mathcal{A}}$ are both compatibly well-posed.

Proof. Since $E_{DB}N + NE_{DB} = 0$ and $E_{DB}^2 = I$ we must have

$$E_{DB} = \operatorname{sgn}(DB) = \begin{bmatrix} 0 & s_{r\perp}(DB)^{-1} \\ s_{r\perp}(DB) & 0 \end{bmatrix}. \tag{8.5.1}$$

Therefore $I - s_{\perp\perp}(DB) = I - s_{rr}(DB) = I$ which is invertible and agree with the inverses on $(\dot{\mathcal{H}}_D^{-\frac{1}{2},+})_\perp$ and $(\dot{\mathcal{H}}_D^{-\frac{1}{2},+})_r$ respectively. Thus $(\mathcal{N})_{L^2}^{\mathcal{A}}$ and $(\mathcal{R})_{L^2}^{\mathcal{A}}$ are both compatibly well-posed. □

We now consider the more general structure of lower triangular matrices. The proof uses the interpolation of the $\dot{\mathcal{H}}_{DB}^s$ spaces even in the case for just well-posedness. Therefore, both well-posedness sand compatible well-posedness are new for lower triangular matrices. The case of the Laplace equation was proven in [7] but we adapt the proof of [6].

Theorem 8.5.6. If $\mathcal{A}$ is lower triangular, then $(\mathcal{N})^{\mathcal{A}}_{L^2}$ is compatibly well-posed, and therefore, also well-posed.

Proof. To prove that $(\mathcal{N})^{\mathcal{A}}_{L^2}$ is compatibly well-posed we will use Proposition 8.5.2 and prove $s_{\perp r}(DB)\colon (\dot{\mathcal{H}}^0_D)_r \to (\dot{\mathcal{H}}^0_D)_\perp$ is an isomorphism. Let $\mathcal{A}_B$ be the diagonal block matrix associated with $\mathcal{A}$. That is

$$\mathcal{A}_B := \begin{bmatrix} A_{\perp\perp} & 0 & 0 \\ 0 & A_{\|\|} & 0 \\ 0 & 0 & a \end{bmatrix} \quad \text{then } B_B := \widehat{\mathcal{A}}_B = \begin{bmatrix} A_{\perp\perp}^{-1} & 0 & 0 \\ 0 & A_{\|\|} & 0 \\ 0 & 0 & a \end{bmatrix}.$$

Then DB_B acts independently on the $\perp$-components and r-components. Therefore, using Proposition 8.3.2 we have

$$DB_B\colon (\dot{\mathcal{H}}^s_{DB_B})_\perp \oplus \{0\} \to \{0\} \oplus (\dot{\mathcal{H}}^{s-1}_{DB_B})_r, \text{ and} \tag{8.5.2}$$

$$DB_B\colon \{0\} \oplus (\dot{\mathcal{H}}^s_{DB_B})_r \to (\dot{\mathcal{H}}^{s-1}_{DB_B})_\perp \oplus \{0\}, \tag{8.5.3}$$

are isomorphisms. Then $s_{\perp r}(DB)\colon (\dot{\mathcal{H}}^0_D)_r \to (\dot{\mathcal{H}}^0_D)_\perp$ being an isomorphism is equivalent to

$$T_r(DB)\colon (\dot{\mathcal{H}}^0_D)_r \to (\dot{\mathcal{H}}^{-1}_D)_r \quad \text{given by } T_r f = \Phi_R DB_B \begin{bmatrix} s_{\perp r}(DB)f \\ 0 \end{bmatrix}$$

being an isomorphism. Now by Proposition 8.5.3 we have

$$T_r(DB)\colon (\dot{\mathcal{H}}^{-\frac{1}{2}}_D)_r \to (\dot{\mathcal{H}}^{-\frac{3}{2}}_D)_r \quad \text{given by } T_r f = \Phi_R DB_B \begin{bmatrix} s_{\perp r}(DB)f \\ 0 \end{bmatrix}$$

is an isomorphism. Now for $f \in (\dot{\mathcal{H}}_D^{-\frac{1}{2}})_r \cap (\dot{\mathcal{H}}_D^{\frac{1}{2}})_r$ we have

$$\begin{aligned}
T_r(DB)f &= \Phi_R DB_B \begin{bmatrix} s_{\perp r}(DB)f \\ 0 \end{bmatrix} \\
&= \Phi_R DB_B \mathbb{P}_r \operatorname{sgn}(DB) \begin{bmatrix} 0 \\ f \end{bmatrix} \\
&= \Phi_R DB \operatorname{sgn}(DB) \begin{bmatrix} 0 \\ f \end{bmatrix} \\
&= \Phi_R \operatorname{sgn}(DB) DB \begin{bmatrix} 0 \\ f \end{bmatrix} \\
&= \Phi_R \operatorname{sgn}(DB) \mathbb{P}_\perp DB_B \begin{bmatrix} 0 \\ f \end{bmatrix} \\
&= s_{r\perp}(DB) \Phi_N DB_B \begin{bmatrix} 0 \\ f \end{bmatrix}.
\end{aligned}$$

Then, as $\Phi_N DB_B \begin{bmatrix} 0 \\ f \end{bmatrix} \in (\dot{\mathcal{H}}_D^{-\frac{1}{2}})_\perp$, using Proposition 8.5.3 and density we have that the operator $T_r(DB) \colon (\dot{\mathcal{H}}_D^{\frac{1}{2}})_r \to (\dot{\mathcal{H}}_D^{-\frac{1}{2}})_r$ is an isomorphism. Now using the boundedness of $T_r(DB)$ on both spaces and the bounded inverse Theorem we have

$$\|T_r(DB)f\|_{(\dot{\mathcal{H}}_{DB_B}^{-\frac{3}{2}})_r} + \|T_r(DB)f\|_{(\dot{\mathcal{H}}_{DB_B}^{-\frac{1}{2}})_r} \eqsim \|f\|_{(\dot{\mathcal{H}}_{DB_B}^{-\frac{1}{2}})_r} + \|f\|_{(\dot{\mathcal{H}}_{DB_B}^{\frac{1}{2}})_r}, \tag{8.5.4}$$

for all $f \in (\dot{\mathcal{H}}_D^{-\frac{1}{2}})_r \cap (\dot{\mathcal{H}}_D^{\frac{1}{2}})_r$. Define $\mathcal{A}_\tau := \tau \mathcal{A}_B + (1-\tau)\mathcal{A}$ for $\tau \in [0,1]$. Note that for $\tau \in (0,1]$ then $\mathcal{A}_\tau$ is lower triangular and so by similar reasoning we have that $T_r(DB_\tau)$ is both bounded and bounded below as in (8.5.4). We know $T_r(DB_B) \colon (\dot{\mathcal{H}}_D^{\frac{1}{2}})_r \cap (\dot{\mathcal{H}}_D^{-\frac{1}{2}})_r \to (\dot{\mathcal{H}}_D^{-\frac{1}{2}})_r \cap (\dot{\mathcal{H}}_D^{-\frac{3}{2}})_r$ is injective. To see that $T_r(DB_B)$ is surjective, recall (8.5.1), and so we

have

$$\begin{bmatrix} 0 \\ T_R(DB_B)f \end{bmatrix} = DB_B \operatorname{sgn}(DB_B) \begin{bmatrix} 0 \\ f \end{bmatrix}.$$

For $g \in (\dot{\mathcal{H}}_D^{-\frac{1}{2}})_r \cap (\dot{\mathcal{H}}_D^{-\frac{3}{2}})_r$ we have $g \in (\dot{\mathcal{H}}_D^{-1})_r$ and so we may define

$$\begin{bmatrix} 0 \\ f \end{bmatrix} := (DB_B)^{-1} \operatorname{sgn}(DB_B) \begin{bmatrix} 0 \\ g \end{bmatrix}.$$

Then $T_r(DB_B)f = g$. That is $T_r(DB_B) \colon (\dot{\mathcal{H}}_D^{\frac{1}{2}})_r \cap (\dot{\mathcal{H}}_D^{-\frac{1}{2}})_r \to (\dot{\mathcal{H}}_D^{-\frac{1}{2}})_r \cap (\dot{\mathcal{H}}_D^{-\frac{3}{2}})_r$ is surjective. Thus, by the method of continuity we have $T_r(DB) \colon (\dot{\mathcal{H}}_D^{\frac{1}{2}})_r \cap (\dot{\mathcal{H}}_D^{-\frac{1}{2}})_r \to (\dot{\mathcal{H}}_D^{-\frac{1}{2}})_r \cap (\dot{\mathcal{H}}_D^{-\frac{3}{2}})_r$ is also an isomorphism.

Then, by Lemma 8.3.4, we have $(\dot{\mathcal{H}}_D^{\frac{1}{2}})_r \cap (\dot{\mathcal{H}}_D^{-\frac{1}{2}})_r$ is dense in $(\dot{\mathcal{H}}_D^{0})_r$ and $(\dot{\mathcal{H}}_D^{-\frac{1}{2}})_r \cap (\dot{\mathcal{H}}_D^{-\frac{3}{2}})_r$ is dense in $(\dot{\mathcal{H}}_D^{-1})_r$. Let $g \in (\dot{\mathcal{H}}_D^{-1})_r$ then there exists $\{g_n\} \subset (\dot{\mathcal{H}}_D^{-\frac{1}{2}})_r \cap (\dot{\mathcal{H}}_D^{-\frac{3}{2}})_r$ converging to g. Then as $T_r(DB) \colon (\dot{\mathcal{H}}_D^{\frac{1}{2}})_r \cap (\dot{\mathcal{H}}_D^{-\frac{1}{2}})_r \to (\dot{\mathcal{H}}_D^{-\frac{1}{2}})_r \cap (\dot{\mathcal{H}}_D^{-\frac{3}{2}})_r$ is an isomorphism so there exists $\{f_n\} \subset (\dot{\mathcal{H}}_D^{\frac{1}{2}})_r \cap (\dot{\mathcal{H}}_D^{-\frac{1}{2}})_r$ such that $f_n = T_r(DB)^{-1} g_n$. Then

$$\|f_n - f_m\|_{DB,0} = \|T_r(DB)^{-1}(g_n - g_m)\|_{DB,0} \lesssim \|g_n - g_m\|_{DB,-1} \to 0,$$

as $\{g_n\}$ is convergent and so Cauchy. Thus, $\{f_n\}$ is a Cauchy sequence and as $(\dot{\mathcal{H}}_D^{0})_r$ is complete then $\{f_n\}$ is convergent. Let $f := \lim_{n\to\infty} f_n$. Then, for arbitrary $n \in \mathbb{N}$, we have

$$\begin{aligned} \|T_r(DB)f - g\|_{DB,-1} &\leq \|T_r(DB)(f - f_n)\|_{DB,-1} + \|T_r(DB)f_n - g\|_{DB,-1} \\ &\lesssim \|f - f_n\|_{DB,0} + \|g_n - g\|_{DB,-1}. \end{aligned}$$

Note that this converges to 0 as $n \to \infty$. Thus, $T_r(DB)f = g$. Thus, $T_r(DB) \colon (\dot{\mathcal{H}}_D^{0})_r \to (\dot{\mathcal{H}}_D^{-1})_r$ is an isomorphism. As required. $\square$

Theorem 8.5.7. If $\mathcal{A}$ is upper triangular, then $(\mathcal{R})_{L^2}^{\mathcal{A}}$ is compatibly well-posed.

Proof. Similar to the lower triangular case where we use

$$T_\perp\colon (\dot{\mathcal{H}}_D^{-\frac{1}{2}})_\perp \to (\dot{\mathcal{H}}_D^{-\frac{3}{2}})_\perp \quad \text{given by } T_\perp f = \Phi_N D B_B \begin{bmatrix} 0 \\ s_{r\perp}(DB)f \end{bmatrix},$$

instead. □

We also consider the self-adjoint case.

Theorem 8.5.8. If $\mathcal{A}$ is self-adjoint, then $(\mathcal{N})^{\mathcal{A}}_{L^2}$ and $(\mathcal{R})^{\mathcal{A}}_{L^2}$ are both compatibly well-posed.

Proof. Define the self-adjoint matrix $\mathcal{A}_\tau = \tau\mathcal{A}+(1-\tau)I$ for $\tau \in [0,1]$ and define $B_\tau = \widehat{\mathcal{A}}_\tau$. Then from Section 6.3 we have B_τ is uniformly elliptic on $\overline{R(D)}$ and uniformly bounded for all $\tau \in [0,1]$. We also have the Rellich estimate

$$\|f\|_2 \eqsim \|f_\perp\|_2 \eqsim \|f_r\|_2 \quad \text{for all } f \in \dot{\mathcal{H}}^{0,+}_{DB_\tau},$$

where the constants are independent of $\tau \in [0,1]$. Therefore, the bounded linear operators

$$\begin{aligned} \Phi_N^{0,\tau}\colon \dot{\mathcal{H}}^{0,+}_{DB_\tau} &\to (\dot{\mathcal{H}}^0_{DB_\tau})_\perp \quad \text{given by } \Phi_N^{0,\tau} f := f_\perp, \\ \Phi_R^{0,\tau}\colon \dot{\mathcal{H}}^{0,+}_{DB_\tau} &\to (\dot{\mathcal{H}}^0_{DB_\tau})_r \quad \text{given by } \Phi_R^{0,\tau} f := f_r, \end{aligned}$$

are bounded below. Also, from the well-posedness in the energy class we have the bounded linear operators

$$\begin{aligned} \Phi_N^{-\frac{1}{2},\tau}\colon \dot{\mathcal{H}}^{-\frac{1}{2}+}_{DB_\tau} &\to (\dot{\mathcal{H}}^{-\frac{1}{2}}_{DB_\tau})_\perp \quad \text{given by } \Phi_N^{-\frac{1}{2},\tau} f := f_\perp, \\ \Phi_R^{-\frac{1}{2},\tau}\colon \dot{\mathcal{H}}^{-\frac{1}{2},+}_{DB_\tau} &\to (\dot{\mathcal{H}}^{-\frac{1}{2}}_{DB_\tau})_r \quad \text{given by } \Phi_R^{-\frac{1}{2},\tau} f := f_r, \end{aligned}$$

are isomorphisms for all $\tau \in [0,1]$. Thus, by the bounded inverse Theorem we have $\Phi_N^{-\frac{1}{2},\tau}$ and $\Phi_R^{-\frac{1}{2},\tau}$ are bounded below.

We proceed with the Neumann problem. Define the bounded linear operator

$$\Phi_N^\tau \colon \dot{\mathcal{H}}^{0,+}_{DB_\tau} \cap \dot{\mathcal{H}}^{-\frac{1}{2},+}_{DB_\tau} \to (\dot{\mathcal{H}}^0_{DB_\tau})_\perp \cap (\dot{\mathcal{H}}^{-\frac{1}{2}}_{DB_\tau})_\perp \quad \text{given by } \Phi_N^\tau f \coloneqq f_\perp.$$

Note that Φ_N^τ is bounded below and so is injective. As $\dot{\mathcal{H}}^0_{DB_\tau} \cap \dot{\mathcal{H}}^{-\frac{1}{2}}_{DB_\tau}$ is dense in both $\dot{\mathcal{H}}^0_{DB_\tau}$ and $\dot{\mathcal{H}}^{-\frac{1}{2}}_{DB_\tau}$, then invertablity of Φ_N^τ is equivalent to compatibility of the inverses of $\Phi_N^{0,\tau}$ and $\Phi_N^{-\frac{1}{2},\tau}$. Note that as $\mathcal{A}_0 = I$ so by Theorem 8.5.5 we have Φ_N^0 is invertable. Then following the procedure in Section 6.3 we have $\Phi_N^\tau \colon \dot{\mathcal{H}}^{0,+}_{DB_\tau} \cap \dot{\mathcal{H}}^{-\frac{1}{2},+}_{DB_\tau} \to (\dot{\mathcal{H}}^0_{DB_\tau})_\perp \cap (\dot{\mathcal{H}}^{-\frac{1}{2}}_{DB_\tau})_\perp$ is an isomorphism. Hence, $(\mathcal{N})^{\mathcal{A}}_{L^2}$ is compatibly well-posed.

The regularity problem follows by a similar argument. $\square$

CHAPTER 9

CONCLUDING REMARKS

We will conclude by summarising some of the key results and then provide some possible generalisations of the work conducted in this thesis.

9.1 Summary of Key results

This thesis was focused on solving boundary value problems for the Schrödinger equation

$$H_{A,a,V}\, u(t,x) := -\operatorname{div}_{t,x} A(x)\nabla_{t,x} u(t,x) + a(x)V(x)u(t,x) = 0$$

on the upper half-space $\mathbb{R}^{n+1}_+ := \{(t,x) \in \mathbb{R} \times \mathbb{R}^n : t > 0\}$, for integers $n \geq 3$, where $A \in L^\infty(\mathbb{R}^{n+1}; \mathcal{L}(\mathbb{C}^{1+n}))$ and $a \in L^\infty(\mathbb{R}^{n+1}; \mathcal{L}(\mathbb{C}))$ are complex, t-independent, and elliptic, and where V is in the reverse Hölder class with exponent $\frac{n}{2}$. The main theorem is Theorem 6.0.1 which states that Neumann and Dirichlet regularity problems are well-posed if A and a are self-adjoint, or the A is of block-type; and the sets of matrices A and a for which the Neumann problem is well-posed and the Dirichlet regularity problem is well-posed, are both open.

To prove this theorem we showed that equation (1.0.1) is equivalent to the first-order system of equations

$$\partial_t F + DBF = 0,$$

where $D\colon \mathscr{D}(D) \subset L^2(\mathbb{R}^n; \mathbb{C}^{n+2}) \to L^2(\mathbb{R}^n; \mathbb{C}^{n+2})$ is a first-order operator associated with

$\Delta + V$, the unperturbed Schrödinger operator, $B\colon L^2(\mathbb{R}^n;\mathbb{C}^{n+2}) \to L^2(\mathbb{R}^n;\mathbb{C}^{n+2})$ is a multiplication operator derived from the perturbations A and a, and $F \in L^2_{\text{loc}}(\mathbb{R}_+;\overline{\mathsf{R}(D)})$, given by $F = \nabla_{\mathcal{A},\mu} u$. In this way, it was possible to solve boundary value problems for the operator $H_{A,a,V}$ by studying the associated initial value problem for the operator DB.

One of the most important properties proven in this thesis was Theorem 3.0.1, which implied that the operator DB has a bounded holomorphic functional calculus, which is equivalent to the following quadratic estimates

$$\int_0^\infty \|tDB(I + t^2 DBDB)^{-1}u\|_2^2 \frac{\mathrm{d}t}{t} \eqsim \|u\|_2^2, \quad \forall u \in \overline{\mathsf{R}(D)}.$$

Here we used a dyadic decomposition which differentiates between big and small cubes, at each scale, depending on the potential V. For small cubes, the arguments from [13] can be adapted. However, as both the reduction to the Carleson measure estimate and the stopping time argument needed to be done on small cubes, a new approach for big cubes was developed based on the Fefferman–Phong inequality.

We also adapted the methods we developed for the electric Schrödinger equation to the magnetic Schrödinger in Theorem 4.0.2. This was done by using a localisation argument on a more sophisticated collection of dyadic cubes which took advantage of the Iwatsuka Gauge transform and allowed the introduction of the magnetic field.

As DB is a bisectorial operator with a bounded holomorphic functional calculus on $\overline{\mathsf{R}(D)}$, it was possible to restrict to a subspace of $\overline{\mathsf{R}(D)}$ for which DB is sectorial. Therefore, DB generates an analytic semigroup $(e^{-tDB})_{t>0}$ on this subspace. We also proved this characterises the solutions to the first-order system. Next, it was shown that well-posedness of the boundary value problem is equivalent to showing that certain mappings into the boundary data space are invertible. We then proved these mappings are indeed invertible.

Another important result was Theorem 7.3.2, which allowed for the control of the non-tangential maximal function of the gradient of solutions

$$\int_0^\infty \left\| t\partial_t \begin{bmatrix} \nabla_{t,x} u \\ |V|^{\frac{1}{2}} u \end{bmatrix} \right\|_2^2 \frac{\mathrm{d}t}{t} \eqsim \|\varphi\|_2^2 \eqsim \left\| \widetilde{N}_* \left(\begin{bmatrix} \nabla_{t,x} u \\ |V|^{\frac{1}{2}} u \end{bmatrix} \right) \right\|_2^2,$$

where u is the solution to $H_{A,a,V}\, u = 0$ for the Neumann boundary data $\varphi \in L^2(\mathbb{R}^n)$. An analogous result holds for the Dirichlet regularity problem. The key proposition here was the weak reverse Hölder inequality of the gradient of solutions in Proposition 7.1.4.

The last key result was that these solutions are unique, in the sense that the solutions which arise from the first-order method are equivalent to the classical (energy) solutions which come from the Lax–Milgram Theorem. Since there is an adapted Sobolev space being used, new trace spaces for this problem were defined and analysed. Then, it was proven that the energy solutions can also be characterised in terms of a semigroup and this allowed us to use the bounded holomorphic functional calculus to prove that these solution coincide.

9.2 Further Work

A main focus for future work would be to consider the boundary value problems for the equation

$$(\nabla + ib)^* A(\nabla + ib)u + aVu = 0, \tag{9.2.1}$$

with both first-order and zeroth-order terms. This would be done by combining the projections from Chapter 3, which are used to proved the quadratic estimates for the purely electric Schrödinger operator, with the localisation onto the set of maximal dyadic cubes from Chapter 4. Once the quadratic estimates have been established then the solvability results and non-tangential maximal function estimates will be the next step. This will include: reducing from the second-order equation to another first-order system; the weak reverse Hölder estimates of the magnetic gradient $\nabla + ib$ of solutions to (9.2.1); and the trace space theory for the adapted Sobolev space $\dot{W}_b^{1,2}(\mathbb{R}^{n+1}_+)$ defined in Section

2.4.

A further generalisation is to prove the boundary value problem for systems of equations rather than a single equation. This would follow the original first-order paper [4] by Auscher, Axelsson, and McIntosh. The results would be new for all the equations considered in this thesis.

Another extension to this work, would be to allow for t-dependence in the matrix $\mathcal{A}$. This would involve adapting the work in [2] by Auscher and Axelsson to include the potentials V and b. In [2] the coefficients A satisfies Dahlberg's small Carleson condition. The methods build upon that of [4] by using the t-independent solution and Duhamel's principle to represent a solution to the t-dependent equation as an integral equation and then use operational calculus to estimate the integral operator.

A final idea for future work based on this thesis, would be to consider boundary data in L^p-spaces for $p \neq 2$. In [27], Frey, McIntosh, and Portal, establish L^p-conical square function estimates for perturbed Hodge–Dirac operators using methods developed from [13]. Here the L^p-conical square function estimate will take the place of the quadratic estimate. Therefore, we would need to adapt the work in [27] to accommodate the potential terms. Once this has been established then we would need to prove the solvability results and non-tangential maximal function estimates in the L^p setting. To do this we would follow the work of Auscher and Stahlhut in [11].

Thus, we see that this thesis can be considered as a prototype for considering other inhomogeneous boundary value problems with singular potentials.

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