AXIAL ALGEBRAS OF MONSTER TYPE $(2\eta, \eta)$

Vijay Joshi

A thesis submitted to
The University of Birmingham
for the degree of
DOCTOR OF PHILOSOPHY

School of Mathematics
Adviser: Prof. Sergey Shpectorov

January 2020

Abstract

Axial algebras are commutative nonassociative algebras generated by a set of special idempotents called axes whose adjoint maps are semisimple. The algebra product is controlled by a fusion law. Jordan algebras, the Griess algebra for the Monster group are some of the examples for axial algebras. This thesis is about study of axial algebras satisfying the Monster like fusion law where non unit elements α, β of the fusion law satisfy the condition $\alpha = 2\beta$. This was an exceptional case in the literature. We show that axes satisfying the above fusion law arise as the sum of two orthogonal axes of Jordan type. These are called the axes of Monster type $\mathcal{M}(2\eta, \eta)$ or the double axes. Then we present the classification of 2-generated subalgebras of the Matsuo algebras generated by double axes. Further we construct two infinite classes of axial algebras for the symmetric group of S_n and for symplectic group $Sp(2n, 2)$ which satisfy fusion law $\mathcal{M}(2\eta, \eta)$.

Acknowledgements

How to say "Thank you" to a great teacher because of whom I have enjoyed every phase of this research work? He is also my mentor whose moral support has helped me in difficult times. I have been really fortunate to learn under the supervision of Sergey Shpectorov towards whom I have immense gratitude and respect.

I thank Alexey Galt, Andrey Mamontov and Alexey Staroletov for the joint work during 2017-2018.

I would like to thank the School of Mathematics for generous support.

Many people have helped me ever since I started my PhD. I am really thankful to Mr. Ranganath Joshi who has been supporting me constantly. It may sound unnatural to formally thank your best friends, still, I want to thank Milind and Shrirang. I am grateful to Mr. Raghavendra Mathad and Mrs. Radhika Mathad who have always welcomed and taken care of me every time I visited Bangalore. I am looking forward to visit them again. My wife's parents, Mr. Narasimha Kalkeri and Mrs. Paddhu Kalkeri, have been really understanding and helping me throughout this research phase. I am very grateful to them.

I am immensely thankful to Mr. Mukund Konnur and his family, without whose support this research work would have been just a dream.

I want to thank my wife for the constant love, support, and encouragement.

Finally, it is my pleasure to thank my parents to whom I am forever indebted. Because of the various sacrifices you have made I have a great life today. You always have my perennial love and respect...

To Srimadh Teekacharyaru.

Contents

List of Tables

List of Figures

Chapter 1

Introduction

This is a study of commutative, non associative algebras in which the product of the subspaces are controlled by what is known as a fusion law and the algebras are called the *axial algebras*. First we give the historical background which describes the motivation for the origin of these algebras.

1.1 Historical context

In 1973, Fischer and Griess independently produced evidence for the existence of the largest sporadic simple group, the Monster. Conway and Norton conjectured that the Monster had a representation of degree 196883 [6]. In particular, Norton noted that such a module would have commutative, non associative algebra structure. Based on Norton's observation, Griess constructed the Monster group in 1982 [13]. He defined a unital commutative, non associative algebra structure on a module B of dimension 196884 for a particular group C, which admitted a symmetric, nondegenerate, associative bilinear form. Then he showed that the Monster group $M = \langle C, \sigma \rangle$ for some $\sigma \in Aut(B)$. The module B is called the Griess algebra.

In 1984, Conway constructed the Monster, using Parker's Moufang loop [5], as the automorphism group of 196884-dimensional algebra $B^\natural$ which is a deformation of the Griess algebra B. In particular, Conway showed the existence of special vectors called *axial vectors* in $B^\natural$.

From the character table of the Monster, it has two classes of involutions, namely, $2A$ and $2B$. Call the involutions of $2A$ the *transpositions*. The product of any two transpositions belongs to exactly one of the following nine classes the Monster.

$$1A,\ 2A,\ 3A,\ 4A,\ 5A,\ 6A,\ 2B,\ 4B, \text{or } 3C. \tag{1.1}$$

The symbol nA represents the class of elements of order n with the largest centraliser order. The next largest centraliser is nB and so on.

The axial vector of an element $m \in M$ is the vector $v \in B^\natural$ which is fixed by the centraliser, $C_M(m)$ of m. For each transposition a, let t_a be the axial vector. Conway showed that the subalgebra generated by two axial vectors t_a, t_b of transpositions a, b, is spanned by various axial vectors v_g, where g belongs to the dihedral group $\langle a, b \rangle$. This shows that, for transpositions a and b, the class of two generated subalgebras $\langle\langle t_a, t_b \rangle\rangle$ is in one to one correspondence with the class of the conjugacy classes given in (1.1). Therefore, it is meaningful to denote the two generated subalgebra $\langle\langle t_a, t_b \rangle\rangle$ by the same symbol as that of the conjugacy class of ab.

In 1984, Frenkel, Lepowsky and Meurman announced the construction a commutative, non associative algebra V, which is a slight variant of the Griess algebra, using vertex operators [9]. The character of V is given by the modular function $J(q) = q^{-1} + 0 + 196884q + \ldots$. Further, they obtained the action of the Monster group on V which was similar to the action of the Monster on the Griess algebra. Borcherds, in1986, introduced *vertex algebras* by defining a product, which was constructed using a generalisation of vertex operators, on the Fock spaces [3]. Building on this work, Frenkel et al. constructed the Griess algebra from vertex operator algebras (VOAs) in [10]. A VOA over the complex field is a non negatively graded vector space $V = \bigoplus_{n=0}^{\infty} V_n$ satisfying certain properties like existence of the identity element, the translation, the locality identity. Here V_n is the weight n subspace of V. The authors showed that the Monster can be realised as the automorphism group of the weight 2 subspace $V_2^\natural$ of the Moonshine VOA, $V^\natural = \bigoplus_{n=0}^{\infty} V_n^\natural$. The subspace $V_2^\natural$ has the structure of commutative, non associative algebra that coincides with the Griess algebra B. Note that, in the Moonshine VOA, $V_0^\natural = 0$ and $\dim V_1^\natural = 1$.

The algebra $V^\natural$ belongs to a larger class of VOA, $V = \bigoplus_{n=0}^{\infty} V_n$ satisfying $V_0 = 0$ and $\dim V_1 = 1$. In this class the weight 2 subspace V_2 always admits structure of commutative algebra similar to the Griess algebra. Therefore, V_2 is called a Griess algebra. Similar to Conway's axial vectors in $B^\natural$, Miyamoto identified the special vectors called *Ising vectors* in V_2 [23]. First, he defined a special element e, called a rational conformal vector, in V_2 of the VOA, $V = \bigoplus_{n=0}^{\infty} V_n$, which defines a representation of the Virasoro algebra with the central charge $1/2$. Then, he showed that, when $\dim V_0 = 0$ and $V_1 = 0$, the element e is a conformal vector with the central charge $1/2$ if and only if $e/2$ is an idempotent of Griess algebra V_2. The idempotent $e/2$ is called an Ising vector. Therefore, the axial vectors are in one to one correspondence with the Ising vectors. Consequently, a subalgebra generated by two Ising vectors is completely determined by the conjugacy classes listed in (1.1). Furthermore, associated to every Ising vector, there exists an automorphism τ_e of V_2 of order at most 2. Call these involutions *τ-involutions*. Note that τ-involutions are precisely the $2A$ involutions of the Monster, the transpositions. Thus, Miyamoto's construction of τ_e is reverse construction of Conway's axial vector.

Sakuma studied all the 2-generated groups by Miyamoto involutions. In particular, he showed that the τ-involutions of V_2 satisfy 6-transposition property, that is, for any two Ising vectors e and f, $|\tau_e\tau_f| \leq 6$. Further, he classified all two generated subalgebras $\langle\langle \tau_e, \tau_f \rangle\rangle$. This classification theorem is known as the Sakuma's theorem. From now on the two generated subalgebras of the Griess algebras will be called as *dihedral algebras*.

1.2 Majorana algebras

As an attempt to generalise the Griess algebra for the Monster, Ivanov axiomatised the properties of the Griess algebra, and defined *Majorana algebras* [20]. A Majorana algebra V is a vector space over reals equipped with a positive definite bilinear form $(\, , \,)$ and a commutative, non associative algebra product such that the form associates and satisfies the Norton inequality. He defined Majorana axes and Majorana involutions which are counterparts of Ising vectors and τ-involutions respectively. For instance, the Griess algebra, B, is a Majorana algebra generated by a set of Majorana axes, $2A$. Note that, with respect to the Majorana axis a, the Griess algebra

Dihedral algebras	Basis	Relations	Inner products
$2B$	$a_0,\ a_1$		$(a_0, a_1) = 0$
$2A$	$a_0,\ a_1,\ a_2$		$(a_i, a_j) = \frac{1}{8}$
$3C$	$a_0,\ a_1,\ a_2$		$(a_i, a_j) = \frac{1}{64}$
$3A$	$a_0,\ a_1,\ a_2,\ s$		$(a_i, a_j) = \frac{13}{256}$
$4A$	$a_{-1},\ a_0,\ a_1,\ a_2,\ s$	$\langle a_i, a_{i+2}\rangle \cong 2B$	$(a_i, a_{i+1}) = \frac{1}{32}$
$4B$	$a_{-1},\ a_0,\ a_1,\ a_2,\ s$	$\langle a_i, a_{i+2}\rangle \cong 2A$	$(a_i, a_{i+1}) = \frac{1}{64}$
$5A$	$a_{-2},\ a_{-1},\ a_0,\ a_1,\ a_2,\ s$		$(a_i, a_j) = \frac{3}{138}$
$6A$	$a_{-2},\ a_{-1},\ a_0,\ a_1,\ a_2,\ s,\ \bar{s}$	$\langle a_i, a_{i+3}\rangle \cong 2A$, $\langle a_i, a_{i+2}\rangle \cong 3A$	$(a_i, a_{i+1}) = \frac{5}{256}$

Table 1.1: Description of the dihedral subalgebras of the Griess algebra

B has the following decomposition.

$$B = B_1^{(a)} \oplus B_0^{(a)} \oplus B_{\frac{1}{2^2}}^{(a)} \oplus B_{\frac{1}{2^5}}^{(a)},$$

where $B_\lambda^{(a)}$ is the λ-eigenspace of the adjoint action. The products of the eigenspaces satisfy the fusion law given in Table 1.2.

$*$	1	0	$\frac{1}{2^2}$	$\frac{1}{2^5}$
1	1	$\emptyset$	$\frac{1}{2^2}$	$\frac{1}{2^5}$
0	$\emptyset$	0	$\frac{1}{2^2}$	$\frac{1}{2^5}$
$\frac{1}{2^2}$	$\frac{1}{2^2}$	$\frac{1}{2^2}$	$1, 0$	$\frac{1}{2^5}$
$\frac{1}{2^5}$	$\frac{1}{2^5}$	$\frac{1}{2^5}$	$\frac{1}{2^5}$	$1, 0, \frac{1}{2^2}$

Table 1.2: Fusion law satisfied by the Griess algebra

According to the fusion law,

$$B_{\frac{1}{2^2}}(a) B_{\frac{1}{2^2}}(a) \subseteq B_1(a) \oplus B_0(a),$$
$$B_1(a) B_{\frac{1}{2^5}}(a) \subseteq B_{\frac{1}{2^5}}(a),$$

and so on. Further, Ivanov et. al. proved the Sakuma's classification theorem in the context of Majorana algebras [19]. That is, a two generated subalgebra by Majorana axes is completely determined by the Conjugacy classes given in (1.1).

1.3 Axial algebras

The class of axial algebras was introduced by Hall, Rehren, and Shpectorov in [16] as a generalisation of the Majorana algebras. These are defined over an arbitrary field $\mathbb{F}$ with char $\mathbb{F} \neq 2$, existence of a bilinear form is not assumed (hence, the conditions like positive definite, Norton's inequality will not arise), fusion law is not restricted to Table 1.2.

We state a few definitions which are essential for the discussion of axial algebras.

Let A be a commutative, non associative algebra over a field $\mathbb{F}$ and char $\mathbb{F} \neq 2$. An adjoint map ad_a of an element $a \in A$ is an endomorphism of the algebra which takes x to ax. For $\lambda \in \mathbb{F}$,

$$A_\lambda(a) = \{x \in A, \mid ax = \lambda x\}$$

is the corresponding eigenspace. Note that $A_\lambda(a) = 0$ if λ is not an eigenvalue of the adjoint map ad_a. A fusion law is a pair $(\mathcal{F}, *)$, where $\mathcal{F} \subseteq \mathbb{F}$ and $*$ is a symmetric map,

$$* : \mathcal{F} \times \mathcal{F} \to 2^{\mathcal{F}},$$

which assigns to a pair (λ, μ) a subset of $\mathcal{F}$. A single instance $(\lambda, \mu) \mapsto S$ of the map $*$ is called a *fusion rule*. Fusion laws are described using tables whose cells are the fusion rules. Some of the fusions laws that appeared in the recent study of axial algebras are given in Table 1.3.

An axial algebra A is a commutative, non associative algebra over the field $\mathbb{F}$ generated by a set of special idempotents called *axes*. With respect to each axis a,

$*$	1	0
1	1	
0		0

$*$	1	0	η
1	1		η
0		0	η
η	η	η	$1, 0$

$*$	1	0	α	β
1	1		α	β
0		0	α	β
α	α	α	$1, 0$	β
β	β	β	β	$1, 0, \alpha$

Table 1.3: $\{0, 1\}$-, $\mathcal{M}(\eta)$-, $\mathcal{M}(\alpha, \beta)$-fusion laws.

the algebra A is the direct sum of the eigenspaces of ad_a such that the product of the eigenspaces is controlled by a fusion law $\mathcal{F}$. For example, the Griess algebra is an axial algebra generated by the set of $2A$-axes, and the product of the eigenspaces is controlled by the fusion law $\mathcal{M}(\frac{1}{4}, \frac{1}{32})$; a Jordan algebra is an axial algebra generated by a set of idempotents, and the product of the eigenspaces is controlled by the fusion law $\mathcal{M}(\frac{1}{2})$.

1.4 About this thesis

Hall, Rehren, and Shpectorov proved the Sakuma's theorem in the context of primitive axial algebras of Jordan type η which have a Frobenius form. In [26], Rehren attempted to prove the Sakuma's theorem for an arbitrary fusion law $\mathcal{M}(\alpha, \beta)$ without assuming the existence of a Frobenius form. He showed that for $\alpha \neq 2\beta$, $\alpha \neq 4\beta$, the dimension of the two generated subalgebra is at most 8 which is the highest dimension of a dihedral algebra. Further, he constructed two generated subalgebras, involving parameters α, β with $\alpha \neq 2\beta$, $\alpha \neq 4\beta$, which are covers of the dihedral algebras.

Description	Basis	Relations
$3A'_{\alpha,\beta}$	b_0, b_1, b_2, s	
$4A_\beta$	b_{-1}, b_0, b_1, b_2, s	$\langle b_i, b_{i+2}\rangle \cong 2B$
$4B_\alpha$	b_{-1}, b_0, b_1, b_2, s	$\langle b_i, b_{i+2}\rangle \cong 2A$
$5A_\alpha$	b_{-2}, b_{-1}, b_0, b_1, b_2, s	
$6A_\alpha$	b_{-2}, b_{-1}, b_0, b_1, b_2, s, $\bar{s}$	$\langle b_i, b_{i+3}\rangle \cong 3C_\alpha$, $\langle b_i, b_{i+2}\rangle \cong 3A_{\alpha,\beta}$

Table 1.4: The covers of dihedral algebras defined by Rehren.

In this thesis, we study axial algebras satisfying the fusion law $\mathcal{M}(\alpha,\beta)$ with the condition $\alpha = 2\beta$ [1]. We establish that an axis satisfying the above mentioned condition arises as the sum of two orthogonal axes satisfying the fusion law $\mathcal{M}(\beta)$. In particular, we prove the following theorem.

Theorem 1.4.1. *Let A be an axial algebra of Jordan type η. Suppose a and b are two axes with $ab = 0$. Then A has the decomposition*

$$A = A_1(a+b) \bigoplus A_0(a+b) \bigoplus A_{2\eta}(a+b) \bigoplus A_\eta(a+b),$$

with respect to $a+b$, which satisfies $\mathcal{M}(2\eta,\eta)$-fusion law.

[1] In an ongoing project, Franchi, Mainardis, and Shpectorov have found an algebra of infinite dimension satisfying the fusion law $\mathcal{M}(4\eta,\eta)$ (specifically, for $\eta = \frac{1}{2}$). Note that, when $\alpha \neq 4\beta$, we suspect that the maximum dimension of a 2-generated algebra is 8.

We call an axis satisfying the fusion law $\mathcal{M}(\eta)$ a *single axis*, and an axis satisfying the fusion law $\mathcal{M}(2\eta, \eta)$ a *double axis*.

A double axis is not primitive in the algebra A, however, it can be primitive in a proper subalgebra of A. We study all the 2-generated subalgebras of A satisfying $\mathcal{M}(2\eta, \eta)$. There are two cases to consider. In the first case, the generating set consists of a single axis and a double axis, in the second case, the generating set consists two double axes. We have found two subalgebras which are different from Rehren's two generated subalgebras. This shows that Rehren's theorem can be generalised for all α, β.

We define a fixed subalgebra of a Matsuo algebra M relative to $H \subseteq Aut(M)$, and show that it is a primitive axial algebra satisfying the fusion law $\mathcal{M}(2\eta, \eta)$. Then we construct fixed subalgebras for symmetric group S_{2n} and fixed subalgebras for symplectic group $Sp(2n, 2)$.

A flip is an automorphism of the diagram defined on the axes set which fixes all the single axes and interchanges the summands of each double axis. Clearly flip has order at most 2. Let h be a flip, then the action of $\langle h \rangle$ on the basis D produces orbits of length at most 2. Note that the double axes come from the orbits of length 2. Thus the algebra generated by all the single axes and double axes in this way is the fixed subalgebra relative to the group $\langle h \rangle$. We have the following theorems.

Theorem 1.4.2. *Let M be the Matsuo algebra for the symmetric group S_{2k} and let*

$$h = (1, 2)(3, 4) \dots (2k - 1, 2k).$$

Then the fixed subalgebara W relative $\langle h\rangle$ is generated by set containing k single axes and $k(k-1)$ double axes. Moreover, the above set of axes span W, therefore, it has dimension k^2.

Call the above fixed subalgebra the k^2-algebra relative to $\langle h\rangle$.

Theorem 1.4.3. *The k^2-algebra for S_{2k}, $k \neq 1, 2$, is not simple for $\eta = -\frac{1}{2(k-1)}$ and $\eta = -\frac{1}{k-2}$.*

Theorem 1.4.4. *Let V be a symplectic space over the field $\mathbb{F}_2$ and $\dim(V) = 2n$. Let $t \in Sp(2n, 2)$ be an inolution of rank l. The dimension of the axial part of the fixed subalgebra $M_{\langle t\rangle}$ relative to $\langle t\rangle$ is given by*

$$\dim M_{\langle t\rangle} = \frac{1}{2}(2^{2n-l} + 2^{\dim V(t)} - 2),$$

where, $V(t) = \{u \in V : (u, u^t) = 0\}$.

This thesis is organised as follows.

In the first three chapters we review some of the results on axial algebras required for this work.

In Chapter 2, we recall the definitions of fusion laws, grading, and axial algebras. We give some examples of axial algebras of Jordan type.

In Chapter 3, we discuss the structural properties of axial algebras. In particular, we define, the radicals, Frobenius forms, and Gram matrices, and prove some fundamental results which will be used in later stage.

In Chapter 4, we discuss the relevant topics of group theory like Coxeter groups, groups of 3-transpositions, groups of symplectic type. Then we define Fischer spaces and Matsuo algebras.

In Chapter 5, we show how the axes of Monster type $(2\eta, \eta)$ arise from the axes of Jordan type η. Then we present a general construction of subalgebras of Matsuo algebras which are always primitive axial algebras of Monster type $(2\eta, \eta)$. In this chapter, we also study all the 2-generated subalgebras by a single axis and a double axis, by two double axes.

In Chapter 6, we construct the fixed subalgebra for the symmetric group S_n, and discuss the radical of it.

In Chapter 7, we construct the fixed subalgebra for symplectic group $Sp(2n, 2)$.

Chapter 2

Preliminaries

Let $\mathbb{F}$ be a field with char $\mathbb{F} \neq 2$ and A be a commutative $\mathbb{F}$-algebra. We do not assume the associative law and existence of the identity element in A.

For an element $s \in A$, define the adjoint endomorphism ad_s of A to be the map

$$\begin{aligned} ad_s : & A \to A \\ & x \mapsto sx. \end{aligned}$$

For a scalar $\mu \in \mathbb{F}$, denote the μ-eigenspace of ad_s by $A_\mu(s)$. That is,

$$A_\mu(s) = \{x \in A | \; sx = \mu x\}.$$

Note that $A_\mu(s) = 0$ whenever μ is not an eigenvalue of ad_s.

2.1 Fusion laws

Definition 2.1.1. *A fusion law is a pair* $(\mathcal{F}, *)$*, where* $\mathcal{F}$ *is a set and* $* : \mathcal{F} \times \mathcal{F} \to 2^{\mathcal{F}}$ *is a symmetric map assigning subset of* $\mathcal{F}$ *to a pair* (λ, μ)*.*

Throughout this work $\mathcal{F}$ is a finite subset of the field $\mathbb{F}$.

Definition 2.1.2. *Let* $A = \bigoplus_{\mathcal{F}} A_i$*. The decomposition is said to satisfy a fusion law* $\mathcal{F}$ *if* $A_i A_j \subseteq \bigoplus_{k \in i*j} A_k$ *for all* $i, j \in \mathcal{F}$*.*

Note that any decomposition of A satisfies the trivial fusion law where $i * j = \mathcal{F}$ for all $i, j \in \mathcal{F}$. However, we are interested in those fusion laws where the cardinality of $i * j$ is minimal but not always the empty set. It is convenient to describe fusion laws using tables whose whose rows and columns correspond to the elements of $\mathcal{F}$.

Some of the important examples of fusion laws which are widely used in this work are given in Table 1.3. Here the set $\mathcal{F}$ is $\{0, 1\}$, $\{0, 1, \eta\}$ and $\{1, 0, \alpha, \beta\}$ respectively, where, $\eta, \alpha, \beta \in \mathbb{F}$. Note that blank entry in a fusion law represents the empty set. The last entry in the second table means $\eta * \eta = \{1, 0\}$.

Suppose A satisfies a fusion law $\mathcal{F}$. Then for any two subsets M and N of $\mathcal{F}$, define $A_M := \bigoplus_{i \in M} A_i$ and $M * N := \bigcup_{i \in M, j \in N} i * j$.

Definition 2.1.3. *Let* A *be a commutative, non-associative algebra over* $\mathbb{F}$ *and* $(\mathcal{F}, *)$ *be a fusion law. An idempotent* $a \in A$ *is said to be an* $\mathcal{F}$*-axis if* ad_a *is semisimple with all its eigenvalues belong to* $\mathcal{F}$*, that is, using the above notation* $A = A_{\mathcal{F}}(a)$ *and the decomposition of* A *into the direct sum of eigenspaces of* A *satisfies the fusion law* $(\mathcal{F}, *)$*.*

Since a is an idempotent, $a \in A_1(a)$, therefore, from now on we assume $1 \in \mathcal{F}$.

Definition 2.1.4. *An axis $a \in A$ is said to be primitive if $A_1(a) = \mathbb{F}a$.*

Definition 2.1.5. *A commutative, non-associative algebra A is said to be a primitive $\mathcal{F}$-axial algebra if it is generated by a set of primitive axes.*

In general, when we consider axes in a primitive axial algebra, we assume that these axes are primitive, unless other otherwise is specified.

Let A be a primitive $\mathcal{F}$-axial algebra and a be an axis. Suppose the elements b and c of A are such that $b \in A_1(a)$ and $c \in A_\lambda(a)$ for $\lambda \neq 0$. Then $b = \mu a$ for some scalar $\mu \in \mathbb{F}$. The product

$$bc = (\mu a)c = \mu(ac) = \mu(\lambda c) = \lambda(\mu c) \subseteq A_\lambda(a).$$

That is, $A_1(a)A_\lambda(a) = A_\lambda(a)$ for $\lambda \neq 0$. In view of this calculations, all fusion laws we consider will satisfy $1 * \lambda = \{\lambda\}$ for all $\lambda \neq 0$. Note that if $\lambda = 0$ then $A_1(a)A_0(a) = 0$, so we can assume that $1 * 0 = \emptyset$ if $0 \in \mathcal{F}$.

Definition 2.1.6. *Let T be a finite abelian group and $(\mathcal{F}, *)$ be a fusion law. Then $\mathcal{F}$ is said to be T-graded if it has a partition $\bigcup \mathcal{F}_s$ with $s \in T$ such that $\mathcal{F}_s * \mathcal{F}_t \subseteq \mathcal{F}_{st}$ for all $s, t \in T$.*

Let A be an $\mathcal{F}$-axial algebra and a be an axis. Suppose $\mathcal{F}$ is T-graded. Then for $t \in T$, define

$$A_t(a) = A_{\mathcal{F}_t}(a) = \bigoplus_{i \in \mathcal{F}_t} A_i(a).$$

Using the above notation A can be written as $A = \bigoplus_{t \in T} A_t(a)$ with $A_t(a)A_s(a) \subseteq A_{ts}(a)$. In general, an $\mathcal{F}$-axial algebra A which has T-graded fusion law is a T-graded algebra. We use the convention $A_t = 0$ whenever $\mathcal{F}_t$ is empty.

Let T^* be the group of all homomorphisms from T to the multiplicative group of $\mathbb{F}$. Such a homomorphism is called a linear character of T over $\mathbb{F}$.

Consider a T-graded algebra A. For an element $a \in A$ and $\kappa \in T^*$, define a map τ_a^κ by $x \mapsto \kappa(t)x$ for all $x \in A_t(a)$. This map is extended to A by linearity.

Proposition 2.1.7. *Let A be a T-graded axial algebra. For an element $a \in A$ and a linear character κ, the map τ_a^κ is an automorphism of A. Furthermore, the map sending κ to τ_a^κ is a homomorphism from T^* to $Aut(A)$.*

Proof. To see that τ_k^a is an automorphism of A, we just need to see that it preserves products. Since eigenvectors of ad_a span A, it suffices to check our claim for products of eigenvectors. Suppose that $u \in A_t(a)$ and $v \in A_s(a)$. Then from the grading we have $uv \in A_{ts}(a)$. Therefore,

$$\tau_a^\kappa(uv) = \kappa(ts)uv = (\kappa(t)\kappa(s))uv = (\kappa(t)u)(\kappa(s)v) = \tau_a^\kappa(u)\tau_a^\kappa(v).$$

Clearly, the map τ_a^κ is onto as for every $u \in A_t(a)$ there exists an element $\kappa^{-1}(t)u$ whose image under the action of κ is u. It is obvious that τ_a^κ is one-one since none of the values of $\kappa(t)$ is zero. This proves the first part of the claim.

Define $\theta : T^* \to Aut(A)$ by $\kappa \mapsto \tau_a^\kappa$. Now suppose $u \in A_t(a)$ then

$$\begin{aligned}\theta(\kappa_1\kappa_2) &= \kappa_1\kappa_2(t)(u)\\ &= \kappa_1(t)(\kappa_2(t)u) = \kappa_1(t)\tau_a^{\kappa_2}(u)\\ &= \tau_a^{\kappa_2}(\kappa_1(t)u)\\ &= \tau_a^{\kappa_2}\tau_a^{\kappa_1}(u) = \theta(\kappa_1)\theta(\kappa_2).\end{aligned}$$

Therefore, the map θ is a homomorphism. $\square$

All the axial algebras considered later in this work are C_2-graded. That is, if we write, $C_2 = \{+, -\}$ then A admits the decomposition $A = A_+(a) \bigoplus A_-(a)$ relative to an axis a, where, $A_+(a) = \bigoplus_{k\in\Lambda_+} A_k(a)$ and $A_-(a) = \bigoplus_{k\in\Lambda_-} A_k(a)$, for disjoint subsets Λ_+ and Λ_- of $\mathcal{F}$ satisfying $\Lambda_+\Lambda_+ \subseteq \Lambda_+$, $\Lambda_-\Lambda_- \subseteq \Lambda_+$, and $\Lambda_-\Lambda_+ \subseteq \Lambda_-$. Since char $\mathbb{F} \neq 2$, we have $T^* = \{\kappa_1, \kappa_{-1}\}$, where κ_1 is the identity. Note that the map $\tau_a^{\kappa_{-1}}$ fixes the $A_+(a)$ part of each element of A and negates the $A_-(a)$ part. In this case $\tau_a^{\kappa_{-1}}$ is nontrivial automorphism if $A_-(a)$ is nonzero and more importantly this involution is known as the Miyamoto involution, which will be discussed in detail in the next chapter.

We prove one more result related to the automorphisms τ_a^κ.

Proposition 2.1.8. *([21], 3.10) Let A be an $\mathcal{F}$-axial algebra generated by the set of axes $\mathcal{A}$. Suppose a is an axis and $W \leq A$ is ad_a-invariant then it is invariant under τ_a^κ for $\kappa \in T^*$.*

Proof. Since W is ad_a invariant, it can be written as $W = \oplus_{\mu\in\mathcal{F}} W_\mu(a)$. But $W_\mu(a) \leq A_\mu(a)$ and τ_a^κ acts as scalar transformation on $A_\mu(a)$. Consequently, every subspace of $A_\mu(a)$ is τ_a^κ-invariant. Therefore, $W^{\tau_a^\kappa} = \oplus_{\mu\in\mathcal{F}} W_\mu(a)^{\tau_a^\kappa} = \oplus_{\mu\in\mathcal{F}} W_\mu(a) = W$. $\square$

2.2 Axial algebras of Jordan type η

Definition 2.2.1. *A primitive axial algebra is said to be of Jordan type* $\eta \in \mathbb{F}$, $\eta \neq 1, 0$, *if it is generated by a set of* $\mathcal{M}(\eta)$*-axes.*

In a fusion law $\mathcal{F}$, an element $u \in \mathcal{F}$ is said to be a *unit* if $u * \lambda = \{\lambda\}$ for all $\lambda \neq 0$.

Note that the $\mathcal{M}(\eta)$ fusion law contains two units, namely, 1 and 0. Such fusion laws are called *Seress fusion laws.* Moreover, the fusion law $\mathcal{M}(\eta)$ is Z_2-graded and hence the algebras with this fusion law inherit Miyamoto involutions.

We conclude this chapter with some examples of axial algebras of Jordan type.

Recall that a commutative algebra J over a field $\mathbb{F}$ is said to be Jordan if

$$(xy)x^2 = x(yx^2),$$

for all $x, y \in J$. The above expression is known as the Jordan identity. Naturally, any associative algebra $(A, \cdot, +)$ becomes a Jordan algebra under the new product

$$x \bullet y = \frac{1}{2}(x \cdot y + y \cdot x),$$

for all $x, y \in A$. Usually, the algebra $(A, +, \bullet)$ is denoted by A^+. Here we show that a Clifford algebra is a primitive axial algebra of Jordan type $\eta = \frac{1}{2}$.

Let V be an n-dimensional vector space over a field $\mathbb{F}$, char $\mathbb{F} \neq 2$, with a quadratic form q on it. Let B be the corresponding bilinear form associated with q. Note that B is symmetric. Now define the Clifford algebra $Cl(V, q)$ as

$$Cl(V, q) = T(V)/\langle u \otimes u - q(u) \rangle,$$

where, $T(V) = \mathbb{F} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \dots$ is the tensor algebra of V and $u \in V$. Note that the Clifford algebra is the largest unital associative algebra generated by V such that $uu = q(u)$ for all $u \in V$. Since $Cl(V, q)$ is associative, it is a Jordan algebra under the new product $\bullet$. Let $Cl^+ = (Cl(V, q), +, \bullet)$.

Now consider a subspace $JCl := \mathbb{F}1 \oplus V$ of $Cl(V, q)$; Then the sum and product on it are as follows.

$$(\alpha 1 + u) + (\beta 1 + v) = (\alpha + \beta)1 + (u + v),$$
$$(\alpha 1 + u) \cdot (\beta 1 + v) = (\alpha\beta + B(u, v))1 + (\alpha v + \beta u)$$

for any $\alpha 1 + u, \beta 1 + v \in JCl$. The subspace is closed for addition and multiplication. Therefore $(JCl, +, \cdot)$ is a subalgebra of $Cl(V, q)$ and moreover, JCl^+ is also a Jordan algebra as Cl^+ is a Jordan algebra.

Proposition 2.2.2. *The algebra JCl^+ is a primitive algebra of Jordan type $\eta = \frac{1}{2}$.*

Proof. First of all, we claim that for any $u \in V$ and $\alpha \in \mathbb{F}$, the vector $a = \alpha 1 + u$ is an idempotent iff either $a \in \{0, 1\}$ or $\alpha = \frac{1}{2}$ and $q(u) = \frac{1}{4}$. To see this consider

$$\begin{aligned} a^2 &= (\alpha 1 + u)(\alpha 1 + u) \\ &= (\alpha^2 + B(u,u))1 + 2\alpha u \\ &= (\alpha^2 + q(u))1 + 2\alpha u. \end{aligned}$$

Thus $a^2 = a$ if and only if $2\alpha u = u$ and $\alpha^2 + q(u) = \alpha$. Now suppose $a \notin \{0, 1\}$, then clearly the above equations imply $\alpha = \frac{1}{2}$ and $q(u) = \frac{1}{4}$.

Thus every non-trivial idempotent in JCl^+ is of the form $a = \frac{1}{2} + u$ where $q(u) = \frac{1}{4}$. Let $J = JCl^+$ and $J_\lambda(a)$ be the λ-eigenspace of ad_a, where, $a \in J$ is an idempotent. Clearly $a \in J_1(a)$. For a non trivial idempotent $a = \frac{1}{2} + u$ of J, define $\bar{a} = \frac{1}{2} - u$; which is also an idempotent in view of the above claim. And also note that

$$a\bar{a} = \left(\frac{1}{2} + u\right)\left(\frac{1}{2} - u\right) = \frac{1}{4} - q(u) = 0.$$

Therefore, $\bar{a} \in J_0(a)$. Also, for $w \in u^\perp$

$$aw = \left(\frac{1}{2} + u\right) w = B(u,w)1 + \frac{1}{2}w = \frac{1}{2}w,$$

which implies $u^\perp \subseteq J_{\frac{1}{2}}(a)$ and similar calculation yields $u^\perp \subseteq J_{\frac{1}{2}}(\bar{a})$.

Since $\dim(J) = \dim(V) + 1$ and $\dim(u^{\perp}) = \dim(V) - 1$, we have

$$
\begin{aligned}
J_1(a) &= \langle a \rangle = J_0(\bar{a}), \\
J_0(a) &= \langle \bar{a} \rangle = J_1(\bar{a}), \\
J_{\frac{1}{2}}(a) &= J_{\frac{1}{2}}(\bar{a}).
\end{aligned}
$$

In other words, ad_a is semisimple.

Let τ_a be the map which act as the identity on $J_1(a) \oplus J_0(a)$ and negative identity on $J_{\frac{1}{2}}(a)$. Clearly τ_a is an involution. This proves that the decomposition

$$J = J_1(a) \bigoplus J_0(a) \bigoplus J_{\frac{1}{2}}(a)$$

is C_2-graded satisfying $\mathcal{J}(\frac{1}{2})$-fusion law. $\square$

Let A be a commutative, non-associative $\mathbb{F}$-algebra over a field of characteristic not equal to 2. Let $a_0 \in A$ be an idempotent and $B = \langle a_0 \rangle$. Clearly a_0 is the only idempotent in B. Indeed if there exists another nonzero idempotent a in B then $a = \beta a_0$ for some $\beta (\neq 0) \in \mathbb{F}$. We have $\beta a_0 = a = a^2 = \beta^2 a_0$, which implies $\beta = 1$. Further, note that $a_0 = 1$. In other words, $B = \mathbb{F}1$ and it is a primitive axial algebra of Jordan type η where $B_0(1) = B_\eta(1) = 0$. We denote this algebra by $1A$.

For an $\eta \in \mathbb{F}$, $\eta \neq 1, 0$, let $3C(\eta)$ be an $\mathbb{F}$-algebra having $S = \{s_0, s_1, s_2\}$ as a basis, where s_0, s_1 and s_2 are nonzero idempotents. Product of the basis elements is given by

$$s_i s_j = \frac{\eta}{2}(s_i + s_j - s_k)$$

for distinct $i, j, k \in \{0, 1, 2\}$. Clearly $3C(\eta)$ is 3-dimensional algebra over $\mathbb{F}$. For simplicity, let $A = 3C(\eta)$. For $\lambda \in \mathbb{F}$, $A_\lambda(s_i)$ denotes the λ-eigenspace of ad_{s_i} for $i \in \{0, 1, 2\}$.

Since $s_i \in S$ is an idempotent, $A_1(s_i) = \mathbb{F}s_i$.

In order to determine the 0-eigenspace of ad_{s_i}, note that

$$\begin{aligned} s_i(\eta s_i - s_j - s_k) &= \eta s_i - \frac{\eta}{2}(s_i + s_j - s_k) - \frac{\eta}{2}(s_i + s_k - s_j) \\ &= \eta s_i - \eta s_i - \frac{\eta}{2}(s_j - s_k + s_k - s_j) = 0, \end{aligned}$$

therefore, $\langle \eta s_i - s_j - s_k \rangle \subseteq A_0(s_i)$.

Similarly, note that

$$\begin{aligned} s_i(s_j - s_k) &= \frac{\eta}{2}(s_i + s_j - s_k) - \frac{\eta}{2}(s_i + s_k - s_j) \\ &= \frac{\eta}{2}(s_i - s_i + s_j + s_j - s_k - s_k) \\ &= \eta(s_j - s_k), \end{aligned}$$

and thus $\langle s_j - s_k \rangle \subseteq A_\eta(s_i)$.

From the dimensions of the eigenspaces, and dimension of $3C(\eta)$, we can conclude that the algebra $3C(\eta) = A_1(s_i) \bigoplus A_0(s_i) \bigoplus A_\eta(s_i)$. In other words, ad_{s_i} is semisimple.

Next we verify that the eigenspace $A_0(s_i)$ is indeed a subalgebra of $3C(\eta)$.

$$\begin{aligned}(\eta s_i - s_j - s_k)(\eta s_i - s_j - s_k) &= \eta s_i(\eta s_i - s_j - s_k) + (-s_j - s_k)(\eta s_i - s_j - s_k)\\ &= 0 - \eta s_j s_i + s_j + s_j s_k - \eta s_k s_i + s_k s_j + s_k\\ &= -\eta s_j s_i - \eta s_k s_i + 2 s_j s_k + s_j + s_k\\ &= -\frac{\eta^2}{2}(s_j + s_i - s_k) - \frac{\eta^2}{2}(s_k + s_i - s_j)\\ &\quad + \eta(s_j + s_k - s_i) + s_j + s_k\\ &= -(\eta^2 + \eta)s_i + (\eta + 1)s_j + (\eta + 1)s_k\\ &= -(\eta + 1)(\eta s_i - s_j - s_k) \in A_0(s_i).\end{aligned}$$

With respect to s_i, an element $x \in 3C(\eta)$ can be written as

$$x = \alpha s_i + \beta(\eta s_i - s_j - s_k) + \gamma(s_j - s_k),$$

for some scalars $\alpha, \beta, \gamma \in \mathbb{F}$. Now define a map $\tau_{s_i} : 3C(\eta) \to 3C(\eta)$ which when acted on x, fixes s_i but switches s_j and s_k in the above expression for x. That is,

$$\begin{aligned}x^{\tau_{s_i}} &= \alpha s_i + \beta(\eta s_i - s_k - s_j) + \gamma(s_k - s_j)\\ &= \alpha s_i + \beta(\eta s_i - s_j - s_k) - \gamma(s_j - s_k).\end{aligned}$$

Clearly, τ_{s_i} is an automorphism of order 2 which fixes $3C(\eta)_+ := A_1(s_i) \bigoplus A_0(s_i)$ and negates $3C(\eta)_- := A_\eta(s_i)$ under its action. Thus $3C(\eta)$ is $\mathbb{Z}_2$-graded and moreover has $\mathcal{J}(\eta)$-fusion law. Hence, the algebra $3C(\eta)$ is a primitive axial algebra of Jordan type η.

Note that the group generated by $\{s_i, s_j\}$ under the above mentioned product is isomorphic to the group S_3. In such case, we say that $S = \{s_i, s_j, s_k\}$, where $s_k = s_i^{s_j}$, forms a Sakuma algebra.

Let $2B$ be a 2-dimensional associative $\mathbb{F}$-algebra spanned by $\{a, b\}$, where a and b are idempotents and $ab = 0$. The only eigenvalues of ad_a(or ad_b) are 0 and 1. Thus $2B_\eta(a) = 0$ for all $\eta(\neq 0, 1)$. The algebra $2B$ has a decomposition $2B = 2B_1(a) \bigoplus 2B_0(a)$ with $2B_1(a) = \mathbb{F}a$ and $2B_0(a) = \mathbb{F}b$. Thus, $2B$ is a primitive axial algebra of Jordan type η for all η.

Chapter 3

Structural properties of axial algebras

We study some of the useful structural properties of axial algebras in this chapter. All the definitions and results of this chapter come from [21].

3.1 Miyamoto groups and closed sets of axes

Let T be an abelian group and A an $\mathcal{F}$-axial algebra where the fusion law is T-graded. Recall that $T^\star$ is the group of all the homomorphisms from T to the multiplicative group of $\mathbb{F}$. If a is an $\mathcal{F}$-axis and and $\kappa \in T^*$, then there exists an automorphism τ_a^κ of A associated with a and moreover the image T_a of the the map sending κ to τ_a^κ is a subgroup of $Aut(A)$. We call the above subgroup the *axis subgroup* corresponding to a.

Definition 3.1.1. *Let A be an $\mathcal{F}$-axial algebra generated by a set of axes $\mathcal{A}$. Then the Miyamoto group $\mathcal{G}(\mathcal{A})$ of the algebra A with respect to the set $\mathcal{A}$ is the group generated by the axis subgroups T_a, $a \in \mathcal{A}$.*

Clearly the Miyamoto group is a subgroup of $Aut(A)$.

We know that the Griess algebra is an $\mathcal{M}(\frac{1}{4}, \frac{1}{32})$-axial algebra generated by $2A$-axes. Since $2A$ involutions generate the Monster group, the Miyamoto group of the Griess algebra with respect to the $2A$ axes is the Monster group.

Proposition 3.1.2. *[21] Suppose A is an $\mathcal{F}$-axial algebra and $f \in Aut(A)$. Let a be axis. Then a^f is also an axis.*

Proof. Since $a \in A$ is an idempotent and $f \in Aut(A)$, a^f is also an idempotent. Let $A_\lambda(a)$ be the λ-eigenspace of ad_a. Suppose $x \in A_\lambda(a)$ then

$$a^f x^f = (ax)^f = (\lambda x)^f = \lambda x^f.$$

So $x^f \in A_\lambda(a^f)$, that is, $(A_\lambda(a))^f \subseteq A_\lambda(a^f)$. On the other hand, if $y \in A_\lambda(a^f)$ then $a^f y = \lambda y$ which implies $ay^{f^{-1}} = \lambda y^{f^{-1}}$. That is, $y^{f^{-1}} \in A_\lambda(a)$ and hence $A_\lambda(a^f) \subseteq (A_\lambda(a))^f$. Therefore, $A_\lambda(a^f) = (A_\lambda(a))^f$. Further, for $\lambda, \mu \in \mathcal{F}$, $A_\lambda(a^f)A_\mu(a^f) = A_\lambda(a)^f A_\mu(a)^f = (A_\lambda A_\mu(a))^f \subseteq A_{\lambda\star\mu}(a)^f = A_{\lambda\star\mu}(a^f)$. In other words, the products of the eigenspaces $A_\lambda(a^f)$ and $A_\mu(a^f)$ are controlled by $\mathcal{F}$-fusion law. Hence a^f is an $\mathcal{F}$-axis. □

Definition 3.1.3. *Let A be an axial algebra and $\mathcal{A}$ a generating set of axes. Then the set $\mathcal{A}$ is said to be closed if $\mathcal{A}^{\mathcal{G}(\mathcal{A})} = \mathcal{A}$, where $\mathcal{G}(\mathcal{A})$ is the Miyamoto group with respect to $\mathcal{A}$.*

Clearly, $\mathcal{A}$ is closed if and only if $\mathcal{A}^\kappa = \mathcal{A}$ for all $\kappa \in T_a$ with $a \in \mathcal{A}$. Further, if $\{\mathcal{A}_x\}$ is a collection of closed sets for some index set I, then $(\bigcap \mathcal{A}_x)^\kappa = \bigcap \mathcal{A}_x^\kappa = \bigcap \mathcal{A}_x$, for $\kappa \in T_a$. That is, intersection of closed sets is again closed. Thus we define *closure* of $\mathcal{A}$ as the unique smallest closed set containing $\mathcal{A}$. And it is denoted by $\bar{\mathcal{A}}$.

Proposition 3.1.4. *([21], 3.5) Let A be an axial algebra generated by a set of axes $\mathcal{A}$ and $\mathcal{G}(\mathcal{A})$ the Miyamoto group with respect to $\mathcal{A}$. Then $\bar{\mathcal{A}} = \mathcal{A}^{\mathcal{G}(\mathcal{A})}$, consequently, $\mathcal{G}(\mathcal{A}) = \mathcal{G}(\bar{\mathcal{A}})$.*

Proof. We prove the result by exhibiting $\mathcal{A}^{\mathcal{G}(\mathcal{A})} \subseteq \bar{\mathcal{A}}$ and $\bar{\mathcal{A}} \subseteq \mathcal{A}^{\mathcal{G}(\mathcal{A})}$.

Note that $\mathcal{G}(\mathcal{A}) \leq \mathcal{G}(\bar{\mathcal{A}})$ as the closure of $\mathcal{A}$ contains all the points of $\mathcal{A}$. Therefore, $\mathcal{A}^{\mathcal{G}(\mathcal{A})} \subseteq \mathcal{A}^{\mathcal{G}(\bar{\mathcal{A}})} \subseteq \bar{\mathcal{A}}$.

To prove the reverse inclusion, first of all we claim that $\mathcal{A}^{\mathcal{G}(\mathcal{A})}$ is closed. For that consider an element $x \in \mathcal{A}^{\mathcal{G}(\mathcal{A})}$. Now $x = a^f$ for some axis a and $f \in \mathcal{G}(\mathcal{A})$. Recall that τ_a^κ is an automorphism of A associated with axis a and the linear character κ. So we have $\tau_{a^f}^\kappa = \tau_a^{\kappa f}$. This implies that the axis subgroup

$$T_x = T_{a^f} = \langle \tau_{a^f}^\kappa \mid \kappa \in T^* \rangle = \langle \tau_a^\kappa \mid \kappa \in T^* \rangle^f = T_a^f.$$

Since T_a is a subgroup of the Miyamoto group $\mathcal{G}(\mathcal{A})$ and $f \in \mathcal{G}(\mathcal{A})$, we see that $T_x = T_a^f \leq \mathcal{G}(\mathcal{A})^f = \mathcal{G}(\mathcal{A})$. Thus $\mathcal{G}(\mathcal{A}^{\mathcal{G}(\mathcal{A})}) = \mathcal{G}(\mathcal{A})$. However, $\mathcal{A}^{\mathcal{G}(\mathcal{A})}$ is invariant under $\mathcal{G}(\mathcal{A})$, hence, it is invariant under $\mathcal{G}(\mathcal{A}^{\mathcal{G}(\mathcal{A})})$. This shows that $\mathcal{A}^{\mathcal{G}(\mathcal{A})}$ is closed and from the definition of closure, it follows $\bar{\mathcal{A}} \subseteq \mathcal{A}^{\mathcal{G}(\mathcal{A})}$.

Further, we have $\mathcal{G}(\bar{\mathcal{A}}) = \mathcal{G}(\mathcal{A}^{\mathcal{G}(\mathcal{A})}) = \mathcal{G}(\mathcal{A})$. $\square$

Definition 3.1.5. *Let A be an axial algebra. Suppose $\mathcal{A}$ and $\mathcal{B}$ are two generating sets of axes. They are said to be equivalent if $\bar{\mathcal{A}} = \bar{\mathcal{B}}$.*

We use the notion $\mathcal{A} \sim \mathcal{B}$ to denote that the sets $\mathcal{A}$ and $\mathcal{B}$ are equivalent.

Proposition 3.1.6. *([21], 3.9) Let $\mathcal{A}$ and $\mathcal{B}$ be two generating sets of axes. Then $\mathcal{A}$ and $\mathcal{B}$ are equivalent if and only if the Miyamoto groups with respect to $\mathcal{A}$ and $\mathcal{B}$ are equal, so $\mathcal{G}(\mathcal{A}) = \mathcal{G}(\mathcal{B}) = \mathcal{G}$, and every element $a \in \mathcal{A}$ is $\mathcal{G}$-conjugate to some element $b \in \mathcal{B}$ and vice versa.*

Proof. Suppose $\mathcal{A}$ and $\mathcal{B}$ are equivalent. Then $\mathcal{G}(\mathcal{A}) = \mathcal{G}(\bar{\mathcal{A}}) = \mathcal{G}(\bar{\mathcal{B}}) = \mathcal{G}(\mathcal{B})$. In view of the above proposition, for any $a \in \mathcal{A}$, we have $a \in \bar{\mathcal{A}} = \bar{\mathcal{B}} = \mathcal{B}^{\mathcal{G}(\mathcal{B})}$. Thus a is $\mathcal{G}$-conjugate to some element $b \in \mathcal{B}$. Similar argument shows that for any $b \in \mathcal{B}$ is $\mathcal{G}$-conjugate to some element $a \in \mathcal{A}$. This proves the first part of the proposition.

Conversely, suppose $\bar{a} \in \bar{\mathcal{A}}$, which implies, $\bar{a} = a^f = (b^g)^f \in \bar{\mathcal{B}}$ for some $a \in \mathcal{A}$, $b \in \mathcal{B}$, $f, g \in \mathcal{G}$. Thus $\bar{\mathcal{A}} \subseteq \bar{\mathcal{B}}$. By symmetry, reverse inclusion follows. This completes the proof. □

Theorem 3.1.7. *([21], 3.12) If $\mathcal{A}$ and $\mathcal{B}$ are two equivalent sets of axes then the axial algebra generated by $\mathcal{A}$ and the axial algebra generated by $\mathcal{B}$ are one and the same.*

Proof. Let A be an axial algebra generated by $\mathcal{A}$ and B an axial algebra generated by $\mathcal{B}$. For every axis $a \in \mathcal{A}$, A is ad_a-invariant. Therefore, from Proposition 3.1.6, A is g-invariant for any $g \in \mathcal{G}(\mathcal{A})$. Since closure of $\mathcal{A}$ is the smallest set with this property, we have $\bar{\mathcal{A}} = \mathcal{A}^{\mathcal{G}(\mathcal{A})} \subseteq A$. On the other hand, B is generated by $\mathcal{B} \subseteq \bar{\mathcal{B}} = \bar{\mathcal{A}} \subseteq A$.

That is to say $B \subseteq A$. Using similar argument, it can be shown $A \subseteq B$. Hence $A = B$. $\square$

3.2 Radicals and Frobenius form

Definition 3.2.1. *Let A be an $\mathcal{F}$-axial algebra generated by a set of primitive axes $\mathcal{A}$. Then the radical $R(A, \mathcal{A})$ of the algebra A with respect to $\mathcal{A}$ is the unique largest ideal of A containing no axes from $\mathcal{A}$.*

It is shown in [21] that indeed such maximal ideals exist.

Theorem 3.2.2. *([21], 4.3) Let A be an axial algebra and $\mathcal{A}$, $\mathcal{B}$ be two equivalent sets of primitive axes. Then $R(A, \mathcal{A}) = R(A, \mathcal{B})$.*

Proof. Since $\mathcal{A}$ and $\mathcal{B}$ are equivalent, $\bar{\mathcal{A}} = \bar{\mathcal{B}}$. By proving $R(A, \mathcal{A}) = R(A, \bar{\mathcal{A}})$, we will have proved the theorem.

Note that the ideal $R(A, \bar{\mathcal{A}})$ does not contain any axis from $\mathcal{A}$. Since $R(A, \mathcal{A})$ maximal with this property, we have $R(A, \bar{\mathcal{A}}) \subseteq R(A, \mathcal{A})$. To see the reverse inclusion, note that every ideal of A is ad_a-invariant and hence $\mathcal{G}(\mathcal{A})$-invariant. Thus $R(A, \mathcal{A})$ contains no axis from $\mathcal{A}^{\mathcal{G}(\mathcal{A})} = \bar{\mathcal{A}}$ as $R(A, \mathcal{A})$ contains no axis from $\mathcal{A}$, which implies $R(A, \mathcal{A}) \subseteq R(A, \bar{\mathcal{A}})$. Hence we have

$$R(A, \mathcal{A}) = R(A, \bar{\mathcal{A}}) = R(A, \bar{\mathcal{B}}) = R(A, \mathcal{B}).$$

$\square$

Definition 3.2.3. *A Frobenius form on an $\mathcal{F}$-axial algebra A is a nonzero bilinear form*

$$(\ , \ ) : A \times A \to \mathbb{F}$$

such that $(a, bc) = (ab, c)$ for all a, b, $c \in A$.

In the following propositions, we assume that A is a primitive $\mathcal{F}$-axial algebra generated by a set of axes $\mathcal{A}$ with a Frobenius form $(\ , \ )$.

Proposition 3.2.4. *[21] For an axis a, eigenspaces $A_\lambda(a)$ and $A_\mu(a)$, $\lambda \neq \mu$, are orthogonal with respect to the Frobenius form $(\ , \ )$.*

Proof. For $x \in A_\lambda(a)$ and $y \in A_\mu(a)$,

$$\lambda(x, y) = (\lambda x, y) = (ax, y) = (x, ay) = (x, \mu y) = \mu(x, y).$$

As $\lambda \neq \mu$, (x, y) must be zero. Thus $A_\lambda(a)$ and $A_\mu(a)$ are orthogonal with respect to the Frobenius form. $\square$

Let a be a primitive $\mathcal{F}$-axis. For any $u \in A$, we have, $u = \sum_{\lambda \in \mathcal{F}} u_\lambda$, where $u_\lambda \in A_\lambda(a)$. Since a is primitive, $u_1 = \mu a$. The vector u_1 is called the projection of u on $A_1(a)$ and is denoted by $\phi_a(u)a$.

Proposition 3.2.5. *([21], 4.5) For an axis a and $u \in A$,*

$$(a, u) = \phi_a(u)(a, a),$$

where $\phi_a(u)a$ is the projection of u on $A_1(a)$.

Proof. Since $A = \oplus_{\lambda \in \mathcal{F}} A_\lambda$, for any $u \in A$ can be written as $u = \sum_{\lambda \in \mathcal{F}} u_\lambda$, where $u_\lambda \in A_\lambda(a)$. Consider

$$(a, u) = (a, \sum_{\lambda \in \mathcal{F}} u_\lambda) = (a, u_1).$$

Note that the last equality follows from Proposition 3.2.4. If $\phi_a(u)$ is the projection of u on a then $(a, u_1) = (a, \phi_a(u)a) = \phi_a(u)(a, a)$. □

Proposition 3.2.6. *([21], 4.6) The Frobenius form is uniquely determined by the values (a, a) on the axes $a \in \mathcal{A}$.*

Proof. Note that every element of A can be expressed as product of axes. Therefore, it is enough to prove that for any $x, y \in A$ that can be chosen to be the product of axes, (x, y) is uniquely determined using that product.

We use induction on the length of x (i.e. the number of axes present when x is written as product of the axes). Suppose x is of length one. Then from 3.2.5, (x, y) is determined by (x, x). Now assume that x has length greater than one. Then x can be written as $x = x_1 x_2$ for some $x_1, x_2 \in A$ with lengths of x_1 and x_2 less than that of x. Thus, $(x, y) = (x_1 x_2, y) = (x_1, x_2 y)$. As the length of x_1 is less than that of x, by induction hypothesis the claim is true for $(x_1, x_2 y)$. This shows that the form is uniquely determined by (a, a). □

Proposition 3.2.7. *[21] The form $(\ ,\ )$ is invariant under the action of the Miyamoto group $\mathcal{G}(\mathcal{A})$ if and only if $(a, a) = (a^f, a^f)$ for all $a \in \mathcal{A}$ and $f \in \mathcal{G}(\mathcal{A})$.*

Proof. Suppose that the form is $\mathcal{G}(\mathcal{A})$-invariant. Then $(a^f, a^f) = (a, a)$ for all $a \in \mathcal{A}$, $f \in \mathcal{G}(\mathcal{A})$.

Assume that $(a,a) = (a^f, a^f)$ for all $a \in \mathcal{A}$, $f \in \mathcal{G}(\mathcal{A})$. To show that the form is $\mathcal{G}$-invariant it is enough to prove $\mathcal{G}$-invariance on products of axes. Note that for any $x, y \in A$ we have $(x,y) = (a,z)$ for some $a \in \mathcal{A}$ and $z \in A$. This is because x, y can be written as product of axes: $x = aa_1 \dots$ and $y = bb_1 \dots$, where $a, a_1, \dots, b, b_1, \dots$ are axes. We have

$$(x,y) = (aa_1 \dots, bb_1 \dots) = (a, a_1 \dots bb_1 \dots) = (a,z)$$

for some $z = a_1 \dots bb_1 \dots$. The last equality comes from the associativity. Thus $(x^f, y^f) = (a^f, z^f)$. It suffices to show that $\phi_{a^f}(z^f) = \phi_a(z)$ as $(a,z) = \phi_a(z)(a,a)$.

Now note that z can be uniquely written as $z = \sum_{\lambda \in \mathcal{F}} z_\lambda$, where $z_\lambda \in A_\lambda(a)$. Under the action of f, $z^f = (\sum_{\lambda \in \mathcal{F}} z_\lambda)^f = \sum_{\lambda \in \mathcal{F}} z_\lambda^f$. On the other hand, z^f can be decomposed with respect to the axis a^f. Then we have $z^f = \sum_{\lambda \in \mathcal{F}} w_\lambda$, where $w_\lambda \in A_\lambda(a^f)$. But we have proved earlier that $A_\lambda(a^f) = A_\lambda(a)^f$. In particular, when $\lambda = 1$; it follows that

$$\phi_a(z)a^f = (z_1)^f = w_1 = \phi_{a^f}(z^f)a^f.$$

Thus we have $\phi_{a^f}(z^f) = \phi_a(z)$. □

Definition 3.2.8. *Let (,) be a Frobenius form on an $\mathcal{F}$-axial algebra A. Then the radical of the form, denoted by $A^\perp$, is*

$$A^\perp = \{x \in A : (x,y) = 0 \textit{ for all } y \in A\}.$$

Suppose x is an element of $A^\perp$ and y, z are elements of A. Then

$$(xy, z) = (x, yz) = 0.$$

That is, xy belongs to $A^\perp$, hence, $A^\perp$ is an ideal of A.

Lemma 3.2.9. *([21], 4.8) Suppose A is an axial algebra with the Frobenius form $(\ ,\ )$ and $A^\perp$ be the radical. A primitive axis a lies in $A^\perp$ if and only if $(a, a) = 0$.*

Proof. Since a is a primitive axis, A can be decomposed as

$$A = A_1(a) \bigoplus_{1 \neq \lambda \in \mathcal{F}} A_\lambda(a)$$

with respect to a. For any $x \in A$, $x = \mu a + \sum_{\lambda \neq 1} x_\lambda$ for some scalar $\mu \in \mathbb{F}$ and $x_\lambda \in A_\lambda(a)$. From the Proposition 3.2.4, $A_1(a)$ is orthogonal to all of $\bigoplus_{1 \neq \lambda \in \mathcal{F}} A_\lambda(a)$. Therefore, $(a, x) = \mu(a, a)$. Hence $a \in A^\perp$ if and only if $(a, a) = 0$. □

Recall that radical of A with respect to an axis set $\mathcal{A}$ is the unique largest ideal $R(A, \mathcal{A})$ that contains no axis of $\mathcal{A}$. The following theorem gives the necessary and sufficient condition for the radical $R(A, \mathcal{A})$ to coincide with the radical of the Frobenius form.

Theorem 3.2.10. *([21], 4.8) Let A be a primitive $\mathcal{F}$-axial algebra generated by a set of axes $\mathcal{A}$ and $(\ ,\ )$ be a Frobenius form on A. Then, the radical $A^\perp = \mathcal{R}(A, \mathcal{A})$ if and only if $(a, a) \neq 0$ for all $a \in \mathcal{A}$.*

Proof. Let $\mathcal{R} = \mathcal{R}(A, \mathcal{A})$. Suppose $A^\perp = \mathcal{R}$, which implies $A^\perp$ does not contain any axis. Therefore, in view of previous lemma, $(a, a) \neq 0$ for all $a \in \mathcal{A}$.

Now assume that $(a, a) \neq 0$ for all $a \in \mathcal{A}$. This means $A^{\perp}$ does not contain any axis. Since $\mathcal{R}$ the unique largest ideal not containing any axis from $\mathcal{A}$, $A^{\perp} \subseteq \mathcal{R}$. To show $\mathcal{R} \subseteq A^{\perp}$, it suffices to show that $(x, \mathcal{R}) = 0$ for all $x \in A$. Again by linearity we can restrict to x being a product of axes. Let $m(x)$ be the length of x, that is, the number of axes in the product-expression for x.

We use induction on $m(x)$ to prove the claim. Assume that $m(x) = 1$. This implies x is an axis of A. Since $\mathcal{R}$ does not contain any axis, from the Lemma 3.2.9, $(x, \mathcal{R}) = 0$. Now assume that the claim is true for all $y \in A$ satisfying $m(y) < m(x)$. We have $x = x_1 x_2$ for some $x_1, x_2 \in A$ with $m(x_1), m(x_2) < m(x)$. Consider $(x, \mathcal{R}) = (x_1 x_2, \mathcal{R}) = (x_1, x_2\mathcal{R})$. Since $\mathcal{R}$ is an ideal, $x_2\mathcal{R} = \mathcal{R}$. Therefore, $(x, \mathcal{R}) = (x_1, \mathcal{R}) = 0$, from the induction hypothesis. It follows that $\mathcal{R} \subseteq A^{\perp}$. Hence $\mathcal{R} = A^{\perp}$. $\square$

Definition 3.2.11. *Let A be a primitive axial algebra and $\mathcal{A}$ be a set of axes. The projection graph Ω is a directed graph on $\mathcal{A}$ such that there is a directed edge from x to y if the projection $\phi_y(x)y$ of x onto $A_1(y)$ is nonzero.*

Since the Miyamoto group $\mathcal{G}(\mathcal{A})$ is a subgroup of $Aut(\Omega)$, we can form a quotient graph $\bar{\Omega} = \Omega/\mathcal{G}(\mathcal{A})$. In that case, the vertices are the orbits of axes, $x^{\mathcal{G}(\mathcal{A})}$ for $x \in \mathcal{A}$. The graph $\bar{\Omega}$ is called *orbit projection graph.*

Proposition 3.2.12. *[21] If A is a primitive axial algebra with a strongly connected orbit projection graph then all the proper ideals are contained in the radical.*

Chapter 4

Fischer spaces and Matsuo algebras

In the first section we recall the basic definitions and some of the results on Coxeter groups and groups of 3-transpositions. Then we define Fischer spaces and related Matsuo algebras.

4.1 Coxeter groups and classification

The notations, definitions and results of this section come from [5].

Definition 4.1.1. *Let $C = (c_{ij})$ be an $n \times n$ symmetric matrix with each entry being a natural number such that $c_{ii} = 1$ and $c_{ij} > 1$ for $i \neq j$. Then the Coxeter group of type C is a quotient of the free group $F(n) = \{r_1, r_2, \ldots, r_n\}$ with n generators.*

That is, the Coxeter group $W(C)$ is

$$W(C) = \langle \{r_1, r_2, \ldots, r_n : (r_i r_j)^{c_{ij}} = 1, i, j \in \{1, 2, \ldots, n\}\}\rangle.$$

The Coxeter diagram associated with $W(C)$ is a graph $\Gamma(C)$ where the vertex set is $\{r_1, r_2, \ldots r_n\}$, and there exists an edge between r_i and r_j if $c_{ij} > 2$. An edge is labelled by c_{ij}. Often the label between r_i, r_j is omitted whenever $c_{ij} = 3$. And a double bond is drawn if $c_{ij} = 4$. Note that there is no bond when $c_{ij} = 2$.

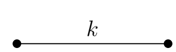
Example 4.1.2. *If $C = \begin{pmatrix} 1 & k \\ k & 1 \end{pmatrix}$ then the Coxeter group $W(C)$ is*

$$W(C) = \langle r_1, r_2 \mid {r_1}^2 = {r_2}^2 = (r_1 r_2)^k = 1\rangle,$$

which is isomorphic to the dihedral group D_{2k}. The matrix

$$\begin{pmatrix} 1 & k \\ k & 1 \end{pmatrix}$$

is denoted by $I_2(k)$. Note that the Coxeter diagram of $W(C)$ is as follows.



In particular the Coxeter group of $I_2(2)$;

$$W(I_2(2)) = \langle r_1, r_2 \mid {r_1}^2 = {r_2}^2 = (r_1 r_2)^2 = 1\rangle$$

is isomorphic to Klein-four group.

i1

Example 4.1.3. *The Coxeter group of the matrix*

$$C = \begin{pmatrix} 1 & 3 & 2 & 2 & \dots & 2 \\ 3 & 1 & 3 & 2 & \dots & 2 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 2 & \dots & 2 & 3 & 1 & 3 \\ 2 & \dots & 2 & 2 & 3 & 1 \end{pmatrix}_{n\times n}$$

is $W(C) = \langle r_1, r_2, \dots, r_n \rangle$*, with generators satisfying*

$$(r_i r_j)^3 = 1 \text{ when } |i-j| = 1,$$
$$(r_i r_j)^2 = 1 \text{ otherwise.}$$

This group is isomorphic to S_{n+1}*.*

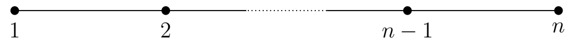
A Coxeter group $W(C)$ is called irreducible if the associated Coxeter diagram is connected.

At this point we state two important results which will be used later.

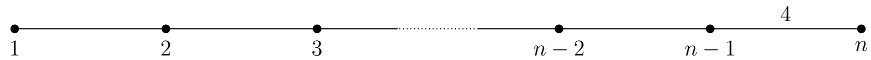
Theorem 4.1.4. *Let* $W(C)$ *be a Coxeter group and* $\Gamma(M_1), \Gamma(M_2), \dots, \Gamma(M_k)$ *be the connected components of* $\Gamma(C)$*. Then* $W(C) \cong W(M_1) \times W(M_2) \times \dots \times W(M_k)$*.*

Theorem 4.1.5. *A finte irreducible Coxeter group is isomorphic to exactly one of the following:*

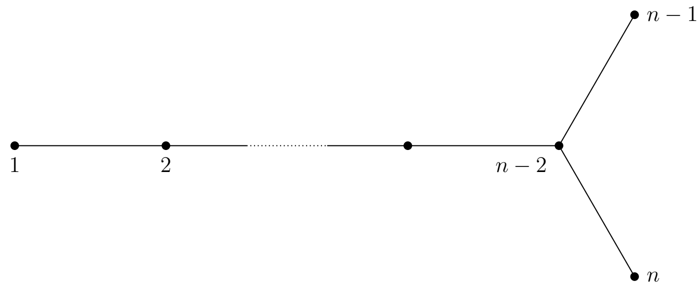
1. $A_n, n \geq 1$*:*



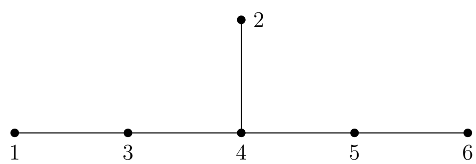
2. $B_n = C_n, n \geq 3$*:*



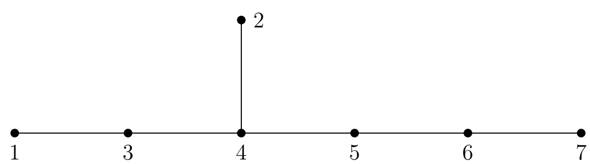
3. $D_n, n \geq 4$*:*



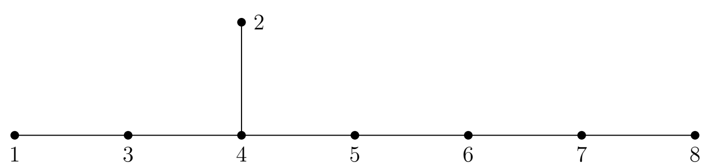
4. E_6*:*



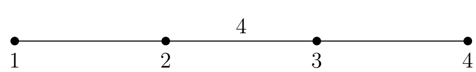
5. E_7*:*



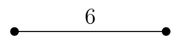
6. E_8*:*



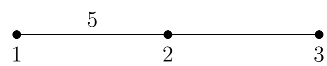
7. F_4*:*



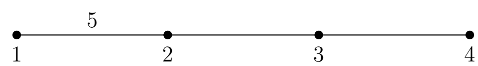
8. G_2*:*



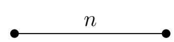
9. H_3*:*



10. H_4*:*



11. $I_2^{(n)}, n \geq 5$*:*



An important class of infinite Coxeter groups are the affine Coxeter groups. These groups have easily realisable finite quotients. We just list some of the infinite Coxeter groups.

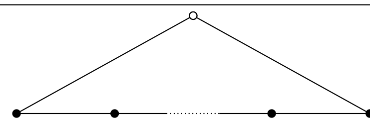
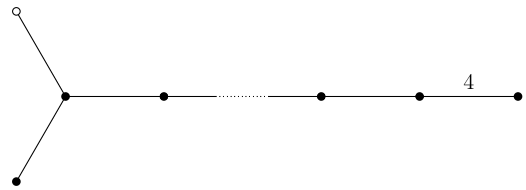
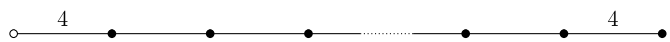
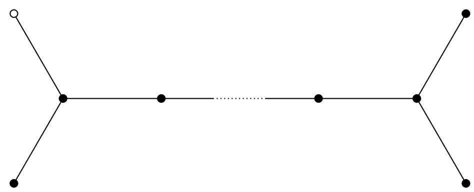
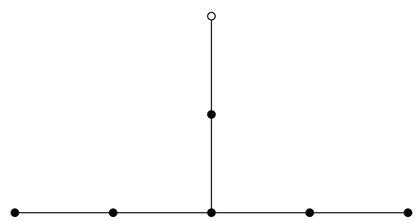
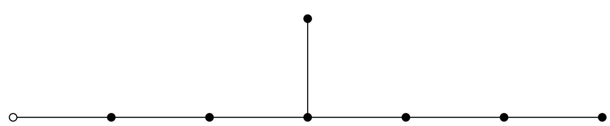
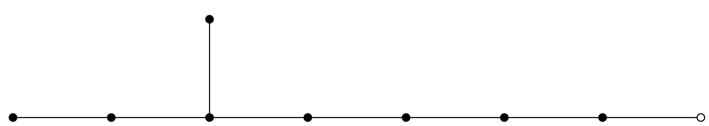
Name	Diagrams
$\tilde{A}_n$	
$\tilde{B}_n$	
$\tilde{C}_n$	
$\tilde{D}_n$	
$\tilde{E}_6$	
$\tilde{E}_7$	
$\tilde{E}_8$	
$\tilde{F}_4$	
$\tilde{G}_2$	
$\tilde{H}_3$	
$\tilde{H}_4$	
$\tilde{I}_1$	

Table 4.1: Some of the infinite Coxeter groups.

4.2 3-transposition groups

Definition 4.2.1. *Let G be a group and D be a normal subset of involutions. Elements of D are said to be* 3*-transpositions if the order of ab is at most 3 for all a,$b \in D$. If D generates G then (G, D) is called a* 3*-transposition group.*

Theorem 4.2.2. *([7], 1.1) Let (G, D) be a* 3*-transposition group. Assume further that it is irreducible. Then, up to a center, we may identify D with one of:*

1. *The transposition class of a symmetric group;*
2. *The transvection class of the isometry group of a nondegenerate orthogonal space over $GF(2)$;*
3. *The transvection class of the isometry group of a nondegenerate symplectic space over $GF(2)$;*
4. *A reflection class of the isometry group of a nondegenerate orthogonal space over $GF(3)$;*
5. *The transvection class of the isometry group of a nondegenerate unitary space over $GF(4)$;*
6. *A unique class of involutions in one of the five groups $P\Omega^+(2) : S_3$, $P\Omega^+(3) : S_3$, Fi_{22}, Fi_{23}, or Fi_{24}.*

Definition 4.2.3. *Let (G, D) be a* 3*-transposition group. The generating set D is said to be of symplectic type if for every $a, b, c \in D$ with $\langle b, c\rangle \cong S_3$, the transposition a commutes with at least one of $\{b, c, b^c = c^b\} = D \cap \langle b, c\rangle$.*

The symmetric group, orthogonal groups over $\mathbb{F}_2$, and symplectic groups $Sp(2n, 2)$ are of symplectic type.

Suppose (G, D) is a 3-transposition group with D being of symplectic type. Then the generating set D is said to be of ADE-type if there is no subgroup $H = \langle D \cap H \rangle$ isomorphic to the central quotient of $W_2(\tilde{D_4})$, where, $\tilde{D}_4$ is the diagram which is the complete bipartite graph $K_{1,4}$ and $W_2(\tilde{D_4}) \cong 2^4 : W(D_4)$.

Now we state two important results on groups of 3-transpositions.

Theorem 4.2.4. *([18], 4.4) Let (G, D) be a 3-transposition group and X be a subset of D. Suppose $H = \langle X \rangle$.*

1. *If the diagram on X is A_n, then H is isomorphic to the Coxeter group $W(A_n) \cong S_{n+1}$.*

2. *If the diagram on X is D_n, then H is isomorphic to a central quotient of the Coxeter group $W(D_n)$. In other words, there are two possibilities for H: It is either isomorphic to $2^{n-1} : S_n$ or isomorphic to $W(D_{2k})/Z(W(D_{2k})) \cong 2^{2k-2} : S_{2k}$.*

3. *If the diagram on X is E_6, then H is isomorphic to the group $W(E_6) \cong O_6^-(2)$.*

4. *If the diagram on X is E_7, then H is either isomorphic to $W(E_7) \cong 2 \times Sp_6(2)$ or isomorphic to the group $W(E_7)/Z(W(E_7)) \cong Sp_6(2)$.*

5. *If the diagram on X is E_8, then H is either isomorphic to $W(E_8) \cong 2 \cdot O_8^+(2)$ or isomorphic to the group $W(E_8)/Z(W(E_8)) \cong O_8^+(2)$, where, $\cdot$ denotes the non split extension.*

Theorem 4.2.5. *([18], 4.7) Let (G, D) be a 3-transposition group where the generating set D is of symplectic type. Then there exists a normal subgroup N of G such that the factor group G/N is isomorphic to one of the groups S_n, $O^{\epsilon}_{2m}(2)$ or $Sp_{2m}(2)$, where, $n \geq 2$, $n \neq 4$ and $m \geq 3$ with $(m, \epsilon) \neq (3, +)$.*

Let a and b be 3-transpositions with $|ab| = 3$. Then the group generated by $\{a, b\}$ is isomorphic to S_3. Naturally, a 3-dimensional group algebra can be constructed on $\langle\langle a, b\rangle\rangle$ as follows (note that the symbol $\langle\langle\ ,\ \rangle\rangle$ means algebra generation). Let $S = \{a, b, c\}$, where $c = a^b = b^a$, and $3C(\eta)$ denotes an $\mathbb{F}$-algebra, where the product is defined as

$$i \cdot j = \frac{\eta}{2}(i + j - k),$$

for distinct $i, j, k \in \{a, b, c\}$ for some $\eta \in \mathbb{F}$. We have shown, in Chapter 1, that $3C(\eta)$ is an axial algebra of Jordan type η generated by the set of axes $\{a, b, c\}$.

Definition 4.2.6. *A partial triple system is an ordered pair $\Gamma = (\mathcal{P}, \mathcal{L})$ of a set $\mathcal{P}$ of points and a collection $\mathcal{L}$ of lines, which are subsets of $\mathcal{P}$, such that every line contains exactly three points and two distinct points belong to at most one line.*

Define a relation $\sim$ on $\mathcal{P}$ as follows. The relation $p \sim q$ holds if there exists a line $l \in \mathcal{L}$ such that $l\{p, q\} \subseteq \mathcal{L}$. Use $p \nsim q$ to indicate that the points p, q are noncollinear. Let $p^{\sim}$ be the set containing all the points which are collinear to p(this includes p itself), and $p^{\nsim}$ be the set containing all the points which are noncollinear to p. Each $p^{\sim}$ is a connected component of Γ.

In a partial triple system if a line $l \in \mathcal{L}$ contains two points a and b then, by definition, it must contain a third point, say, c. Sometimes we use a^b to denote the third point.

Let $\mathcal{P}' \subseteq \mathcal{P}$, $\mathcal{L}' \subseteq \mathcal{L}$. Then the ordered pair $(\mathcal{P}', \mathcal{L}')$ is a subspace of $(\mathcal{P}, \mathcal{L})$ if $x, y \in \mathcal{P}'$ are on a line l then $l \in \mathcal{L}'$.

Definition 4.2.7. *For two intersecting lines l_1, l_2, the subspace generated by $l_1 \cup l_2$ is called a plane.*

Definition 4.2.8. *A morphism $\theta : (\mathcal{P}, \mathcal{L}) \to (\mathcal{P}', \mathcal{L}')$ is a map $\theta : \mathcal{P} \to \mathcal{P}'$ with the property $\theta\mathcal{L} \subseteq \mathcal{L}'$.*

Two important partial triple systems used in this work are the following:

The dual affine plane of order two $DA(2, 2)$ is the projective plane of order two with a point such that all the lines through it are removed. Thus $DA(2, 2)$ has six points and four lines.

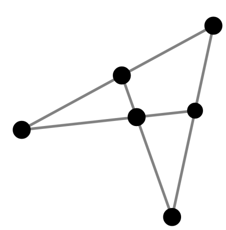


Figure 4.1: Dual affine plane of order 2

The affine plane of order three $AG(2, 3)$ has nine points and twelve lines. Each point is a vector and each line is the 1-flat of $\mathbb{F}_3^2$.

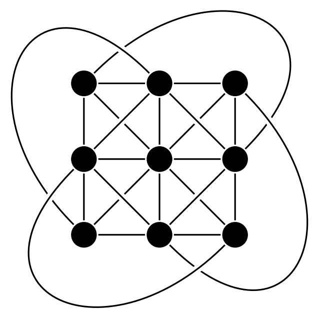


Figure 4.2: The affine plane of order 3

Definition 4.2.9. *A Fischer space is a partial triple system in which every plane is isomorphic to either the dual affine plane of order two, $DA(2,2)$ or the affine plane of order three, $AG(2,3)$.*

The following theorems due to Buekenhout show that Fischer spaces arise from the the group of 3-transpositions.

Theorem 4.2.10 ([1], [4])**.** *Let (G, D) be a 3-transposition group. Let $\Gamma(G, D)$ be the partial triple system with point set D, and line set $\{(a, b, a^b) \mid a, b \in D, |ab| = 3\}$. Then $\Gamma(G, D)$ is a Fischer space and G acts on it by conjugation on the point set.*

Theorem 4.2.11 ([1], [4])**.** *Let $\Gamma = \Gamma(\mathcal{P}, \mathcal{L})$ be a Fischer space. For each point $p \in \mathcal{P}$, define τ_p be the permutation of Γ fixing $p \cup p^{\sim}$ and interchanging the two points different from p on any line through p. Let $F = \{\tau_p \mid p \in \Gamma\}$. Then $F \subseteq Aut(\Gamma)$, and $(\langle F \rangle, F)$ is a group of 3-transpositions with trivial centre. Moreover, $\Gamma(\langle F \rangle, F) \cong \Gamma$.*

4.3 The Matsuo Algebras

Definition 4.3.1. *Let $\Gamma(\mathcal{P}, \mathcal{L})$ be a partial triple system, $\mathbb{F}$ be a feild with char $\mathbb{F} \neq 2$, and $\eta \in \mathbb{F}$, $\eta \neq 0, 1$. Then the associated Matsuo algebra is the $\mathbb{F}$-space $\bigoplus_{p \in \mathcal{P}} \mathbb{F}a_p$ with the product of the basis elements as follows:*

$$a_p a_q = \begin{cases} a_p & \text{if } p = q, \\ 0 & \text{if } p \nsim q, \\ \frac{\eta}{2}(a_p + a_q - a_r) & \text{if } \{p, q, r\} \text{ is a line,} \end{cases}$$

For each line $\{p, r, s\} \in \Gamma$, let $a_{prs} = \eta a_p - a_r - a_s$ and $b_{prs} = a_r - a_s$.

Now we show that the Matsuo algebras are axial algebras of Jordan type η.

Lemma 4.3.2. *([16], 6.2) Let $\Gamma(\mathcal{P}, \mathcal{L})$ be a Fischer space, and $M = M(\Gamma, \eta)$ be the associated Matsuo algebra. Then the endomorphism ad_{a_p} is semisimple.*

Proof. Let $M_\lambda(a_p)$ be the λ-eigenspace of ad_{a_p}. Clearly, from the definition of the Matsuo product, for each $p \in \mathcal{P}$, a_p is an idempotent, consequently, 1 is an eigenvalue, and $a_p \in M_1(a_p)$. Furthermore, $a_q \notin M_1(a_p)$ for either $q \in p^{\nsim}$ or $\{p, q, r\}$ in $\mathcal{L}$. Hence,

$$M_1(a_p) \supseteq \mathbb{F}a_p.$$

In order to determine $M_0(a_p)$, we consider two possibilities, namely, $q \nsim p$ or $q \sim p$. Suppose that $q \nsim p$. Then by the definition of the algebra product, $a_q \in M_0(a_p)$. Now assume that $q \sim p$. From the definition of partial triple system, there exists a third point r such that $\{p, q, r\}$ is a line. Observe that

$$\begin{aligned}
a_p a_{pqr} &= a_p(\eta a_p - a_q - a_r) \\
&= \eta a_p - \frac{\eta}{2}(a_p + a_q - a_r) - \frac{\eta}{2}(a_p + a_r - a_q) \\
&= \eta a_p - \eta a_p - \frac{\eta}{2}(a_q - a_q + a_r - a_r) = 0.
\end{aligned}$$

Therefore, $a_{pqr} \in M_0(a_p)$.

To see the elements of $M_\eta(a_p)$, consider

$$\begin{aligned}
a_p b_{pqr} &= a_p(a_q - a_r) \\
&= \frac{\eta}{2}(a_p + a_q - a_r) - \frac{\eta}{2}(a_p + a_r - a_q) \\
&= \eta(a_q - a_r) = \eta b_{pqr}.
\end{aligned}$$

That is, $b_{pqr} \in M_\eta(a_p)$.

From the above expressions, we have

$$\begin{aligned}
M_1(a_p) &\supseteq \mathbb{F} a_p \\
M_0(a_p) &\supseteq \bigoplus_{q \nsim q} \mathbb{F} a_q \bigoplus_{q \in l} \mathbb{F} a_{pqr}, \\
M_\eta(a_p) &\supseteq \bigoplus_{p \sim q} \mathbb{F}(a_q - a_r) = \bigoplus_{q \in l} \mathbb{F} b_{pqr}.
\end{aligned}$$

For each $p \in \mathcal{P}$ and for each line $\{p, q, r\} \in \mathcal{L}$, the set $\{a_p, a_{pqr}, b_{pqr}\}$ is linearly independent and spans $\langle a_p, a_q, a_r \rangle$. Therefore the pair $\{a_q, a_r\}$ in $\mathcal{A}$ can be replaced by $\{a_{pqr}, b_{pqr}\}$. Hence ad_{a_p} is semisimple, that is

$$M(\Gamma,\eta) = \mathbb{F}a_p \bigoplus_{q \nsim q} \mathbb{F}a_q \oplus \bigoplus_{q\in l} \mathbb{F}a_{pqr} \bigoplus_{q\in l} \mathbb{F}b_{pqr}$$
$$= M_1 \bigoplus M_0 \bigoplus M_\eta.$$

□

We state a few results related to the Matsuo algebras. Proofs of the following results can be found in [16].

Theorem 4.3.3. *Let $\Gamma(\mathcal{P},\mathcal{L})$ be a Fischer space, and $M = M(\Gamma,\eta)$ be the associated Matsuo algebra. Then M is a primitive axial algebra of Jordan type η generated by the set of axes $\mathcal{A} = \{a_p \mid p \in \mathcal{P}\}$. Furthermore, for every $p \in \mathcal{P}$, there exists a unique Miyamoto involution $\tau(a_p)$.*

Theorem 4.3.4. *Let $\Gamma(\mathcal{P},\mathcal{L})$ be a partial triple system. Then the associated Matsuo algebra $M = M(\Gamma,\eta)$ is an axial algebra of Jordan type η if and only if $\Gamma(\mathcal{P},\mathcal{L})$ is a Fischer space.*

Remark 4.3.5. *The Matsuo algebra $M = M(\Gamma,\eta)$ admits a Frobenius form: For the axes a_p and a_q,*

$$[a_p, a_q] = \begin{cases} 1 & \text{if } p{=}q, \\ 0 & \text{if } q \nsim p, \\ \frac{\eta}{2} & \text{if } q \in p^{\sim}. \end{cases}$$

Therefore, the Gram matrix of the form with respect to the basis $\{a_p \mid p \in \mathcal{P}\}$ *is*

$$F_{[,]} = I + \frac{\eta}{2} M_{[,]},$$

where I *is the identity matrix and* $M_{[,]}$ *is the adjacency matrix of the non-commuting graph on the basis.*

Chapter 5

Axes of Monster type

Recall that an axial algebra of Monster type (α, β) is generated by a set of primitive $\mathcal{M}(\alpha, \beta)$-axes. Here we show that an axis of Monster type $(2\eta, \eta)$ arises as the sum of two orthogonal axes of Jordan type η. Call an axis of Jordan type a single axis and an axis of Monster type $(2\eta, \eta)$ a double axis. Then we study subalgebras in Matsuo algebras generated by one single axis, one double axis; and two double axes.

$*$	1	0	α	β
1	1		α	β
0		0	α	β
α	α	α	$1, 0$	β
β	β	β	β	$1, 0, \alpha$

Table 5.1: Fusion law $\mathcal{M}(\alpha, \beta)$

5.1 Fusion laws for double axes

Let A be an axial algebra satisfying a Seress fusion law $\mathcal{F}$. That is, $\mathcal{F}$ has two units, 1 and 0. In this section we first analyse the decomposition of A induced by a pair of orthogonal axes. As a particular case, we compute the fusion law arising from sum of two orthogonal axes of Jordan type η.

Proposition 5.1.1. *Let A be an axial algebra. If W is ad_a-invariant subspace of A for some axis a then ad_a is semisimple on W.*

Proof. Let $p(x) \in \mathbb{F}[x]$ be the minimum polynomial of ad_a. Consider the endomorphism $T := ad_a\big|_W$ on W. Clearly T satisfies $p(x)$, and so if $q(x)$ is the minimum polynomial of T then $p(x)$ divides $q(x)$. In particular, if λ is an eigenvalue of ad_a then $(x - \lambda)$ divides $q(x)$ and moreover, the multiplicity of $(x - \lambda)$ is 1. Hence, W can be written as direct sum of λ-eigenspaces of ad_a. $\square$

For orthogonal axes a and b of an axial algebra A, denote

$$A_{\mu,\lambda}(a, b) := A_\mu(a) \cap A_\lambda(b),$$

where $A_\mu(a)$, $A_\lambda(b)$ are the μ-, λ-eigenspaces respectively. Similarly, for two sets $M, N \subset \mathcal{F}$,

$$A_{M,N}(a, b) := A_M(a) \cap A_N(b).$$

Lemma 5.1.2. *Let A be an axial algebra satisfying a Seress fusion law $\mathcal{F}$. Suppose a and b are two orthogonal axes. Then for $\lambda \in \mathcal{F}$*

$$A_\lambda(a) = \bigoplus_{\mu \in \mathcal{F}} A_{\lambda,\mu}(a,b).$$

Proof. Since ad_b is semisimple on A and also on all ad_b-invariant subspaces of A, it suffices to show that $A_\lambda(a)$ is ad_b-invariant for all $\lambda \in \mathcal{F}$.

The axes a and b are orthogonal, i.e., $ab = 0$, which implies $b \in A_0(a)$. Consequently,

$$A_\lambda(a)b \subseteq A_\lambda(a)A_0(a) \subseteq A_\lambda(a).$$

The last inclusion is due to the fact that $\mathcal{F}$ is Seress with 0 being one of the units. Hence $A_\lambda(a)$ is ad_b-invariant. □

Corollary 5.1.3. *If a and b are two orthogonal axes of an axial algebra A which satisfies Seress fusion law $\mathcal{F}$ then*

$$A = \bigoplus_{\lambda,\mu \in \mathcal{F}} A_{\lambda,\mu}(a,b).$$

Lemma 5.1.4. *If A is an axial algebra of Jordan type η, and a, b are orthogonal axes then the subspaces $A_{1,1}(a,b), A_{1,\eta}(a,b)$ and $A_{\eta,1}(a,b)$ are zero.*

Proof. Since a and b are primitive, $A_{11}(a,b) = \mathbb{F}a \cap \mathbb{F}b = 0$.

Let $y \in A_1(a) \cap A_\eta(b)$, $y \neq 0$. Then $ay = y$ and $by = \eta y$ for some $\eta \neq 0{,}1$. Substituting for y in the second equation and applying $ab = 0$ gives $\eta = 0$ which is a contradiction. Therefore $A_{1,\eta}(a,b) = 0$. Similarly $A_{\eta,1}(a,b) = 0$. □

Lemma 5.1.5. *Let A be an axial algebra satisfying a fusion law $\mathcal{F}$ and a, b be orthogonal axes. Then the endomorphism ad_{a+b} acts on $A_{\lambda,\mu}(a,b)$ as scalar $\lambda+\mu$.*

Proof. Let $x = a+b$. Then for $u \in A_{\lambda,\mu}(a,b)$, we have,

$$xu = (a+b)u = au + bu = \lambda u + \mu u = (\lambda+\mu)u$$

as $u \in A_\lambda(a)$ and $u \in A_\mu(b)$. □

Lemma 5.1.6. *Let A be an axial algebra satisfying a fusion law $\mathcal{F}$. Suppose a and b are orthogonal axes. Then $A_{\alpha,\beta}(a,b)A_{\lambda,\mu}(a,b) \subseteq A_{\alpha*\lambda,\beta*\mu}(a,b)$ for $\alpha,\beta,\lambda,\mu \in \mathcal{F}$.*

Proof. We just apply the fusion law term wise to get the result. Consider

$$\begin{aligned} A_{\alpha,\beta}(a,b)A_{\lambda,\mu}(a,b) &= (A_\alpha(a) \cap A_\beta(b))(A_\lambda(a) \cap A_\mu(b)) \\ &\subseteq A_\alpha(a)A_\lambda(a) \cap A_\beta(b)A_\mu(b) \\ &= A_{\alpha*\lambda}(a) \cap A_{\beta*\mu}(a,b) = A_{\alpha*\lambda,\beta*\mu}(a,b). \end{aligned}$$

□

We now assume that a and b are orthogonal axes of Jordan type η. Note that $a+b$ is an idempotent. The following theorem shows that $a+b$ is an axis of Monster type $(2\eta,\eta)$.

Theorem 5.1.7. *Let A be an axial algebra of Jordan type η. Suppose a and b are two axes with $ab = 0$. Then A has the decomposition*

$$A = A_1(a+b) \bigoplus A_0(a+b) \bigoplus A_{2\eta}(a+b) \bigoplus A_\eta(a+b),$$

with respect to $a+b$, which satisfies the $\mathcal{M}(2\eta,\eta)$-fusion law.

Proof. From Proposition 5.1.1, Lemma 5.1.2, 5.1.4, we have,

$$A = A_{1,0}(a,b) \oplus A_{0,1}(a,b) \oplus A_{0,0}(a,b) \oplus A_{\eta,\eta}(a,b) \oplus A_{0,\eta}(a,b) \oplus A_{\eta,0}(a,b).$$

From Lemma 5.1.5,

$$\begin{aligned} A_1(a+b) &\supseteq A_{10}(a,b) \oplus A_{0,1}(a,b), \\ A_0(a+b) &\supseteq A_{00}(a,b), \\ A_{2\eta}(a+b) &\supseteq A_{\eta\eta}(a,b), \\ A_\eta(a+b) &\supseteq A_{0\eta}(a,b) \oplus A_{\eta 0}(a,b). \end{aligned}$$

That is, $A = A_1(a+b) \oplus A_0(a+b) \oplus A_{2\eta}(a+b) \oplus A_\eta(a+b)$. Thus ad_{a+b} is semisimple. And all the above inclusions are equal.

The subset $A_0(a+b) = A_{00}(a,b) = A_0(a) \cap A_0(b)$, being intersection of two subalgebras of A, is a subalgebra of A.

Now it remains to prove that the above mentioned decomposition of A satisfies $\mathcal{M}(2\eta,\eta)$-fusion law. For that we compute $\lambda * \mu$ for all $\lambda, \mu \in \{1, 0, 2\eta, \eta\}$. The algebra A is commutative, therefore, $\lambda * \mu = \mu * \lambda$.

For simplicity, set $A_{\lambda\mu} = A_{\lambda,\mu}(a,b)$.

We use Lemma 5.1.6 several times without explicitly quoting.

Note that $A_{\emptyset\lambda} = \emptyset$ for all $\lambda \in \mathcal{M}(2\eta,\eta)$.

We see that

$1 * 1 = \{1\}$:

$$
\begin{aligned}
A_1(a+b)A_1(a+b) &= (A_{10} + A_{01})(A_{10} + A_{01}) \\
&\subseteq A_{10} + A_{01} \\
&= A_1(a+b).
\end{aligned}
$$

$1 * 0 = \emptyset$:

$$
\begin{aligned}
A_1(a+b)A_0(a+b) &= (A_{10} + A_{01})A_{00} \\
&\subseteq A_{1*0\ 0*0} + A_{0*0\ 1*0} \\
&= \emptyset.
\end{aligned}
$$

$1 * 2\eta = 2\eta$:

$$
\begin{aligned}
A_1(a+b)A_{2\eta}(a+b) &= (A_{10} + A_{01})A_{\eta\eta} \\
&\subseteq A_{1*\eta\ 0*\eta} + A_{0*\eta\ 1*\eta} \\
&\subseteq A_{\eta\eta} = A_{2\eta}(a+b).
\end{aligned}
$$

$1 * \eta = \eta$:

$$\begin{aligned} A_1(a+b)A_\eta(a+b) &= (A_{10} + A_{01})(A_{\eta 0} + A_{0\eta}) \\ &\subseteq A_{\eta 0} + A_{\emptyset\ \eta}(a,b)A_{\eta\ \emptyset} + A_{0\eta} \\ &= A_{\eta 0} + A_{0\eta} = A_\eta(a+b). \end{aligned}$$

$0 * 0 = 0$:

$$A_0(a+b)A_0(a+b) = A_{00}A_{00} \subseteq A_{00} = A_0(a+b).$$

$0 * 2\eta = 2\eta$:

$$A_0(a+b)A_{2\eta}(a+b) = A_{00}A_{\eta\eta} \subseteq A_{\eta\eta} = A_{2\eta}(a+b).$$

$0 * \eta = \eta$:

$$\begin{aligned} A_0(a+b)A_\eta(a+b) &= A_{00}(A_{\eta 0} + A_{0\eta}) = A_{00}A_{\eta 0} + A_{00}A_{0\eta} \\ &\subseteq A_{0*\eta\ 0*0} + A_{0*0\ 0*\eta} \\ &= A_{\eta 0} + A_{0\eta} = A_\eta(a+b). \end{aligned}$$

$2\eta * 2\eta = \{1, 0\}$:

$$
\begin{aligned}
A_{2\eta}(a+b)A_{2\eta}(a+b) &= A_{\eta\eta}A_{\eta\eta} \subseteq A_{\eta*\eta\ \ \eta*\eta} = A_{\eta*\eta}(a) \cap A_{\eta*\eta}(b) \\
&= (A_1(a) + A_0(a)) \cap (A_1(b) + A_0(b)) \\
&= A_{11} + A_{10} + A_{01} + A_{00} \\
&= A_1(a+b) + A_0(a+b).
\end{aligned}
$$

$2\eta * \eta = \eta$:

$$
A_{2\eta}(a+b)A_{\eta}(a+b) = A_{\eta\eta}(A_{\eta 0} + A_{0\eta}); \tag{5.1}
$$

We compute each summand separately and substitute in the above expression.

$$
\begin{aligned}
A_{\eta\eta}A_{\eta 0} &\subseteq A_{\eta*\eta\ \ \eta*0} = A_{\{1,0\}\ \ \eta} \\
&= (A_1(a) + A_0(a)) \cap A_{\eta}(b) \\
&= A_{1\eta} + A_{0\eta} = \emptyset + A_{0\eta},
\end{aligned}
$$

$$
\begin{aligned}
A_{\eta\eta}A_{0\eta} &\subseteq A_{\eta*0\ \ \eta*\eta} = A_{0\ \ \{1,0\}} \\
&= A_{\eta}(a) \cap (A_1(b) + A_0(b)) \\
&= A_{\eta 1} + A_{\eta 0} = \emptyset + A_{\eta 0}.
\end{aligned}
$$

Substituting these results back in equation 5.1, we get the claimed result.

$\eta * \eta = \{1, 0, 2\eta\}$:

$$A_\eta(a+b)A_\eta(a+b) = (A_{\eta 0} + A_{0\eta})(A_{\eta 0} + A_{0\eta})$$
$$\subseteq A_{\eta 0}A_{\eta 0} + A_{\eta 0}A_{0\eta} + A_{0\eta}A_{0\eta}.$$

But

$$A_{\eta 0}A_{\eta 0} \subseteq A_{\eta*\eta\ \ 0*0} = A_{\{1,0\}\ \ 0}$$
$$= A_{\{1,0\}}(a) \cap A_0(b) = (A_1(a) + A_0(a)) \cap A_0(b)$$
$$= A_{10} + A_{00};$$

similarly, $A_{0\eta}A_{0\eta} \subseteq A_0(a) \cap A_{\{1,0\}}(b) = A_{01} + A_{00}$;

$$A_{\eta 0}A_{0\eta} \subseteq A_{\eta*0\ \ 0*\eta} = A_{\eta\eta}.$$

On substituting back, we get

$$A_\eta(a+b)A_\eta(a+b) \subseteq A_{10} + A_{00} + A_{01} + A_{00} + A_{\eta\eta}$$
$$= A_1(a+b) + A_0(a+b) + A_{2\eta}(a+b).$$

From the above computation we see that the axis $a+b$ satisfy the fusion law given by Table 5.2.

□

$*$	1	0	2η	η
1	1		2η	η
0		0	2η	η
2η	2η	2η	$1,0$	η
η	η	η	η	$1,0,2\eta$

Table 5.2: Fusion law satisfied by $a+b$.

Remark 5.1.8. *The decomposition*

$$A = A_1(a+b) \oplus A_0(a+b) \oplus A_{2\eta}(a+b) \oplus A_\eta(a+b)$$

is C_2-graded. Let

$$\begin{aligned} A_+(a+b) &= A_0(a+b) \oplus A_1(a+b) \oplus A_{2\eta}(a+b), \\ A_-(a+b) &= A_\eta(a+b). \end{aligned}$$

Then, from Table 5.2, we have $A_\epsilon(a+b)A_\delta(a+b) \subseteq A_{\epsilon\delta}(a+b)$ for $\epsilon, \delta \in \{+,-\}$. Therefore, naturally A inherits the Miyamoto involution.

Definition 5.1.9. *An axis of Jordan type η is called a single axis. The sum of two orthogonal single axes is called a double axis.*

Note that a single axis satisfies the Monster fusion law $\mathcal{M}(2\eta,\eta)$ where the 2η-eigenspace is zero. Therefore, an algebra generated by a set of single and double axes satisfies the fusion law $\mathcal{M}(2\eta,\eta)$.

Lemma 5.1.10. *Suppose $a+b$ is a double axis in A. Then $\tau_{a+b} = \tau_a\tau_b$.*

Proof. Consider an element $x \in A$. Since $a+b$ in an $\mathcal{M}(2\eta,\eta)$-axis, x can be expressed as $x = x_1 + x_0 + x_{2\eta} + x_\eta$, where, $x_i \in A_i(a+b)$ for $i \in \{1, 0, 2\eta, \eta\}$. The Miyamoto involution τ_{a+b} when acted on x, it fixes x_1, x_0 and $x_{2\eta}$ and negates x_η.

As x_1 and x_0 belong to different combinations of intersections of the subspaces $A_1(a)$, $A_0(a)$, $A_1(b)$ and $A_0(b)$ which are fixed by τ_a and τ_b, they are fixed by $\tau_a\tau_b$. Next consider $x_{2\eta} \in A_{\eta\eta}(a,b) = A_\eta(a) \cap A_\eta(b)$. Clearly $x_{2\eta}$ is negated under the action of τ_a and τ_b, consequently, it is fixed under the action of $\tau_a\tau_b$.

To see the action of $\tau_a\tau_b$ on $x_\eta \in A_{\eta 0}(a,b) + A_{0\eta}(a,b)$, let $x_\eta = y + z$, where $y \in A_0(a) \cap A_\eta(b)$ and $z \in A_\eta(a) \cap A_0(b)$. Therefore, $x_\eta^{\tau_a} = y - z$ and

$$(x_\eta^{\tau_a})^{\tau_b} = -y - z = -x_\eta.$$

That is, $\tau_a\tau_b$ negates x_η. □

5.2 Fixed subalgebras

A double axis $a+b$ may not be primitive in the algebra A. However, it can be primitive in a proper subalgebra of A. In what follows, we study a subalgebra of A in which double axes of the generating set are primitive.

Let A be an axial algebra of Jordan type η and H be a subgroup of $Aut(A)$. Consider a subset W of A which contains all those elements u such that $u^h = u$, for all $h \in H$. Suppose u, v are in W then $(uv)^h = u^h v^h = uv$. That is, W is closed for multiplication. And also, $(u+v)^h = u^h + v^h$, closed for addition. In this section we study some of the properties of these subalgebras.

Definition 5.2.1. *Let A be an axial algebra of Jordan type η. Suppose $H \leq Aut(A)$ is a subgroup. Then a subalgebra W of A satisfying $w^h = w$ for all $w \in W$ and $h \in H$ is called the fixed subalgebra with respect to H.*

In order to understand the dimension and structure of the fixed subalgebra relative to H, H has to permute all the axes of a generating set $\mathcal{A}$ of the axial algebra A. This happens in the case of Matsuo algebras. Therefore, from this point onwards, we assume A to be a Matsuo algebra.

Let $M = M_\eta(G, D)$ be the Matsuo algebra associated with the Fischer space Γ_G of a 3-transposition group (G, D). Here, for simplicity, we abuse the notation and use D to denote the basis of M. Thus, the elements of D are vectors as well.

Proposition 5.2.2. *Let $W \leq M$ be a fixed subalgebra with respect to $H \leq Aut(M)$. Then the dimension of W is equal to the number of H-obits on W.*

Proof. Let H act on D and suppose that $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_k$ are the orbits of the action. Then the condition, $w^h = w$ for all $w \in W$ and $h \in H$, implies that all coefficients of basis elements of an orbit $\mathcal{O}_i$, for $i \in \{1, 2, \ldots, k\}$, are equal. Now let

$$e_i = \sum_{d \in \mathcal{O}_i} d$$

for all $i \in \{1, 2 \ldots, k\}$. The set $\{e_1, e_2, \ldots, e_k\}$ spans W, and moreover, it is linearly independent. Thus the set $\{e_1, e_2, \ldots, e_k\}$ forms a basis for W. □

Corollary 5.2.3. *An element w of W can be written as*

$$w = \sum_{i=1}^{k} \alpha_i \sum_{d \in \mathcal{O}_i} d,$$

where k is the number of H-orbits of the action.

Corollary 5.2.4. *Let $h \in Aut(M)$ be an involution. Assume that there exists $a \in D$ such that a, a^h are orthogonal. Then the double axis $a + a^h$ is primitive in the fixed subalgebra W relative to $H = \langle h \rangle$.*

Proof. It is given that a and a^h are orthogonal, therefore, h does not fix a, consequently, $a + a^h$ is a double axis of the fixed subalgebra W. Further, as $\{a, a^h\}$ is a single orbit, a and a^h do not belong to W seperately. Hence, 1-eigenspace of ad_{a+a^h} of W is,

$$W_1(a + a^h) = M_1(a + a^h) \cap W = \langle a, a^h \rangle \cap W = \langle a + a^h \rangle.$$

This shows that $a + a^h$ is primitive in W. $\square$

Remark 5.2.5. *For an involution $h \in Aut(M)$, the action of $\langle h \rangle$ on D has three types of orbits: The first type orbits are of length one. In this case $a^h = a$. Each such orbit gives a single axis of W. Second type orbits are of length two satisfying the condition $aa^h = 0$, for $a \in D$. From each such orbit, we have a primitive double axis $a + a^h$ of W. The third type orbits are also of length two, however, $aa^h \neq 0$ for $a \in D$. Axes arising from these orbits are called extras.*

5.3 Diagrams

Definition 5.3.1. *Let M be the Matsuo algebra for the group (G, D). Suppose x is an axis of M. Then the support of x is the corresponding set $\{a\} \subseteq D$ whenever $x = a$ is a single axis and the support of x is the set $\{a, b\} \subseteq D$ whenever $x = a + b$ is a double axis. Then the support of a set X of axes is the union of supports of all the axes present in X.*

Definition 5.3.2. *The axes x and y are said to be independent if their supports are disjoint. A set of axes is said to be independent if every pair of axes in the set is independent.*

From now on $\mathcal{B}$ represents an independent set of axes.

Definition 5.3.3. *For an independent set of axes $\mathcal{B}$, the diagram $\Lambda_{\mathcal{B}}$ is a graph whose vertex set is the support of $\mathcal{B}$ and two vertices a, b have an edge in $\Lambda_{\mathcal{B}}$ if a and b are collinear in the Fischer space Γ_G.*

Definition 5.3.4. *Let $\Lambda_{\mathcal{B}}$ be the diagram of the independent set $\mathcal{B}$. Let θ be the permutation of $\mathcal{B}$ which fixes all the points corresponding single axes and interchanges the points of each support set that correspond to a double axis. If θ is an automorphism of $\Lambda_{\mathcal{B}}$ then it is called the flip of the diagram.*

We say that the diagram $\Lambda_{\mathcal{B}}$ has the flip whenever the flip θ exists. On the other hand, if θ is not a flip then, we say, $\Lambda_{\mathcal{B}}$ has no flip.

In the following theorem we prove the relation between a diagram having the flip and the primitivity of the axes.

Lemma 5.3.5. *Let A be an axial algebra satisfying the fusion law $\mathcal{F}$. Let x be an axis and W be an ad_x-invariant subspace of A. Assume that an element $y \in W$ has the following decomposition with respect to x,*

$$y = \sum_{\lambda \in \mathcal{F}} y_\lambda,$$

where $y_\lambda \in A_\lambda(x)$ for all $\lambda \in \mathcal{F}$. Then each component y_λ belongs to W.

Proof. Since W is ad_x-invariant, any polynomial in ad_x when acted on W leaves it invariant. In particular, consider the polynomial

$$f_\lambda(t) = \prod_{\lambda \neq \mu \in \mathcal{F}} (t - \mu),$$

plugging ad_x in place of t and letting the polynomial $f(ad_x)$ act on y,

$$f_\lambda(ad_x)y = (\prod_{\lambda \neq \mu \in \mathcal{F}} (\lambda - \mu))y_\lambda.$$

As the left hand side of the above expression belongs to W so is y_λ. $\square$

Now we return to considering single and double axes in a Matsuo algebra M.

Theorem 5.3.6. *Let $\mathcal{B}$ be an independent set of axes in M and $\Lambda_{\mathcal{B}}$ be the corresponding diagram. Then the subalgebra W generated by $\mathcal{B}$ is primitive only if $\Lambda_{\mathcal{B}}$ has the flip.*

Proof. Clearly W is ad_x-invariant for all $x \in \mathcal{B}$. If $\mathcal{B}$ contains only single axes then the diagram $\Lambda_{\mathcal{B}}$ admits the flip, namely, the identity map. In this case, W is primitive. Now assume that there exists at least one double axis, say $x = a + b$ in $\mathcal{B}$. For any other axis $y \in \mathcal{B}$, its decomposition with respect to x is

$$y = y_1 + y_0 + y_{2\eta} + y_\eta,$$

where $y_\lambda \in A_\lambda(x)$ for $\lambda \in \{1, 0, 2\eta, \eta\}$. Using Lemma 5.3.5, the component y_1 can be expressed in terms of x and y using the polynomial $f(t) = t(t - 2\eta)(t - \eta)$. In particular,

$$\begin{aligned} f_1(ad_x)y &= (1 - 2\eta)(1 - \eta)y_1 \\ &= (1 - 2\eta)(1 - \eta)(\alpha a + \beta b). \end{aligned}$$

The last equality follows from $y_1 \in A_1(x) = \mathbb{F}a + \mathbb{F}b$. Note that the scalars α and β are the projections of y on the axes a and b respectively. Therefore, $\alpha = (y, a)$ and $\beta = (y, b)$, where $(\ , \ )$ denotes the Frobenius form on M. The term $\alpha a + \beta b$ belongs to W as $f_1(ad_x)y$ belongs to W. For the primitivity of the axis x in W, we must have

$$W_1(x) = W \cap \langle a, b \rangle = \langle x \rangle.$$

Hence, x is imprimitive unless $\alpha = \beta$ or equivalently $(y, a) = (y, b)$. That is, the points a and b should have same number of neighbours within the support of any other axis $y \in \mathcal{B}$. If y is a single axis, say $y = c$, then the condition $(y, a) = (y, b)$ implies that either there exist edges between a, c and b, c or no edge between a, c and b, c. If y is a double axis, say $y = c + d$, then repeating the above process for

y, we see that, for the primitivity of y, the axes c and d must have same number of neighbours within the support of $x \in \mathcal{B}$. That is, the nonsymmetric subdiagrams shown in Figure 5.1 will not lead to primitivity of the axes. Hence, the subalgebra

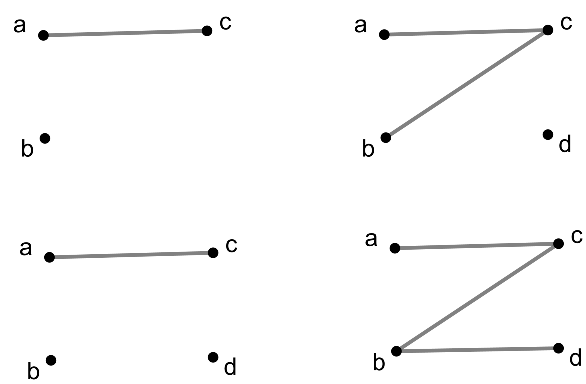


Figure 5.1: Diagrams not having the flip.

W is primitive only if the diagram $\Lambda_{\mathcal{B}}$ has the flip. $\square$

We give a partial converse of Theorem 5.3.6.

Theorem 5.3.7. *Let the diagram $\Lambda_{\mathcal{B}}$ have the flip θ which extends to an automorphism of the algebra M. Then the algebra $\langle\langle \mathcal{B} \rangle\rangle$ is primitive.*

Proof. Note that θ is an automorphism of order two which fixes $\langle\langle \mathcal{B} \rangle\rangle$. That is, $\langle\langle \mathcal{B} \rangle\rangle$ is the fixed subalgebra relative to $\langle \theta \rangle$. Hence, it is primitive. $\square$

5.4 2-generated subalgebras

Recall that M is the Matsuo algebra associated with the Fischer space $\Gamma(G, D)$ of 3-transposition group (G, D). Let x, y be axes of M and $\mathcal{B} = \{x, y\}$. Clearly, $\mathcal{B}$ is independent, and the diagram $\Lambda_{\mathcal{B}}$ is a subgraph induced on the support set of $\mathcal{B}$ by the diagram on D. The subgroup $F := \langle \Lambda_{\mathcal{B}} \rangle$ is a quotient of the Coxeter group with diagram $\Lambda_{\mathcal{B}}$. The elements of D in F form a conjugacy class of F. If T denote the conjugacy class then (F, T) is a 3-transposition group. Let Γ_F be the Fischer space of (F, T).

In the following section we study two-generated subalgebras $\langle\langle x, y \rangle\rangle$ of M. The subalgebras generated by $\{x, y\}$ where x is a single axis and y is a double axis will be called MA-type. The subalgebras generated by $\{x, y\}$ where both x and y are double axes will be called MB-type.

Let $x = a$ and $y = b + c$. All possible diagrams on $\mathcal{B} = \{x, y\}$ that have flips are given in Figure 5.2.

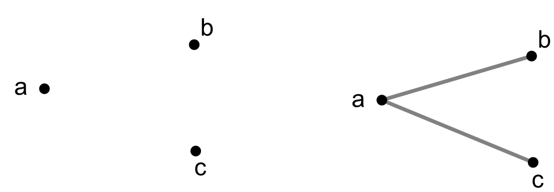


Figure 5.2: Diagram A_1 and diagram A_2.

Lemma 5.4.1. *The algebra arising from the diagram A_1 is isomorphic to $2B$.*

Proof. The Coxeter group $\hat{G}$ of the diagram A_1 is an elementary abelian 2-group of rank 3. Therefore, (F, T) can be $\hat{G}$ or a factor group of it. In either case $\langle\langle x, y \rangle\rangle$ satisfies the relations

$$x^2 = x, \ y^2 = y, \ xy = 0$$

as $ab = ac = bc = 0$. Therefore, it is isomorphic to $2B$. This is an associative algebra whose only idempotents are x, y. And the identity element is $x + y$. $\square$

Lemma 5.4.2. *The algebra arising from diagram A_2 is primitive, and has two single axes and two double axes.*

Proof. The Coxeter group $\hat{G}$ of the diagram A_2 is S_4. Since (F, T) can not be isomorphic to the factor groups S_3 and $\mathbb{Z}_2$ of $\hat{G}$, we conclude $F = \hat{G}$. The Fischer space of S_4 is given in Figure 5.3.

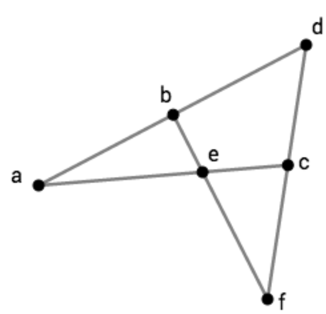


Figure 5.3: Fischer space Γ_F of S_4.

We have $x = a$ and $y = b + c$. Therefore,

$$\begin{aligned} w := \frac{2}{\eta}(xy) &= a(b+c) \\ &= a + b - d + a + c - e \\ &= 2a + (b+c) - (d+e). \end{aligned}$$

If τ_x and τ_y are the Miyamoto involutions with respect to x and y. Then

$$\begin{aligned} x^{\tau_y} &= a^{\tau_b \tau_c} = f := s, \\ y^{\tau_x} &= (b+c)^{\tau_a} = d + e := t. \end{aligned}$$

Thus we have

$$\begin{aligned} w - x + x^{\tau_y} &= 2a + (b+c) - (d+e) - a + f \\ &= x + y - y^{\tau_x} + x^{\tau_y}. \end{aligned}$$

The above computation implies that

$$xy = \frac{\eta}{2}(2x + y - t).$$

Therefore, y is primitive in $\langle\langle x, y \rangle\rangle_{A_2}$. Furthermore, the set $\{x, y, s, t\}$ is linearly independent in $\langle\langle x, y \rangle\rangle_{A_2}$ as they are linearly independent in M. Next we show that the above set is closed for the algebra product by computing all possible products of the basis elements. From the diagram we observe the following.

$xs = 0.$

$xt = a(d+e) = \frac{\eta}{2}(a+d-b)+\frac{\eta}{2}(a+e-c) = \frac{\eta}{2}(2a+d+e-(c+b)) = \frac{\eta}{2}(2x+t-y).$

Product of y and s:

$$\begin{aligned} ys &= (b+c)f = bf + cf \\ &= \frac{\eta}{2}(b+f-e) + \frac{\eta}{2}(c+f-d) \\ &= \frac{\eta}{2}(2f+b+c-(d+e)) \\ &= \frac{\eta}{2}(2s+y-t). \end{aligned}$$

Product of y and t:

$$\begin{aligned} yt &= (b+c)(d+e) = bd + be + cd + ce \\ &= \frac{\eta}{2}(b+d-a+b+e-f+c+d-f+c+e-a) \\ &= \eta(b+c+d+e-a-f) \\ &= \eta(y+t-x-s). \end{aligned}$$

Product of s and t:

$$
\begin{aligned}
st &= f(d+e) = fd + fe \\
&= \frac{\eta}{2}(f + d - c + f + e - b) \\
&= \frac{\eta}{2}(2s + t - y).
\end{aligned}
$$

Thus $\langle\langle x, y\rangle\rangle_{A_2}$ is a 4-dimensional algebra generated by the set of axes $\{x, y, s, t\}$, and the product of the axes is given by Table 5.3. $\square$

	x (single)	y (double)	s (single)	t (double)
x	x	$\frac{\eta}{2}(2x+y-t)$	0	$\frac{\eta}{2}(2x+t-y)$
y	$\frac{\eta}{2}(2x+y-t)$	y	$\frac{\eta}{2}(2s+y-t)$	$\frac{\eta}{2}(y+t-s-x)$
s	0	$\frac{\eta}{2}(2s+y-t)$	s	$\frac{\eta}{2}(2s+t-y)$
t	$\frac{\eta}{2}(2x+t-y)$	$\frac{\eta}{2}(y+t-s-x)$	$\frac{\eta}{2}(2s+t-y)$	t

Table 5.3: Multiplication table for $\langle\langle x, y\rangle\rangle_{A_2}$.

Remark 5.4.3. *To identify this algebra with the 2-generated family of algebras appeared in Rehren's work [26], we need to compute $n = |x^F \cup y^F|$. Clearly, $x^F = \{x, s\}$ and $y^F = \{y, t\}$. Therefore $n = 4$. The only covers with coefficient 4 are $4A(\eta)$ and*

$4B(\eta)$, *and both are* 5*-dimensional. Hence,* $\langle\langle x, y\rangle\rangle_{A_2}$ *does not belong to the family of covers.*

Proposition 5.4.4. *The algebra* $\langle\langle x, y\rangle\rangle_{A_2}$ *has the following decomposition with respect to* y*:*

$$\langle\langle x, y\rangle\rangle_{A_2} = \mathbb{F}y \oplus \mathbb{F}(x - 2\eta y + s + 2t) \oplus \mathbb{F}(x - \eta y + s + 2t) \oplus \mathbb{F}(x + s + 2t).$$

In particular, y *is primitive.*

Proof. We just determine 0-, 2η- and η-eigenspaces of ad_y. Note that

$$\begin{aligned}
y(x - 2\eta y + s + 2t) &= xy - 2\eta y + ys + 2yt \\
&= \frac{\eta}{2}(2x - 2t + 2s + 2t - 2s) - 2\eta y + 2\eta = 0, \\
y(x + s + 2t) &= yx + ys + 2yt \\
&= \frac{\eta}{2}(2x - t + 2s - t + 2t - 2s - 2x) + 2\eta y = 2\eta y, \\
y(x - \eta y + s + 2t) &= yx - \eta y + ys + 2yt \\
&= \frac{\eta}{2}(2x - t + 2s - t + 2t - 2s - 2x) + 2\eta y - \eta y = \eta y.
\end{aligned}$$

Comparing the dimension of $\langle\langle x, y\rangle\rangle_{A_2}$ with the dimensions of the eigenspaces, the claim follows. $\square$

Proposition 5.4.5. *The algebra* $\langle\langle x, y\rangle\rangle_{A_2}$ *has a nontrivial radical only for* $\eta = -\frac{1}{2}$*. The dimension of the radical is* 1*.*

Proof. We show that the determinant of the Gram matrix of $\langle\langle x, y\rangle\rangle_{A_2}$ vanishes when $\eta = -\frac{1}{2}$.

First we change the basis $\{x, s, y, t\}$ to a suitable basis with respect to which the Gram matrix is easy to analyse. Note that change of basis alter the value of the determinant, however, we are looking for those values of η for which the determinant becomes zero. and this will not change under change of basis.

Let $\{x, s, y, t\}$ change to $\{x, s, z, w\}$, where $z = y + t$ and $w = y - t$. Then transform $\{x, s, z, w\}$ to $\{x, s, \tilde{z}, w\}$, where, $\tilde{z} = z - 2\eta x - 2\eta s$. It is clear that $\{x, s, \tilde{z}, w\}$ is linearly independent.

Note that $\tilde{z}$ is perpendicular to $\langle x, s, w\rangle$:

$$
\begin{aligned}
(x, \tilde{z}) &= (x, y + t - 2\eta x - 2\eta s) = \eta + \eta - 2\eta + 0 = 0, \\
(s, \tilde{z}) &= (s, y + t - 2\eta x - 2\eta s) = \eta + \eta + 0 - 2\eta = 0, \\
(w, \tilde{z}) &= (y - t, y + t - 2\eta x - 2\eta s) \\
&= (y + t, y - t) - 2\eta(x + s, y - t) = 0 - 2\eta(\eta - \eta + \eta - \eta) = 0.
\end{aligned}
$$

Further note that

$$
\begin{aligned}
(\tilde{z}, \tilde{z}) &= 2(2 + 2\eta - 4\eta^2) = 4(1 + \eta - 2\eta^2), \\
(w, w) &= 4 - 2(y, t) = 4(1 - \eta), \\
(x, w) &= (s, w) = \eta - \eta = 0.
\end{aligned}
$$

Therefore, the Gram matrix of $\langle\langle x, y\rangle\rangle_{A_2}$ with respect to $\{x, s, \tilde{z}, w\}$ is

$$Gr = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4(1+\eta-2\eta^2) & 0 \\ 0 & 0 & 0 & 4(1-\eta) \end{pmatrix}.$$

Now $\det Gr = 0$ implies $16(1+\eta-2\eta^2)(1-\eta) = 16(2\eta+1)(1-\eta)^2 = 0$. Since $\eta \neq 1$, it follows that $\eta = -\frac{1}{2}$. Note that the rank of the Gram matrix is 2. Hence the dimension of the radical is 1. $\square$

Now we consider MB-type algebras,[1] that is, $\mathcal{B} = \{x, y\}$, where $x = a + b$ and $y = c + d$. All possible diagrams on $\mathcal{B}$ are given by Figure 5.4.

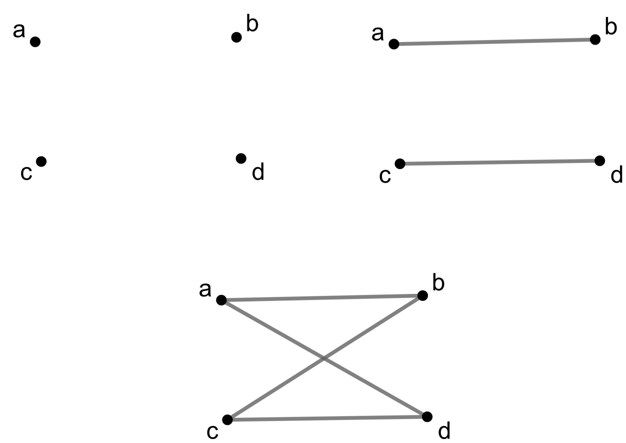


Figure 5.4: Diagram B_1, diagram B_2 and digram B_3.

[1] A. Staroletov independently constructed all MB-type algebras for $\eta = 3$ using GAP.

Lemma 5.4.6. *The algebra $\langle\langle x, y\rangle\rangle_{B_1}$ is isomorphic to $2B$.*

Lemma 5.4.7. *The algebra $\langle\langle x, y\rangle\rangle_{B_2}$ is isomorphic to $3C(\eta)$.*

Proof. The Coxeter group of B_2 is $\hat{G} = S_3 \times S_3$. In this case also, we see, $F = \hat{G}$. For simplicity we embed F into S_6. Therefore, we may assume $x = (1, 2) + (3, 4)$ and $y = (2, 5) + (4, 6)$. The product xy is,

$$\begin{aligned} xy &= \frac{\eta}{2}((1, 2) + (2, 5) - (1, 5) + (3, 4) + (4, 6) - (3, 6)) \\ &= \frac{\eta}{2}(x + y - (1, 5) - (3, 6)). \end{aligned}$$

Set $z = (1, 5) + (3, 6)$. Thus we have $xy = \frac{\eta}{2}(x + y - z)$. That is, $\{x, y, z\}$ is a line in the Fischer space. Consequently, $x^{\tau_y} = y^{\tau_x} = z$. Further,

$$\begin{aligned} xz &= \frac{\eta}{2}(x + z - y), \\ yz &= \frac{\eta}{2}(y + z - x). \end{aligned}$$

The parameter $n = |x^F \cup y^F| = 3$. Hence, the algebra $\langle\langle x, y\rangle\rangle_{B_2} \cong 3C(\eta)$. $\square$

The Coxeter group $\hat{G}$ of B_3 is an infinite group. Therefore, F is a quotient of $\hat{G}$. From [17], we have $F = W_p(\tilde{A}_3) \cong p^3 : S_4$ where $p = |a^d b^c|$, and $p \in \{1, 2, 3\}$. Let B_3^1 represent the case where the order of $p = 1$. Similarly, B_3^2 and B_3^3 represent cases where the orders of p are 2 and 3 respectively.

Lemma 5.4.8. *The algebra $\langle\langle x, y\rangle\rangle_{B_3^1}$ is isomorphic to $3C(2\eta)$.*

Proof. In this case $F \cong S_4$. The Fischer space Γ_F is given in Figure 5.5.

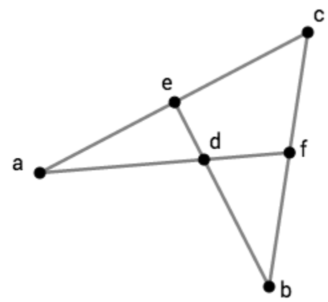


Figure 5.5: Fischer space associated with B_6^1

From Γ_F, we have

$$\begin{aligned} xy &= (a+b)(c+d) \\ &= \frac{\eta}{2}(a+c-e+a+d-f+b+c-f+b+d-e) \\ &= \eta(x+y-z), \end{aligned}$$

where, $z := e+f$. Since e and f are not collinear in Γ_F, they are orthogonal in M_F. Therefore, z is a double axis. We compute the products xz, yz, and determine the dimension of the algebra.

$$\begin{aligned}
xz &= (a+b)(e+f)\\
&= \frac{\eta}{2}(a+e-c+a+f-d+b+e-d+b+f-c)\\
&= \eta(x+z-y),\\
yz &= (c+d)(e+f)\\
&= \frac{\eta}{2}(c+e-a+c+f-b+d+e-b+d+f-a)\\
&= \eta(y+z-x).
\end{aligned}$$

Therefore, $\{x,y,z\}$ is a line in the Fischer space. Therefore, $\langle\langle x,y\rangle\rangle_{B_3^1} \cong 3C(2\eta)$. $\square$

In the next case, we obtain an algebra that belongs to the class of 2-generated algebras introduced by Rehren in [26].

Lemma 5.4.9. *The algebra* $\langle\langle x,y\rangle\rangle_{B_3^2} \cong 4A(\eta)$.

Proof. The group F is isomorphic to $2^3 : S_4$. Let $T = \{a,b,c,d,e,f,a',b',c',d',e',f'\}$ be the generating set of F. The Fischer space Γ_F is a geometry on 12 points. Some of the planes of the Fischer space are shown in Figure 5.7. For simplicity, we change the notations. Let $x = a + f'$ and $y = c + d$, so that $a^{cd}f' = b^d f' = ff' = 0$. Thus

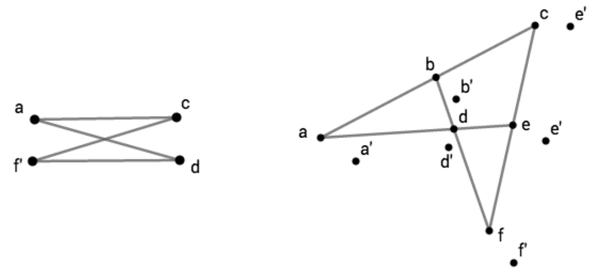


Figure 5.6: Fischer space Γ_F

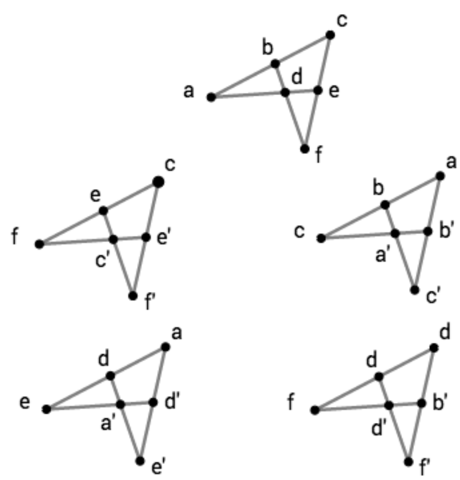


Figure 5.7: The planes of Fischer space used in the construction of $\langle\langle x, y\rangle\rangle_{B_3^2}$.

$a^{cd}f' = 2$. Products of the axes are as follows.

$$
\begin{aligned}
xy &= (a + f')(c + d) \\
&= \frac{\eta}{2}(a + c - b + a + d - e + f' + c - e' + f' + d - b') \\
&= \eta(x + y) - \frac{\eta}{2}(b + e + b' + e') \\
&= \eta(x + y) + \sigma,
\end{aligned}
$$

where $\sigma = -\frac{\eta}{2}(b+e+b'+e') = xy - \eta(x+y)$.

Note that

$$x^{\tau_y} = (a+f')^{\tau_c\tau_d} = (b+e')^{\tau_d} = f + a' := s,$$
$$y^{\tau_x} = (c+d)^{\tau_a\tau_f'} = (b+e)^{\tau_f'} = d' + c' := t.$$

Consider

$$\begin{aligned}
xs &= (a+f')(a'+f) = aa' + af + f'a' + ff' = 0\\
yt &= (c+d)(c'+d') = cc' + cd' + dc' + dd' = 0,\\
xt &= (a+f')(c'+d')\\
&= \frac{\eta}{2}(a+c'-b'+a+d'-e'+f'+c'-e+f'+d'-b)\\
&= \eta(a+f'+c'+d') - \frac{\eta}{2}(b+e+b'+e')\\
&= \eta(x+t)+\sigma,
\end{aligned}$$

$$\begin{aligned}
x\sigma &= -\frac{\eta}{2}(a+f')(b+e+b'+e')\\
&= -\frac{\eta^2}{4}(a+b-c+a+e-d+a+b'-c'+a+e'-d')\\
&\quad -\frac{\eta^2}{4}(f'+b-d'+f'+e-c'+f'+b'-d+f'+e'-c)\\
&= -\frac{\eta^2}{4}(4(a+f')-2(c+d+c'+d')+2(b+e+b'+e'))\\
&= -\eta^2 x + \frac{\eta^2}{2}(y+t) + \eta\sigma,
\end{aligned}$$

$$
\begin{aligned}
y\sigma &= -\frac{\eta}{2}(c+d)(b+e+b'+e') \\
&= -\frac{\eta^2}{4}(c+b-a+c+e-f+c+b'-a'+c+e'-f') \\
&\quad -\frac{\eta^2}{4}(d+b-f+d+e-a+d+b'-f'+d+e'-a') \\
&= -\frac{\eta^2}{4}(4(c+d)-2(a+f'+a'+f)+2(b+e+b'+e') \\
&= -\eta^2 y+\frac{\eta^2}{2}(x+s)+\eta\sigma, \\
s\sigma &= -\frac{\eta}{2}(a'+f)(b+e+b'+e') \\
&= -\frac{\eta^2}{4}(a'+b-c'+a'+e-d'+a'+b'-c+a'+e'-d) \\
&\quad -\frac{\eta^2}{4}(f+b-d+f+e-c+f+b'-d'+f+e'-c') \\
&= -\frac{\eta^2}{4}(4(a'+f)-2(c+d+c'+d')+2(b+e+b'+e')) \\
&= -\eta^2 s+\frac{\eta^2}{2}(y+t)+\eta\sigma, \\
t\sigma &= -\frac{\eta}{2}(c'+d')(b+e+b'+e') \\
&= -\frac{\eta^2}{4}(c'+b-a'+c'+e-f'+c'+b'-a+c'+e'-f) \\
&\quad -\frac{\eta^2}{4}(d'+b-f'+d'+e-a'+d'+b'-f+d'+e'-a) \\
&= -\frac{\eta^2}{4}(4(c'+d')-2(a+f'+a'+f)+2(b+e+b'+e') \\
&= -\eta^2 t+\frac{\eta^2}{2}(x+s)+\eta\sigma,
\end{aligned}
$$

$$
\begin{aligned}
ys &= (c+d)(a'+f) \\
&= \frac{\eta}{2}(c+a'-b'+c+f-e+d+a'-e'+d+f-b) \\
&= \eta(c+d+a'+f) - \frac{\eta}{2}(b+e+b'+e') \\
&= \eta(y+s)+\sigma, \\
ts &= (c'+d')(a'+f) \\
&= \frac{\eta}{2}(c'+a'-b+c'+f-e'+d'+a'-e+d'+f-b') \\
&= \eta(c'+d'+a'+f) - \frac{\eta}{2}(b+e+b'+e') \\
&= \eta(t+s)+\sigma, \\
\sigma^2 &= \frac{\eta^2}{4}(b+e+b'+e')(b+e+b'+e') \\
&= \frac{\eta^2}{4}(b+e+b'+e') \\
&= -\frac{\eta}{2}\sigma.
\end{aligned}
$$

The multiplication table for $\langle\langle x, y\rangle\rangle_{B_3^2}$ is given in Table 5.4. It remains to determine the parameter n. From the Fischer space, we have

$$
\begin{aligned}
x^{\tau_y} = s, \ s^{\tau_y} = x, \\
y^{\tau_y} = t, \ t^{\tau_x} = y.
\end{aligned}
$$

Thus $n = |x^F \cup y^F| = |\{x, s, y, t\}| = 4$. We identify that the algebra $\langle\langle x, y\rangle\rangle_{B_3^2}$ is isomorphic to $4A(\eta)$. Note that all the axes of the algebra are double axes. $\square$

	x	y	s	t	σ
x	x	$\eta x + \eta y + \sigma$	0	$\eta x + \eta t + \sigma$	$-\eta^2 x + \frac{\eta^2}{2}(y+t) + \eta\sigma$
y		y	$\eta y + \eta s + \sigma$	0	$-\eta^2 y + \frac{\eta^2}{2}(x+s) + \eta\sigma$
s			s	$\eta s + \eta t + \sigma$	$-\eta^2 s + \frac{\eta^2}{2}(y+t) + \eta\sigma$
t				t	$-\eta^2 t + \frac{\eta^2}{2}(x+s) + \eta\sigma$
σ					$-\frac{\eta}{2}\sigma$

Table 5.4: Multiplication table of $\langle\langle x, y\rangle\rangle_{B_3^2}$

Now we consider the last case, that is, $p = |a^d b^c| = 3$. Note that in this case the group F is isomorphic to $3^3 : S_4$. We realise F as follows. [2] Let V be a 4-dimensional vector space over field $GF(3)$ with a basis $\{e_1, e_2, e_3, e_4\}$ as the permutation module for S_4. Let $W = \langle e_1 + e_2 + e_3 + e_4 \rangle^{\perp}$. Clearly W is invariant under the action of S_4. Take F to be the semi direct product of W and S_4. Thus, $F \cong 3^3 : S_4$. Denote by

$$
\begin{aligned}
r_{ij} &= ((i,j),\ 0),\\
r'_{ij} &= ((i,j),\ e_i - e_j),\\
r''_{ij} &= ((i,j),\ e_j - e_i),
\end{aligned}
$$

for $i, j \in \{1, 2, 3, 4\}$ and $i < j$. From now on till the end of this section, all the subscripts belong to $\{1, 2, 3, 4\}$ unless otherwise is stated. From the definition of r_{ij}, the lines $\{r_{ij}, r_{ik}, r_{kj}\}$ and $\{r_{ij}, r_{il}, r_{lj}\}$ generate a 6-point plane in the Fischer space Γ_F.

Lemma 5.4.10. *For $i < j$, $\{r_{ij}, r'_{ij}, r''_{ij}\}$ is a line.*

Proof. Consider

$$
\begin{aligned}
(r'_{ij})^{r_{ij}} &= r_{ij}\ r'_{ij}\ r_{ij}\\
&= ((i,j),\ 0)((i,j),\ e_i - e_j)((i,j),\ 0)\\
&= ((i,j),\ (e_i - e_j)^{(i,j)})\\
&= ((i,j),\ e_j - e_i) = r''_{ij}.
\end{aligned}
$$

[2] I am grateful to A. Galt, A. Mamontov, A. Staroletov for suggesting me this realisation of F.

Similarly, $(r_{ij})^{r'_{ij}} = r''_{ij}$. Hence, $\{r_{ij}, r'_{ij}, r''_{ij}\}$ is a line in the Fischer space Γ_F. $\square$

Lemma 5.4.11. *For mutually disjoint pairs of indices, the elements r_{ij}, r'_{kl}, and r''_{mn} pairwise commute.*

Proof. In this case, the semidirect product reduces to the direct product. Therefore, the claim follows. $\square$

Now we construct the multiplication table for $\langle\langle x, y\rangle\rangle_{B_3^3}$. Let $a_1 = r_{13} + r_{24}$ and $a_2 = r_{12} + r'_{34}$. Clearly,

$$a^d = r_{13}^{r'_{34}} = r'_{14},$$
$$b^c = r_{24}^{r_{12}} = r_{14},$$

and $|a^d b^c| = |r'_{14} r_{14}| = |(1,\ e_4 - e_1)| = 3$. Therefore, the algebra generated by $\{a_1, a_2\}$ is isomorphic to B_3^3. Consider

$$a_3 := a_1^{\tau_{a_2}} = (r_{13} + r_{24})^{\tau_{r_{12}}\tau_{r'_{34}}} = r''_{13} + r'_{24},$$
$$a_4 := a_2^{\tau_{a_1}} = (r_{12} + r'_{34})^{\tau_{r_{13}}\tau_{r_{24}}} = r_{34} + r'_{12}.$$

Clearly, a_3 and a_4 are axes.

Lemma 5.4.12. *Let $a_5 = r'_{13} + r''_{24}$ and $a_6 = r''_{12} + r''_{34}$. Then $\langle\langle a_1, a_3, a_5\rangle\rangle$ and $\langle\langle a_2, a_4, a_6\rangle\rangle$ are isomorphic to $3C(\eta)$.*

Proof. From the definitions of a_1, a_3, a_2, and a_4, the diagrams on $\{a_1, a_3\}$ and $\{a_2, a_4\}$ are given in Figure 5.8.

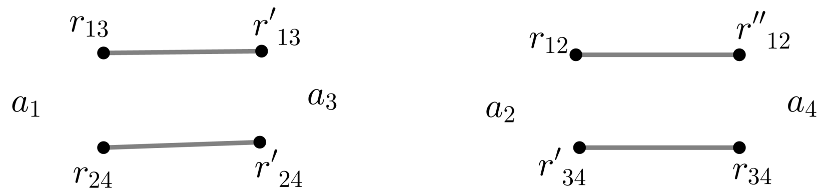


Figure 5.8: Diagrams on $\{a_1, a_3\}$ and $\{a_2, a_4\}$.

Therefore, we have

$$\begin{aligned}
a_1 a_3 &= (r_{13} + r_{24})(r''_{13} + r'_{24}) \\
&= \frac{\eta}{2}(r_{13} + r''_{13} - r'_{13} + r_{24} + r'_{24} - r''_{24}) \\
&= \frac{\eta}{2}(a_1 + a_3 - a_5), \\
a_2 a_4 &= (r_{12} + r'_{34})(r'_{12} + r_{34}) \\
&= \frac{\eta}{2}(r_{12} + r'_{12} - r''_{12} + r'_{34} + r_{34} - r''_{34}) \\
&= \frac{\eta}{2}(a_2 + a_4 - a_6).
\end{aligned}$$

Hence, from Lemma 5.4.7, $\langle\langle a_1, a_3, a_5 \rangle\rangle$ and $\langle\langle a_2, a_4, a_6 \rangle\rangle$ are isomorphic to $3C(\eta)$. □

Lemma 5.4.13. *Suppose that* $a_7 = r''_{14} + r''_{23}$*, then* $\langle\langle a_1, a_6, a_7 \rangle\rangle$*,* $\langle\langle a_2, a_5, a_7 \rangle\rangle$*, and* $\langle\langle a_3, a_4, a_7 \rangle\rangle$ *are isomorphic to* $3C(2\eta)$*.*

Proof. Again consider the diagrams on $\{a_1, a_6\}$, $\{a_2, a_5\}$ and $\{a_3, a_4\}$ which are given in Figure 5.9. In each case we observe that the condition $|a^d b^c| = 1$ holds for axes

$a+b$ and $c+d$. That is,

$$|(r_{13})^{r''_{34}}\ (r_{24})^{r''_{12}}| = |r''_{14}r''_{14}| = 1,$$

$$|(r_{12})^{r''_{24}}\ (r'_{34})^{r'_{13}}| = |r''_{14}r''_{14}| = 1,$$

$$|(r''_{13})^{r_{34}}\ (r'_{24})^{r'_{12}}| = |r''_{14}r''_{14}| = 1.$$

From Lemma 5.4.8, each algebra $\langle\langle a_1, a_6\rangle\rangle$, $\langle\langle a_2, a_5\rangle\rangle$, and, $\langle\langle a_3, a_4\rangle\rangle$ is isomorphic to $\langle\langle x, y\rangle\rangle_{B_3^1}$. Hence, the assertion follows.

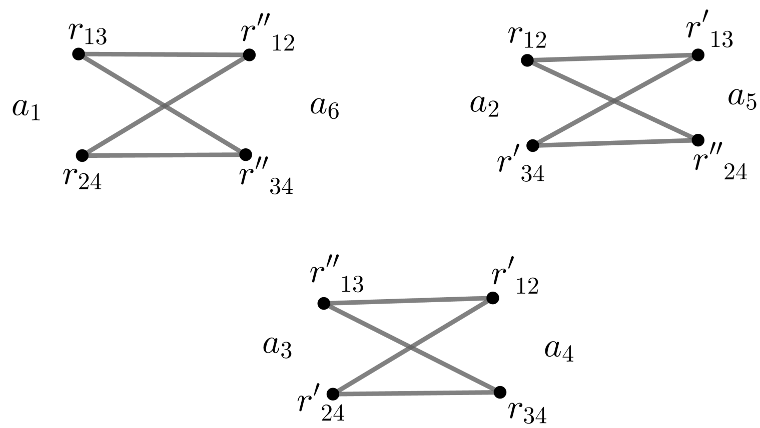


Figure 5.9: Diagrams on $\{a_1, a_6\}$, $\{a_2, a_5\}$, and $\{a_3, a_4\}$.

$\square$

Lemma 5.4.14. *For two double axes u and v of a primitive axial algebra of Monster type $(2\eta, \eta)$, set*

$$\sigma(u, v) = uv - \eta(u + v).$$

Then $\sigma(u^{\tau_v}, v) = \sigma(u, v) = \sigma(u, v^{\tau_u})$.

Proof. Consider the decomposition of u with respect to v,

$$u = \alpha v + v_0 + v_{2\eta} + v_\eta,$$

where, $v_\lambda \in A_\lambda(v)$ for all $\lambda \in \{0, 2\eta, \eta\}$. Substituting for u in the expression of $\sigma(u, v)$,

$$\begin{aligned}
\sigma(u, v) &= (\alpha v + v_0 + v_{2\eta} + v_\eta)v - \eta(\alpha v + v_0 + v_{2\eta} + v_\eta + v) \\
&= \alpha v + 2\eta v_{2\eta} + \eta v_\eta - (\eta\alpha + \eta)v - \eta v_0 - \eta v_{2\eta} - \eta v_\eta \\
&= (\alpha - \eta\alpha - \eta)v - \eta v_0 + \eta v_{2\eta}.
\end{aligned}$$

Therefore, $\sigma \in A_1(v) \bigoplus A_0(v) \bigoplus A_{2\eta}(v) = A_+(v)$. Since τ_v is an automorphism which fixes $A_+(v)$ and negates $A_-(v) = A_\eta(v)$, we have,

$$\sigma(u, v) = \sigma(u, v)^{\tau_v} = \sigma(u^{\tau_v}, v).$$

Similarly, by decomposing v with respect to u, we see that $\sigma(u, v) = \sigma(u, v^{\tau_u})$. $\square$

Lemma 5.4.15. *For* $(u, v) \in \{(a_1, a_2), (a_1, a_4), (a_2, a_3), (a_3, a_6), (a_4, a_5), (a_5, a_6)\}$,

$$\sigma(u, v) = -\frac{\eta}{2}(r_{14} + r_{23} + r'_{14} + r''_{23}),$$

furthermore,

$$\begin{aligned}\sigma(u, v)u &= -\eta^2 u + \frac{\eta^2}{2}(v + v^{\tau_u}) + \eta\sigma,\\ \sigma(u, v)v &= -\eta^2 v + \frac{\eta^2}{2}(u + u^{\tau_v}) + \eta\sigma.\end{aligned}$$

Proof. We compute $\sigma = \sigma(a_1, a_2)$ then repeated application of Lemma 5.4.14 on σ yields the remaining cases. Consider

$$\begin{aligned}a_1 a_2 &= (r_{13} + r_{24})(r_{12} + r'_{34})\\ &= \frac{\eta}{2}(2(r_{13} + r_{24}) + 2(r_{12} + r'_{34}) - (r_{14} + r_{23} + r'_{14} + r''_{23}))\\ &= \eta(a_1 + a_2) - \frac{\eta}{2}(r_{14} + r_{23} + r'_{14} + r''_{23}).\end{aligned}$$

By the definition of σ, we have $\sigma = -\frac{\eta}{2}(r_{14} + r_{23} + r'_{14} + r''_{23})$. We have already seen that $a_3 = a_1^{\tau_{a_2}}$ and $a_4 = a_2^{\tau_{a_1}}$. From the definitions of a_i for $i \in \{1, 2, 3, 4, 5, 6\}$, we

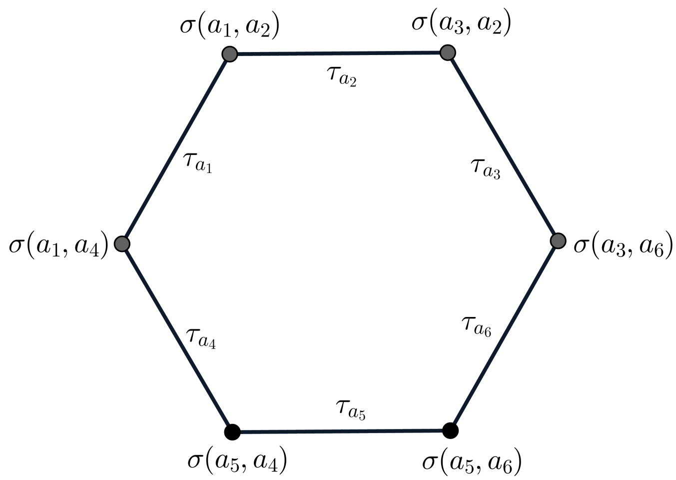


Figure 5.10: Actions of Miyamoto involutions on $\sigma(a_i, a_j)$.

have

$$
\begin{aligned}
{a_2}^{\tau_{a_3}} &= (r_{12} + r'_{34})^{\tau_{r''_{13}+r'_{24}}} = (r''_{23} + r_{14})^{\tau_{r'_{24}}} = r''_{34} + r''_{12} = a_6, \\
a_3^{\tau_{a_6}} &= (r''_{13} + r'_{24})^{\tau_{r''_{12}+r''_{34}}} = (r_{23} + r_{14})^{\tau_{r''_{34}}} = r''_{24} + r'_{13} = a_5, \\
a_6^{\tau_{a_5}} &= (r''_{12} + r''_{34})^{\tau_{r'_{13}+r''_{24}}} = (r''_{23} + r_{14})^{\tau_{r''_{24}}} = r_{34} + r'_{12} = a_4, \\
a_5^{\tau_{a_4}} &= (r'_{13} + r''_{24})^{\tau_{r'_{12}+r_{34}}} = (r_{23} + r_{14})^{\tau_{r_{34}}} = r_{24} + r_{13} = a_1.
\end{aligned}
$$

Thus $\sigma(a_1, a_2) = \sigma(a_1, a_2^{\tau_{a_1}}) = \sigma(a_1, a_4)$, which implies, $\sigma(a_1, a_4) = \sigma(a_1^{\tau_{a_4}}, a_4) = \sigma(a_5, a_4)$. Similarly, other cases follow. Images of $\sigma(a_i, a_j)$ under the actions of the various Miyamoto involutions are shown in Figure 5.10.

We prove the second part of the claim for a_1 and a_2, remaining cases follow from the actions of Miyamoto involutions on σa_1 and σa_2. Consider

$$
\begin{aligned}
\sigma a_1 &= -\frac{\eta}{2}(r_{14}+r_{23}+r'_{14}+r''_{23})(r_{13}+r_{24}) \\
&= -\frac{\eta^2}{4}(2(r_{14}+r_{23}+r'_{14}+r''_{23})+4(r_{13}+r_{24})-2(r_{34}+r'_{34}+r_{12}+r'_{12})) \\
&= -\eta^2 a_1+\frac{\eta^2}{2}(a_2+a_4)+\eta\sigma \\
&= -\eta^2 a_1+\frac{\eta^2}{2}(a_2+a_2^{\tau_{a_1}})+\eta\sigma, \\
\sigma a_2 &= -\frac{\eta}{2}(r_{14}+r_{23}+r'_{14}+r''_{23})(r_{12}+r'_{34}) \\
&= -\frac{\eta^2}{4}(2(r_{14}+r_{23}+r'_{14}+r''_{23})+4(r_{12}+r'_{34})-2(r_{13}+r_{24}+r''_{13}+r'_{24})) \\
&= -\eta^2 a_2+\frac{\eta^2}{4}(a_1+a_3)+\eta\sigma \\
&= -\eta^2 a_2+\frac{\eta^2}{4}(a_1+a_1^{\tau_{a_2}})+\eta\sigma.
\end{aligned}
$$

□

Note that $s = r_{14} + r_{23}$ and $t = r'_{14} + r''_{23}$ are double axes and moreover $\langle\langle s, t, a_7 \rangle$ are isomorphic to $3C(\eta)$. Therefore,

$$\begin{aligned}
\sigma^2 &= \frac{\eta^2}{4}(r_{14} + r_{23} + r'_{14} + r''_{23})^2 = \frac{\eta^2}{4}(s+t)^2 \\
&= \frac{\eta^2}{4}(s + t + \frac{\eta}{2}(s + t - a_7)) \\
&= -\frac{\eta}{2}(1 + \frac{\eta}{2})\sigma - \frac{\eta^3}{8}a_7, \\
\sigma a_7 &= -\frac{\eta^2}{4}(s+t)a_7 = -\frac{\eta^2}{4}(s + a_7 - t + t + a_7 - s) = -\frac{\eta^2}{2}a_7.
\end{aligned}$$

Theorem 5.4.16. *The two-generated algebra $\langle\langle x, y \rangle\rangle_{B_6^3}$ is an 8-dimensional primitive axial algebra of Monster type $(2\eta, \eta)$, and product of the basis elements is given by Table 5.5.*

Thus the multiplication table of $\langle\langle x, y \rangle\rangle_{B_3^3}$ is as follows.

	a_1	a_2	a_3	a_4	a_5	a_6	a_7	σ
a_1	a_1	$\eta(a_1+a_2)+\sigma$	$\frac{\eta}{2}(a_1+a_3-a_5)$	$\eta(a_1+a_4)+\sigma$	$\frac{\eta}{2}(a_1+a_5-a_3)$	$\eta(a_1+a_6-a_7)$	$\eta(a_1+a_7-a_6)$	$-\eta^2 a_1+\frac{\eta^2}{2}(a_2+a_4)+\eta\sigma$
a_2		a_2	$\eta(a_2+a_3)+\sigma$	$\frac{\eta}{2}(a_2+a_4-a_6)$	$\eta(a_2+a_5-a_7)$	$\frac{\eta}{2}(a_2+a_6-a_4)$	$\eta(a_2+a_7-a_5)$	$-\eta^2 a_2+\frac{\eta^2}{2}(a_1+a_3)+\eta\sigma$
a_3			a_3	$\eta(a_3+a_4-a_7)$	$\frac{\eta}{2}(a_3+a_5-a_1)$	$\eta(a_3+a_6)+\sigma$	$\eta(a_3+a_7-a_4)$	$-\eta^2 a_3+\frac{\eta^2}{2}(a_2+a_6)+\eta\sigma$
a_4				a_4	$\eta(a_4+a_5)+\sigma$	$\frac{\eta}{2}(a_4+a_6-a_2)$	$\eta(a_4+a_7-a_3)$	$-\eta^2 a_4+\frac{\eta^2}{2}(a_1+a_5)+\eta\sigma$
a_5					a_5	$\eta(a_5+a_6)+\sigma$	$\eta(a_5+a_7-a_2)$	$-\eta^2 a_5+\frac{\eta^2}{2}(a_4+a_6)+\eta\sigma$
a_6						a_6	$\eta(a_6+a_7-a_1)$	$-\eta^2 a_6+\frac{\eta^2}{2}(a_3+a_5)+\eta\sigma$
a_7							a_7	$-\frac{\eta^2}{2}a_7$
σ								$-\frac{\eta}{2}(1+\frac{\eta}{2})\sigma-\frac{\eta^3}{8}a_7$

Table 5.5: Multiplication table of $\langle\langle x,y\rangle\rangle_{B_3^3}$

Chapter 6

Fixed subalgebras for S_n

6.1 Algebra of dimension k^2

Let (G, D) be a 3-transposition group and Γ be the Fischer space on (G, D). Let $M = M(\Gamma, \eta)$ be the associated Matsuo algebra. Recall that for an involution $h \in Aut(M)$ fixing D, the fixed subalgebra $W \leq M$ relative to $\langle h \rangle$ is a primitive axial algebra of Monster type $(2\eta, \eta)$. Earlier we showed that W arises from the $\langle h \rangle$-orbits on D. Note that an orbit $\{a, a^h\}$ of length two is the support of a double axis if $aa^h = 0$. Any other vector arising from an orbit of length two is called an *extra*.

The elements of D denote the basis elements of M, in order to distinguish between the algebra product with the group product, we use $\cdot$ to denote the algebra product.

In this chapter, we study the fixed subalgebra for the symmetric group S_{2k} and its radical.

Definition 6.1.1. *Let $h \in S_{2k}$ be an involution. Then the rank of h is the number of 2-cycles in it.*

Theorem 6.1.2. *Let M be the Matsuo algebra for the symmetric group S_{2k} and $h = (1,2)(3,4)\dots(2k-1,2k)$ be an involution in S_{2k}. Then the fixed subalgebra relative to $\langle h \rangle$ is generated by the set containing k single axes and $k(k-1)$ double axes. Furthermore, these set of axes form a basis for the fixed subalgebra, therefore, the dimension is k^2.*

Proof. Since D denotes the set of 3-transpositions and is a basis for M, we use $a \cdot b$ to denote the algebra product of two elements $a, b \in D$. Let $\langle h \rangle$ act on D by conjugation. Then the union of all $\langle h \rangle$-orbits of length 1 is the centraliser of h in D which is

$$C_h(D) = \{(1,2), (3,4)\dots, (2k-1, 2k)\}.$$

Clearly, each element of $C_h(D)$ is a single axis of the fixed subalgebra. Therefore, there are k single axes.

To see the $\langle h \rangle$-orbits of length 2, first we consider the action of rs for some $r = (i,j)$, $s = (m,n)$ present in the expression of h. The action of rs on 2-cycles (i,m) and (i,n) belong to D are respectively,

$$(i,m)^{rs} = (j,n),$$
$$(i,n)^{rs} = (j,m).$$

Since rs is an involution, $(j,n)^{rs} = (i,m)$ and $(j,m)^{rs} = (i,n)$. Furthermore, the algebra product, $(i,m) \cdot (i,m)^{rs} = 0 = (i,n) \cdot (i,n)^{rs}$ as the pairs (i,m), $(i,m)^{rs}$

and (i,n), $(i,n)^{rs}$ commute. Note that rs fixes all the remaining 2-cycles in D. Therefore, for every disjoint pair r,s there exist two orbits of length 2 such that each orbit corresponds to a double axis. In the above case, the double axes are $(i,m)+(j,n)$ and $(i,n)+(j,m)$. Since there are $k(k-1)/2$ such pairs in h, we see that, the action of $\langle h\rangle$ on D has $k(k-1)$ orbits of length 2, and each such orbit constitute a double axis. Hence, the total number of double axes is $k(k-1)$. Consequently, the dimension of the fixed subalgebra for S_{2k} is k^2. □

Call the above algebra the k^2-algebra for S_{2k}[1].

Corollary 6.1.3. *A 2-generated subalgebra of the k^2-algebra is isomorphic to exactly one of the following.*

$$2B,\ 3C(\eta),\ \textit{or}\ 3C(2\eta).$$

Proof. All possible diagrams on the support set of $\mathcal{B}=\{x,y\}$, where x and y are axes of k^2-algebra, is given in Figure 6.1. From the diagrams the assertion follows. □

Definition 6.1.4. *Let W be the fixed subalgebra of a Matsuo algebra M relative to $\langle h\rangle$, for an involution $h\in Aut(M)$. Then the axial part of W is the subalgebra containing no extras. That is, the axial part is generated by a set containing only single axes and double axes.*

Theorem 6.1.5. *Let M_n be the Matsuo algebra for the symmetric group S_n, $n=2k+1$. Then the axial part of the fixed subalgebra relative to $\langle h\rangle$, where*

$$h=(1,2)(3,4)\dots(2k-1,2k),$$

[1]A. Galt, A. Mamontov, A. Staroletov constructed this algebra independently.

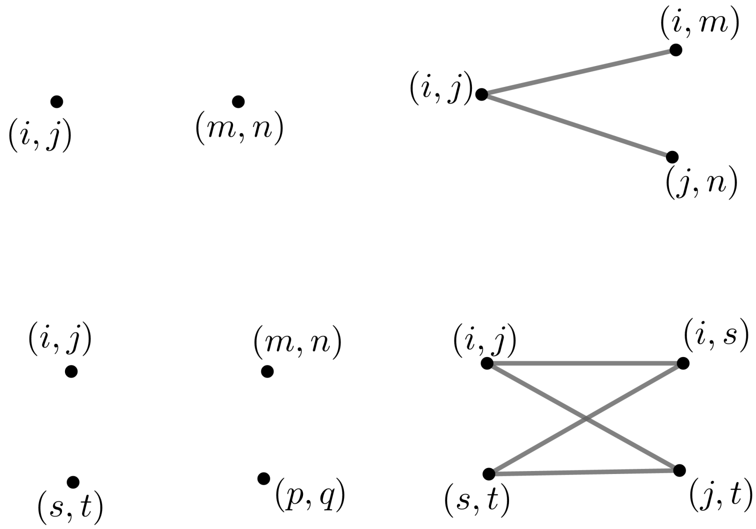


Figure 6.1: Diagrams on support set containing two axes of k^2-algebra.

for S_n is the k^2-algebra.

Proof. Note that the action of h, $(i, 2k+1)^h = (j, 2k+1)$ for any $i \leq 2k$. Clearly, $(i, 2k+1)^h \cdot (j, 2k+1) \neq 0$. Hence, we do not have any new axis from the above mentioned 2-cycles. In other words, the action of h on $(i, 2k+1)$ yields only extras. □

6.2 Radical of the k^2-algebra

Let W be the fixed subalgebra for S_{2k} relative to $\langle h \rangle$ for a rank k involution $h \in S_{2k}$. Let $\mathcal{S}$ and $\mathcal{D}$ be the sets of single axes and double axes respectively. In order to determine the radical of W, we compute the determinant of the Gram matrix with respect to a suitable basis. So first, we change the basis $\mathcal{S} \cup \mathcal{D}$.

Note that each element of $\mathcal{S}$ is a 2-cycle $a_i := (2i-1, 2i)$ for $i = 1, 2, \ldots, k$. For each pair a_i, a_j, $i < j$, let

$$d_{ij} = (2i-1, 2j-1) + (2i, 2j),$$
$$\bar{d_{ij}} = (2i-1, 2j) + (2j-1, 2i)$$

be the corresponding double axes. For i, j, let

$$C_{ij} = d_{ij} + \bar{d_{ij}},$$
$$D_{ij} = d_{ij} - \bar{d_{ij}}.$$

Clearly, from the definitions of a_i, C_{ij}, and D_{ij}, the set

$$B = \{a_i, C_{ij}, D_{ij} \mid i, j \in \{1, 2, \ldots, k\}, i < j\}$$

is a basis for W. Moreover, the basis elements have the following properties which are presented as lemmas.

Lemma 6.2.1. *The subspace $\langle D_{ij} \mid i, j \in \{1, 2, \ldots k\}\rangle$ is orthogonal to the subspace $\langle a_i, C_{mn} \mid i, m, n \in \{1, 2, \ldots k\}\rangle$ with respect to the Frobenius form $(\ ,\ )$ on M.*

Proof. If i, j, l, m and n all are distinct then the claim holds as the 2-cycles of D_{ij}, a_l, and C_{mn} are disjoint and their inner product is zero. Now we compute the inner products of the basis elements which have some of the common indices. From the diagrams on the corresponding support sets, we have

$$(a_i, D_{ij}) = (a_i, d_{i,j} - \bar{d}_{ij}) = (a_i, d_{ij}) - (a_i, \bar{d}_{ij}) = \eta - \eta = 0,$$

$$(C_{ij}, D_{ij}) = (d_{ij} + \bar{d}_{ij}, d_{ij} - \bar{d}_{ij}) = 2 - 2\eta + 2\eta - 2 = 0,$$

$$(C_{ij}, D_{ix}) = (d_{ij} + \bar{d}_{ij}, d_{ix} - \bar{d}_{ix}) = \eta - \eta + \eta - \eta = 0.$$

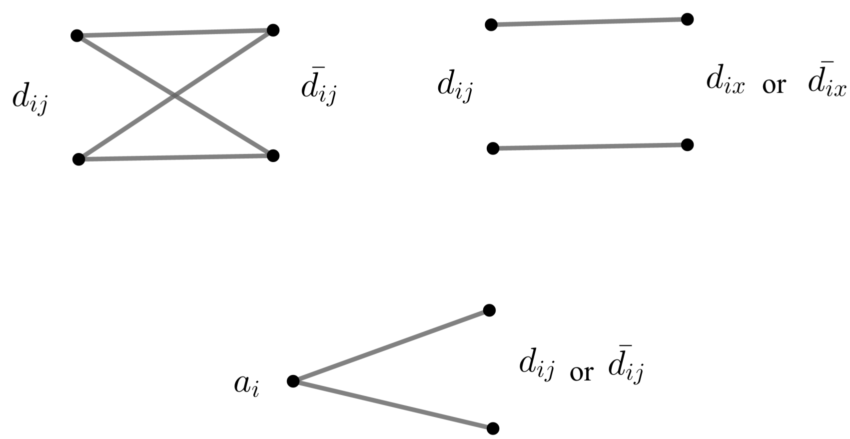


Figure 6.2: Diagrams on single and double axes of B

Hence, the subspaces mentioned in the statement are orthogonal. □

Lemma 6.2.2. *For $i < j, x$, $(D_{ij}, D_{ij}) = 4(1-\eta)$ and $(D_{ij}, D_{i,x}) = 0$.*

Proof. Again from Figure 6.2,

$$(D_{ij}, D_{ij}) = (d_{i,j} - \bar{d}_{ij}, d_{i,j} - \bar{d}_{ij}) = 2 - 2\eta - 2\eta + 2 = 4(1-\eta),$$

$$(D_{ij}, D_{ix}) = (d_{i,j} - \bar{d}_{ij}, d_{i,x} - \bar{d}_{ix}) = \eta - \eta - \eta + \eta = 0.$$

$\square$

For $i, j \in \{1, 2, \ldots, k\}$, define

$$\tilde{C}_{ij} = C_{ij} - 2\eta a_i - 2\eta a_j.$$

Clearly, the set containing a_i, D_{jl} and $\tilde{C}_{mn}$ for all $i, j, l, m, n \in \{1, 2, \ldots, k\}$ is a basis for W. And moreover, D_{kl} is orthogonal to $\tilde{C}_{ij}$.

Lemma 6.2.3. *For distinct* $i, j, x \in \{1, 2, \ldots, k\}$,

$$(a_i, \tilde{C}_{ij}) = (a_j, \tilde{C}_{ij}) = 0$$
$$(\tilde{C}_{ij}, \tilde{C}_{ij}) = 4(1 + \eta - 2\eta^2),$$
$$(\tilde{C}_{ij}, \tilde{C}_{ix}) = 4\eta(1 - \eta).$$

Proof. From the diagrams, we have

$$(a_i, C_{ij}) = (a_j, C_{ij}) = \eta + \eta = 2\eta,$$
$$(C_{ij}, C_{ij}) = (d_{ij} + \bar{d_{ij}}, d_{ij} + \bar{d_{ij}}) = 2 + 2\eta + 2\eta + 2 = 4(1 + \eta),$$
$$(C_{ij}, C_{ix}) = (d_{ij} + \bar{d_{ij}}, d_{ix} + \bar{d_{ix}}) = \eta + \eta + \eta + \eta = 4\eta.$$

Now consider,

$$
\begin{aligned}
(a_i, \tilde{C}_{ij}) &= (a_j, \tilde{C}_{ij}) = 2\eta - 2\eta = 0, \\
(\tilde{C}_{ij}, \tilde{C}_{ij}) &= (C_{ij} - 2\eta a_i - 2\eta a_j, C_{ij} - 2\eta a_i - 2\eta a_j) \\
&= (C_{ij}, C_{ij}) - 4\eta(a_i, C_{ij}) - 4\eta(a_j, C_{ij}) + 8\eta^2 \\
&= 4(1 + \eta - 2\eta^2), \\
(\tilde{C}_{ij}, \tilde{C}_{ix}) &= (C_{ij} - 2\eta a_i - 2\eta a_j, C_{ix} - 2\eta a_i - 2\eta a_x) \\
&= (C_{ij}, C_{ix}) - 2\eta(a_i, C_{ij}) - 2\eta(a_i, C_{ix}) + 4\eta^2 \\
&= 4\eta - 4\eta^2 - 4\eta^2 + 4\eta^2 = 4\eta(1 - \eta).
\end{aligned}
$$

□

Let $X = \{a_i, | \ i \in \{1, 2, \ldots, k\}\}$, $Y = \{D_{ij}, | \ i, j \in \{1, 2, \ldots, k\}, i < j\}$, and $Z = \{\tilde{C}_{ij}, | \ i, j \in \{1, 2, \ldots, k\}, i < j\}$. Then the set $B' := X \cup Y \cup Z$ is a basis for the k^2-algebra. We compute the determinant of the Gram matrix with respect to B'. Note that change of basis may alter the eigenvalues of the Gram matrix, however, we are interested in those values of η for which the determinant of the Gram matrix is zero.

Theorem 6.2.4. *The Gram matrix of the k^2-algebra with respect to the basis B' is*

$$
Gr_{B'} = \begin{pmatrix} I_{|X|} & & \\ & 4(1-\eta)I_{|Y|} & \\ & & Gr_Z \end{pmatrix},
$$

where I is the identity matrix, Gr_Z is the Gram matrix with respect to Z.

Since $\eta \neq 1$, to determine the radical of the k^2-algebra, it is enough to find the radical of the subspace generated by Z. For that we define a graph Ω_Z on Z and notice that it is distance regular. Then we can find the eigenvalues of the adjacency matrix of Ω_Z using known results.

Let Ω_Z be a graph on Z whose vertex set is Z and two distinct vertices $\tilde{C}_{ij}$ and $\tilde{C}_{lm}$ connected by an edge if $|\{i,j\} \cap \{l,m\}| = 1$.

We have referred [12] for the results related to distance regular graphs.

Definition 6.2.5. *Let Ω be a regular graph that is neither complete nor empty. Then Ω is said to be strongly regular with parameters (μ, K, a, c) if it is K-regular, every pair of adjacent vertices has a common neighbours, and every pair of nonadjacent vertices has c common neighbours.*

Clearly the graph Ω_Z is a strongly regular graph with parameters

$$(\mu, K, a, c) = (k(k-1)/2, 2k-4, k-2, 4). \tag{6.1}$$

Remark 6.2.6. *From Lemma 6.2.1, Lemma 6.2.3, the Gram matrix G_Z is*

$$Gr_Z = 4(1+\eta-2\eta^2)I + 4\eta(1-\eta)\mathcal{M}_{\Omega_Z},$$

where, I is the identity matrix and $\mathcal{M}_{\Omega_Z}$ is the adjacency matrix of a graph on Z. Furthermore, for $\theta = 4(1+\eta-2\eta^2)$ and $\tau = 4\eta(1-\eta)$, $\theta + \Psi\tau$ is an eigenvalue of Gr_Z, where Ψ is an eigenvalue of $\mathcal{M}_{\Omega_Z}$.

We use the following result to find the eigenvalues of $\mathcal{M}_{\Omega_Z}$.

Theorem 6.2.7. *Let Γ be a strongly regular graph with parameters (μ, K, a, c). Then the eigenvalues of the adjacency matrix of Γ are*

$$\begin{aligned}
X_0 &= K, \\
X_1 &= \frac{(a-c)+\sqrt{\Delta}}{2}, \\
X_2 &= \frac{(a-c)-\sqrt{\Delta}}{2},
\end{aligned}$$

where, $\Delta = (a-c)^2 + 4(K-c)$.

From the above theorem, one of the eigenvalues of $\mathcal{M}_{\Omega_Z}$ is $X_0 = K = 2n-4$. To see the remaining two eigenvalues are we first compute Δ:

$$\begin{aligned}
\Delta &= (a-c)^2 + 4(K-c) \\
&= (k-6)^2 + 4(2k-4-4) \\
&= k^2 - 4k + 4 = (k-2)^2.
\end{aligned}$$

Therefore, the remaining eigenvalues are

$$\begin{aligned}
X_1 &= \frac{(k-6)+(k-2)}{2} = k-4, \\
X_2 &= \frac{(k-6)-(k-2)}{2} = -2.
\end{aligned}$$

Recall that we are looking for those values of η for which 0 is the eigenvalue of $Gr_{B'}$. However, the eigenvalues of Gr_Z are of the form $\theta + \Psi\tau$, where,

$$\theta = 4(1 + \eta - 2\eta^2),$$
$$\tau = 4\eta(1 - \eta),$$
$$\Psi \in \{X_0, X_1, X_2\}.$$

Therefore, we find the values of η for which $\theta + \Psi\tau = 0$, $\Psi \in \{X_0, X_1, X_2\}$.

Lemma 6.2.8. *The equation $\theta + X_0\tau = 0$ has the root for $\eta = -\frac{1}{2(k-1)}$.*

Proof. The expression $\theta + X_0\tau = 0$ implies $4(1+\eta-2\eta^2)+4\eta(1-\eta)2(k-2) = 0$. On rearranging the terms, we get $(-2k+2)\eta^2 + (2k-3)\eta + 1 = 0$, a quadratic equation in η. The discriminant of the above quadratic equation is

$$B^2 - 4AC = (2k - 3)^2 + 4(2k - 2) = 4k^2 - 4k + 1 = (2k - 1)^2.$$

Clearly, the roots of the quadratic equation are 1 and $-\frac{1}{2(k-1)}$. However, $\eta \neq 1$. Hence, $\eta = -\frac{1}{2(k-1)}$ is the only root. □

Lemma 6.2.9. *The root of the expression $\theta + X_1\tau = 0$ is $-\frac{1}{k-2}$.*

Proof. Consider $\theta + X_1\tau = 0$, which implies $4(1 + \eta - 2\eta^2) + 4\eta(1 - \eta)(k - 4) = 0$. On rearranging the terms,

$$-(k - 2)\eta^2 + (k - 3)\eta + 1 = 0, \tag{6.2}$$

which has the discriminant

$$B^2 - 4AC = (k-3)^3 + 4(k-2) = k^2 - 2k + 1 = (k-1).$$

Therefore, the roots are $\eta = -\frac{1}{k-2}$, $k \neq 2$ and $\eta = 1$. Again, as $\eta \neq 1$, $\eta = -\frac{1}{k-2}$ is the only root of the quadratic equation. $\square$

Lemma 6.2.10. *The expression $\theta + X_2\tau = 0$ has no roots.*

Proof. Note that the expression

$$\theta + X_2\tau = 4(1 + \eta - 2\eta^2) + 4\eta(1-\eta)(-2) = 0,$$

implies $4(1-\eta) = 0$. Since $\eta \neq 1$, the claim follows. $\square$

Recall that the projection graph Ω of a primitive axial algebra A with a set of axes $\mathcal{A}$ is a directed graph on $\mathcal{A}$ such that there is a directed edge from x to y if the projection $\phi_y(x)y$ of x onto $A_1(y)$ is nonzero. Orbit projection graph is obtained from factor of Ω by the Miyamoto group.

In view of Proposition 3.2.12, Ω_Z is strongly connected orbit projection graph and therefore, all the proper ideals are contained in the radical. Thus, we have following theorem on the radical of the k^2-algebra.

Theorem 6.2.11. *The k^2-algebra for S_{2k}, $k \neq 1, 2$, is not simple for $\eta = -\frac{1}{2(k-1)}$ and $\eta = -\frac{1}{k-2}$.*

Proof. Lemma 6.2.8, Lemma 6.2.9 and Lemma 6.2.10. $\square$

Chapter 7

Fixed subalgebras for $Sp(2n, 2)$

Let V be a nondegenerate symplectic space over field $\mathbb{F}_2$. Let $(\ ,\ )$ denote the symplectic bilinear form on V. Recall that $Sp(2n, 2)$ is the group of all isometries of the above bilinear form. Furthermore, the group $Sp(2n, 2)$ is a group of 3-transpositions generated by set of all transvections. In this chapter we mainly describe the fixed subalgebra of the Matsuo algebra M for $Sp(2n, 2)$ relative to $\langle t \rangle$, where t is an involution of $Sp(2n, 2)$.

7.1 Transvections

Recall that, a subspace $W \leq V$ of codimension 1 is called a hyperplane of V.

Definition 7.1.1. *A linear transformation* $1 \neq r \in Sp(2n, 2)$ *is said to be a transvection if there exists a hyperplane* W *which is fixed by* r *and for any* $v \in V \setminus W$, $v^r - v \in W$. *In this case* W *is called the fixed hyperplane of* r.

Theorem 7.1.2. *For $0 \neq u \in V$ there exists a transvection r_u such that*

$$v^{r_u} = v + (v, u)u,$$

for all $v \in V$. Moreover, r_u fixes $u^\perp$.

From now on V is a nondegenerate symplectic space over $\mathbb{F}_2$ of dimension $2n$ unless otherwise is stated.

We show that every transvection is an involution and further the set of all transvections is closed for conjugation.

Lemma 7.1.3. *For $0 \neq u \in V$, r_u is an involution.*

Proof. For any vector $x \in V$, consider

$$\begin{aligned} x^{r_u^2} &= (x + (x, u)u)^{r_u} = x + (x, u)u + (x + (x, u)u, u)u \\ &= x + (x, u)u + (x, u)u + (x, u)(u, u)u = x, \end{aligned}$$

since the field is $\mathbb{F}_2$, $2(x, u)u = 0$, and also $(u, u) = 0$, for all $u \in V$. Therefore, $r_u^2 = 1$, hence, it is an involution. □

From now on when we use the notation r_u, we always mean $u \neq 0$.

Lemma 7.1.4. *For $\sigma \in Sp(2n, 2)$, $r_u^\sigma = r_{u^\sigma}$ for all $u \in V$. In particular, the set of all transvections is closed for conjugation.*

Proof. Note that $r_u^\sigma = \sigma^{-1} r_u \sigma$. Therefore, for any any $x \in V$,

$$\begin{aligned} x^{r_u^\sigma} =& (x^{\sigma^{-1}})^{r_u \sigma} = (x^{\sigma^{-1}} + (x^{\sigma^{-1}}, u)u)^\sigma \\ &= x + (x^{\sigma^{-1}}, u)u^\sigma = x + (x, u^\sigma)u^\sigma = x^{r_{u^\sigma}}. \end{aligned}$$

Since, x is an arbitrary element, the maps r_u^σ and r_{u^σ} are equal. □

7.2 Classes of involutions

Our approach used in this section is motivated by the one used in [2]. However, we slightly deviate from that eventually.

Definition 7.2.1. *Consider an involution $t \in Sp(2n, 2)$. The commutator space $[V, t]$ is*

$$[V, t] = \langle v^t - v \mid v \in V \rangle.$$

Then the rank of t is the dimension of the commutator space $[V, t]$.

Equivalently, the rank of t is the number of blocks of size 2 in the Jordan form of the matrix of t. We define a new bilinear form, $(\ ,\ )_B$ on V from which a quadratic form naturally arises.

Definition 7.2.2. *Let $(\ ,\ )_B : V \times V \to \mathbb{F}_2$ be the map defined as*

$$(u, v)_B := (u, [v, t]) = (u, v^t - v) = (u, v) + (u, v^t),$$

for all $u, v \in V$.

Clearly, $(\ ,\ )_B$ is a bilinear form on V.

Proposition 7.2.3. *The form $(u, v)_B$ is symmetric on V.*

Proof. Since $t \in Sp(2n, 2)$ is an isometry and also an involution, we have

$$(u^t, v) = ((u^t)^t, v^t) = (u, v^t).$$

Therefore,

$$(u, v)_B = (u, v) + (u, v^t) = (v, u) + (u^t, v) = (v, u) + (v, u^t) = (v, u)_B.$$

□

From Proposition 7.2.3, there exists a quadratic form $q(u) = (u, u)_B$ on V. For an involution $t \in Sp(2n, 2)$, define

$$V(t) = \{u \in V \mid q(u) = 0\} = \{u \in V \mid (u, u^t) = 0\}.$$

That is, $V(t)$ is the set of the singular points of quadratic form q.

Proposition 7.2.4. *Suppose $t \in Sp(2n, 2)$ is an involution. Then $V(t)$ is a subspace of codimension at most 1.*

Proof. Define a map $\phi : V \to \mathbb{F}_2$ via $v \mapsto (v, v^t)$. Clearly the map ϕ is linear, since for any $\alpha \in \mathbb{F}_2$ and $u, v \in V$, we have, $\phi(\alpha u) = \alpha^2(u, u^t) = \alpha(u, u^t) = \alpha\phi(u)$ and

$$\begin{aligned}
\phi(u+v) &= (u+v, u^t+v^t) \\
&= (u, u^t) + (u, v^t) + (v, u^t) + (v, v^t) \\
&= (u, u^t) + (u, v^t) + (v^t, u) + (v, v^t) \\
&= (u, u^t) + (v, v^t) = \phi(u) + \phi(v).
\end{aligned}$$

Note that $V(t)$ being the kernel of ϕ is a subspace of V. From rank-nullity, it follows that codimension of $V(t)$ is at most 1. □

Recall that the centraliser of t in V is $C_V(t) = \{v \in V : v^t = v\}$.

Proposition 7.2.5. *Suppose that $t \in Sp(2n, 2)$ is an involution. Then*

$$[V, t] \leq C_V(t) \leq V(t).$$

Furthermore, $C_V(t) = [V, t]^\perp$.

Proof. For $u \in C_V(t)$, $(u, u^t) = (u, u) = 0$. Therefore, $u \in V(t)$ which means $C_V(t) \leq V(t)$.

Let $x \in [V, t]$, then $x = u^t - u$ for some $u \in V$. Then

$$x^t = (u^t - u)^t = (u^t + u)^t = u + u^t = u^t - u = x.$$

Note that, since the field is F_2, $u^t - u = u^t + u$. Therefore, $x \in C_V(t)$. Consequently, $[V, t] \leq C_V(t)$.

To see that $C_V(t) = [V, t]^\perp$, define a map $\psi : C_V(t) \to V$ via $v \mapsto [v, t]$. Since

$$\begin{aligned}\psi(\alpha u + \beta v) &= [\alpha u + \beta v, t] \\ &= (\alpha u + \beta v)^t - (\alpha u + \beta v) = \alpha(u^t - u) + \beta(v^t - v) \\ &= \alpha[u, t] + \beta[v, t] = \alpha\psi(u) + \beta\psi(v),\end{aligned}$$

the map ψ is linear, further note that the $\mathrm{Ker}(\psi) = [V, t]$. Therefore, by rank+nullity, $\dim([V, t])$ is equal to the codimension of $C_V(t)$. On the other hand, if $x \in C_V(t)$, then for any $u \in V$,

$$(x, u^t - u) = (x, u^t) - (x, u) = (x^t, u^t) - (x, u) = (x, u) - (x, u) = 0.$$

That is, $C_V(t) \leq [V, t]^\perp$. On combining both the arguments, it follows that, $C_V(t) = [V, t]^\perp$. □

A subspace $W \leq V$ is said to be totally isotropic if $(u, v) = 0$ for all $u, v \in W$.

Corollary 7.2.6. *Let $t \in Sp(2n, 2)$ be an involution. Then rank $(t) \leq n$.*

Proof. Let U be a maximal totally isotropic subspace of V. Then $\dim U = n$. We show that $[V, t]$ is totally isotropic, consequently, it follows that $\dim [V, t] \leq n$. Consider $x, y \in [V, t]$. Then $y \in [V, t] \leq C_V(t) = [V, t]^\perp$. Therefore, $(x, y) = 0$. Since, $x, y \in [V, t]$ are arbitrary, $[V, t]$ is totally isotropic. □

In what follows, we prove some of the results related to hyperplanes and restrictions on the dimensions of totally isotropic subspaces of V. These will be used to describe the classes of involutions.

Proposition 7.2.7. *Let U be a totally isotropic subspace of V. Consider the map τ_U defined as*

$$\tau_U = \prod_{0 \neq u \in U} r_u.$$

The map $\tau_U = 1$ if $\dim U \geq 3$. Furthermore, τ_U is an involution when $\dim U \leq 2$.

Proof. First we claim that for a hyperplane W of U, $\sum_{u \in U \setminus W} u = 0$ whenever $\dim U \geq 3$. Indeed, the complement $U \setminus W$ is a disjoint union of cosets $\{u, w+u\}$ of $\langle w \rangle = \{0, w\}$ for any fixed $w \in W \setminus \{0\}$. The sum of the two vectors in the coset is w. If $\dim U = k$ then there are 2^{k-2} cosets $\{u, w+u\}$. Therefore, the above sum is,

$$\sum_{u \in U \setminus W} u = 2^{k-2} w.$$

Since U is defined over $\mathbb{F}_2$, $2^{k-2} = 0$ unless $k \leq 2$.

Now we prove the lemma. Consider an element $x \in V$. If $x \in U^\perp$ then $x^{\tau_U} = x$, therefore, the assertion is true in this case. Now assume that $x \in V \setminus U^\perp$. Let $T = U \cap x^\perp$. Note that T is a hyperplane of U. The action of τ_U on x is,

$$x^{\tau_U} = x + \sum_{0 \neq u \in U} (x, u) u = x + \sum_{u \in U \setminus T} u,$$

as the inner product $(x, u) = 0$ for all $u \in T$ and $(x, u) = 1$ for all $u \in U \setminus T$.

Suppose that $\dim U = k$, then from the previous paragraph, $\sum_{u \in U \setminus T} u = 0$ whenever $k \geq 3$. In other words, whenever $\dim U \geq 3$, τ_U fixes all $x \in V$, hence, it is the identity.

Let $\dim U = 1$, say, $U = \langle u \rangle$. Then $\tau_U = r_u$, clearly an involution. Assume that $\dim U = 2$, say, $U = \langle u, v \rangle$. Then $\tau_U = r_u r_v r_{u+v}$. Then the action of τ_U on x is

$$x^{\tau_U} = x + (x, u)v + (x, v)u := y$$

Consider

$$y^{\tau_U} = x + (x, u)v + (x, v)u + (x, u)v + (x, v)u = x.$$

Therefore, τ_U is an involution. $\square$

Note that the action of τ_U on x does not depend on the choice of the basis $\{u, v\}$. If $x \in U^{\perp}$ then x is fixed by τ_U. If $x \notin U^{\perp}$ then τ_U maps x to $x + u'$ where u' is the only nonzero vector in U perpendicular to x.

Corollary 7.2.8. *Let U and W be totally isotropic and orthogonal subspaces of V with $\dim U = \dim W = 2$. Then τ_U and τ_W commute.*

Proof. Since $W \subseteq U^{\perp}$, τ_U fixes W. Hence, for any $w \in W$,

$$r_w^{\tau_U} = r_{w^{\tau_U}} = r_w.$$

Since w is an arbitrary element of W, it follows that τ_U and τ_W commute. $\square$

The classes of involutions of classical groups over the field of even characteristic are discussed in [2]. In particular, the authors show the existence of the Suzuki symplectic basis using which the matrix of an involution $t \in Sp(2n, 2)$ and the respective bilinear form are described. From result (7.6) of [2] it follows that,

Theorem 7.2.9. *Let $t_l \in Sp(2n, 2)$ be an involution with rank l. Then exactly one of the following holds.*

1. *l is even and either $V = V(t_l)$ or the codimension of $V(t_l)$ is 1.*

2. *l is odd and the codimension of $V(t_l)$ is 1.*

Now we claim the following results on the classes of involutions.

Theorem 7.2.10. *Let U be a maximal totally isotropic subspace of V with a basis $\{u_1, u_2, \dots u_n\}$. Define $\tau_k = r_{u_1} r_{u_2} \dots r_{u_k}$ for $k = 1, 2, \dots, n$. Then $V \neq V(\tau_k)$, in particular, $V(\tau_k) = (\sum_{i=1}^{k} u_i)^{\perp}$. Furthermore, the rank of τ_k is k.*

Proof. We show $V \neq V(\tau_k)$ by proving $V(\tau_k) = (\sum_{i=1}^{k} u_i)^{\perp}$. Consider $x \in V$, then

$$\begin{aligned}(x, x^{\tau_k}) &= (x,\ x + \sum_{i=1}^{k} (x, u_i) u_i) \\ &= (x, x) + \sum_{i=1}^{k} (x, u_i)^2 = \sum_{i=1}^{k} (x, u_i).\end{aligned}$$

That is, $x \in V(\tau_k)$ if and only if $\sum_{i=1}^{k} (x, u_i) = 0$, or, $(x, \sum_{i=1}^{k} u_i) = 0$ if and only if $x \in (\sum_{i=1}^{k} u_i)^{\perp}$. Hence, $V(\tau_k) = (\sum_{i=1}^{k} u_i)^{\perp}$.

Note that $[V, \tau_k] \leq \langle u_1, u_2, \dots, u_k \rangle$ from Corollary 7.2.6. To see the reverse inclusion, consider $u \in \langle u_1, u_2, \dots, u_k \rangle$, then $u = \sum_{i=1}^{k} \lambda_i u_i$ for some scalars λ_i. Since the set $\{u_1, u_2, \dots u_k\}$ is linearly independent in V and the form $(\ ,\ )$ is nondegenerate in V, there exists $v \in V$ such that $\lambda_i = (v, u_i)$ for all $i \in \{1, 2, \ldots, k\}$. It follows that

$$v^{\tau_k} - v = \sum_{i=1}^{k} (v, u_i) u_i = \sum_{i=1}^{k} \lambda_i u_i = u.$$

That is, $u \in [V, \tau_k]$. Therefore, $[V, \tau_k] = \langle u_1, u_2, \ldots, u_k \rangle$. Consequently, τ_k has the rank k. □

Theorem 7.2.11. *Let U be a maximal totally isotropic subspace of V with a basis $\{u_1, u_2, \ldots u_n\}$ and let $U_i = \langle u_{2i-1}, u_{2i} \rangle$ for $i = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$. Define*

$$\sigma_s = \tau_{U_1} \tau_{U_2} \ldots \tau_{U_s}$$

for $s = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$. Then $V = V(\sigma_s)$ and the rank of σ_s is always even. In particular, rank $(\sigma_s) = 2s$.

Proof. Let $v \in V$. Then for any $s \in \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$, we have,

$$\begin{aligned}
(v, v^{\sigma_s}) = (v,\ v + \sum_{i=1}^{s} (v, u_{2i-1}) u_{2i} + \sum_{i=1}^{s} (v, u_{2i}) u_{2i-1}) \\
(v, v) + (v,\ \sum_{i=1}^{s} (v, u_{2i-1}) u_{2i} + \sum_{i=1}^{s} (v, u_{2i}) u_{2i-1}) \\
= \sum_{i=1}^{s} (v, u_{2i-1})(v, u_{2i}) + \sum_{i=1}^{s} (v, u_{2i})(v, u_{2i-1}) \\
= 2 \sum_{i=1}^{s} (v, u_{2i-1})(v, u_{2i}) = 0,
\end{aligned}$$

which implies $V \subseteq V(\sigma_s)$ for all $s \in \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$. That is $V = V(\sigma_s)$ for all s in the above mentioned range.

In order to determine the rank of σ_s, we first note that $[V, \sigma_s] \leq \langle u_1, u_2, \ldots, u_{2s} \rangle$. Now consider $u \in \bigoplus_{i=1}^{s} U_i$. Let $u = \sum_{i=1}^{2s} \lambda_i u_i$ for some scalars λ_i. As the set $\{u_1, u_2, \ldots, u_{2s}\}$ is linearly independent in V and the form is nondegenerate in V, there exists $v \in V$ such that $\lambda_{2i} = (v, u_{2i-1})$ and $\lambda_{2i-1} = (v, u_{2i})$ for all $i \in \{1, 2, \ldots, s\}$. Therefore,

$$v^{\sigma_s} - v = \sum_{i=1}^{s} (v, u_{2i-1}) u_{2i} + \sum_{i=1}^{s} (v, u_{2i}) u_{2i-1} = \sum_{i=1}^{2s} \lambda_i u_i = u.$$

Thus, $u \in [V, \sigma_s]$ and the rank of σ_s is $2s$. This completes the proof. $\square$

7.3 Fixed sublagebras

Two vectors $u, v \in V$ are said to form a hyperbolic pair if $(u, v) = 1$. In this case, the subspace $\langle u, v \rangle$ is called a hyperbolic line. Note that V can be expressed as the orthogonal direct sum of n hyperbolic lines.

Let D be set of all transvections r_u, $u \in V$. Define a geometry on Γ_D with points being transvections and two transvections r_u, r_v forming an edge if $\langle u, v \rangle$ is a hyperbolic line. Equivalently, $\{r_u, r_v, r_{u+v}\}$ is a line in Γ_D. Since, $Sp(2n, 2)$ is a group of 3-transpositions, Γ_D is a Fischer space. Therefore, the Matsuo algebra $M = M(\Gamma_D, \eta)$ for the group $Sp(2n, 2)$, $\eta \in \mathbb{F}$ and $\eta \neq 0, 1$, is as follows.

For any $r_u, r_v \in D$ the algebra product is

$$r_u r_v = \begin{cases} r_u & \text{if } \langle u \rangle = \langle v \rangle, \\ 0 & \text{if } \langle u \rangle \neq \langle v \rangle,\ (u,v) = 0, \\ \frac{\eta}{2}(r_u + r_v - r_{u+v}) & \text{if } \langle u, v \rangle \text{ is a hyperbolic line,} \end{cases}$$

for all $u, v \in V$.

Note that the underlying field $\mathbb{F}$ of the algebra $M = M(\Gamma_D, \eta)$ is different from the underlying field $\mathbb{F}_2$ of the symplectic space V. In particular, it has odd characteristic.

Theorem 7.3.1. *Let $t \in Sp(2n, 2)$ be an inolution of rank l. The dimension of the axial part of the fixed subalgebra $M_{\langle t \rangle}$ relative to $\langle t \rangle$ is given by*

$$\dim M_{\langle t \rangle} = \frac{1}{2}(2^{2n-l} + 2^{\dim V(t)} - 2),$$

where, $V(t) = \{u \in V : (u, u^t) = 0\}$.

Proof. Consider the action of $\langle t \rangle$ on the set of all 1-spaces $\langle u \rangle$ of V. Since t in an involution, length of any $\langle t \rangle$-orbit is at most 2.

An orbit $\mathcal{O}_1$ of size one consists of those $\langle w \rangle$'s such that $w^t = w$. Therefore total number of orbits of size one is $|C_V(t)|$. Note that $w \in V(t)$.

There are two types of orbits of size two: An orbit $\mathcal{O}_2$ of first type consists of those $\langle u \rangle^t$ such that $(u, u^t) = 0$ (in this case $u \in V(t)$). An orbit of second type consists of those $\langle v \rangle^t$ with $(v, v^t) \neq 0$ (which implies that $v \notin V(t)$).

Thus, all single and double axes of fixed subalgebra essentially come from $V(t)$.

The number of single axes is $|C_V(t)|$. From Corollary 7.2.6, $\dim C_V(t) = 2n - l$ Therefore, $|C_V(t)| = 2^{2n-l} - 1$ which is the number of single axes. Consequently, the number of double axes is

$$\frac{1}{2}(2^{\dim V(t)} - 2^{2n-l}).$$

Finally, the total number of axes of the axial part of the fixed subalgebra relative to $\langle t \rangle$ is

$$\frac{1}{2}(2^{2n-l} + 2^{\dim V(t)} - 2).$$

$\square$

Note that in the above theorem if t is a representative of a conjugacy class D, the class of 3-transpositions that generates the group $G = Sp(2n, 2)$, then the action of $\langle t \rangle$ on the set of all 1-spaces of V will not yield any double axis. Hence, in this case, the fixed subalgebra relative to $\langle t \rangle$ is the Matsuo algebra M for $Sp(2n, 2)$.

We have the isomorphisms, $Sp(2, 2) \cong S_3$ and $Sp(4, 2) \cong S_6$. Therefore, the fixed subalgebras for the above groups relative to $\langle t \rangle$, where t is an involution, are known from the previous chapter.

Chapter 8

Conclusion

In this research work, we have studied axial algebras of Monster type (α, β) satisfying the condition $\alpha = 2\beta$ which is an exceptional case in Rehren's attempt of classifying all the 2-generated subalgebras of Monster type (α, β). In particular, we have shown that the covers of dihedral algebras of the Griess algebra can be defined without imposing the conditions proposed by Rehren.

We also constructed new infinite series of axial algebras of Monster type $(2\eta, \eta)$, namely, the fixed subalgebra of Matsuo algebra for S_n, the axial part of the fixed subalgebra for the symplectic group $Sp(2n, 2)$. And both these algebras are 1-closed.

Based on our observations, we have the following meaningful questions in this direction of research.

Is the maximum dimension of 2-generated subalgebra of an arbitrary axial algebra A of Monster type $(2\eta, \eta)$ is equal to 8? This is true for a Matsuo algebra.

Can the underlying Fischer space of an axial algebra A of Monster type $(2\eta, \eta)$ be recovered. In other words, an axis of Monster type $\mathcal{M}(2\eta, \eta)$ be split into sum of two axes of Jordan type η. Can the simple case to start with in this direction would be algebras which are 1-closed.

We also like to investigate the values of η for which the axial part of the fixed subalgebra for $Sp(2n, 2)$ is not simple. Further in this case, we conjecture that the above subalgebra relative $\langle t \rangle$, where t is an involution and $V \neq V(t)$, has 1-dimensional radical.

The k^2-algebra and the axial part of the fixed algebra for $Sp(2n, 2)$ are constructed for 3-transposition groups of symplectic type. Every plane of the Fischer space of a group of symplectic type is isomorphic to dual affine plane of order two. Therefore, describing a method of constructing the fixed subalgebra of Matsuo algebra for group of symplectic type is an interseting task.

Bibliography

[1] M. Aschbacher, *3-transposition groups*, Cambridge University Press, 1997.

[2] M. Aschbacher, G.M. Seitz, *Involutions in Chevally groups over fields of even order*, Nagoya Math. J. **63**(1976), 1–91.

[3] R. Borcherds, *Vertex algebras, Kac–Moody algebras, and the Monster*, Proc. Natl. Acad. Sci. USA, **83**(1986), 3068–3071.

[4] F. Buekenhout, *La géometrie des groupes de Fischer*, Free University of Brussels, unpublished notes, 1974.

[5] A.M. Cohen, *Coxeter Groups and three Related Topics*, Generators and Relations in Groups and Geometries, **333**(1991).

[6] J.H. Conway, S.P. Norton, *Monstrous moonshine*, Bull. Lon. Math. Soc., **11**(1979), 308–339.

[7] H. Cuypers, J.I. Hall, *The 3-transposition groups with trivial center*, J. Algebra. **178** (1995), 149–193.

[8] B. Fischer, *Finite groups generated by 3-transpositions*, I, Invent. Math. **13** (1971), 232–246.

[9] I. Frenkel, J. Lepowsky, A. Meurman, *A natural representation of the Fischer-Griess Monster with the modular function J as character*, Proc. Natl. Acad. Sci. USA, **81**(1984), 3256–3260.

[10] I. Frenkel, J. Lepowsky, A. Meurman, *Vertex operator algebras and the Monster*, Pure and Applied mathematics, **134**,(1988), Academic Press, Inc.

[11] A. Galt, V. Joshi, A. Mamontov, S. Shpectorov, A. Staroletov, *Double axes and partition subalgebras in Matsao algebras*, manuscript.

[12] C. Godsil, G.F. Royle, *Algebraic graph theory*, Springer-Verlag New York, 2001.

[13] R.L. Griess Jr., *The friendly giant*, Invent. Math. **69**(1982), 1–102.

[14] L.C. Grove, *Classical Groups and Geometric Algebra*, American Mathematical Society, (Graduate Studies in Mathematics, vol 39).

[15] J.I. Hall, *The general theory of 3-transposition groups*, Math. Proc. Camb. Phil. Soc. **114** (1993), 269–294.

[16] J.I. Hall, F. Rehren, S. Shpectorov, *Primitive axial algebras of Jordan type*, J. Algebra, **437** (2015), 79–115.

[17] J.I. Hall, L.H. Soicher, *Presentations of some 3-transposition groups*, Comm. Algebra. **23** (1995), 2517–2559.

[18] J.I. Hall, Y. Segev, S. Shpectorov, Miyamoto involutions in axial algebras of Jordan type half, *Israel J. Math.*, **223** (2018), 261–308.

[19] A.A. Ivanov, D.V. Pasechnik, Á. Seress, S. Shpectorov, *Majorana representations of the symmetric group of degree 4*, J. Algebra, **324** (2010), 2432–2463.

[20] A.A. Ivanov, *The Monster group and Majorana involutions*, Cambridge Tracts in Mathematics, vol. 176, 2009.

[21] S.M.S. Khasraw, J. M^{c}Inroy, S. Shpectorov, *On the structure of axial algebras*, Trans. Amer. Math. Soc. (to appear).

[22] K. McCrimmon, *A Taste of Jordan Algebras*, Springer, New York, 2004.

[23] M. Miyamoto, *Griess algebras and conformal vectors in vertex operator algebras*, J. Algebra, **179** (1996), 523–548.

[24] T.D. Medts, F. Rehren, *Jordan algebras and 3-transposition groups*, J. Algebra, **478** (2017), 318–340.

[25] F.G. Rehren, *Generalised dihedral subalgebras from the Monster*, Trans. Amer. Math. Soc. **369** (2017), 6953–6986.

[26] F.G. Rehren, *Axial algebras*, PhD thesis, 2015.

[27] S. Sakuma, 6-*Transposition property of* τ*-involutions of vertex operator algebras*, Int. Math. Res. Not. IMRN 2007 (2007), rnm030, 19 pp.

[28] R.D. Schafer, *An Introduction to Nonassociative Algebras*, Stillwater, Oklahoma, 1961.

[29] V.G. Tkachev, *The universality of one half in commutative nonassociative algebras with identities*, arXiv:1808.03808v1, 29 pages, Aug 2018.